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In this paper, we consider a history-dependent mixed shock model which is a combination of the history-dependent extreme shock model and the history-dependent -shock model. We assume that shocks occur according to the generalized Polya process that contains the homogeneous Poisson process, the non-homogeneous Poisson process and the Polya process as the particular cases. For the de ned survival model, we derive the corresponding survival function, the mean lifetime and the failure rate. Further, we study the asymptotic and monotonicity properties of the failure rate. Finally, some applications of the proposed model have also been included with relevant numerical examples.

Reliability, -shock model, extreme shock model, optimal replacement policy, optimal mission duration

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Most of the systems that are used in reality are directly or indirectly affected by some 'harmful' 'instantaneous' events (shocks of various nature), which either cause the system failure or decrease the system's lifetime. Thus, the study of system's lifetime subject to external shocks is one of the important problems in reliability theory. A large number of studies on different shock models and their applications could be found in the literature.

The existing shock models are mostly classified into four broad categories: extreme shock models, cumulative shock models, run shock models and δ-shock models. In the extreme shock model, a system fails when a single shock occurs with a critical magnitude (see, for instance, Gut and Husler [14, 15], Shanthikumar and Sumita [26, 27], Cha and Finkelstein [6], and the references there in). In the cumulative shock model, a system fails when the aggregate damage due to shocks exceeds the predetermined threshold value (see A-Hameed and Proschan [1], Esary et al. [11], Gut [13], to name a few). Further, in the run shock model, a failure of the system occurs when the magnitudes of consecutive shocks exceed the prefixed threshold value (see, e.g., Mallor and Omey [22], Ozkut and Eryilmaz [24]). Lastly, in the δ-shock model, a system fails if the time lag between two successive shocks is less than the predetermined threshold value δ, i.e., the recovery time of a system from a shock is δ (see Li, Chan and Yuan [17], Li, Huang and Wang [18], Li and Kong [19], and the references there in). It is worthy to mention that the δ-shock model is different, in nature, from other shock models because the δ-shock model deals mostly with the frequency of shocks, whereas magnitudes of shocks play the key role in other shock models. Apart from these shock models, there are various mixed shock models, which are the combinations of two or more shock models, namely, the extreme shock model with the cumulative shock model (Cha and Finkelstein [2]), the extreme shock model with the run shock model (Eryilmaz and Tekin [10]), the extreme shock model with the δ-shock model (Wang and Zhang [31], Parvardeh and Balakrishnan [25]), the cumulative shock model with the run shock model (Mallor et al. [23]), the cumulative shock model with the δ-shock model (Parvardeh and Balakrishnan [25]), the run shock model with the δ-shock model (Eryilmaz [7]), etc.

Even though the classical extreme shock model was intensively studied in the literature, its applications were limited due to the restrictive assumption that the system's survival probability at any time does not depend on the history of a shock process. Therefore, a history-dependent extreme shock model was proposed by Cha and Finkelstein [3]. In this model, the survival probability of a system at a given time depends on the number of shocks that the system has experienced in the past. Cha and Finkelstein [4] further generalized this model by considering that both the system s survival probability and the shock process depend on this history.

Similar to the classical extreme shock model, the classical- shock model has also been generalized in di erent directions in the literature. Wang and Zhang [31] have extended it by considering two types of failures. Further, Parvardeh and Balakrishnan [25] have studied this model based on a renewal process. Eryilmaz and Bayramoglu [8] have studied a -shock model based on the renewal process with uniformly distributed inter-arrival times. Wang and Peng [30] have considered a generalized -shock model with two types of shocks with two di erent threshold values and . Eryilmaz [9] have studied the -shock model based on the Polya process of shocks. Tuncel and Eryilmaz [29] considered the -shock model with non-identically distributed inter-arrival times. Recently, Lorvand et al. [21, 20] have generalized the mixed -shock model to the multi-state systems.

