

Reliability analysis of load-sharing systems with spatial dependence and proximity effects

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ABSTRACT

In the operating process of multi-component systems, components may interact with each other in a way that the load of a failed component is taken up by its nearby components at varying proportions. As a result, the failure rates of the proximate components are accelerated while distant components are not affected. This paper develops a novel spatial model for estimating the reliability of a load-sharing system considering spatial dependence and proximity effects. The spatial model is applicable to systems with heterogeneous or homogeneous components and is suitable for systems with or without distance information. To investigate the importance and significance of the spatial effect, we compare our spatial model with an equal load-sharing through numerical examples. Our results demonstrate that our model is more accurate than standard load sharing models in evaluating system reliability when spatial effects exist.

KEY WORDS: Reliability analysis; load-sharing; multi-component systems; proximity model; spatial dependence

1 Introduction

1.1 Context

Load-sharing dependence occurs when the load imposed on a system is shared by multiple components in the system (Ye et al., 2014). Examples of such systems, referred to as load-sharing systems, include the operating process of cables in a suspension bridge (Suprasad et al., 2008), pipelines used to transport water, oil, or gas (Taghipour and Kassaei, 2015), and water treatment sites (Ye et al., 2014). In a load-sharing system, if a component fails the remaining components share its load and the increased load on the remaining components induces higher failure rates due to the increased load (Kvam and Peña, 2005). The components of a load-sharing system are

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stochastically dependent on each other due to the load-sharing mechanism (Taghipour and Kassaei, 2015). Improper treatment of such dependency can lead to misleading reliability assessment (Kvam and Peña, 2005; Suprasad et al., 2008).

A common feature in a load-sharing system, is that all components in the system comply by a certain set of load-sharing rule by which the load of a failed component is automatically redistributed to the remaining components (Suprasad et al., 2008). Some common load-sharing rules include equal load-sharing (Liu et al., 2016), monotone load-sharing (Suprasad et al., 2008), and local load-sharing rules (Kvam and Peña, 2005). Equal load-sharing rule assumes that the same workload is equally shared by all the remaining components (Liu et al., 2016). Monotone load-sharing rule indicates that the load on the working components is non-decreasing even after the repair of the failed component (Suprasad et al., 2008) while local load-sharing rule implies that the load on a failed component is transferred to adjacent components (Kvam and Peña, 2005). Since the first work on load-sharing systems by Daniels (1945), a considerable number of studies have appeared in literature to investigate load-sharing systems for reliability assessment (Wang et al., 2019; Franco et al., 2020; Zhang et al., 2020), and maintenance optimization (Taghipour and Kassaei, 2015; Wang et al., 2015; Zhang et al., 2017).

Due to the ease of application, numerous studies of load-sharing systems assume that components are identically distributed. Gupta (2002) discussed load-sharing effects on reliability of k -out-of- n : G system. Tang and Wang (2005) studied a load-sharing repairable parallel system with time-varying failure rates. Yun et al. (2012) considered a consecutive k -out-of- n : F system composed of n identical components with exponential failure distribution. Kim and Kvam (2004) proposed a maximum likelihood estimation approach for a load-sharing system with equal and monotone load-sharing rule. The work was further extended by considering a parallel system with Weibull distributed components (Park, 2010). Singh and Gupta (2012) developed a Bayesian estimation approach to estimate the parameters of a load-sharing system where the lifetime of each component follows a Lindley distribution. Zhao et al. (2018) developed a method to estimate parameters of a parallel load-sharing system with continuously degrading components. Asha et al. (2018) proposed a load share model with frailty to explain the dependence between failure times of components in a two-component system. Franco et al. (2020) developed a method to find the maximum likelihood estimates of components of a two-component load sharing system whose lifetime was modelled by generalized Freund bivariate class of distributions. Zhang et al. (2020) developed reliability model of the load-sharing k -out-of- n : F system subject to discrete external load in which the occurrence of the external load follows a non-homogeneous Poisson process.

In practice, components may not be identically distributed. In literature, there are studies considering non-identically distributed components. Yinghui and Jing (2008) studied a k -out-of- n load-sharing system with different components. The components were assumed to be non-repairable with exponentially distributed lifetimes with different failure rates. Huang and Xu (2010) estimated the reliability of a load-sharing system with general life distributions. Jain and Gupta (2012) studied

the reliability of a load-sharing system with common cause failures. Singh et al. (2008) developed a Bayesian method to estimate the parameters of a load-sharing parallel system with non-identical components. They assumed that some of the components have constant failure rates while the others have linearly increasing failure rates.

1.2 Motivating Problem

This paper develops a novel load sharing system with spatial dependence and proximity effect. If a component fails, its neighbours will take up extra load if they are close enough. Each working component's performance depends on its spatial neighbours state, proximity, and the spatial pattern among components. Unlike equal load-sharing structure, where the load of a failed component is taken up by all the remaining working components, in a system with spatial dependence, the load of a failed component is taken up either by its nearest neighbouring components (while non-neighbours operate at their normal rate) (Wang, Yang and Tian, 2017) or there may exist a proximity effect that measures the degree to which components in the system affect each other depending on how close they are (Suprasad et al., 2008).

We present two examples to illustrate this occurrence. First, a water utility has a network of pipes that collectively serve the purpose of delivering water from a treatment site to houses and businesses. An individual failure causes a drop in pressure but does not cause the system to fail. If sufficient failures occur, a sufficient drop in pressure results in failure and an increased load on neighbouring pipes. The distribution of that load is not uniform across the entire network. Instead, neighbouring pipes will pick up that additional pressure, and hence have an increased failure rate. Second, a radar is made up of many, sometimes thousands of, small sensors. If a single sensor fails, the radar can continue to operate. However, the 'load' is shifted to nearby sensors that are operated more intensely. This will cause their failure rate to increase as their operational profile changes.

1.3 Overview of Load Sharing Spatial Models

Durham et al. (1997) and Ibnabdeljalil and Curtin (1997) considered spatial dependence effect in the form of localized load-sharing. Durham et al. (1997) studied the failure mechanism of fibres under five load-sharing rules: equal load-sharing which assumes that load of broken fibre is shared equally by all working components; tapered load-sharing which assumes that when a fibre breaks, one-fourth of its load is distributed to each of its four nearest unbroken neighbour; local load-sharing in which when a fibre breaks, one-half of its load is distributed to its nearest unbroken neighbours; nearest-neighbour load-sharing in which when a fibre breaks, its load is assigned to its nearest unbroken neighbour; and a hybrid load-sharing rule 'which is the same as nearest-neighbour except that when a tie occurs, the extra load is divided equally among the tied neighbours as in the local rule. Ibnabdeljalil and Curtin (1997) studied the reliability and strength of fibre-reinforced

composites under a local load-sharing condition in which they assumed that stress from broken fibres is transferred predominantly to the nearby unbroken fibres. Durham et al. (1997) and Ibnabdeljalil and Curtin (1997) both used monte carlo simulation for system reliability analysis. Although the studies by Ibnabdeljalil and Curtin (1997), and Durham et al. (1997) were first to consider load-sharing system with spatial dependence their models were developed for specific application to fibres. Wang and Si (2014), Wang, Tian and Pei (2017), and Wang, Yang and Tian (2017) proposed markov models to study the availability of markov repairable load-sharing systems with spatial dependence and considered circular, star, and lattice system structures. They assumed that if a component fails, its left and right neighbours will detect the change and take up its load. There is no clear mathematical method that captures the effect of each load change on the failure rate of a working spatial neighbour rather deterministic failure rate values were assumed. Recently, Guo et al. (2020) developed a reliability model for a system with local load-sharing whose components are subject to degradation and external shocks. Both papers considered adjacent neighbour interaction. They assumed that the extra load added to distant components are very small and can be ignored.

Most of the recent studies on load sharing systems focus on the determination of reliability or availability function, statistical inference, and maintenance of systems with equal load-sharing, very few are focused on local-load sharing. Some recent works have been conducted on systems with equal load sharing (Zhang et al., 2017; Zhao et al., 2018; Asha et al., 2018; Wang et al., 2019; Zhang et al., 2019; De Paula et al., 2019; Franco et al., 2020; Zhang et al., 2020; Sharifi et al., 2021). In contrast, only a handful of recent works have looked at reliability study on local load-sharing problems (Guo et al., 2020).

1.4 Rationale of Proposed Load Sharing Spatial Model

Local load-sharing models have been studied in the context of systems with spatial dependence, however, these works focused on local load-sharing without considering proximity effect (Wang and Si, 2014; Wang, Tian and Pei, 2017; Wang, Yang and Tian, 2017; Guo et al., 2020). In addition, none of these works considered studying the consequence of ignoring spatial effect if it exists. Furthermore, some of the papers either do not have generic models (e.g., Durham et al. (1997); Ibnabdeljalil and Curtin (1997)) or have no clear mathematical method that captures the effect of each load change on the failure rate of a working spatial neighbour (Wang and Si, 2014; Wang, Tian and Pei, 2017; Wang, Yang and Tian, 2017). The contribution of this paper is two fold. First a generic model for estimating the reliability of a load-sharing system with spatial dependent components is developed. Second, using the developed model, the importance and significance of the spatial effect is studied. The proposed model captures the load-life relationship of a component considering spatial effect. In the setup of the present paper, the proximity effect in local load-sharing is considered. That is, we consider the case that the redistribution of a failed component's load depends on proximity to the failed component.

An extension of the capacity flow model is used to characterize the relationship between the failure rate of a component and the load imposed on the component. A Markov model is used to characterize the deterioration process of the entire system. Markov model evaluates the probability of jumping from one known state into the next logical state. The process continues until the system being considered has reached the final failed state or until a particular mission time is achieved (Ambani et al., 2009). In particular, Markov model have been found to be useful and suitable for modelling the distribution of time to failure of load sharing systems in various papers (Wang and Si, 2014; Xiao et al., 2016; Wang, Tian and Pei, 2017; Wang, Yang and Tian, 2017; De Paula et al., 2019; Sharifi et al., 2021). A modified Euler’s method is employed to derive the system state probabilities and the associated system reliability. Finally, the developed reliability model is illustrated using numerical examples.

