Defect-Mediated Turbulence in Lasers with Injected Signals

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Tito Arecchi has been a pioneer in the theory and experiments of laser operation and nonlinear dynamics. Lasers with an injected signal are beautiful prototypes for some of the richest dynamical behaviours ever observed in physics. Focusing on the simplest model of a class A laser with injected signal in the presence of transverse diffraction, we describe locked and unlocked states and the mechanisms of instability when decreasing the intensity of the injected signal. An unexpected transition to defect mediated turbulence is demonstrated in the unlocked regime in spite of the presence of the driving frequency of the injection. This optical turbulence state can lead to defect–mediated rogue waves and to a process of crystallisation when the control parameter is reversed.

I. INTRODUCTION

The development, understanding and application of lasers own immensely to the work and genius of Tito Arecchi. For a physics student at the University of Florence in the late 70s - early 80s, Tito and his group at the Istituto Nazionale di Ottica (INO) were a beacon of novelty, imagination and international reputation. The scientific debate and development of new physics ideas at INO were unparalleled in Florence at the time if one excludes statistical physics. Tito's visitors included Bloembergen, Feigenbaum, Hellwarth, Narducci, Jacobs, Abrahahm, Coullet, Brewer, Vorontsov, Bonifacio, De Giorgio and Lugiato. Every day something new to discuss and to think about. This fact combined with Tito's wide knowledge of science, literature, art and culture made the INO a unique and fabulous place for learning and education. My main gratitude to Tito is for the creation of such a buzzing place of scientific debate and technological innovation during years of unimaginable boredom due to nuclear and particle physics in Florence.

This paper reviews the temporal and spatio-temporal dynamics of lasers with injected signals (LIS), where external laser light is injected into the cavity of another laser for its stabilization and/or to shift the frequency of its output. In a LIS there are two fundamental parameters for control: the intensity of the injection and the detuning between the frequency of the external laser and that of the initially unperturbed laser. We start with purely temporal dynamics of the LIS system in Section II. In particular we describe the unexpected phenomenon of frequency pulling where the frequency of the injected laser moves away from (instead of towards) that of the external laser even at the point where two lasers lock with each other and the output becomes stationary instead of oscillating [1]. This work was done in collaboration with Antonio Politi and Tito Arecchi and opened up an entire new field of predicted and observed dynamical features in LIS, including the unintuitive coexistence of conservative and dissipative dynamics [2]. Section II concludes with the temporal dynamics of lasers where the evolutions of the material variables are slaved to that of the laser field.

In Section III, we introduce a simple spatio-temporal model for the description of LIS in the presence of spatial coupling in the form of diffraction or dispersion. We study the homogeneous steady states, the locking region and the appearance of periodically modulated solutions in space (patterns).

In Section IV we discuss the core element of this contribution, defect-mediated turbulence (DFM) in LIS. We compare the transition to DFM in LIS to that originally described by Coullet, Gil and Lega in the late 80s [3] for the complex Ginzburg-Landau equation. Unexpectedly, in the region before the locking, DFM takes place in the LIS models even without diffusion and in the presence of an external driving such as the laser injection. These intriguing phenomena can lead to crystallisation processes and to output pulses of very large intensities (rogue waves). These are a culmination of the research initiated with Tito Arecchi more than 35 years ago. Section V contains the conclusions and suggestions for the application of the transitions observed in LIS models to statistical physics of systems outside thermodynamic equilibrium.

II. TEMPORAL DYNAMICS OF LASERS WITH INJECTED SIGNALS

During the early 80s, the main focus at INO had been the first clear-cut experimental observation of subharmonic bifurcations and deterministic chaos in CO_2 lasers in a collaboration between Tito, Jorge Tredicce, Gian Piero Puccioni and Riccardo Meucci [4]. Together with Gian Luca Lippi, their interest in unstable behaviours had then extended to lasers with injected signals, the main topic of this paper, where external light is injected in a laser for stabilization and/or to shift the frequency of its output [5, 6]. Antonio Politi and myself became interested in the nonlinear dynamics of the Arecchi-Bonifacio (also known as Maxwell-Bloch [7]) equations of a class B [8] LIS. In particular we were intrigued by the oscillatory dynamics of the LIS after the adiabatic elimination of the material polarization [9] and in the limit of long-lived metastable transitions. Without injection the dynamics of the output intensity of the laser is equivalent to that of particle in a Toda potential [10]. With the addition of the optical injection but still in the limit of long-lived metastable transitions, a peculiar system of ordinary differential equations can be derived [2]. The very first evidence of coexistent conservative and dissipative behaviour in science was found in this numerical model [2] leading to well-hidden acclaim and world-wide virtual unrest.