As far as we know and follows from the forgoing discussion, the xed was considered in all -shock models developed so far in the literature. In other words, the recovery time of a system from the damage of a shock, arrived at any time, is assumed to be xed. However, the assumption of a constant is non-realistic at many practical instances. Indeed,due to deterioration of a system, its recovery time from the damage caused by a shock often gradually increases as the number of shocks increases. Furthermore, it is assumed, in most of the studies, that shocks occur according to the homogeneous Poisson process (HPP) or the non-homogeneous Poisson process (NHPP). Note that these processes are characterized by independent increments,which is not the case at many instances in practice. For instance, the larger number of shocks in the past often implies the larger number of shocks in the future (positive dependence). Therefore, the main goal of this paper is to study a history-dependent mixed shock model which is a combination of the history-dependent extreme shock model and the history-dependent -shock model. The novelty in this paper is as follows.

( ) We consider a history-dependent -shock model, i.e., the recovery time of a system from a shock depends on the number of shocks arrived in the past.

( ) We combine this history-dependent -shock model with the history-dependent extreme shock model.

( ) Finally, we consider a more general shock process, with dependent increments, namely, the generalized Polya process (GPP) that contains the HPP, the NHPP and the Polya process as the particular cases.

The rest of the paper is organized as follows. In Section 2, we rst provide some preliminaries and then describe the model. In Section 3, we derive the survival function, the mean lifetime and the failure rate of a system. Further, we study the long-run behaviour and monotonicity properties of the corresponding failure rate. Some applications of the proposed model are discussed in Section 4. Finally, the concluding remarks are given in Section 5.

For any random variable , we denote the cumulative distribution function by ( ), the survival function by ( ), the probability density function (if exists) by ( ) and the failure rate function by ( ); thus, ( ) 1 ( ) and ( ) ( ) ( ).

Denote by ( ) 0 the point process of shocks, where ( ) is the number of shocks that have occurred by time . Assume that our system is absolutely reliable in the absence of shocks. Cha and Finkelstein [3] have studied a history-dependent extreme shock model when each shock results in its failure with probability (( )) and has no e ect with probability (( )) = 1 (( )). Thus, the survival probability of a system depends on the number of shocks ( ) occurred in [0). Further, it was assumed that (( )) = 1 (( )) = ( ) (( )), where ( ) is a decreasing function of ( ), for each xed t, and ( ) is the realization of ( ). On the other hand, Li et al. [17] studied the -shock model as discussed in the introduction section. In this paper, extending the previous studies, we consider a generalized version of the -shock model to be called the as depends on the history of the shock process. Moreover, we combine the history-dependent extreme shock model with the history-dependent -shock model and study the lifetime behaviour of a system under this mixed shock model governed by the GPP. In what follows in this section, we give the formal de nition of the GPP that contains the NHPP, the HPP and the Polya process as the particular cases. Due to the dependent increment property, the GPP can present a more adequate model in practice as compared with counting processes with independent increments.

( ) 0

( ) 0 0

( ) (0) = 0

( ) = (( ) +) ( )

( ) ( ) ( ) = 0 0 = 1

( ) ( ) 0 = 1

( )

( ) ( ) ( ) = 1 (+) 0 = 1

Let denote the lifetime of a system that has started operation at time = 0. The system is subject to external shocks that arrive according to the GPP with the set of parameters ( ) . Let 0 = be a sequence of the corresponding arrival times of shocks, and let = , = 1 2 , be the inter-arrival time between the -th and the (1)-th shocks. Let : 0 [0) be an increasing function of its argument, where ( ) = and is the set of natural numbers. Thus, the recovery time after a shock increases with the number of shocks occurred previously describing system s deterioration under shocks. In accordance with the suggested model, we assume that the system fails at the -th shock (0) when:

( ) ,

( ) (with probability ( )).