The remainder of the paper is outlined as follows. Section 2 describes the load-sharing system and presents the assumptions. Section 3 presents the spatial dependence and proximity model. Section 4 analyses the system state transition model when the components of the system are homogeneous and heterogeneous respectively. Two numerical examples are presented in Sections 5 and 6 to illustrate the model. We use simulation to model two different systems, introducing spatial dependency for two purposes. First, we assess the accuracy of our developed model against the simulated output. Second, we compare our developed model with existing literature to determine the impact of not capturing spatial dependency. Finally, Section 7 presents the concluding remarks and future research directions.

2 System description and load-sharing rule

Consider a load-sharing system that consists of n components connected in parallel. The following assumptions are made to better position our study:

- Each component can either be in a working or failed state while the system is multi-state and can function at different performance levels depending on the states of its components.
- The lifetimes of the components are load dependant and follow exponential distributions.
- The system fails if the sum of the loads on each working component at time t is less than the total system load L .
- When a component fails, its working spatial neighbours in any direction (as long as they are in close proximity) can take up the failed component’s load.

Exponential lifetime distribution is assumed primarily for two reasons. First, the assumption of an exponential distribution for component lifetime modelling fits the Markov model whose conditional distribution of a future state is independent of the past states and the sojourn time of

each state is exponentially distributed. Use of exponential lifetime distribution facilitates model establishment and reliability calculation. In addition, in the literature, exponential distribution has been widely used to describe the failure process of components in a load-sharing system, e.g., computing systems (Xiao et al., 2016), intelligent air condition systems (Wang, Tian and Pei, 2017), and water transition systems (De Paula et al., 2019; Sharifi et al., 2021).

The components are spatial dependent and the system structure is known beforehand. The number of spatial neighbours a component has is determined by the number of links with other components. The system operates such that at time $t = 0$ the total load L of the system is shared by all the components. The components share the constant system load L in the proportion $\gamma_1, \gamma_2, \dots, \gamma_n$, where γ_i is the proportion taken by component i at time $t = 0$ and n is the number of components in the system. If the components equally share the load, then we can simply let $\gamma_i = \gamma_j$ for any $1 \leq i, j \leq n$. Let i denote a working component, and j denote the index of a failed neighbour of component i . The load z_i taken by component i at initial time $t = 0$ is given by (Yinghui and Jing, 2008):

$$z_i = \frac{\gamma_i L}{\sum_{i=1}^n \gamma_i} \quad (1)$$

Denote $z_i(j)$ as load on component i given that its neighbour component j has failed. We have

$$z_i(j) = z_i + \vartheta_{ij} z_j \quad (2)$$

where ϑ_{ij} denotes the ratio of a failed component's load that affects the failure rate of a working adjacent component. It can be viewed as the proportion of a failed component's load that its working spatial neighbour will take on. The proportion of extra load is a function of whether or not a failed component is proximate to a working component. ϑ_{ij} takes values between 0 and 1 as a result. If $\vartheta_{ij} = 0$ a working component is not impacted by the failure of component j . We will refer to ϑ_{ij} as the *proximity effect* and refer to the expression $z_i(j)$ as the *interacting load function*.

Similarly, the load on component i after the failure of its second neighbour k , $z_i(j,k)$, is given by

$$z_i(j,k) = z_i(j) + \vartheta_{ik} z_k$$

The system fails when the sum of loads on the working components is less than the load placed on the system, i.e., $\sum_{i=1}^n z_i < L$.

Remark 1. *We focus on the parallel systems because parallel structures are more common in practice for load-sharing systems. However, the parallel system considered here is different from traditional parallel systems. Traditional parallel systems fail when all components fail, while the system under investigation here fails when the system cannot bear a specified load. As such, our system is more analogous to a performance-based system that fails when it cannot sustain its performance (load). As reported in Krivtsov et al. (2018), load-sharing systems in series configuration*

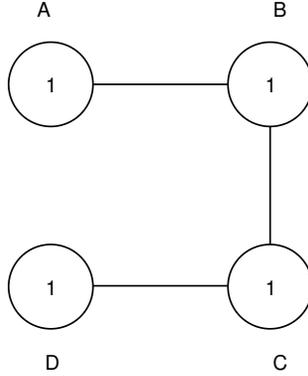


Figure 1: Visual depiction of four component system

also exist although are not as popular as parallel structures. As a series system fails when any of the components fails, it is unnecessary to consider the spatial dependence and proximity effects to evaluate system reliability. We could, however, model the spatial effect on the load distribution among the components in a series system. Different from the equal load sharing rule, the load allocated to each component could vary due to the spatial effect. Research could be conducted to answer interesting decision-making questions such as “the optimal number of components in the series system” and “optimal location for each component”, etc. In addition, it would be interesting to investigate the spatial dependence and proximity effects for a series-parallel system, where the spatial dependence can be considered in the parallel subsystems.

3 Modelling spatial dependence and proximity effect

In order to account for components’ interaction with each other, we will define spatial dependence in terms of a given system structure. We assume that if we know the spatial arrangement of components and connections between them, we may be able to infer their dependency. We introduce a conceptual dependence model to account for spatial dependence between components. The dependence model is based on the assumption that pairs of components with a direct link (in the form of solid lines) between them are close enough to interact while pairs of components without a link (i.e, have no line) are independent even if they are positioned next to each other.

To illustrate the dependence concept, we consider the four-component system depicted in Figure 1. Note that the system is in a parallel structure, and the links between the components indicate proximity of the components. The system is composed of four components indexed as A, B, C and D. Components A, B, and C have solid lines between them while A and D have no line indicating that even though components A and D are spatial neighbours in terms of their position next to each other, they do not influence each other. By definition, component B is spatially dependent on A and C while component A is only spatially dependent on component B. Component C is

spatially dependent on component B and D while component D is only spatially dependent on component C. While the performance of components B and C could be influenced by its two spatial neighbours, components A and D's performance is only influenced by one spatial neighbour. In contrast, component pair A and D in the two structures are not spatially dependent so if one of them is in failed state, the other one's reliability is not affected.

3.1 Modelling the proximity effect

In this section, we will assume that the spatial arrangement (spatial pattern) of components in the system is known and that there is no information about the distance between components. In section 3.2 we will develop proximity models assuming that we know the distance between components. We assume that one can derive the proximity effect (which describes the proportion of load that a working proximate component takes from a failed component) between components using their spatial arrangement and the link between them (that is, the spatial dependence). An example of a load-sharing system that the method could be applied to is an intelligent air conditioning system. In the literature, the lattice, star and circular structures of the intelligent air conditioning has been used to infer spatial dependence and load distribution between components (see (Wang and Si, 2014; Wang, Tian and Pei, 2017; Wang, Yang and Tian, 2017)). Let θ_{ij} represent spatial dependence of two components such that if they have a direct link $\theta_{ij} = 1$ otherwise $\theta_{ij} = 0$. We will introduce a dependence matrix to capture each θ_{ij} .

Let us assume that θ_{ij} is updated according to the state of the components then at time $t = 0$ when all components are working, the dependence matrix $[\theta_{ij}]$ will be

$$[\theta_{ij}] = \begin{bmatrix} \theta_{11} & \theta_{12} & \dots \\ \vdots & \ddots & \\ \theta_{n1} & & \theta_{nn} \end{bmatrix}$$

where the j^{th} column of matrix $[\theta_{ij}]$ denotes the index of a failed component while the i^{th} row is the index of its working neighbours.

In order to ensure that the sum of proportion of the j^{th} failed component's load taken up by all its working neighbours at time t sum up to one, the elements of each column in matrix $[\theta_{ij}]$ will be column normalized. If we assume that the j^{th} column of $[\vartheta_{ij}]$ represents the proportion of its load that a failed component j will transfer to its working proximate neighbours i at time t , then the normalization equation will be:

$$\vartheta_{ij} = \frac{\theta_{ij}}{\sum_{i=1}^n \theta_{ij}} \quad (3)$$

After column normalizing, we would derive a new matrix $[\vartheta_{ij}]$ with normalized values ϑ_{ij} which represents the proximity effect, i.e.,

$$[\vartheta_{ij}] = \begin{bmatrix} \vartheta_{11} & \vartheta_{12} & \dots \\ \vdots & \ddots & \\ \vartheta_{n1} & & \vartheta_{nn} \end{bmatrix}$$

where $\vartheta_{ii} = 0$ and ϑ_{ij} is non-negative and the values in each column have unit sum, i.e., $\sum_{i=1}^n \vartheta_{ij} = 1$, for any $j = 1, \dots, n$.

3.2 Proximity models with distance information

For load-sharing systems with spatial dependent components, Suprasad et al. (2008) defined the load-sharing rule as a rule in which the load on a failed component is transferred to proximate components, and the proportion of the load that the working components inherit depends on their distance to the failed component. They mentioned that examples of this kind of systems include cables supporting bridges and other structures, composite materials with bounding matrix joints, and transmission. Following the load distribution rule, a working component will be affected by its failed neighbour j with an increased load $z_{i(j)}$ as

$$z_{i(j)} = z_0(1 + p_{ij}) \quad (4)$$

where p_{ij} describes how much a failed component's load affects its working proximate neighbours. If $p_{ij} = 0$, it means two components are not close enough to have load-sharing interaction whereas $p_{ij} \leq 1$ represents the degree to which two proximate component affect each other's performance.

In this section we consider the case that there is information about the distance d_{ij} between components and one can derive the proximity values p_{ij} which would describe how much a failed component's load affects its working proximate neighbours. According to Suprasad et al. (2008), the proportion of the load inherited by the working component depends on their distance to the failed component thus we will assume that p_{ij} is a function of the distance d_{ij} between components. We will introduce two distance decay spatial weights matrix methods applied in geostatistics for deriving the proximity values. We shall refer to the distance decay spatial weights methods as *constant proximity model* and an *exponential proximity model*. The methods are based on the idea that areas closer to an area of interest have more of an influence than those further away and thus are weighted as such in (Chen, 2012).

