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Balancing load and performance for different failure models

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Abstract. In this paper, we discuss an optimal loading for items with lifetimes described by failure models that are popular in reliability and statistics. The obtained results can be relevant, e,g., for production/manufacturing systems. The expected productivity and the mission success probability are maximized with respect to the value of a load. It is shown that the optimal load for the considered settings is not necessarily the load that maximizes the production rate. The crucial function in our discussion is the production rate over the load. It is shown that, depending on the model, the optimal load can be equal, smaller or larger than the value of the load that achieves the maximum of this function. The accelerated life and the proportional hazards failure models are considered, as well as the additive hazards model. Illustrative examples confirm our theoretical findings.

Keyword: Optimal load; expected productivity; mission success probability; accelerated life model; proportional hazards model

1. Introduction

We consider systems with lifetimes that along with their reliability characteristics (e.g., the expected lifetime or the survival function) are described also by the quality of performance indices. For example, various production systems, chemical and other processes, transportation systems (e.g., gas, oil), systems performing certain additive operations (e.g., computations) can be characterized by the productivity/efficiency indices such as the production rate (production in a unit interval of time) or expected production till failure. The latter is already a complex characteristic that also takes into account loading/regime of a system. We will use the terms "load" and "regime" interchangeably. Obviously, at most instances, the heavier load results in a larger productivity, but at the same time, it leads to reduction of a system's lifetime. Thus, for achieving maximal production, or accomplishing a relevant mission task, a *balance between reliability and productivity* should be achieved by optimizing the operational load. The discussion of the corresponding optimal loading problem, its properties and relevant comparisons is the goal of this paper.

We suggest and describe the *new approach* to considering the described problem that employs specific lifetime models that are popular in reliability and statistics. We will mostly deal with two important and widely used in practice models that describe the impact of a regime/loading on the

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lifetimes of systems, namely, the ALM (accelerated life model) and the PH (proportional hazards) model (see, e.g., Meeker and Escobar[1], Bagdonavicius and Nikulin[2], Finkelstein[3] and Nelson[4]), whereas the additive hazards model will be also briefly discussed. As already mentioned, loading also impacts the corresponding productivity, i.e., the production rate (PR) becomes a function of a load, which is natural for the considered class of systems. This function should be specified for relevant systems, however, its monotonicity properties are rather general, which is important for the corresponding analysis to follow. It is natural to assume that, at first, for small values of a load, production cannot be put into effect due to technical and economic reasons, therefore, it is set as 0 in this interval (although, not necessarily, and for modeling purposes it can start increasing from 0, as we will have also in our examples). Then it starts to increase, reaches some maximum and then usually decreases or stays on the maximal level for a specified bounded load range defined by technical specifications. It can also continue to increase in this range but usually with a smaller 'slope' than previously. At some instances, the maximal allowed load can also result in the maximal PR. Thus, for many production systems, the PR function or its analogue should be a part of their technical specifications or technological description. Combining this specified pattern with the impact that the load has on the corresponding lifetimes, we will formulate the optimal problem of maximizing the expected production of a system until its failure or until completion of a mission/task.

In Finkelstein et al[5], the cost-wise optimization problem was discussed for missions with a possibility of abortion and change in load at inspection. The important feature of this model was that the mission *time was fixed*, whereas the amount of work to be accomplished was not set. However, at many instances this time is not fixed and depends on how effective/productive the system operates, whereas the productivity depends on the loading (regime). A mission is accomplished when a certain amount of work, A (e.g., operations) is completed, whereas the corresponding optimal strategy should maximize the probability $P(A)$ of this event. The failure of a non-repairable system during the mission results in the failure of a mission (the accomplished work is discarded).

In the literature, one can find quite a number of publications (mostly of a computational, algorithmic nature) devoted to optimization of load for maximizing reliability or other performance measures in complex binary and multistate systems. For instance, Levitin and Amari (2009) present an algorithm for determining an optimal loading of elements in series-parallel systems. Levitin and Finkelstein[6] deal with a similar problem for system subject to shocks. Levitin et al[7] considers an optimal choice of load of components in the 1 out of N cold standby system. It is assumed that each component can operate under different levels of load and that the "lifetime acceleration factor" of each component is also affected by load. Xiao et al.[8] discussed how to balance the load and protection of system's elements along with the relevant reliability characteristics (see also Xiao et al.[9] and Gajpal and Nourelfath[10] for other algorithmic solutions for the optimal loading in specific multicomponent systems). Some general discussions can be found in Amari et al[11], Iyer and Rosetti[12], Kapur and Lambersom[13].

However, to the best of our knowledge, this problem was not approached so far in the suggested general way employing the ALM and the PH model as the basic failure models and comparing optimal solutions with the value of load that achieves the maximum of the introduced simple characteristic function that is defined as the PR over the load. The latter also *provides some simple bounds* for the optimal load that can be effectively used in practice as rough estimates. Distinct from the referenced papers, we are considering a system as a whole and are interested in establishing the inherent stochastic properties of the models and of the corresponding optimal solutions (see also Filus[14], where a special case of an exponential lifetime distribution and a power function for the effect of a load on the corresponding failure rate were considered).