While Tito preferred to leave us alone when studying the boundary of conservative and dissipative systems, its mathematics and numerical simulation, his interest in the complex dynamical behaviour of the LIS and possible experimental observation did not abate. In a seminal paper in 1986, Tito, Antonio, Gian Luca and myself studied the superbly counter-intuitive phenomenon of frequency pushing in LIS [1].

When two oscillators operate at different frequencies and one oscillator is used as the external driving of the other (injection), a locking is expected where the driven (injected) oscillator starts to operate at the frequency of the driver. This is generally referred as Adler frequency locking [11]. If the intensity of the driver is increased from zero up to the locking value, it is generally expected that the frequency of the driven oscillator progressively moves towards that of the driver. This is known as 'frequency pulling'. Frequency pushing is its exact OPPOsite. When increasing the intensity of the driver, one observes the frequency of the injected laser to move away from that of the injection up to, even more surprisingly, the critical value of locking. This can be explained by the nature of the bifurcation when reaching the steady state of the locked solution. In Fig. 1 the S-shaped stationary intensity $|E_S|^2$ is reported as a function of the injection intensity E_{IN}^2 for an injection that is detuned from the cavity resonance. The middle and the lower branches of this curve are always unstable; note that the lower branch stems out of the zero-intensity point of the non-injected laser that is unstable for lasers above threshold. The point of intersection between the unstable middle branch and the stable upper branch corresponds to the onset of locking. For low injections, there is a region where there are no stable stationary states and where the injected laser is not yet slaved to the injection.

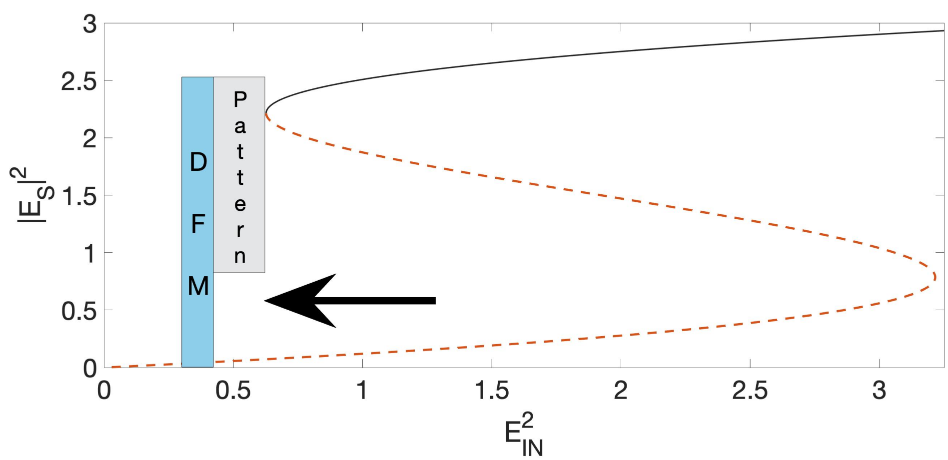

FIG. 1. (Color online) Typical S-shaped curve of the stationary intensity $|E_S|^2$ versus the injection intensity E_{IN}^2 for a LIS where the injection is detuned from the cavity resonance. The solid black line indicates the stable locked state, red dashed curves corresponds to unstable states. The gray and pale blue box indicate the ranges where pattern and DFM are observed when decreasing (see arrow) the injection intensity from the locked state.