Let us assume that one can represent all the distances between components as elements of a distance matrix $[D_{ij}]$ given as

$$[D_{ij}] = \begin{bmatrix} d_{11} & d_{12} & \dots \\ \vdots & \ddots & \\ d_{n1} & & d_{nn} \end{bmatrix}$$

where $d_{11} = d_{22} = \dots = d_{nn} = 0$. Let us assume that distance is an important criterion of the degree of influence between components and that there is a threshold distance beyond which there

is no influence between components. Let α denote the threshold distance. First, we introduce the *constant proximity model* which assumes that all component pairs within a distance threshold α have the same proximity effect p_{ij} regardless of their distance apart while component pairs with distance more than the threshold do not interact (Chen, 2012). Each p_{ij} is derived by:

$$p_{ij} = \begin{cases} c, & 0 \leq d_{ij} \leq \alpha \\ 0, & d_{ij} > \alpha \end{cases} \quad (5)$$

where c is a predefined constant value and is the same for all pair of components within the threshold distance α . All other components outside the threshold distance have $p_{ij} = 0$.

In contrast, the exponential proximity model assumes a diminishing proximity effect which reduces as the distance between components increase up to the threshold α . The exponential proximity model could find application to a cable-strut system in suspension bridge where the booms far from the failed one are only subjected to the indirect force and the transfer effect on them is less compared with those on the proximate booms (Guo et al., 2020). The exponential proximity model unlike the constant proximity model allows for variability of the p_{ij} . The exponential proximity model is given by

$$\hat{p}_{ij} = e^{-\frac{d_{ij}}{\alpha}} \quad (6)$$

where α is the threshold and d_{ij} is the distance between component i and j . If we take the limit of the exponential proximity model as d_{ij} is close to zero i.e.,

$$\lim_{d_{ij} \rightarrow 0} e^{-\frac{d_{ij}}{\alpha}} = 1 \quad (7)$$

As the distance between two components is reduced to zero, the influence component j can have on a neighbour i increases to one. In contrast, if we take the limit of the exponential proximity model as d_{ij} tends to ∞ , i.e.,

$$\lim_{d_{ij} \rightarrow \infty} e^{-\frac{d_{ij}}{\alpha}} = 0 \quad (8)$$

In the operation of industrial systems in real-life, highly distant components would barely interact directly. As a result, we set every element $\hat{p}_{ij} = 0$ for all ($d_{ij} > \alpha$). In this work, we will consider the exponential model. We also assume that the threshold α can be derived as the average of pairs of distances in the system (without repetition) and $\alpha = \frac{\sum d_{ij}}{l}$. l is the number of inter-component links and $d_{ij} = d_{ji}$. α ensures that all components j whose distances d_{ij} from a component i less than or equal to mean distance ($d_{ij} \leq \alpha$) are influenced by i while components j with distances greater than the mean distance ($d_{ij} > \alpha$) are not affected. α is calculated by taking an average of the upper or lower triangular matrix instead of the entire matrix as the distance values will be doubled.

After deriving each proximity value \hat{p}_{ij} using an exponential proximity model, we represent the proximity effects between components of a system as elements of a proximity matrix

$$[\hat{P}_{ij}] = \begin{bmatrix} 1 & \hat{p}_{12} & \dots \\ \vdots & \ddots & \\ \hat{p}_{n1} & & 1 \end{bmatrix}$$

The diagonal elements \hat{p}_{ii} of the $[\hat{P}_{ij}]$ matrix contains unit values indicating that a component is in close proximity with itself. If the unit values of each \hat{p}_{ii} elements are left, they would affect how the load of a failed component is shared by its working neighbours. In order to remove the unit values, we will subtract unit matrix from $[\hat{P}_{ij}]$ so that we have:

$$[P_{ij}^*] = \begin{bmatrix} 1 & \hat{p}_{12} & \dots \\ \vdots & \ddots & \\ \hat{p}_{n1} & & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & \dots \\ \vdots & \ddots & \\ 0 & & 1 \end{bmatrix} = \begin{bmatrix} 0 & p_{12}^* & \dots \\ \vdots & \ddots & \\ p_{n1}^* & & 0 \end{bmatrix}$$

Similar to θ_{ij} in section 3.1, p_{ij}^* functions according to the state of the components and the matrix $[P_{ij}^*]$ provides proximity information at time $t = 0$ when all components are working.

Similar to $[\theta_{ij}]$, the proximity elements of the $[P_{ij}^*]$ matrix will be column normalized, assuming that the j^{th} column of $[P_{ij}^*]$ contains all proximity effects of neighbours influenced by the failed state of component j . The normalization equation is given by:

$$p_{ij} = \frac{p_{ij}^*}{\sum_{i=1}^n p_{ij}^*} \quad (9)$$

After column normalizing, we derive a new matrix $[P_{ij}]$ with normalized proximity values of p_{ij} . The proximity values in each column are normalized to have unit sum, i.e.,

$$\sum_{i=1}^n p_{ij} = 1, j = 1, \dots, n \quad (10)$$

where $p_{ii} = 0$ and p_{ij} is non-negative.

3.3 A simple example: derivation of load function for a multi-component system

We use a multi-component system to illustrate the derivation of the load functions shown in Eq 4. Let us consider a system that consists of n components connected in parallel. Without generality, consider the case that the components share the constant system load L in the proportion $\gamma_1, \gamma_2, \dots, \gamma_n$, where γ_i is the proportion taken by component i at time $t = 0$. From Eq 1 the system load is derived as:

$$L = \frac{\sum_{i=1}^n \gamma_i}{\gamma_i} z_i \quad (11)$$

where z_i is the load taken by component i at initial time $t = 0$. Let $\gamma_i = \gamma_j$ for any $1 \leq i, j \leq n$ then for $t \geq 0$ the system load can be rewritten as:

$$L = \sum_{i=1}^{n-j} z_{i(j)} = z_{i(j)} \times (n - j) \quad (12)$$

where i denotes a working component, and j denotes the number of failed components in the system. $z_{i(j)}$ is the load of the i^{th} working component after j number of components have failed. Eq 12 is the system load when we consider that the components equally share the system load. When $j = 0$ Eq 12 and Eq 11 are equal.

If we consider that an increased load on each working component i for $i = 1, 2, \dots, n - j$ is a function of both its own independent load z_i and the load of a failed component z_j then $z_{i(j)}$ can be written as:

$$z_{i(j)} = z_i + l_j \quad (13)$$

where l_j is the proportion of the load of the j^{th} component taken up by component i . l_j is given by:

$$l_j = \frac{\gamma_i}{\sum_{i \neq j} \gamma_i} z_j \quad (14)$$

When the first component failure occurs, $z_{i(j)}$ equals:

$$z_{i(1)} = z_i + l_1$$

When the second component failure occurs, $z_{i(j)}$ will become:

$$z_{i(2)} = z_{i(1)} + l_2 = z_i + l_1 + l_2$$

After the j^{th} component failure has occurred, $z_{i(j)}$ can be written as:

$$\begin{aligned} z_{i(j)} &= z_{i(j-1)} + l_j = z_i + \sum_j l_j \\ &= z_i + \frac{\gamma_i}{\sum_{i \neq j} \gamma_i} \sum_j z_j \end{aligned} \quad (15)$$

Let $e_{ij} = \frac{\gamma_i}{\sum_{i \neq j} \gamma_i}$ be the proportion of the load of component j taken by component i . From Eq 15, e_{ij} does not have a spatial element. However, consider the load-sharing rule such that an increased load on a component is a function of both its own independent load z_i and the load of its influencing proximate component z_j . To introduce spatial dependence between components in $z_{i(j)}$ we set $e_{ij} = p_{ij}$ if distance information is known otherwise $e_{ij} = \vartheta_{ij}$.

4 Formulation of system state transition

4.1 Failure rate function for homogeneous components

When a component fails, the load of the failed component is added to the proximate components and causes an increase in their failure rates. In this paper, the relation between component i and the load of its failed proximate neighbour j is defined as follows:

$$\lambda_{i(j^r)} = \begin{cases} \lambda_0, & \text{if } j = 0, \\ \lambda_0(z_{i(j)})^\beta = \lambda_0(z_0 + \vartheta_{ij}z_0)^\beta, & j \neq 0, \end{cases} \quad (16)$$

where r is the number of working neighbours of component j that shares its load. z_0 represents the load imposed on components i and j when $z_i = z_j$. λ_0 is the baseline failure rate (also called hazard rate) function. We use the power beta function as it has been widely used in the literature to account for the effect of load on a component failure rate (see for example, Suprasad et al. (2008); Krivtsov et al. (2018); Sharifi et al. (2021)). In this paper, we aim to highlight the importance of spatial effect and therefore employ the commonly used power beta function for illustration. If the failure mechanism is known or real data is available to fit the associated load functions, a more sophisticated load function can be used instead. Actually, our model is generic that can be easily adapted if alternative functions are used to describe the effect of load on the failure rate. Whenever a working neighbour of component j fails after component j has failed, the load of component j is redistributed to the remaining working neighbours and r changes. We assume that the initial load imposed on each working component i cannot be less than 1. Otherwise, the load shared between components will have a reducing effect on a working component's failure rate. β is the load factor which describes the influence of the increased load on the failure rate of a working component. When $\beta = 0$, an increased load has no impact on a component's failure rate. The higher β is, the higher the impact of load on the working neighbours; thus, the working neighbours have higher failure rates. At time $t = 0$, we assume that all the components have the same initial failure rate given as $\lambda_i = \lambda_0$. Obviously we have $\lambda_i \leq \lambda_{i(j^r)}$.