A proportional load-productivity relationship can be appropriate, e.g., for a pipe transmitting fluid (e.g., oil) in a laminar flow mode, where the pipe pressure is proportional to the velocity of the flow. However, the pressure of the pipe and the velocity of the flow can have a nonlinear relationship if the fluid is in a turbulent flow mode (Levitin and Amari[15]). Similar considerations can be applied (with appropriate changes) to gas pipes. There can be numerous other examples involving load-productivity relationships, however, at many instances in practice the load is not variable and therefore, the corresponding optimization problem is not relevant.

The paper is organized as follows. In Section 2 optimal loading for maximizing the expected production is discussed for the ALM and PH model. Section 3 considers maximization of the corresponding mission success probability. Concluding remarks are given in Section 4.

2. Optimal load for maximizing expected production

The effect of a regime/loading on a lifetime of a system can be modeled in different ways. The most popular and effective models in reliability and statistical analysis are the ALM (accelerated life model), which is, in fact, a time scale transformation and the proportional hazards (PH) model. The latter describes the direct impact of the load on the failure rate of a system. The larger load usually results in a shorter lifetime, but at the same time, in a larger productivity. Therefore, the corresponding optimization problem (optimal value of load) is relevant.

2.1. Accelerated life model

Assume that the cumulative distribution function (Cdf) of a lifetime T in the baseline regime/load is $F(t)$ with the pdf $f(t)$. Let the impact of other regimes be modeled by the ALM $F(rt)$, where formally $0<r<\infty$. Note that, in practice, there always exists a maximal load for operation of a system, i.e., $0<r<r_{\max}<\infty$. When $0<r<1$ the regime is lighter than the baseline and, obviously,

$$\begin{aligned} &F(rt) \le F(t), 0<r<1, \\ &F(rt) \ge F(t), 1 \le r<\infty. \end{aligned} \tag{1}$$

We are considering an item/system that is described by a certain productivity in a unit interval of time $\pi(r)$, i.e., the *production rate* (PR). For instance, in our example in the Introduction, it can be an amount of gas or oil transmitted by a pipe in a minute. Assume the same linear relationship (as for the scale transformation factor in the ALM) holds for productivity rate as a function of load, i.e., $\pi(r)=kr\pi, k>0$, where k is a constant of proportionality. For convenience of notation, let $k=1$. On the other hand, the mean time to failure of a system under the ALM is

$$E[T_r]=\int_0^\infty \bar{F}(ry)dy=\frac{\int_0^\infty \bar{F}(y)dy}{r}$$

where, T_r is the lifetime of a system with the Cdf $F(rt)$ ($T_1 \equiv T; \bar{F} \equiv 1-F$). Then the *expected* production in $[0,\infty)$, $\Pi(r)$ is the product $\pi r E[T_r]$. Thus, obviously, in this specific case, it does not depend on the regime, i.e.,

$$\Pi(r) \equiv \Pi = \pi \int_0^\infty \bar{F}(y)dy. \tag{2}$$

Note that this type of proportionality is rather specific, however, can happen in practice, at least, approximately.

Example 1. Weibull distribution. In this case, the baseline Cdf and the corresponding hazard (failure) rate are given by $F(t) = 1-\exp\{-(\lambda t)^\alpha\}$ and $\lambda(t) = (\alpha\lambda)(\lambda t)^{\alpha-1}$, whereas the corresponding life expectancy is

$$E[T] = \int_0^\infty \bar{F}(y)dy = \frac{1}{\lambda}\Gamma\left(1+\frac{1}{\alpha}\right), \lambda, \alpha > 0,$$

and $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$. Thus, the baseline scale parameter λ becomes $r\lambda$, whereas, obviously, the shape parameter α is not affected by this scale transformation in accordance with the ALM.

Although the lifetimes in different regimes in practice are often modeled by the linear ALM, productivity often is not necessarily proportional to *r,* as the PR starts to depend on r, i.e., $\pi(r)$. In this case, the expected productivity, that already depends on *r*, similar to (2), is

$$\Pi(r) = \pi(r)E[T_r] = \frac{\pi(r)}{r}\int_0^\infty \bar{F}(y)dy \tag{3}$$

It is important to note also that monotonicity analysis of $\Pi(r)$ as a function of *r* does not depend on the lifetime distribution $F(t)$.

Thus, the optimal load r_0 that maximizes $\Pi(r)$ in (3) should be obtained as the maximum of the following function

$$\pi(r_0)/r_0 = \max(\pi(r)/r), 0 < r < r_{\max} \tag{4}$$

The function $\pi(r)/r$ will play a pivotal role in our paper. Therefore, the next subsection is devoted to its analysis. Thus, our first obvious result can be formulated as the following proposition:

Proposition 1. *The maximal expected productivity of a system with the PR* $\pi(r)$ *under the described ALM is attained at the value of load that maximizes the function* $\pi(r)/r$.

2.2. Production rate and $\pi(r)/r$

The production rate $\pi(r)$

of $\pi(r)/r$. Each system can have its own PR function, however there are some general monotonicity patterns that are relevant for further analyses.