If there is frequency pulling (normal behaviour) in the unlocked region, the stable locked state is reached through a saddle-node bifurcation where the frequency difference between injection and injected lasers smoothly approaches zero while the intensity of periodic or aperiodic pulses suddenly goes to zero at the threshold. If there is frequency pushing (ABnormal behaviour) in the unlocked region, the stable locked state is reached through a reverse Hopf bifurcation where the amplitude of laser light oscillations smoothly approaches zero while the frequency difference between injection and injected lasers suddenly goes to zero at the threshold. The observation of pulling or pushing depends critically on the value of the initial frequency difference when the injection amplitude is almost zero [1]. This means that there are critical values of the detuning and injection intensity where the bifurcation to the locked state changes from a saddle-node to a Hopf bifurcation. In particular there is a value, known as a co-dimension two bifurcation, where the saddle-node and the Hopf bifurcations take place at the same values of detuning and injection intensity [1]. In the mathematical theory of bifurcations in dynamical systems, it is known that close to co-dimension two points, the dynamics of the nonlinear system is extremely rich with great variation on the nature of the oscillations and steady states via very small changes of the parameter values.

The nonlinear dynamics of the co-dimension two of the LIS system was exposed in its full glory in 1994 [12] and in

1997 [13] and extended few years later to the full analysis of global bifurcations [14]. These papers have not obtained the full recognition that they deserve. By adding an extra term to the equations to describe the linewidth enhancement factor of semiconductor lasers, many, if not all, dynamical results first predicted in LIS in these contributions have been replicated and then successfully compared with experimental results [15]. For the reader interested in the mathematics of the unfolding of co-dimension two bifurcations in LIS, we recommend the careful reading of [12–14].

III. SPATIO-TEMPORAL MODELS OF CLASS A LASERS WITH AN INJECTED SIGNAL

We move now to the spatio-temporal dynamics of LIS. We start from the LIS equations taken from [1] and then include diffraction for the light propagation in a cavity as done in [16–18]:

$$\begin{aligned}\partial_t E &= -k\left[(1-i\omega)E - Q - E_{IN} - i\nabla^2 E\right]\\ \partial_t Q &= -\gamma_\perp\left[Q - E\Delta\right]\\ \partial_t \Delta &= -\gamma_{||}\left[\Delta - P + (1/6)(EQ^* + E^*Q)\right]\end{aligned} \tag{1}$$

where t is the time, F, Q and Δ are, respectively, the cavity electric field, the material polarisation and the population inversion. k, $\gamma_\perp$ and $\gamma_{||}$ are the field, material polarisation and population inversion decay rates, respectively. The detuning ω is the difference between the injection frequency and the closest cavity resonance normalised to k. E_{IN} is the (real) amplitude of the injection field. For zero injections, we consider here a laser where the separation of cavity frequency and material resonance is proportional to $|\omega|$. Finally, P is the external (incoherent) pump and $i\nabla^2 E$, with ∇^2 being the Laplacian operator in the transverse plane (x, y), describes diffraction in the laser cavity. Space is normalised to $\sqrt{L\lambda/(4\pi)}$, where L is the cavity length and λ the laser wavelength, and we are considering an optical cavity with plane mirrors and wide transverse size [19]. The character $*$ represents the operation of complex conjugation and the factor 1/6 instead of the usual 1/2 is for convenience since the field intensity can be renormalised (see later).

Here we consider class A lasers [8] where $\gamma_\perp$ and $\gamma_{||}$ are much bigger than k, i.e. an unperturbed laser with a high quality factor. Following the guidelines of [9], we then adiabatically eliminate the fast dynamics of the material polarization Q and population inversion Δ. This does not mean that these variables cannot oscillate in space and time; they simply follow the dynamics of the slowest variable, the cavity electric field E. Setting their temporal partial derivative to zero, one obtains:

$$\begin{aligned} Q = E\Delta \qquad\qquad \Delta &= \frac{P}{1+(1/3)|E|^2}\\ \partial_\tau E = E_{IN} - (1-i\omega)E &+ \frac{PE}{1+(1/3)|E|^2} + i\nabla^2 E\end{aligned} \tag{2}$$

where $\tau = kt$ is a non-dimensional time. By using the standard Kerr-like approximation for relatively small intensities of the electric field in the cavity, we obtain

$$\partial_\tau E = E_{IN} + (P - 1 + i\omega)E - (P/3)\,|E|^2 E + i\nabla^2 E \tag{3}$$

This partial differential equation is extremely simple, is the core model of this paper and has been used successfully in [20, 21]. The reason for the factor 1/3 in front of the non linearity is that Eq. (3) is also capable to describe the spatio-temporal dynamics of a singly resonant optical parametric oscillator where the nonlinear term is in the form $sinc^2(|E|)$ [22]. The interpretation of each term of Eq. (3) is straightforward: E_{IN} is the injection amplitude, $(P-1)E$ describes the difference between the gain and the losses, $i\omega E$ is the frequency detuning of the cavity and the external signal, $-(P/3)\,|E|^2 E$ describes the intensity dependent saturation mechanism of the gain and $i\nabla^2 E$ the diffraction in the laser cavity.