To illustrate the load-sharing concept, let us consider a four-component system whose components are indexed by A, B, C, and D as seen in Figure 2. Using the spatial dependence concept introduced in section 3 component pairs (A, B), (A, D), (B, C), and (C, D) are spatially dependent while (A, C) and (B, D) are independent. If component B first failed at time t_1 then its two working neighbours component A and C will have increased loads of $z_{A(B)}$ and $z_{C(B)}$ given by:

$$z_{A(B)} = z_0 + \vartheta_{AB}z_0$$

and

$$z_{C(B)} = z_0 + \vartheta_{CB}z_0$$

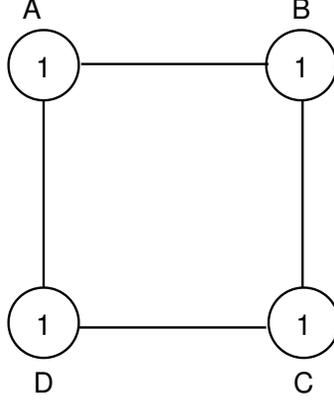


Figure 2: The four component system structure

Their failure rate will become $\lambda_{A(B^2)} = (z_0 + \vartheta_{AB}z_0)^\beta \lambda_0$ and $\lambda_{C(B^2)} = (z_0 + \vartheta_{CB}z_0)^\beta \lambda_0$ respectively.

On the other hand, component D will remain as z_0 with failure rate λ_D since it is not spatially dependent on component B.

If one of components A or C fails at time t_2 given that component B has failed, the dependence matrix is updated and the only working neighbour of B will take all the load of B. The load and failure rate of the working component will be:

$$z_{i(B)} = z_0 + \vartheta_{i(B)}z_0$$

and

$$\lambda_{i(B^1)} = (z_0 + \vartheta_{i(B)}z_0)^\beta \lambda_0$$

where i indicates component A or C, and $\vartheta_{i(B)} = 1$.

However, if components A and C are still working and both have two neighbours in failed state, say component D has failed at time t_2 after component B failed, then components A and C will have increased load and failure rate functions of:

$$z_{i(B,D)} = z_{i(B)} + \vartheta_{iD}z_D = z_0 + \vartheta_{iB}z_0 + \vartheta_{iD}z_0$$

and $\lambda_{i(B^2,D^2)} = (z_0 + z_0\vartheta_{iB} + z_0\vartheta_{iD})^\beta \lambda_0$, where i denotes node A and C.

On the other hand, if we assume that there is information about the distance d_{ij} between components. Then given the distance values d_{ij} between each pair of components one can derive proximity values p_{ij} and the failure rate in Eq 16 will become:

$$\lambda_{i(j^r)} = (z_0 + p_{ij}z_0)^\beta \lambda_0$$

4.2 System state transition with homogeneous components

In order to describe the behaviour of the load-sharing system with proximity and spatial dependence, we will apply the aforementioned proximity concept and failure rate model in a Markov model. The proximity model will be used to infer proximity between components, while the failure rate model will be used to account for effect of component load on the transition rates in the Markov model. Markov models are frequently used in reliability and maintainability work where events, such as the failure or repair of a component, can occur at any point in time. The Markov model evaluates the probability of jumping from one known state into the next logical state. The process continues until the system being considered has reached the final failed state or until a particular mission time is achieved. The model assumes that the conditional distribution of a future state is independent of the past states of the process i.e., the behaviour of the system in each state is memoryless (Ambani et al., 2009). Thus, the sojourn time of each state is exponentially distributed and the transition probability to each state is independent of the process history. An advantage of Markov models is that they are simple to generate even though they require a complicated mathematical approach.

To apply the Markov model, we will construct a state diagram of load-sharing system representing all possible states of the system. The transition from one state to another will be specified by an arrow whose direction indicates the direction of transition. The failure rate expressions in section 4.1 will denote the rate parameter of the transition from one state to another.

Consider the four-component system in Figure 2. Consider that the system's state can be in any one of a discrete set of states S_0, S_1, \dots, S_5 at time t . Let each system's state S_i for $i = 0, 1, \dots, 5$ be described by the vector of its components' states. The definitions of state S_i are given as follows (see Figure 3). Let a vector of zeros and ones represent the states of components in a system such that a working component is denoted by 1 and a failed component is denoted by 0. If we let the vector also denote the arrangement of components in the system and assume that the order of components in the system is a function of the first component to fail. Therefore (1,1,1,1) will denote that all components in the system are working. If one component fails, say component A first fails, then (0,1,1,1) will denote that component A is in failed state while components B, C, and D are working. Otherwise if any other component failed first, say component C, then (0,1,1,1) will mean that the working components are D, A and B in that order. If two components fail, say components A and B, then (0,0,1,1) denotes that the two proximate components are in failed state. If two non spatially dependent components are in failed state, say components A and C, then (0,1,0,1) will denote failed state of the two components. (0,0,0,1) denotes that three spatially dependent components A, B and C are in failed state while component D is still working. (0,0,0,0) denotes that all components have failed. For other forms of system states, (0,1,0,1) and (1,0,1,0) indicates that two components are in failed state and there is a working component between the two failed components. (0,1,1,0) denotes that two proximate working components each have one

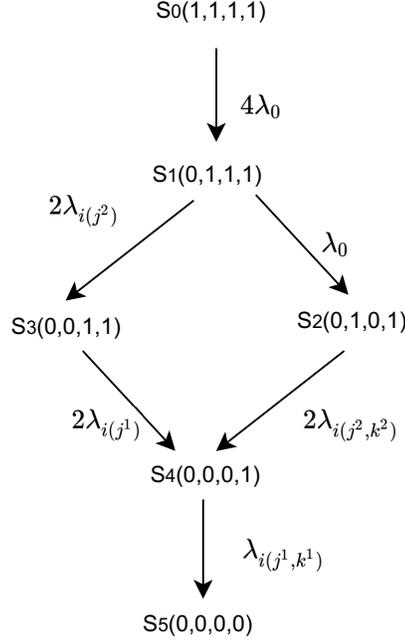


Figure 3: State transitions for the four-component system with homogeneous components

failed neighbour.

The four-component system's evolution is determined by the transitions among states. State transitions of the system goes as follows. The transition from state $(0,1,1,1)$ to state $(0,0,1,1)$ means that one of the two working components that is spatially dependent on the failed component has now failed. The transition from state $(0,1,1,1)$ to state $(0,1,0,1)$ means that one working component which is not spatially dependent on the failed component has now failed. If a component has one failed spatial dependent neighbour, its transition rate will become $\lambda_{i(j^r)}$. So the transition from state $(0,1,1,1)$ to state $(0,0,1,1)$ will have a transition rate of $\lambda_{i(j^2)}$. For example, since component B is spatially dependent on component A then the transition from state $(0,1,1,1)$ to $(0,0,1,1)$ will have a transition rate of $\lambda_{i(j^2)}$ while the transition from state $(0,1,1,1)$ to state $(0,1,0,1)$ would have a transition rate of λ_0 because component C is not spatially dependent on component A. If we consider a transition from state $(0,1,0,1)$ to state $(0,0,0,1)$, it shows that the working component between two failed neighbours has now failed. Since the component has two failed neighbours, its transition rate will be $\lambda_{i(j^2,k^2)}$. Transition from state $(0,1,1,0)$ to state $(0,0,1,0)$ denotes that one of two working components with a failed neighbour has failed with transition rate $\lambda_{i(j^1)}$.

The states where the sum of loads on the working components are less than the load placed on the system (i.e. $\sum_{i=1}^n z_i < L$) are system failure states. For instance, if we consider transition from state $(0,1,1,0)$ to state $(0,0,1,0)$. The transition denotes that one of the last two working components in the system has failed. Thus, the last working component will take up all the load of its two proximate neighbour and the load of all other failed components will be lost. Due to the

load distribution, the total load on the working component in state $(0,0,1,0)$ will be less than the required and the system fails.

4.3 Failure rate function for heterogeneous components

This section considers the load-sharing system with heterogeneous components, which is very common in reality. An example of a system with non-identically distributed components is a tri-engine airplane which can operate when at least both of its wing engines or its central engine are working. The wing engines and the central engine are not necessarily identical (Taghipour and Kassaei, 2015). A representative sample of such study is Yinghui and Jing (2008) that suggested a new model for load-sharing k -out-of- n : G system with different component. The model was developed as an extension of the capacity flow model. The model describes the increase of the component's failure rates under different loads. Jain and Gupta (2012) extended the model by introducing common cause failure. An example can be observed in a load-sharing system with identical types of components, but having a mix of old (used) and new components. This section extends the work of Yinghui and Jing (2008) and Jain and Gupta (2012) that both studied systems with equal load-sharing and heterogeneous components by considering spatial dependence and proximity effects in load-sharing systems. Thus we extend the load-sharing model developed in the previous sections by considering systems with heterogeneous components.

Assume that there is no information about the distance between components and one cannot estimate how close the components are to each other. Whilst other assumptions are maintained, we assume that the life times of the components are load dependant and follow the non-identical exponential distributions. In this case, at time $t = 0$ all components have the different initial failure rates given by

$$\lambda_i = \lambda_i^* \quad (17)$$

where $\lambda_i \neq \lambda_j$ for i and $j = 1, 2, \dots, n$.

If we assume for example that all the components in the system take up varied load proportions z_i derived from Eq 1, then when one component fails, its neighbours will have an increased failure rate of

$$\lambda_{i(j^r)} = (z_{i(j)})^\beta \lambda_i^* = (z_i + \vartheta_{ij} z_j)^\beta \lambda_i^* \quad (18)$$

where $\lambda_i < \lambda_{i(j^r)}$.