Let, for convenience, the normalized value $r=1$ results in the maximum of the initially increasing (which is a natural assumption) PR function. In some situations, it can be the 'full load' (as, e.g. the number of operating computers in a computational network) and, therefore, the values $r>1$ are not even defined. At other instances (e,g., the electrical load), the values $r>1$ are defined, but the PR is not increasing in this interval (for instance, it can decrease showing that $r=1$ is indeed optimal when not taking into account reliability characteristics). However, at some instances it can also increase. In any case, the support of $\pi(r)$ is bounded from above and this boundary is defined by actual properties of a system that can operate under different values of load. We will describe now two model cases that describe the pattern of the increasing $\pi(r)$ in $[0,1)$. Note that, for $r>1$, where applicable, we mostly assume that the PR is constant, i.e., $\pi(r)=\pi(1)$, or decreasing. On the other hand, just for completeness, the increasing $\pi(r), r>1$, will be also considered at some instances.

In the first example to follow, we assume that $\pi(r)$ is increasing from $\pi(0)$ and *convex* in $[0,1)$. In this case, it can be easily shown that the function $\pi(r)/r$ is increasing $[0,1]$, as, obviously, e.g., for $\pi(r)=r^k, k>1$. Thus, $r=1$ is the load that maximizes $\Pi(r)$ in (3) in this interval. The second practical example describes the function that is already not convex $[0,1]$. These two patterns of $\pi(r)$ will be used as basic for the specific failure models throughout our paper.

Example 2a. It can be reasonable to assume that $\pi(r)=0$ for $r\in[0,r_{\min}]$ and then it starts to increase, e.g., linearly to $\pi(1)$. Then it can still increase linearly or be constant, i.e., $\pi(r)=\pi(1)$ for $r>1$. The latter has a practical meaning as further increase of load does not increase productivity rate at many instances. Obviously, $\pi(r)/r$ is then decreasing in this interval. In accordance with the given description:

$$\pi(r)=\begin{cases}0, & 0<r\le a<1\\ k(r-a), & a<r\le 1, k>0\\ \tilde{k}(r+b), & 1<r, 0\le\tilde{k}<k,\end{cases} \tag{5}$$

where $b=\frac{k}{\tilde{k}}(1-a)-1$

Thus, $\pi(r)/r$ for the $\pi(r)$ given by (4) is 0 in the first interval, is increasing in the second interval and is:

1. Decreasing in the third interval if $k(1-a)-\tilde{k}>0$ (when $\tilde{k}=0$, we must set $\pi(r)=\pi(1)$.

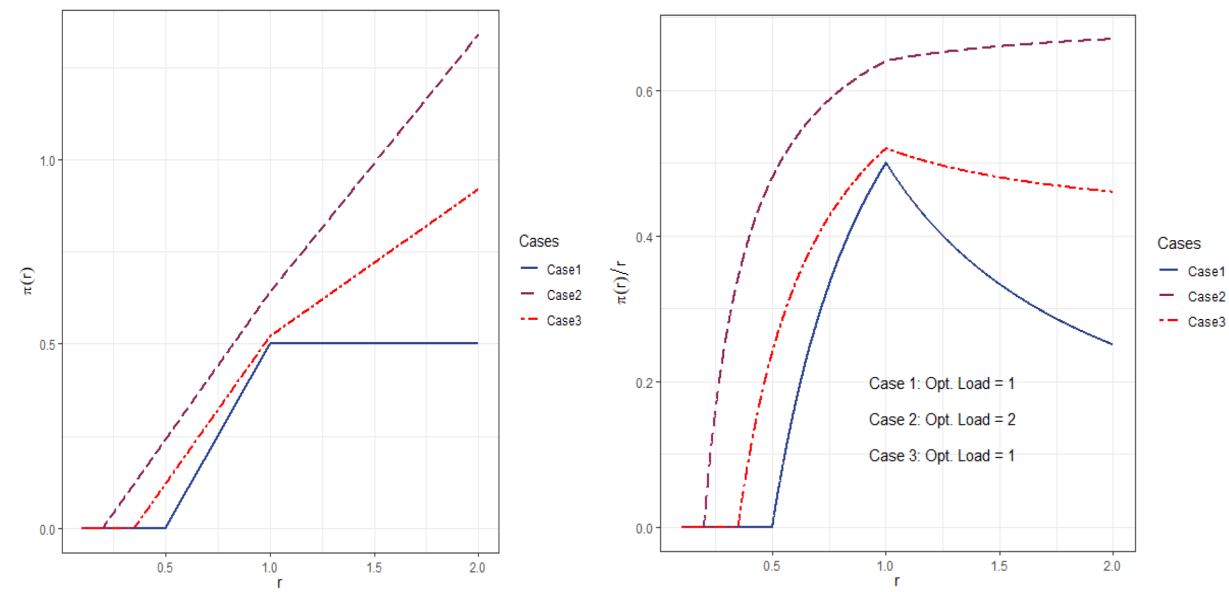
2. Increasing in the third interval when $k(1-a)-\tilde{k}<0$ as b is negative in this case. Thus, the obvious decision in this formal case is to apply the maximal technologically allowed load.

See, the corresponding illustration for the specific values of parameters in Fig. 1 (left)

Example 2b. As another possibility for illustrations, suppose that $\pi(r)$ follows a 'half-bell-shaped' curve. Note that, distinct from the previous case, this function is not convex in [0,1] (Fig.2 left)

$$\pi(r) = \begin{cases} 0, & 0 < r \le a \\ k \exp\left(-\frac{(r-1)^2}{\tilde{k}^2}\right), & a < r \le 1,\ k, \tilde{k} > 0 \\ k, & 1 < r,\ k, \tilde{k} > 0 \end{cases} \tag{6}$$

Figs 1 and 2 illustrate the functions $\pi(r)$ (left) and $\pi(r)/r$ (right) for the specific values of parameters for both examples accordingly. We see that for the PR given by (5), $\pi(r)/r$ attains its maximum in [0,1] at r=1 (see also Remark 2 to follow), whereas for (6) this maximum is smaller than r=1 (as the half-bell-shaped curve is not convex in this interval).