IV. SPATIAL STRUCTURES AND DEFECT MEDIATED TURBULENCE IN LASERS WITH AN INJECTED SIGNAL

The purely temporal dynamics of Eq. (3) was studied in details in [23]. Fig. 1 shows the homogeneous steady states (HSS) of Eq. (3) for $\omega = 0.53$ and $P = 4$. The HSS have the characteristic S-shape of a LIS where the solid black line corresponds to locked states where the frequency of the injected laser is the same of that of the injection. The red dashed line corresponds to unstable HSS; before the locking region there are no stable HSS because of a detuning ω different from zero. When decreasing the injection intensity below the locking point and into the unlocked region (see arrow in Fig. 1), one observes first the appearance of patterns and then of DMT because of the presence of diffraction and spatio-temporal coupling. Fig. 2 (A) shows the transition lines when decreasing the injection amplitude in the P vs E_{IN}^2 plane for $\omega = 0.53$, first from the locked state to the pattern and then from the pattern to DMT. The threshold corresponding to the continuous blue line in Fig. 2 (A) has been obtained analytically via:

$$|E_S|^2_{Lock} = (1/P)\left[2(P-1) + \sqrt{(P-1)^2 - \omega^2}\right]$$
$$|E_{IN}|^2_{Lock} = |E_S|^2_{Lock}\left[\left(1 - P + P|E_S|^2_{Lock}/3\right)^2 + \omega^2\right] \tag{4}$$

while the second one is a fit through numerically determined points (red squares). It is the latter threshold between a transverse pattern and DMT when decreasing the injection intensity, the key objective of this paper. The pattern has a honeycomb-like [24] structure in the light intensity as shown in Fig. 2 (B) for $P = 4$, $\omega = 0.53$ and $E_{IN}^2 = 0.435$. Note that the intensity scale reported with the colour map on the right of Fig. 2 (B) is everywhere well above zero. Spatial wavelengths for the intensity of these structures in broad area lasers with plane mirrors typically vary between 0.75 and 4 μm [19]. When decreasing E_{IN}^2 to 0.40 after a transient of 2000 time units, the pattern starts to oscillate (see Fig. 2 (C)) and then DMT sets in through the appearance of point vortices of zero intensity (see Fig. 2 (D)).