When the second neighbour of a working component fails, the working component will have an increased load of $\lambda_{i(j^r, k^r)}$ given by

$$\lambda_{i(j^r, k^r)} = (z_{i(j)} + \vartheta_{ik} z_k)^\beta \lambda_i^* \quad (19)$$

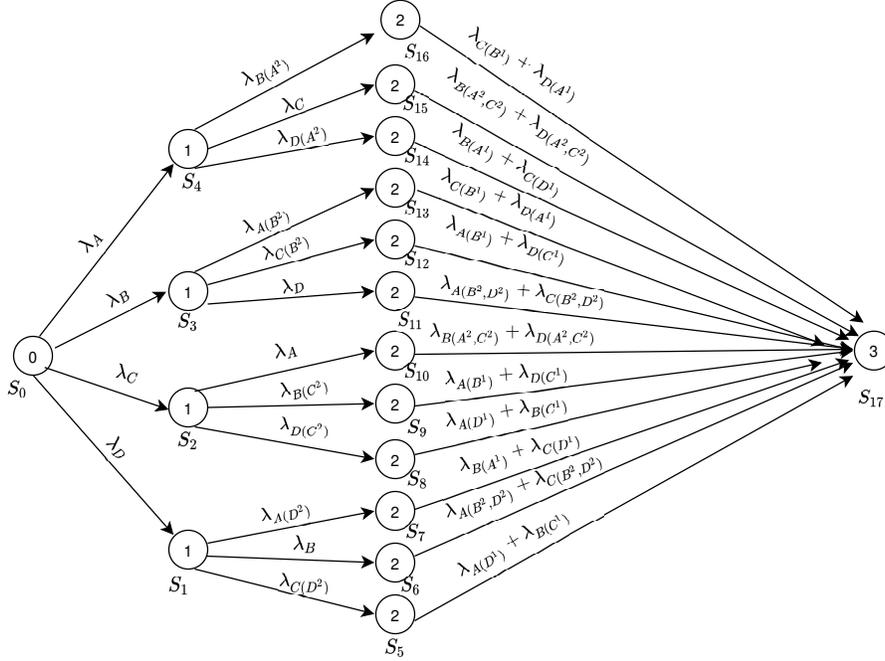


Figure 4: State transitions for the four-component system with heterogeneous components.

4.4 System state transition with heterogeneous components

In this section, we will formulate a method of accounting for the states of the system with heterogeneous components using Markov model. We will then integrate the formulated failure rates into the Markov model. To illustrate the method, we will consider the four-component system in Figure 2 with heterogeneous components indexed as A, B, C, and D. The definition of each state of the system will be as follows (see Figure 4). If any one component in the system is in failed state, say component B, then components A and C will have increased failure rates of $\lambda_{A(B^2)}$ and $\lambda_{C(B^2)}$ respectively while component D remains the same as λ_i since it is not spatially dependent on component B. If component A has two spatial dependent components in failed state, say component B and D, then component A and C will have increased failure rates of $\lambda_{A(B^2,D^2)}$ and $\lambda_{C(B^2,D^2)}$ respectively.

The four-component system's evolution will be determined by the transitions among states. When all components are working, we indicate this initial state by 0 meaning that zero components have failed. When one component fails we denote this state by 1. The procedure is continued until the system fails.

The transition rate from 0 to 1 when one component has failed is $\lambda_A + \lambda_B + \lambda_C + \lambda_D$ implying that anyone of the four components could fail. The transition rate of one possible path from 1 to 2 is $2\lambda_{i(j^2)} + \lambda_i$. For example, using the four-component system, if component B first failed, then anyone of component A, C and D could be the next to fail with transition rate $\lambda_{A(B^2)} + \lambda_D + \lambda_{C(B^2)}$.

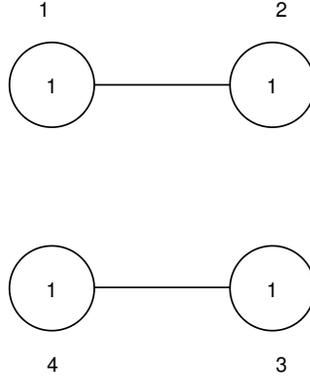


Figure 5: The four-component systems with two links

Likewise, if we assume that component C is the second component to have failed, then anyone of component A, or D could be the next to fail with transition rate $\lambda_{A(B^1)} + \lambda_{D(C^1)}$. Here component A is not affected by the failed state of component C, likewise component D is also not affected by the failed state of component B. In contrast, component A and D are affected by the failed state of components B and C in a way that A and D take up 100 percent of their load. When either component A or D fails, the system fails because the sum of loads on the working component will be less than the load placed on the system. The process is applied to derive all possible state transition equations for the system.

5 Numerical example 1: a simple four-component system

In this section we use a four-component system with a simple structure to illustrate the developed model. The structure of the system is depicted in Figure 5. Note that the components are connected in parallel, while the links between the components indicate spatial dependence. The system has four components indexed as component 1, 2, 3, and 4. The components are linked together in a way that the system is composed of two subsystems (that is, subsystem having components 1 and 2 and another subsystem having components 3 and 4). The system fails whenever the sum of loads on the working components is less than the load placed on the system. When one component fails, its immediate working neighbour takes up all the load so that the system still works. If a third failure occurs, two components in one of the subsystems must have failed, hence the system fails. We assume that there is no information about the distance between components in the system so that the component arrangement will be used to infer component proximity. In the analysis, we will consider two systems; one with homogeneous components and another with heterogeneous components. We assume that the parameters to be used for the numerical analysis are known. The parameters for this numerical analysis are taken from Jain and Gupta (2012).

We examine the effect of the load factor on the reliability estimations of a load-sharing sys-

tem with spatial dependence and proximity effect. Consider a system composed of homogeneous components with equal load allocations at time $t = 0$. Since we aim to highlight the importance of spatial effect and demonstrate the proposed spatial model through numerical example, we do not assume any unit for time. The state transitions are presented as Figure 6. The system failed or "down" states are S_3 and S_4 while the system "up" states are S_0 , S_1 , and S_2 . Let $Q_i(t) = \Pr(\text{System is in state } S_i \text{ at time } t)$. In order for the system to be in state S_0 at time $t + \Delta t$, the system must be in state S_0 at time t , and no transition occurs from state S_0 in time $(t, t + \Delta t)$. Thus, we have

$$\begin{aligned} Q_0(t + \Delta t) &= Q_0(t)(1 - 4\lambda_0\Delta t) \\ \lim_{\Delta t \rightarrow 0} \frac{Q_0(t + \Delta t) - Q_0(t)}{\Delta t} &= -4\lambda_0 Q_0(t) \\ \frac{d}{dt} Q_0(t) &= Q'_0(t) = -4\lambda_0 Q_0(t). \end{aligned} \quad (20)$$

Likewise for the system to be in state S_1 at time $t + \Delta t$, either the system is in state S_1 at time t and no transition occurs during $(t, t + \Delta t)$ or the system is in state S_0 at time t and a transition $S_0 \rightarrow S_1$ occurs in $(t, t + \Delta t)$.

$$\begin{aligned} Q_1(t + \Delta t) &= Q_1(t)(1 - (2\lambda_0 + \lambda_{i(j^1)})\Delta t) + Q_0(t)4\lambda_0\Delta t \\ \lim_{\Delta t \rightarrow 0} \frac{Q_1(t + \Delta t) - Q_1(t)}{\Delta t} &= -(2\lambda_0 + \lambda_{i(j^1)})Q_1(t) + 4\lambda_0 Q_0(t) \\ Q'_1(t) &= -(2\lambda_0 + \lambda_{i(j^1)})Q_1(t) + 4\lambda_0 Q_0(t) \end{aligned} \quad (21)$$

Using the same logic, we can derive the corresponding differential equation for $Q'_2(t)$ as follows

$$Q'_2(t) = -2\lambda_{i(j^1)}Q_2(t) + 2\lambda_0 Q_1(t) \quad (22)$$

$Q_3(t)$ and $Q_4(t)$ are not needed for calculating the system reliability because they are the probabilities of the system being in failed states S_3 and S_4 .

In contrast, if we assume that the components in the system are heterogeneous (non-identically distributed), then the state transitions can be derived as shown in Figure 7. The system failed or "down" states are S_5 , S_8 , S_{13} , S_{16} , and S_{17} while the system "up" states are S_0 , S_1 , S_2 , S_3 , S_4 , S_6 , S_7 , S_9 , S_{10} , S_{11} , S_{12} , S_{14} , S_{15} . The set of differential equations derived for the up states are given as

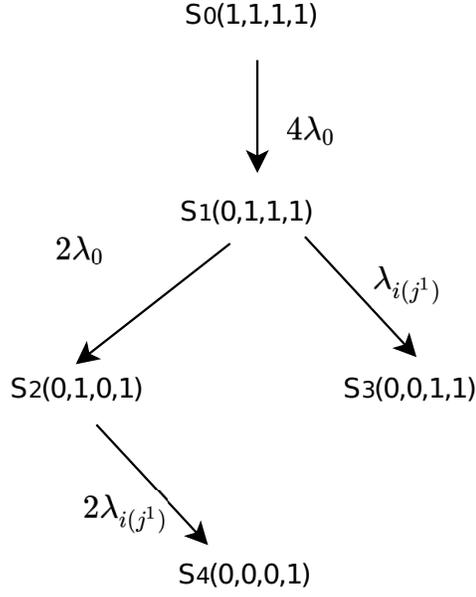


Figure 6: State transitions for the four-component system with homogeneous components.

$$\begin{aligned}
 Q'_0(t) &= -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)Q_0(t) \\
 Q'_1(t) &= -(\lambda_1 + \lambda_2 + \lambda_{3(4^1)})Q_1(t) + \lambda_4Q_0(t) \\
 Q'_2(t) &= -(\lambda_1 + \lambda_2 + \lambda_{4(3^1)})Q_2(t) + \lambda_3Q_0(t) \\
 Q'_3(t) &= -(\lambda_{1(2^1)} + \lambda_3 + \lambda_4)Q_3(t) + \lambda_2Q_0(t) \\
 Q'_4(t) &= -(\lambda_{2(1^1)} + \lambda_3 + \lambda_4)Q_4(t) + \lambda_1Q_0(t) \\
 Q'_6(t) &= -(\lambda_{1(2^1)} + \lambda_{3(4^1)})Q_6(t) + \lambda_2Q_1(t) \\
 Q'_7(t) &= -(\lambda_{2(1^1)} + \lambda_{3(4^1)})Q_7(t) + \lambda_1Q_1(t) \\
 Q'_9(t) &= -(\lambda_{1(2^1)} + \lambda_{4(3^1)})Q_9(t) + \lambda_2Q_2(t) \\
 Q'_{10}(t) &= -(\lambda_{2(1^1)} + \lambda_{4(3^1)})Q_{10}(t) + \lambda_1Q_2(t) \\
 Q'_{11}(t) &= -(\lambda_{1(2^1)} + \lambda_{3(4^1)})Q_{11}(t) + \lambda_4Q_3(t) \\
 Q'_{12}(t) &= -(\lambda_{1(2^1)} + \lambda_{4(3^1)})Q_{12}(t) + \lambda_3Q_3(t) \\
 Q'_{14}(t) &= -(\lambda_{2(1^1)} + \lambda_{3(4^1)})Q_{14}(t) + \lambda_4Q_4(t) \\
 Q'_{15}(t) &= -(\lambda_{2(1^1)} + \lambda_{4(3^1)})Q_{15}(t) + \lambda_3Q_4(t)
 \end{aligned} \tag{23}$$

We will use the derived equations to estimate the system reliability for both the homogeneous and heterogeneous systems.