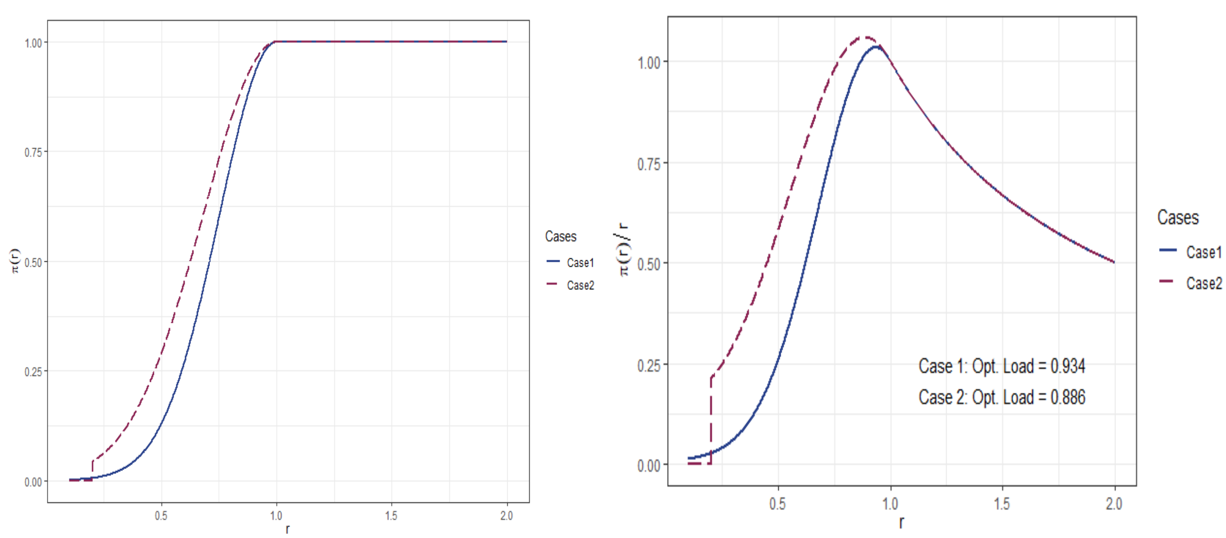


Fig. 2. $\pi(r)$ values (left) for (6), $a = 0$, $k = 1$, $\tilde{k} = 0.35$ (Case 1), and $a = 0.2$, $k = 1$, $\tilde{k} = 0.45$ (Case 2); $\pi(r)/r$ values (right) for the same cases.

The following remark, which is *important for all our further discussions,* refers to the shape of $\pi(r)$ when attaining its maximum.

Remark 2. In (5), for convenience of illustration, the step-linear functions $\pi(r)$ was considered. This means that the derivative of the function $\pi(r)/r$ is not continuous at change points. Specifically, when it reaches its maximum at r_0. Therefore, when $\pi(r)/r$ is multiplied by another monotone function of r (as will be done at many occasions in what follows), the analysis of this product needs additional assumptions. On the other hand, in practice, as for our second example (6), there is no reason (except for simplicity of modeling) to assume that $\pi(r)$ is 'edgy' as the corresponding processes are "smooth". Therefore, *we will assume in what follows that* $(\pi(r)/r)'$ *is differentiable* (or, equivalently that $\pi(r)/r$ is twice differentiable). In this case, e.g., if $\pi(r)/r$ is multiplied by the decreasing (increasing) function, the corresponding maximum of the product will be at the smaller (larger) value of load i.e., r_s ,i.e., $r_s < r_0 (r_s > r_0)$.

Assume now that we want to maximize the 'remaining expected production' in (x,∞) and not on the whole interval $(0,\infty)$. The reasons for that can be various, e.g., tests, experimental operation of a system in $[0,x)$, the increased demand for a product or expected increase in its price in (x,∞), etc. When considering the interval $(0,\infty)$, the loading that resulted in the maximum of $\pi(r)/r$ in (3) was optimal. Whether it is optimal in this setting? To answer this question, consider the remaining lifetime T_x for the considered ALM, that is, when the lifetime distribution is defined by $\bar{F}(rt)$. Therefore, the survival function of T_x is given by

$$\Pr[T_x > t] \equiv \bar{F}_x(t) = \frac{\bar{F}(r(x+t))}{\bar{F}(rx)}. \tag{7}$$

Thus, the remaining expected productivity can be defined as

$$\Pi_x(r) = \frac{\pi(r)\int_{rx}^{\infty}\bar{F}(y)dy}{r\bar{F}(rx)}. \tag{8}$$

Comparing (8) with (3), we see that now the optimal r obtained from (3) not necessarily maximizes (8) and, as it will be shown, a switch to a *lighter regime* could increase the remaining expected productivity.

Proposition 2. *Let* $F(t)$ *belong to the DMRL (decreasing mean remaining lifetime) class. Then the maximum of the remaining expected productivity* $\Pi_x(r)$ *under the ALM is attained at the value smaller than that for the function* $(\pi(r)/r)$.