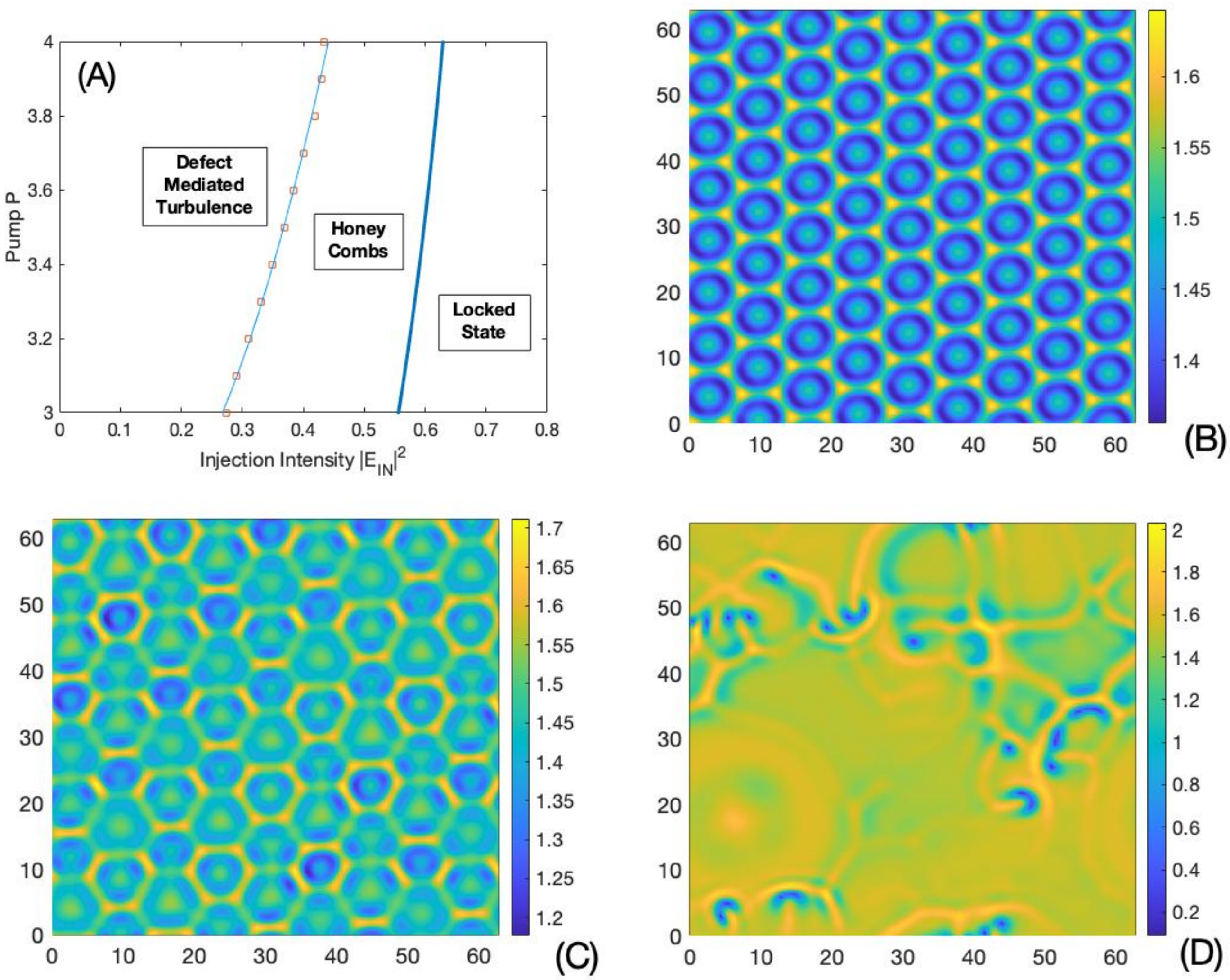

FIG. 2. (A) Regions of locked states (on the right), honeycomb-like patterns (centre) and DMT (left) for $\omega = 0.53$ in the (E_{IN}^2, P) parameter space; (B) Steady honeycomb-like patterns of the transverse intensity in the (x, y) plane for $P = 4$, $\omega = 0.53$ and $E_{IN}^2 = 0.435$ after transients have been discarded; (C) Transverse intensity as in part (B) but for $E_{IN}^2 = 0.40$ after 2000 time units and while oscillating; (D) Transverse intensity as in part (C) but after 4000 time units, displaying DMT.

But the important question is, is this regime a real laser DMT? The relevance of vortex defects to disordered states in nonlinear optics intrigued Tito Arecchi and his research group in the 90's [25, 26]. By using an optically pumped photorefractive crystal in a ring cavity, two-dimensional optical chaos in the presence of up to 100 vortex defects was experimentally observed in regimes of large Fresnel number [25] where the dynamics is bulk-controlled [26]. This research was initiated by the seminal work of Coullet, Gil and Lega on the onset of DMT in Ginzburg-Landau models [3]. There was the hope at the time to observe the transition to DMT through phase and amplitude instabilities as described in [3] in broad area lasers. Although the Arecchi-Bonifacio equations for lasers bare a strong resemblance to Ginzburg-Landau models, the instability leading to DMT imposes conditions on the complex coefficients of the spatial (Laplacian) terms [3] that were challenging to be met in laser devices. At the time, lasers with injected signals were not considered because of the idea that an external driver like an injection would have increased, instead of decreased, spatial order at the expense of turbulent regimes. Here we show that this is not the case.