The failure rate of the component depends on the load function and proximity effect of the components. To derive the proximity effect of a failed component on the working components, the column normalized values of the working components are derived. Therefore, the spatial dependence matrix of components in the system is given as matrix $[\theta_{ij}]$,

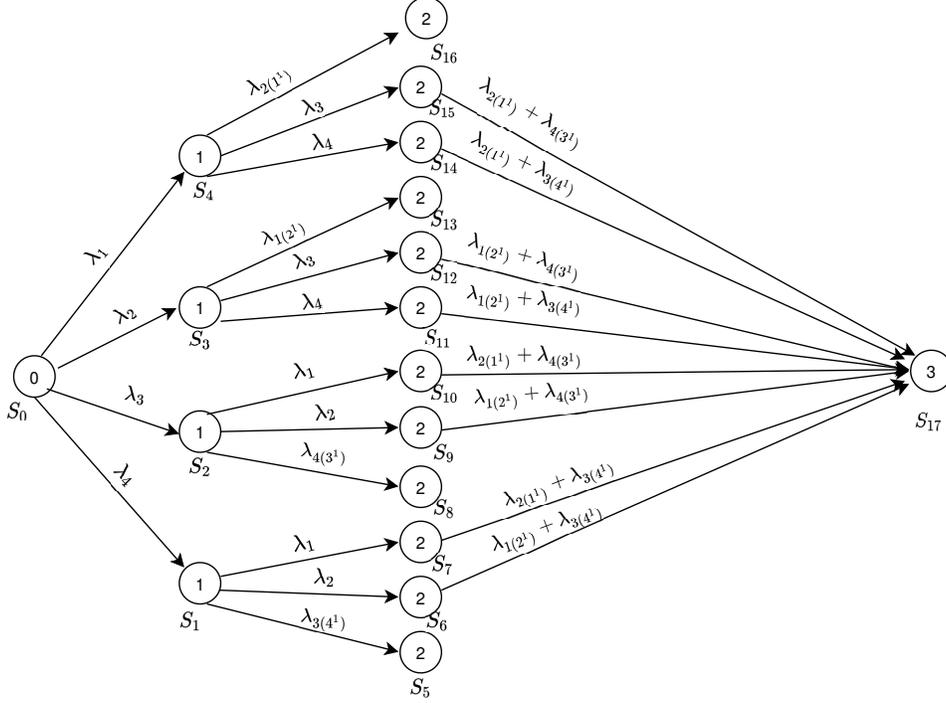


Figure 7: State transitions for the four-component system with Heterogeneous components.

$$[\theta_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The corresponding proximity matrix $[\vartheta_{ij}]$ for components at time $t = 0$ is given by

$$[\vartheta_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Consider the failure rates for the components if they are homogeneous. When all components are working, their failure rate is $\lambda_i = \lambda_0$ for $i = 1, 2, 3, 4$. Using the proximity effects of ϑ_{ij} derived in the column normalized matrix, when one component in the system has failed, the failure rate of the neighbour in the same subsystem becomes:

$$\lambda_{i(j^1)} = (z_0 + \vartheta_{ij} \times z_0)^\beta \lambda_0$$

where $\vartheta_{ij} = 1$ for $i = 1, 2, \dots, 4$, and where $z_0 = \frac{\gamma_i L}{\sum_{i=1}^4 \gamma_i} = L/n$ where $\gamma_i = \gamma_j$ for any $1 \leq i, j \leq n$.

If the working components are non-neighbours of the failure component their failure rate would remain as λ_0 .

For the system with heterogeneous components, when all four components are in a working state, their failure rate is λ_i^* where $i = 1, 2, 3, 4$ and $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_4$. When one component has failed, the failure rate of its spatial neighbour is given by one of the following failure rates:

$$\begin{aligned}\lambda_{1(2^1)} &= (z_1 + \vartheta_{12}z_2)^\beta \lambda_1^* \\ \lambda_{2(1^1)} &= (z_2 + \vartheta_{21}z_1)^\beta \lambda_2^* \\ \lambda_{3(4^1)} &= (z_3 + \vartheta_{34}z_4)^\beta \lambda_3^* \\ \lambda_{4(3^1)} &= (z_4 + \vartheta_{43}z_3)^\beta \lambda_4^*\end{aligned}$$

where $z_i = \frac{\gamma_i L}{\sum_{i=1}^4 \gamma_i}$ and $\gamma_i \neq \gamma_j$ for any $1 \leq i, j \leq n$.

From this, we can estimate the reliability of the system for varied values of β . Let $R(t)$ be the probability that the system is functioning at time t , the reliability of the system is given by

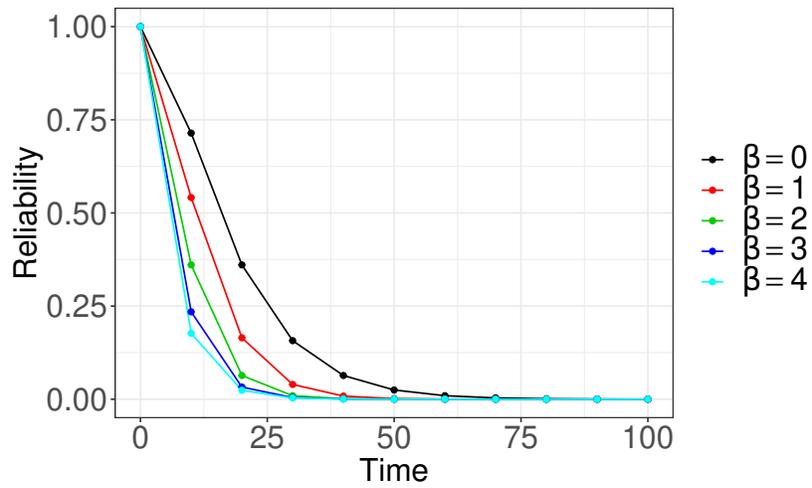
$$R(t) = \sum_{i=0}^{s-h} Q_i(t) \quad (24)$$

where s is the number of system states, h is the number of system down states and $Q_i(t)$ are the state probabilities for the system up states. The state probabilities $Q_i(t)$ can be computed by solving the Kolmogorov equations (i.e., the set of differential equations) expressed in Eq. 20, 21, 22, and 23. Let the initial conditions be set as $Q_0(0) = 1$ and $Q_i(0) = 0$ for $i = 1, 2, \dots, s$. Calculating the system's reliability could be cumbersome depending on the method applied and the number of components considered. The main challenge lies in the exponential explosion of number of states. The large number of states makes it difficult to calculate system reliability. Appropriate methods should be used to ease the computational burden for a large-scale system. Analytical methods such as Laplace transform can be applied to derive the system's reliability function from the Kolmogorov equation. However, the derivation of the analytic expression of the reliability function could be time consuming and cumbersome when large number of states are involved. On another hand, numerical methods such as the modified Euler's method can be used to easily derive the system state probabilities and corresponding reliability estimate. Sur and Sarkar (1996) and Sharma and Bansal (2017) used a numerical method to solve a system of differential-equations and demonstrated that the approximate results obtained by numerical method match considerably well with the results obtained by laplace transform technique. We will implement the method using deSolve package in R. Table 1 presents the parameters used to study the effects of β where β values will be made to vary between 0 to 4 by steps of 1.

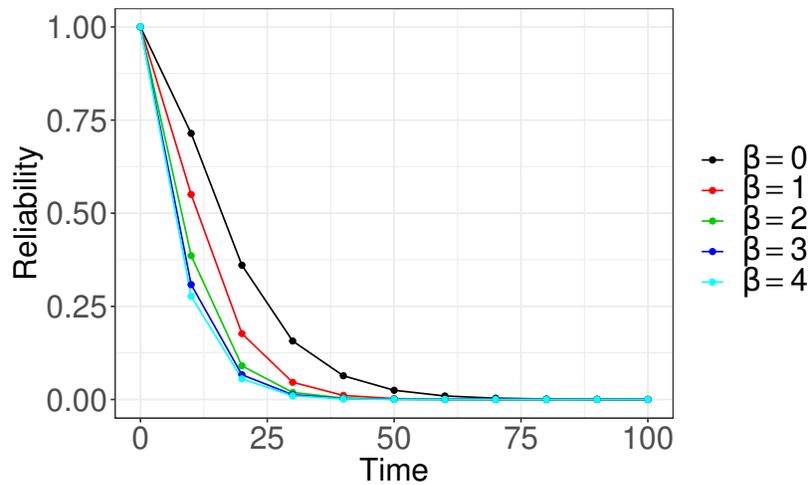
Figures 8 and 9 show the variation of system reliability with respect to β . Figures 8a and 8b present two cases for systems with homogeneous components. Figure 8a represents reliability estimations when the system load is assumed to be equally distributed to the components at time $t = 0$. Figure 8b represents reliability estimations when the system load is not equally distributed.