Accordingly, the conditional survival function is

$$( ) = ( ) ( )$$

$$= ( ) ( ) (+) \qquad (2.1)$$

where ( ) 1 when ( ) = 0, and ( ) is an indicator function, i.e.,

$$(\quad + \quad) = \begin{cases} 1 & \text{if} \quad + \\ 0 & \text{otherwise.} \end{cases}$$

Note that the history dependent extreme shock model (i.e., = 0, for all 0) and the history dependent -shock model (i.e., ( ) 1) are the particular cases of this model. Further, when ( ) 1 and = , for all 0 , then this model reduces to the classical -shock model with the constant recovery time .

In this section, we discuss some reliability characteristics of a system under the de ned mixed shock model.

We begin this subsection with the following lemma obtained in Cha [5]. This lemma will be used in proving the main result of this subsection.

(( )) 0 0

(( ))

$$(\quad) = \frac{(\quad + \;)}{(\quad)} \quad (\;) \exp \quad (\;) \quad \exp \quad (\; + \quad) \; (\;)$$

0 = 0 1 2

In the following theorem, we derive the survival function of a system for the mixed shock model as discussed above.

( )

0 0

$$(\;) = \exp \quad (\;) + \frac{(\;)}{\exp \; (\; + \quad) \; (\;)} \frac{(\quad + \;)}{(\quad)}$$

$$(\;) \; (\;) \exp \quad (\;)$$

$$(\;) = \max \quad 1 \qquad (\;) = \quad (\;) \qquad (\;) = \quad (\;)$$

Note that the system survives shocks in [0) provided + + . This implies that . In other words, if (or, ( )), then the probability of the event the system survives shocks till time is zero. Now,

( ) = ( )

= ( )

= ( ) ( ) (+)

( )

= exp ( ) + ( ) / exp (+) ( ) (+) / ( )

( ) ( ) exp ( )

where the third equality follows from (2.1) and the last equality follows from Lemma 3.1.

The following corollary follows from Theorem 3.1 by using Remark 2.1(b).

( ) =

0 = 1 0

( ) 1

( ) = exp 1 / log 1 + / ! 1 / log

( )

The next corollary is an immediate consequence of Theorem 3.1. Here we assume that shocks occur according to the GPP with the set of parameters ( ) = . Note that this GPP contains the HPP (( ) = 0 and = 1) and the Polya process (= 1) as the particular cases.

( ) =

0 0 0

( ) = / + 1 + ( ) (+) / ( ) / +

( )

( ) ( )

Consider now some special cases of corollary 3.2.

( ) When ( ) is a periodic function with the periodicity (i.e., (+) = ( ) for any 0) and = for all 0 , the survival function of the system is given by

$$( ) = \frac{}{+} \quad 1 + \quad ( ) \frac{(\quad + \quad)}{! \; (\quad)} \quad \frac{}{+} \qquad ( )$$

( ) When ( ) is an exponential function (i.e., ( ) = for some 0 1), the survival function of the system is

$$( ) = \frac{}{+} \quad 1 + \quad ( ) \frac{(\quad + \quad)}{! \; (\quad)} \quad \frac{}{+} \qquad \frac{1}{\log(\;)} \tag{3.1}$$

( ) When ( ) 1 (i.e., the system follows the history-dependent -shock model), the survival function of the system is given by

$$( ) = \frac{}{+} \quad 1 + \quad \frac{(\quad + \quad)}{(\quad)} \quad \frac{}{+} \quad \frac{}{!}$$

( ) When the -th recovery time is de ned as = for all 0 , 1, the survival function of the system is

$$( ) \; = \; \frac{}{+} \quad 1 + \quad ( ) \frac{(\quad + \quad)}{(\quad)} \quad \frac{}{+}$$

$$(\tfrac{}{}) \qquad \frac{}{} \qquad ( )$$

where $( ) = \frac{\overline{\quad}}{\quad}$.