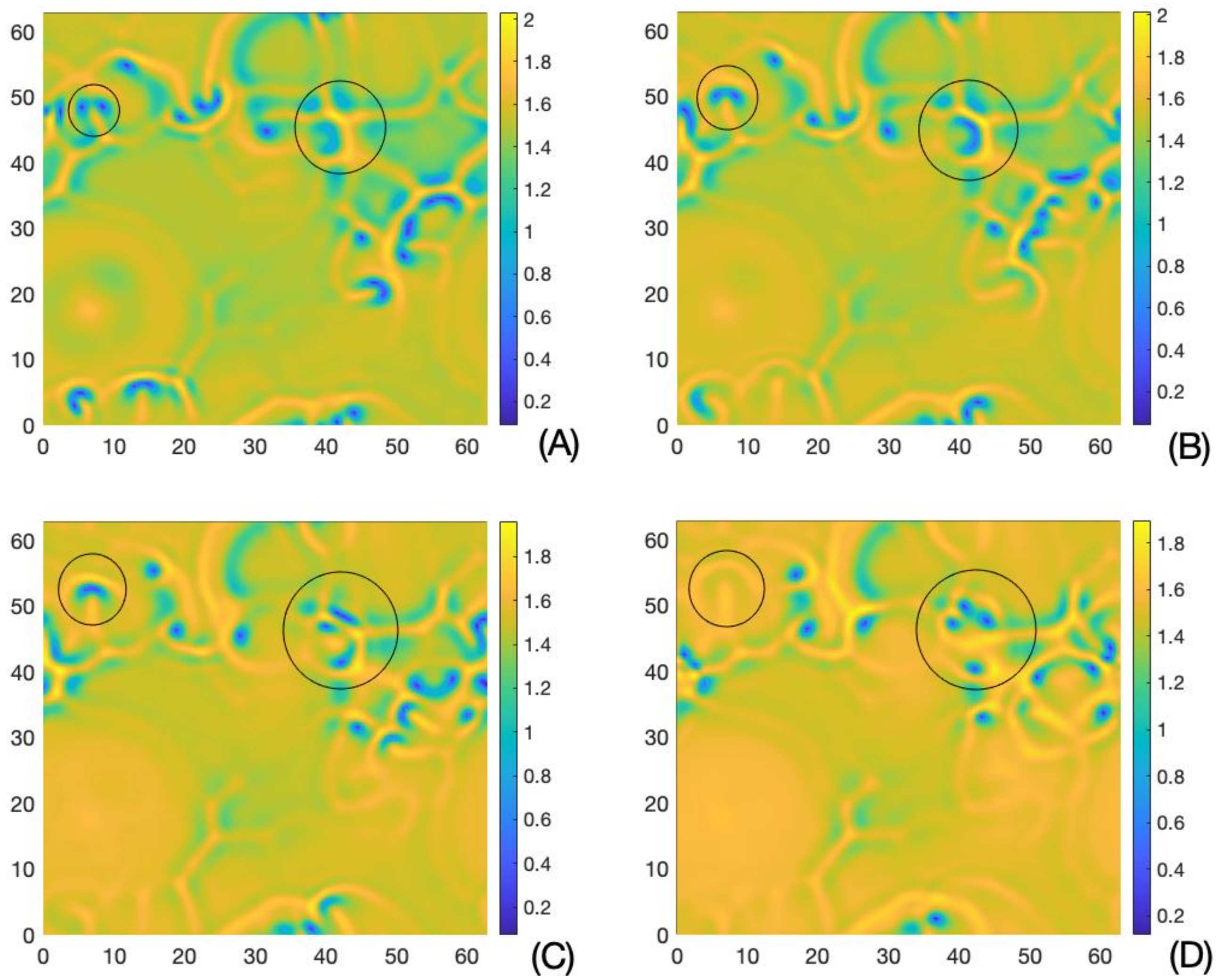

FIG. 3. (A) Transverse intensity for $P = 4$, $\omega = 0.53$ and $E_{IN}^2 = 0.40$; (B) Transverse intensity as for (A) but after 0.8 time units; (C) Transverse intensity as for (A) but after 1.6 time units; (D) Transverse intensity as for (A) but after 2.4 time units. The circles on the left (right) hand side identify a process of vortex annihilation (generation).