Proof. Note that the mean remaining lifetime (MRL) of the baseline distribution $F(t)$ is given by

$m(t)=\frac{\int_t^\infty \bar{F}(y)dy}{\bar{F}(t)}$. Then, $\Pi_x(r)$ in (8) can be written as

$$\Pi_x(r)=\frac{\pi(r)}{r}m(rx).$$

Note that, if $F(t)$ belong to DMRL class, $m(t)$ is decreasing in t and thus

$$m(rx)=\frac{\int_{rx}^\infty \bar{F}(y)dy}{\bar{F}(rx)}$$

is decreasing in r. Thus, *depending on parameters,* the value of load which is smaller than that for (3) is optimal. Indeed, for $r>r_0$ (see Remark 2), both functions (multipliers) $\pi(r)/r$ and $\int_{rx}^\infty \bar{F}(y)dy/\bar{F}(rx)$, respectively are decreasing in r. When $(\pi(r)/r)'$ is differentiable, as follows from Remark 1, this property does not depend on parameters.

■

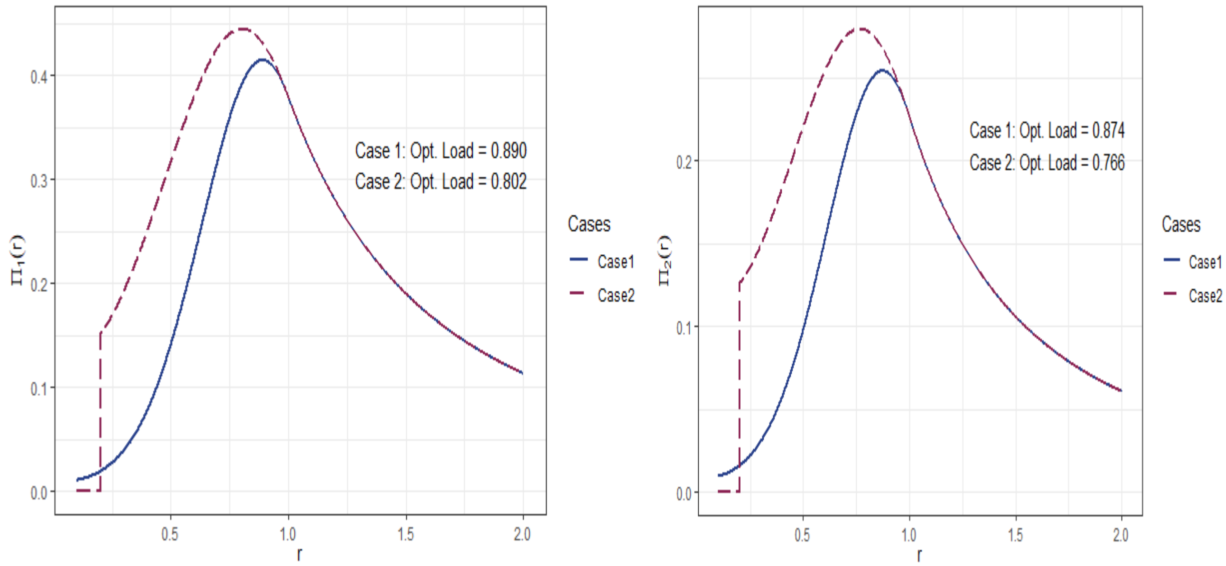
Remark 3. For the case when $(\pi(r)/r)'$ is not differentiable, for the optimal value to be smaller than that for (3), the following sufficient condition for the fixed x and $r=r_0$ should hold: $\frac{\partial \Pi_x(r)}{\partial r}<0$. Otherwise, the maximum is the same as for $\pi(r)/r$, i.e., r_0. Note that, the partial derivative should be taken from the left, as $[\pi(r)/r]'$ cannot be continuous at $r=r_0$.

Thus, the main message of the foregoing discussion is that the optimal value for the remaining expected productivity can differ (smaller under the considered assumptions) from that obtained by maximizing (3). The following example for the half-bell-shaped PR (differentiable $(\pi(r)/r)'$) illustrates our reasoning.

Example 3. We consider Weibull distribution for the system's lifetime in the same notation as in Example 1. Thus, in this case

$$\int_{rx}^\infty \bar{F}(y)dy/\bar{F}(rx)=\frac{1}{\alpha}\exp\{\tau\}\Gamma\left(1+\frac{1}{\alpha}\right)\left[1-\Gamma_\tau\left(\frac{1}{\alpha}\right)\right]\Gamma\left(\frac{1}{\alpha}\right);\ \tau=(\lambda rx)^\alpha\ , \qquad (9)$$

where $\Gamma_\tau(t)=\int_0^\tau x^{t-1}e^{-x}dx$ is the incomplete gamma-function. It can be shown that (9) is decreasing in x (and, therefore, in r as well) for $\alpha>1$ (when the corresponding failure rate is increasing) (Lai and Xie[16]).



Comparing Fig. 3 with Fig 2 (right), we see that, as stated in Proposition 2, the optimal values of load for the expected remaining productivity are smaller than those for the initial expected productivity. Numerical experiments for the PR given by (5) are also similar. We also performed experiments for other values of x that prompt that the expected remaining productivity is decreasing in x for the IFR lifetimes distributions. Although it looks intuitively plausible, we were not able to prove this fact analytically so far. Thus, for the given criterion of maximal remaining expected productivity, the initially optimal load ($x=0$) can be adjusted with the increase of the performance time without failures.