( ) When shocks occur according to the HPP with intensity 0, the survival function of the system is given by

$$( ) = \exp \quad 1 + \quad ( ) \qquad ( )$$

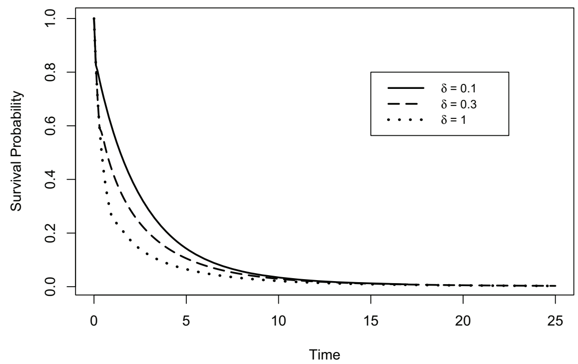

Figure 1: Plot of system's survival function against $t \in [0, 25]$, for fixed $\sigma = 0.9$, $\rho = 0.95$, $\beta = 2$, $\alpha = 1$, and $b = 1$.

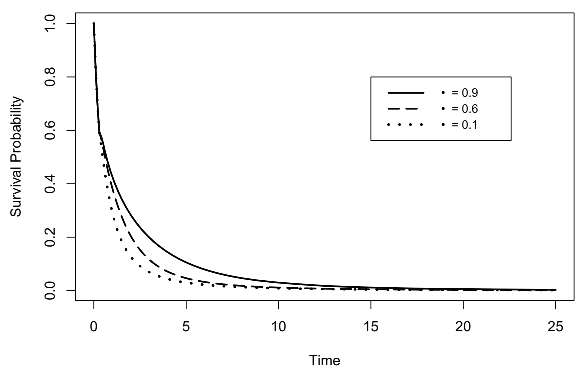

Figure 2: Plot of system's survival function against $t \in [0, 25]$, for fixed $\delta = 0.3$, $\rho = 0.95$, $\beta = 2$, $\alpha = 1$, and $b = 1$.

To illustrate the result in (ii), we plot the corresponding survival functions. Figure 1 shows that the system's survivability decreases as the recovery time δ increases (which is obvious). On the other hand, Figure 2 (by comparison with Figure 1) shows that the system's survivability increases as σ increases. This holds because an increment in σ implies a corresponding increment in the probability of the system's survivability after a shock.

3.2 Mean lifetime

In this subsection, we derive relationships for the mean lifetime of a system for the above discussed mixed shock model.

( )

0 0

( ) = exp ( ) + – ( ) / exp (+) ( ) (+) / ( )

– ( ) ( ) exp ( )

Since the system survives -shocks till time , we have . On using this, we can write

( ) = ( )

= (( ) = 0) + [(( ) = 0) + (( ) = 1)]

+ [(( ) = 0) + (( ) = 1) + (( ) = 2)] +

= (( ) = 0) + (( ) = 1) + (( ) = 2) +

= (( ) = 0) + – (( ) =) (3.2)

Further, from Theorem 3.1, we have

(( ) = 0) = exp ( ) (3.3)

and, for 1,

(( ) =) = ( ) / exp (+) ( ) (+) / ( )

– ( ) ( ) exp ( ) (3.4)

On using the above equalities in (3.2), we get the required result.

0

( ) = 0 1

( ) = 1/– 1 + ( ) – exp

( ) = —/–

From (3.3) and (3.4), we get

$$(\quad ( ) = 0) = \exp$$

and, for $\quad 1$,

$$(\quad ( ) = \quad) = \quad (\ ) \exp \quad \frac{}{!} \quad ^{-} \quad \frac{^{-} \quad 1}{\log(\ )}$$

On using the above equalities in (3.2), we get

$$(\ ) = \frac{1}{} + \quad _{-} \quad \exp \quad \frac{(\ )}{!} \frac{}{\log(-)} \quad ^{-} \quad 1 \quad ^{-}$$

$$= \frac{1}{} + \quad \frac{(\ )}{!} \frac{}{\log(-)} \quad ^{-} \quad \exp \quad \exp \quad 1$$

(3.5)

Now, consider

$$= \quad \exp \quad 1$$

Note that $\quad = 1 \quad$. Integrating by parts, we get

$$= \frac{\log(-)}{+ \quad \log(-)} \quad = \frac{1}{} \quad \frac{! \ \log(-)}{\log(-) \ +}$$

On using the above equality in (3.5), we get the required result.