Fig. 3 shows four snapshots of a typical DMT regime for the parameter values specified in Fig. 2 (C)–(D). The DMT regime follows a phase instability of the honeycomb-like pattern leading to an amplitude instability and then to DMT as shown in [20] and in Fig. 2 (C)–(D). In the core of the defect vortices, the phase is undefined and the light intensity is zero. Inside the circles on the left of the four panels of Fig. 3, one can see a clear process of vortex annihilation while inside the circles on the right hand side of the same four panels, two processes of generation of two vortices of opposite topological charge are observed. Repeated processes of birth and death of couples of vortices take place in random positions in the transverse space ad infinitum as originally described for DMT regimes in [3]. In the particular case of Fig. 3 the average number of vortices is 18 with a standard deviation of 9. Although the instability mechanism for the occurrence of DMT is different from that of the Ginzburg-Landau model [3], it is clear that the DMT regimes are the same. The positive elements of the instability leading to DMT in LIS when compared with that in the Ginzburg-Landau model, are that there are no requirements on the spatial coefficients of the Laplacian, there is no need of diffusion and a one-to-one correspondence with real laser devices is possible. The key element of these instabilities is the unlocked region of the LIS where there are no stable steady states and where the work in collaboration with Tito Arecchi focused in 1986 [1].

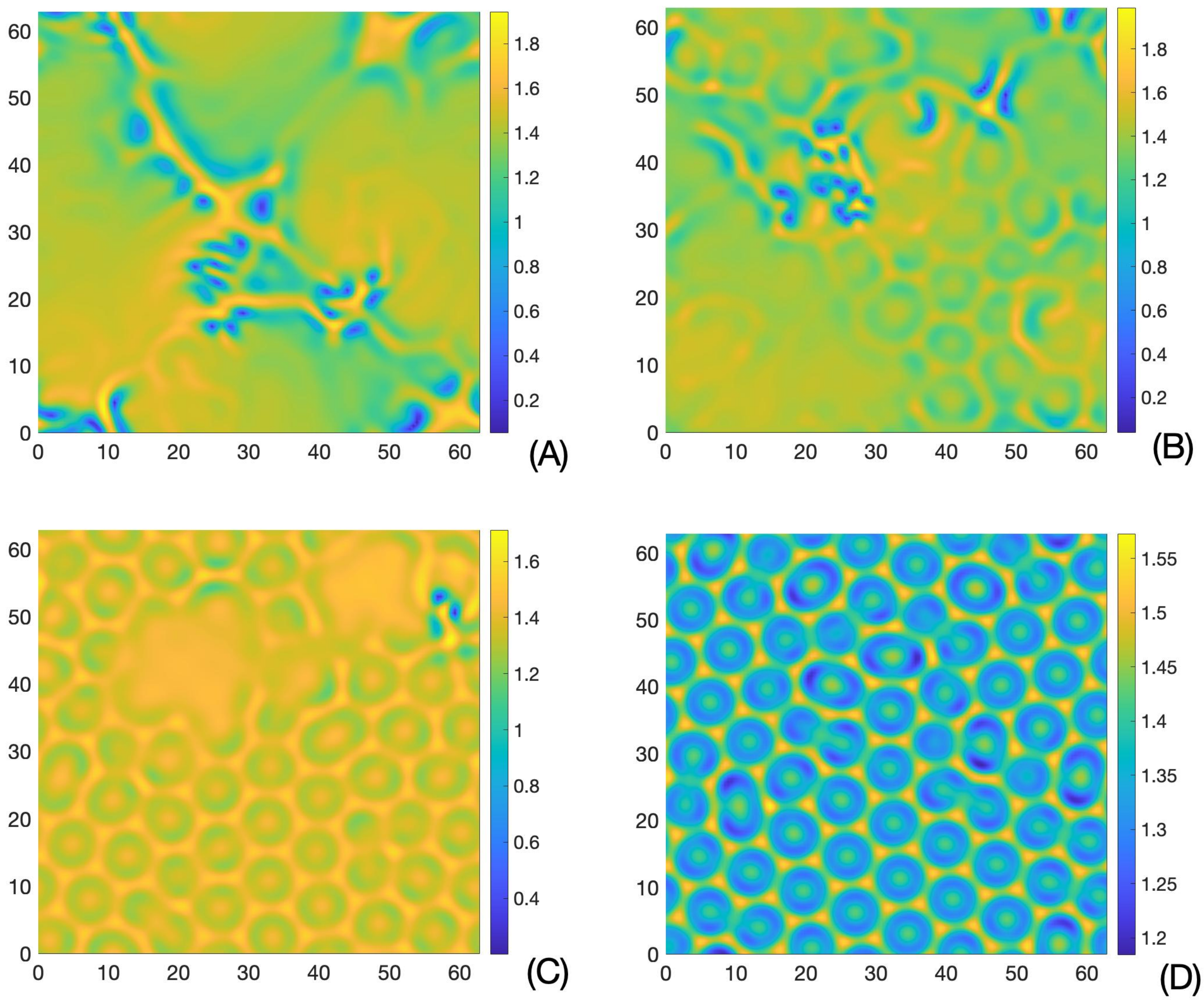

FIG. 4. (A) Transverse intensity in the DMT regime for $P = 3$, $\omega = 0.53$ after a quick quench from $E^2_{IN} = 0.273$ to 0.40; (B) Transverse intensity as for (A) with $E^2_{IN} = 0.40$ and after 410 time units; (C) Transverse intensity as for (A) with $E^2_{IN} = 0.40$ and after 700 time units; (D) Transverse intensity as for (A) with $E^2_{IN} = 0.40$ and after 1100 time units. Patches of the honeycomb-like pattern already appear in (B) while an irregular pattern with no vortex defects is well formed in (D).

For completeness we show in Fig. 4 the dynamics of the LIS when crossing the boundary between DMT and the pattern state by a sudden increase of the injection intensity from 0.273 to 0.40. At the beginning, DMT persists (see Fig. 4 (A)) but around 410 time units patches of the honeycomb-like patterns can be detected (see Fig. 4 (B)) and progressively grow until vortex defects are confined to very narrow regions of the transverse plane (see Fig. 4 (C) after 700 time units). Finally an irregular honeycomb-like pattern takes over (see Fig. 4 (D) after 1100 time units) with the laser intensity well above zero over the entire transverse plane. No defect vortices reappear at any later time of the simulation. The evolution to a regularly spaced honeycomb-like pattern takes a much longer time than the disappearance of the vortex defects. Moving from a spatio-temporal disordered state to a regular structure can be seen as a process of crystallisation in a non-equilibrium system.