2.3. Proportional and additive hazards models

Another very popular in reliability and statistics model that describes the impact of the regime/load on a lifetime of a system is the proportional hazards (PH) model when environment acts directly on the failure rate as a multiplier. In fact, in all referenced in the Introduction and computational in nature papers some relevant specific cases of this model were employed. Thus, in accordance with this model, the expected production in $[0,\infty)$ is (compare with (3)),

$$\Pi(r) = \pi(r)\int_0^\infty (\bar{F}(x))^r\,dx\,. \tag{10}$$

For instance, in the simplest case, when $\pi(r)=0$ in $[0,a)$, and it is constant afterwards, the optimal solution maximizing (10) is just $r=a$, as further increase in r only decreases the integral. Distinct from (3) the *analysis* of $\Pi(r)$ in (10) depends now on the lifetime distribution and, therefore, its maximal value can be achieved at values different from that for the function $\pi(r)/r$. For conformity with (3) and for convenience in further reasoning, let us write (10) as

$$\Pi(r) = \frac{\pi(r)}{r}\int_0^\infty r(\bar{F}(x))^r\,dx\,. \tag{11}$$

Denote the cumulative failure rate that corresponds to $F(t)$ by $\Lambda(t) = \int_0^t \lambda(u)du$.

Proposition 3. *The maximum of the expected productivity* $\Pi(r)$ *under the PH model is attained at the value larger than that for the function* $(\pi(r)/r)$.

Proof. Using the foregoing notation, the integral in (11) can be written as

$$\int_0^\infty r(\bar{F}(x))^r dx = \int_0^\infty \frac{r\lambda(x)\exp\{-r\Lambda(x))}{\lambda(x)}dx .$$

Note that, the nominator in the integrand is the density that corresponds to the survival function $(\bar{F}(x))^r$ and, therefore, its integral is 1. Assume, as in the case of the ALM, that $F(t)$ is IFR (strictly monotone failure rate). Then, for the fixed *r*, due to the weighted mean value theorem for integrals (the integral in (10) is finite), there exists some $0 < x_r < \infty$ such that

$$\int_0^\infty \frac{r\lambda(x)\exp\{-r\Lambda(x)\}}{\lambda(x)}dx = \frac{1}{\lambda(x_r)} . \tag{12}$$

With the increase of *r*, the failure rate that corresponds to $(\bar{F}(x))^r$ (i.e., $r\lambda(x)$) is increasing, thus the density becomes more skewed to the origin. On the other hand, the function $1/\lambda(x)$ is decreasing. Therefore, the corresponding mean value of this function is increasing in *r* and so does the integral $\int_0^\infty r(\bar{F}(x))^r dx$. Thus, the larger optimal value of load than the load that maximizes the function $\pi(r)/r$ is now optimal ($(\pi(r)/r)'$). This, obviously, can be applied to the settings when this larger load can be administered in practice. Otherwise, the value that results in the maximum of $\pi(r)$ should be taken as an operational one (it was $r = 1$ in Example 2).

■

The additive hazards model is probably less applied in reliability modeling; however, an impact of environment or regime can be described effectively with this model as well. In our notation, the failure rate for this model is defined as

$$\lambda(x,r) = \lambda(x) + cr , \tag{13}$$

where $c > 0$ is the coefficient of proportionality. Accordingly, (11) can be written as

$$\Pi(r) = \frac{\pi(r)}{r}\int_0^\infty r\bar{F}(x)\exp\{-crx\}dx , \tag{14}$$

where $F(x)$ is the baseline Cdf with the failure rate $\lambda(x)$. Therefore,

$$\Pi(r) = \frac{\pi(r)}{r}\int_0^\infty \bar{F}(x)\frac{cr\exp\{-crx\}}{k}dx. \tag{15}$$

As $cr\exp\{-crx\}$ is the density of the exponential distribution and $\bar{F}(x)$ is decreasing, using the same arguments as when considering the PH model, we see that the integral in (15) is increasing in r. Thus, the larger optimal value of load than the load that maximizes the function $\pi(r)/r$ is again optimal ($(\pi(r)/r)'$ is differentiable). Thus, we have the following proposition.

Proposition 4. *The maximum of the expected productivity* $\Pi(r)$ *under the additive hazards model is attained at the value larger than that for the function* $(\pi(r)/r)$.

Example 4. We will illustrate numerically only the PH model for the PR given by (6) (differentiable $(\pi(r)/r)'$) and the Weibull lifetime model of the previous examples. Then (10) can be written as

$$\Pi(r) = \pi(r)\int_0^\infty \exp(-r(\lambda y)^\alpha)\,dy.$$

Fig. 4 plots the corresponding curves. We see that the maximums are attained at the values larger than those for the function $\pi(r)/r$ (see Fig. 2 right).