Table 1: Parameters used to study the effect of β

| | β | λ_0 | λ_1^* | λ_2^* | λ_3^* | λ_4^* | L | γ_1 | γ_2 | γ_3 | γ_4 |
|---------------|---------|-------------|---------------|---------------|---------------|---------------|---|------------|------------|------------|------------|
| Homogeneous | (0, 4) | 0.05 | - | - | - | - | 4 | 0.5 | 1 | 1.5 | 2 |
| Heterogeneous | (0, 4) | - | 0.05 | 0.10 | 0.15 | 0.12 | 4 | 0.5 | 1 | 1.5 | 2 |

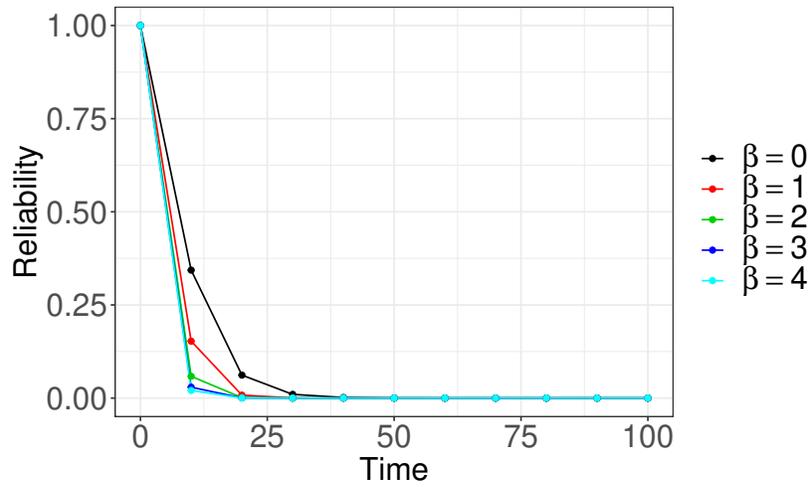


(a) Components share equal initial load

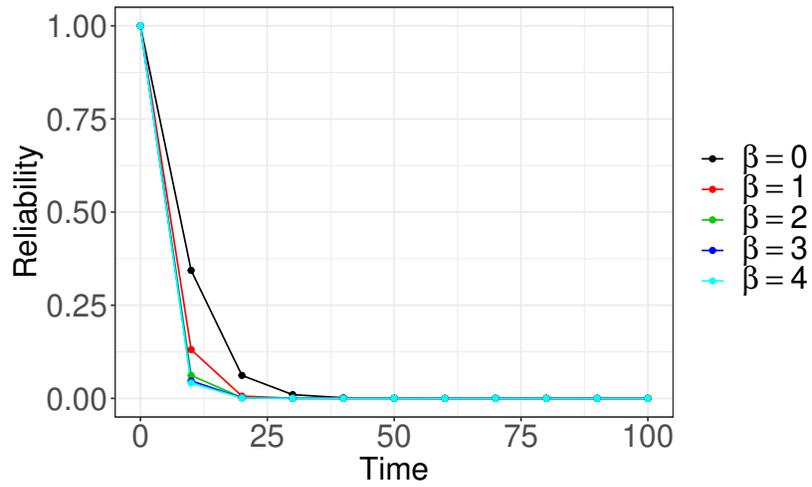


(b) Components share different initial load

Figure 8: Variation of system reliability with load factor β considering homogeneous components



(a) Components share equal initial load



(b) Components share different initial load

Figure 9: Variation of system reliability with load factor β considering heterogeneous components.

For the case of equal initial load distribution, when $\beta = 0$, which indicates that the system is not load-sharing, the system reliability hits zero at time $t = 60$. However, when $\beta = 1$ indicating the presence of load sharing, the system's reliability reduces. As β is increased, indicating that the impact of load-sharing between components is increased, the system's fails faster. For the case of unequal initial load distribution, a similar pattern is observed.

In Figure 9a and 9b we present systems with heterogeneous components. In both cases, we observe that $\beta = 0$, indicating that the system is not load-sharing, the system reliability hits zero at time $t = 27$. However, as the impact of load-sharing between components is increased, the system's reliability reduces sharply. When compared with the reliability of the system with homogeneous components, the reliability of a system with heterogeneous components decreases sharply over time.

We compare our developed models with existing models to investigate the importance and significance of the spatial effect. We compare our spatial model with capacity flow models developed for a k -out-of- n : G systems, in the setting of equal load-sharing with homogeneous components (Yinghui and Jing, 2008) and heterogeneous components (Yinghui and Jing, 2008; Jain and Gupta, 2012). The four-component system fails when two components in the same subsystem fails, i.e. components 1 and 2, or components 3 and 4. This behaviour can be likened to those of k -out-of- n : G system in which the system works when k components work and the system fails when $n - k + 1$ components have failed. The case where the system described in Figure 5 fails due to the failure of two components in the same subsystem can be likened to 3-out-of-4: G system. Likewise, the case where the system fails when three components fail can be likened to 3-out-of-4: G system. As a result, we will compare reliability estimation from our spatial model with models of equal load-sharing for 2-out-of-4 and 3-out-of-4: G systems. We will examine the performance of the three models for reliability estimation of the four-component system with homogeneous and heterogeneous components. We use Monte Carlo simulation (MCS) to estimate the system reliability and compare with our models. The simulation model is used to compare the underlying data generating process with the modelling approaches. Using simulation, we are able to assess the accuracy of our model, while "real data" does not allow doing so. Another advantage of simulation is that it allows to change the settings and test our model in various ways. Using MCS, it is possible to modify the failure behaviour of the working components after one of the proximate components have failed. The algorithm for generating the system lifetime T_{sys} is described as follows.

Algorithm 1 Algorithm for generating lifetime data of a system with load-sharing and spatial dependence

Require: Load factor β ; component failure rate λ ; system load L ; number of components n ; proximity matrix $[\vartheta_{ij}]$.

Ensure: System lifetime T_{sys} at the end of the observation period.

- 1: start at $k = 1$.
- 2: **repeat**

```

3:   for all working components  $i \in \{k, k + 1, \dots, n\}$  do
4:     calculate  $t_i = \frac{-\ln(U_{0,1})}{\lambda_i}$  where  $U_{0,1}$  is a random value drawn from a uniform distribution.
5:   end for
6:   let the  $k$ th component failure time  $T_k = \min(t_k, t_{k+1}, \dots, t_n)$ ;
7:   calculate  $z_{i(j)} = (z_i + \sum_{j \neq i} \vartheta_{ij} z_j)$  for proximate neighbours of failed component  $j$ .
8:   replace  $\lambda_i$  by  $\lambda_{i(j^r)} = (z_i + \sum_{j \neq i} \vartheta_{ij} z_j)^\beta \lambda_i$  for proximate neighbours of failed component  $j$ .
9:   update the proximity matrix  $[\vartheta_{ij}]$  by excluding the failed component.
10:  set  $k = k + 1$ .
11: until  $\sum_{i=1}^{n-k} z_i < L$ ;
12: return  $T_{sys} = \sum_k T_k$ .
    
```

If one generates a large number of system lifetimes T_{sys} , the cumulative distribution function of the system can be evaluated. Using the above algorithm, 1000 system lifetimes T_{sys} are generated and used to evaluate the system's reliability $R_{sys}(t) = \frac{m(t)}{1,000}$ where $m(t)$ denotes the number of times the system survived beyond time t . Table 2 presents the parameters for the two forms of the four-component systems used for the comparison. Figure 10 and 11 present system reliability when homogeneous and heterogeneous components are considered respectively. In addition, Table 3 presents root mean squared error (RMSE) of reliability estimations from the three compared models and the MCS predictions. It can be observed that the spatial model's estimation error is way less than those by the models that assume equal load-sharing. In other words, the simulation results validate the results from the spatial model. In addition, predictions by the models which assumes equal load-sharing for 2-out-of-4 and 3-out-of-4 systems either overestimate or underestimate the reliability of the four component system indicated by the MCS predictions.

As a result, an application of models which assume equal load-sharing for reliability analysis of a load-sharing system when spatial dependence exists could lead to misleading predictions and thus ineffective maintenance strategies.

Table 2: parameters used for model comparison

| | β | λ_0 | λ_1^* | λ_2^* | λ_3^* | λ_4^* | L | γ_1 | γ_2 | γ_3 | γ_4 |
|---------------|---------|-------------|---------------|---------------|---------------|---------------|---|------------|------------|------------|------------|
| Homogeneous | 1 | 0.05 | - | - | - | - | 4 | - | - | - | - |
| Heterogeneous | 1 | - | 0.05 | 0.10 | 0.15 | 0.12 | 4 | 0.5 | 1 | 1.5 | 2 |

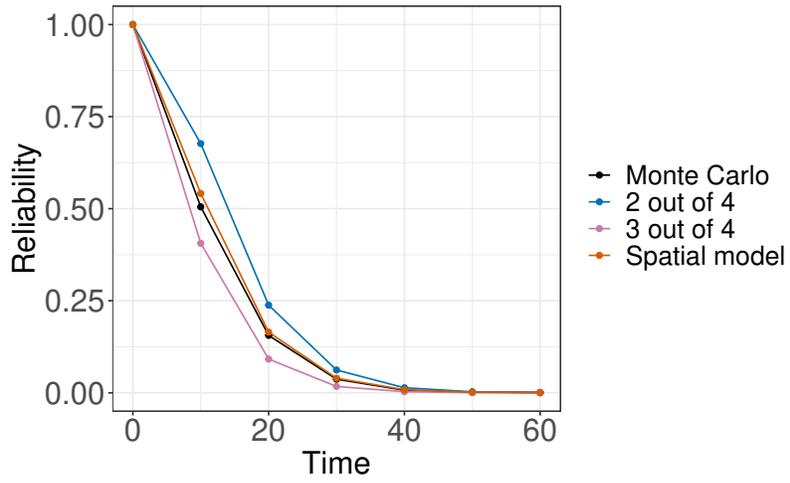


Figure 10: Comparison of reliability estimation using monte carlo simulation, the spatial model and existing models considering homogeneous components