V. CONCLUSIONS

Following a single logical thread that started from the joint work with Tito Arecchi [1] to today, we have seen that the unlocked region of a LIS is full of extraordinary dynamical and disordered regimes. From frequency pushing states [1] to coexistence of conservative and dissipative behaviours [2], from the unfolding of co-dimension two bifurcations [12] to defect mediated turbulence [20]. In particular we have demonstrated here that there exist a clear transition line from a pattern state to DMT in simple LIS models for class A lasers. This may appear counter intuitive because an injection is expected to lock and regularise the dynamics instead of creating a turbulent regime that is not present for the unperturbed laser. Note that Chaté, Pikovsky and Rudzick had already noticed in 1999 that DMT regimes of the Ginzburg-Landau equation can extend to injections different from zero [27]. The LIS case is even more striking since the DMT regime here cannot be obtained as an extension of the DMT observed in the Ginzburg-Landau case in view of the absence of a diffusion term. The DMT regime in LIS is intrinsically related to the instabilities of the unlocked state.

Identifying DMT regimes in LIS is not simply a mathematical exercise. In [20, 21] it was shown that the high density of defect vortices in DMT regimes of almost constant total intensity implies the presence of rare, short-lived and extraordinarily high peaks of light known as optical rogue waves. Here we have shown that the inverse process of pattern crystallisation from DMT is possible when increasing the injection intensity. Analogues of processes of melting and crystallisation have recently been extended to systems outside the thermodynamic equilibrium with energy inputs and dissipation such as lasers. The LIS equation (3) can become the prototype model to investigate, for example, novel physical properties of the photon fluid, analogues of non-equilibrium BKT transitions [28] and, possibly, negative temperature states [29, 30]. That would complete for me the parabola of the interconnections between statistical and laser physics that Antonio Politi, Roberto Livi, Stefano Ruffo and, of course, Tito Arecchi taught me so many years ago in Florence.

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