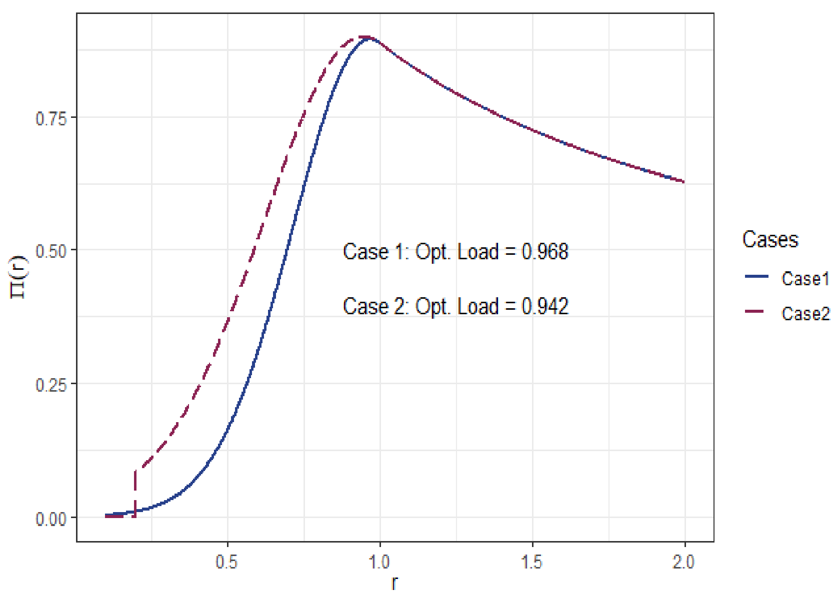


Fig. 4. $\Pi(r)$ for the half-bell-shaped $\pi(r)$ with $a = 0$, $k = 1$, $\tilde{k} = 0.35$ (Case 1), and $a = 0.2$, $k = 1$, $\tilde{k} = 0.45$ (Case 2) , for $\lambda = 1$, $\alpha = 2$.

3. Maximizing MSP

Maximization of the expected productivity considered in the previous sections is important for many applications. However, at many instances, we are interested also in maximization of the *mission success probability* (MSP), which is the probability of executing the predetermined amount of work A. What are the properties of the optimal regime for that?

Suppose that the operational regime of a production system is described by the linear PR, i.e., $\pi(r) = kr$, then we need the time A / kr to accomplish the mission, whereas the corresponding MSP for the ALM failure model, can be defined as

$$\bar{F}(rA / kr)) = \bar{F}(A / k)$$

and, similar to (2), it obviously does not depend on *r*.

However, if a more realistic PR characterizes the production pattern of a system, the MSP can depend on *r*. Thus, $\bar{F}(rA / \pi(r)) = \bar{F}((r / \pi(r))A)$ should be maximized, and because $\bar{F}(t)$ is monotonically decreasing, its maximum is achieved when the function $\pi(r) / r$ achieves its maximum (or, equivalently, $r / \pi(r)$ achieves its minimum), Thus, similar to Proposition 1, we have the following result

Proposition 5. *The maximal MSP of a system with a fixed amount of work and the PR* $\pi(r)$ *under the ALM is attained at the value of a load that maximizes the function* $\pi(r) / r$.

On the other hand, for the PH model, the maximum of the function $(\bar{F}(A / \pi(r)))^r$ should be achieved. Thus,

$$(\bar{F}(A / \pi(r)))^r = \exp\{-r\Lambda(A / \pi(r))\} = \exp\left\{-\frac{Ar}{\pi(r)} \frac{\Lambda(A / \pi(r))}{A / \pi(r)}\right\} \tag{16}$$

attains maximum at some value r_s and it is equal to the value that brings the maximum of $\pi(r) / r$ only for the case of the constant failure rate i.e., $\lambda(t) = \lambda$. Note that by $\Lambda(\cdot)$, as previously, we denote the corresponding cumulative failure rate.

Proposition 6. *Let the lifetime of a system performing a fixed work A be IFR. Then its maximal MSP under the PH model is attained at the value of a load larger than that for the function* $(\pi(r) / r)$.

Proof. As the baseline Cdf is assumed to be IFR, it also belongs to the IFRA (increasing failure rate in average). It is well-known that in this case, the function $\Lambda(t) / t$ is increasing in *t*. This means that $\Psi(r) = \dfrac{\Lambda(A / \pi(r))}{A / \pi(r)}$ as a function of r in (16) is decreasing when *r* is increasing (as $\pi(r)$ is increasing in *r*). Thus, as $r / \pi(r)$ has a minimum at r_0 and $\Psi(r)$ is decreasing, the maximum of $(\bar{F}(A / \pi(r)))^r$ can be a achieved at *larger values* r_s, i.e., $r_s > r_0$ ($(\pi(r) / r)'$ is differentiable). This is similar to what we had for optimization of the expected production and differs from the case of the ALM, as now the optimal loading can be larger than that for the ALM. Note that, obviously, the corresponding decision can be implemented in the systems where larger than r_0 values of the load can be realized in practice.

■

Denote by $F(x, r)$ the Cdf for the additive hazards rate model defined by the failure rate (13). Then the corresponding MSP takes the form

$$\bar{F}(A/\pi(r),r)=\exp\left\{-\int_0^{A/\pi(r)}(\lambda(x)+cr)dx\right\}=\exp\left\{-\left(\Lambda(A/\pi(r)+cAr/\pi(r)\right)\right\}. \qquad (17)$$

Thus, as $\Lambda(A/\pi(r)$ is decreasing in r the maximum of $\bar{F}(A/\pi(r),r)$ is also attained at a larger values than the maximum of $\pi(r)/r$ (minimum of $r/\pi(r)$ in (17)). It is remarkable that we do not need the IFR requirement here.