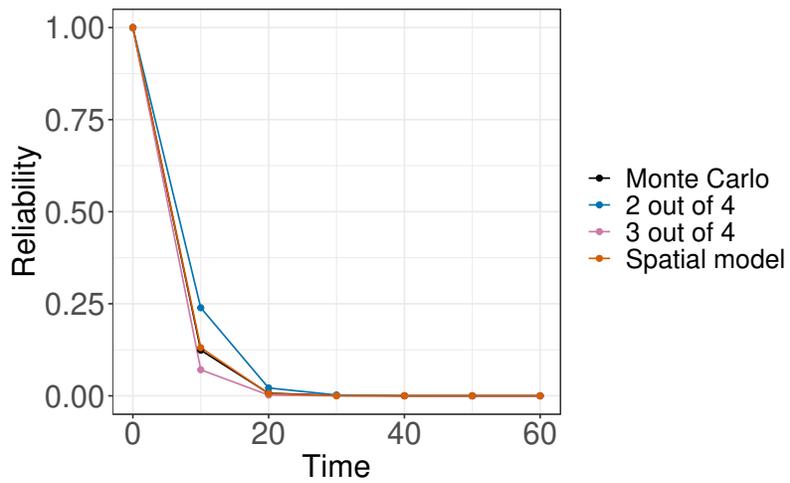


Figure 11: Comparison of reliability estimation using monte carlo simulation, the spatial model and existing models considering heterogeneous components

Table 3: Root mean squared error of the reliability estimations by the models

| | | Homogeneous | | |
|------|--|----------------------|---------|------------|
| | | 2-out-of-4 | Spatial | 3-out-of-4 |
| RMSE | | 0.0495 | 0.0105 | 0.0684 |
| | | Heterogeneous | | |
| | | 2-out-of-4 | Spatial | 3-out-of-4 |
| | | 0.0509 | 0.0036 | 0.019 |

6 Numerical example 2: a five-component system with complex spatial structure

We extend the example in section 5 to consider a more complex spatial structure with 5 components. For brevity, we only consider homogeneous components for the system depicted in Figure 12, however, the modelling could be easily extended for heterogeneous components. The components are linked together in a way that the system is composed of one dominant component (component 1) and four secondary components (component 2, 3, 4, and 5). As indicated by Figure 12, the system fails when the dominant component and at least three secondary components fail. In this case the load on the only working component is less than the load placed on the system. We assume that there is no information about the distance between components in the system so that the component arrangement will be used to infer component proximity. To investigate the importance and significance of the spatial effect, we compare our spatial model with an equal load-sharing model for a 1-out-of-5 system.

Table 4 presents the parameters used in the model while the system state transition diagram is presented in Figure 13.

Table 4: Parameters used for model comparison on the five-component system

| | β | λ_0 | L |
|-------------|---------|-------------|---|
| Homogeneous | 1 | 0.05 | 5 |

Table 5 presents RMSE from comparing the spatial model with the 1-out-of-5 model. We observe that our spatial model's estimation error is less than those by the 1-out-of-5 model that assumes equal load-sharing. Figure 14 present the system reliability predictions by our spatial model and the 1-out-of-5 model where the predictions from the 1-out-of-5 model at time $t = 1$

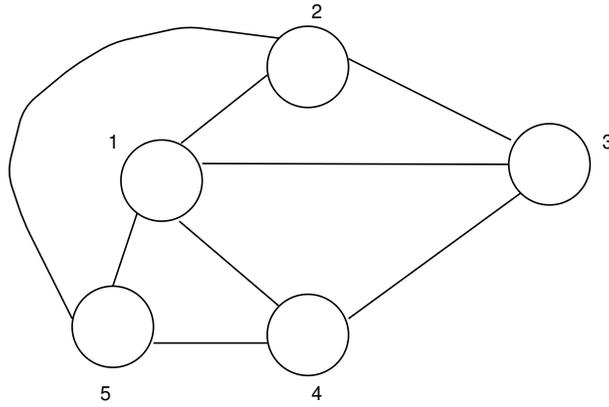


Figure 12: The five-component system

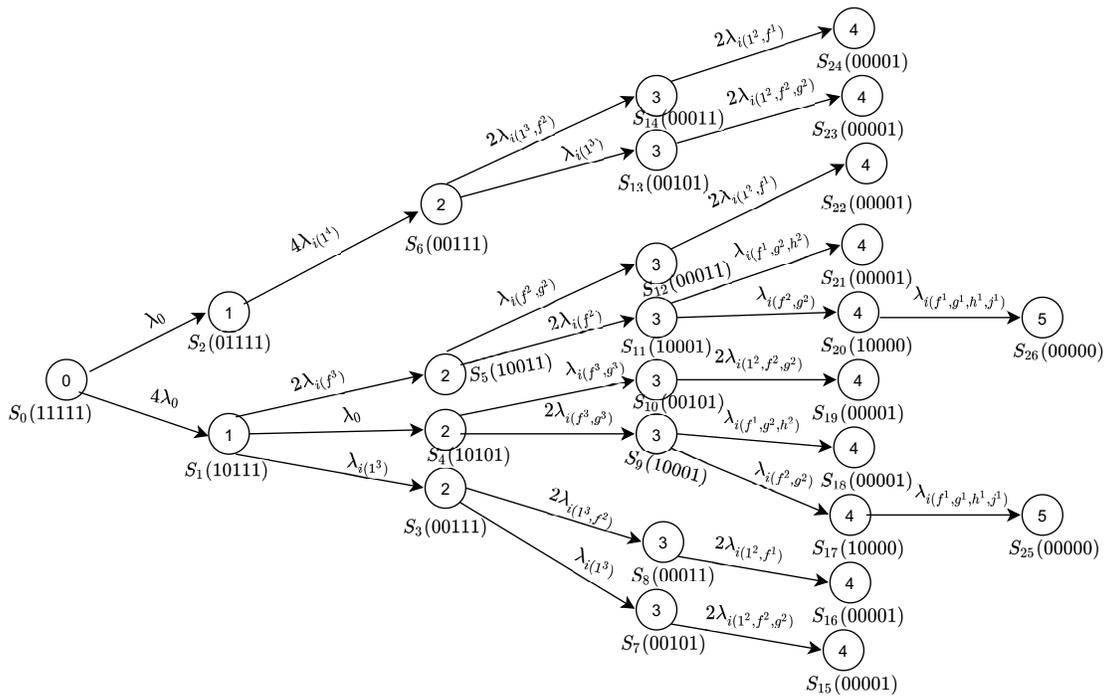


Figure 13: Five-component system state transition diagram.

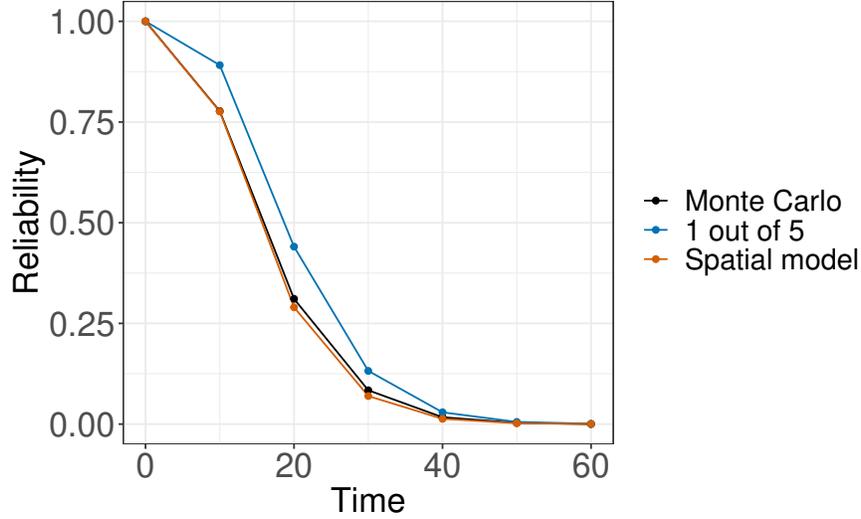


Figure 14: Comparison of reliability estimation using the spatial model and capacity flow model.

and $t = 30$ is far from the MCS prediction compared to the spatial model whose prediction closely matches the MCS. Hence, it suggests that our spatial model is more accurate at modelling these more complex scenarios. Our results demonstrate that our model is more accurate than standard load sharing models in evaluating system reliability when spatial effects exist.

Table 5: Root mean squared error of the reliability estimations by the models

| RMSE | Homogeneous | |
|------|-------------|-------------|
| | 1-out-of-5 | Spatial |
| | 0.07939322 | 0.004913213 |

7 Conclusions

This paper developed and evaluated a load-sharing system with spatial dependence and proximity effects. We have assumed that a system exists that operates in a way that the load on a failed component is taken up only by its working spatial neighbours in close proximity. We developed a model to evaluate system reliability, extending the capacity flow model to take into account load-sharing interactions and proximity effects. The model was developed for both homogeneous or heterogeneous components and illustrated through two numerical examples. Sensitivity analysis of the load factor was conducted to examine its effect on reliability estimations. In addition, we examined the impact of using a wrong model for reliability estimation of a four-component load-sharing system with spatial dependent components. The analysis was conducted by comparing our spatial

model with existing equal load-sharing models for 2-out-of-4 and 3-out-of-4 systems. Monte Carlo simulation was used to generate cumulative distribution function of the four-component system and to validate the reliability estimation from the compared models. We found that an application of equal load-sharing model for reliability analysis of a load-sharing system with spatial dependence could lead to either overestimated or underestimated reliability prediction. A similar analysis was made on a five-component system with a more complex structure. Comparison was made between the spatial model and an equal load-sharing model for 1-out-of-5. It was found that the equal load model had overestimated reliability assessment. The importance and significance of the spatial and proximity effect was highlighted through comparison with existing models. Our results demonstrate that our model is more accurate than standard load sharing models in evaluating system reliability when spatial effects exist.

There are avenues of future research that could be considered. For example, maintenance models for k-out-of-n load-sharing system with an equal load-sharing rule have been studied in the literature (Taghipour and Kassaei, 2015). However, preventive replacement and periodic inspection modelling of load-sharing systems with spatial dependence and proximity effects have not yet been considered in the literature. It is of interest to develop a maintenance model for the system investigated in this study. In addition, the components in this study only have two states and their lifetime follow an exponential distribution. This assumption can be further relaxed by considering a more realistic but complicated failure process. Another improvement can be achieved by considering a more elaborate proximity model. The impact of spatial distance on load reallocation can be investigated in a more detailed way, in addition to the constant and exponential proximity models as described in this study.

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