Example 5. As previously, we will illustrate numerically only the PH model for the PR given by (6) (differentiable $(\pi(r)/r)'$) and the Weibull lifetime model ($\lambda=1$, $\alpha=2$). Denote in this case,

$$\Omega(r)=\frac{Ar}{\pi(r)}\frac{\Lambda(A/\pi(r))}{A/\pi(r)}=A\frac{r}{\pi(r)}\lambda\frac{A}{\pi(r)}.$$

Fig. 5 illustrates Proposition 6, showing that the optimal values of load in the considered in this section case are larger than those for the function $\pi(r)/r$ (compare with Fig.2 right). The minimum obviously does not depend on A and the three curves are just given for better visibility.

Note that our Examples 3-5 are just illustrative for the range of parameters involved, because the statements in the corresponding Propositions 2-6 are proved. On the other hand, using a function $\pi(r)/r$ as a rough bound for obtaining approximate values for the optimal load is very appealing as it does not involve any parameters of the lifetime model. The practical aspects of the possibility of this approximation including sensitivity analyses should be investigated in the future applied research. Here, on the other hand we have developed the theoretical basis for this.

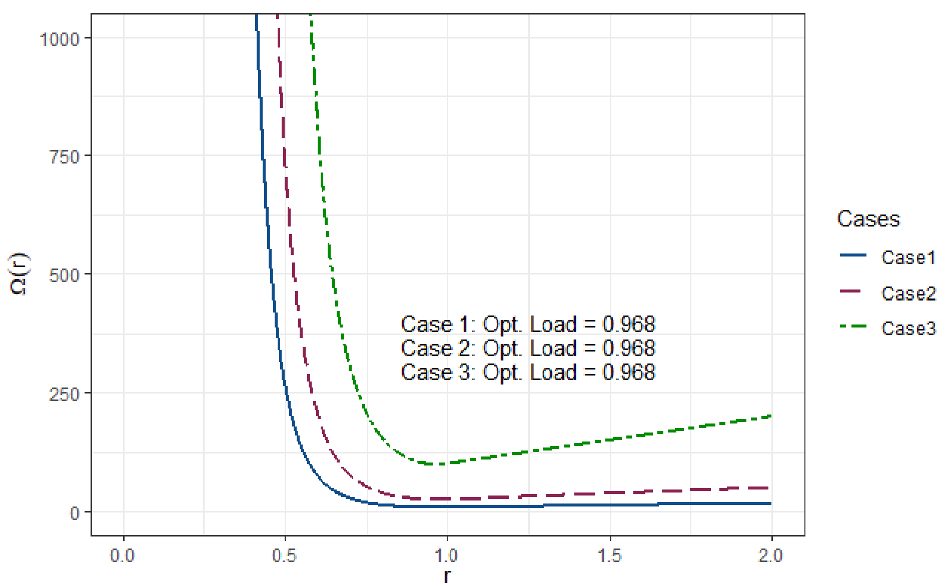


Fig. 5. $\Omega(r)$ values for half-bell-shaped $\pi(r)$, $A=3$, $a=0$, $k=1$, $\tilde{k}=0.35$ (Case 1), $A=5$, $a=0$, $k=1$, $\tilde{k}=0.35$ (Case 2), $A=10$, $a=0$, $k=1$, $\tilde{k}=0.35$ (Case 3), for $\lambda=1$, $\alpha=2$.

4. Concluding remarks

We consider systems that are characterized by their productivity. The larger load results in a larger productivity but, at the same time, it can lead to reduction of system's lifetimes. Thus, for achieving maximal production or accomplishing a mission task, a balance between reliability and productivity should be achieved by optimizing the operational load. In this paper, we discuss some basic underlying properties of the corresponding optimal solutions for the cases when the impact of a load on lifetimes is described by the popular in practice accelerated life model, the proportional hazards model and the additive hazards model. As far as we know, the optimal loading problem in the suggested set-up was not considered in the literature so far.

Under the described models, the expected productivity and the mission success probability are studied and some properties of optimal solutions for the load are discussed. The pivotal function in our analysis is the production rate over the load, i.e., $\pi(r)/r$. It is shown that, depending on the model, the optimal load can be equal, smaller or larger than the value of the load that achieves the maximum of $\pi(r)/r$. Specifically, for maximization of the expected remaining productivity under the ALM, the optimal load is smaller than the value that maximizes this function. On the contrary, for maximization of the expected productivity and the mission success probability for the PH model, the optimal value for $\pi(r)/r$ turns to be the lower bound for the optimal load. The latter is very appealing in practice, as it defines *simple bounds* that does not depend on other parameters.

The practical aspects of the possibility of the foregoing approximation including sensitivity analyses should be investigated in the future applied research, whereas our paper has developed the theoretical basis for this analysis and has provided some illustrative examples.

There are some questions of a theoretical nature that have to be also addressed. For instance, although intuitively clear (and the performed numerical experiments justify it) that it is likely that the expected remaining productivity under the ALM is decreasing with the time of observation, we were not able to prove it analytically.

There can be other, e.g., environmental factors (covariates) that influence productivity and reliability of systems. Taking them into account in the framework of the ALM and PH model can constitute a topic for further research. Another direction of subsequent studies can include the relevant cost-wise analysis (e.g., maintenance costs) for obtaining more general optimal solutions.

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