To illustrate the result given in Corollary 3.3, Figure 3 shows that the mean lifetime changes as varies over (0 1].

$$0$$

$$= 1 \quad ( ) = 1 \quad =$$

$$(\ ) = \frac{1}{(1 \quad \exp \quad)}$$

In this section, we discuss the corresponding failure rate for the de ned mixed shock model governed by the NHPP.

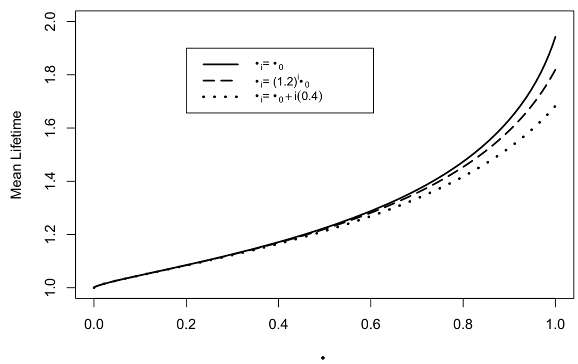

Figure 3: Plot of mean lifetime against $\sigma \in (0, 1]$, for $\lambda = 1$, $\delta_0 = 0.5$ and $\rho(i) = 0.8$, for all i.

Theorem 3.3 *Let shocks occur according to the NHPP with intensity $\lambda(t)$. Assume that $\delta_i = \delta$ for all $i \in \mathbb{N} \cup \{0\}$ and $\rho(j) = \rho$ for all $j \in \mathbb{N}$. Then the failure rate of a system for the defined mixed shock model is given by*

$$r_L(t) = \begin{cases} \lambda(t), & \text{if } 0 < t < \delta \\ \lambda(t)\left[1 - \rho q(t) \exp\left\{-\int_{t-\delta}^{t} \lambda(x)dx\right\} \frac{\bar{F}_L(t-\delta)}{\bar{F}_L(t)}\right], & \text{if } t \geq \delta. \end{cases}$$

Proof: From Theorem 3.1, we have

$$\bar{F}_L(t) = \exp\{-\Lambda(t)\} + \sum_{n=1}^{\lfloor \frac{t}{\delta} \rfloor} \rho^n \exp\{-\Lambda(t)\} \int_{n\delta}^{t} \int_{(n-1)\delta}^{t_n - \delta} \cdots \int_{\delta}^{t_2 - \delta} \left(\prod_{i=1}^{n} q(t_i)\lambda(t_i)\right) dt_1 \ldots dt_n,$$

where $\Lambda(t) = \int_0^t \lambda(x)dx$. Now consider the following cases.

Case I: Let $0 < t < \delta$. Then $\bar{F}_L(t) = \exp\{-\Lambda(t)\}$, and hence $r_L(t) = \lambda(t)$.

Case II: Let $\delta \leq t < 2\delta$. Then $\bar{F}_L(t) = \exp\{-\Lambda(t)\} + \rho \exp\{-\Lambda(t)\} \left(\int_\delta^t q(x)\lambda(x)dx\right)$, which gives

$$\begin{aligned} f_L(t) &= -\left[-\lambda(t)\exp\{-\Lambda(t)\} - \rho\lambda(t)\exp\{-\Lambda(t)\}\left(\int_\delta^t q(x)\lambda(x)dx\right) + \rho\exp\{-\Lambda(t)\}q(t)\lambda(t)\right] \\ &= \lambda(t)\bar{F}_L(t) - \rho\exp\{-\Lambda(t)\}q(t)\lambda(t) \\ &= \lambda(t)\bar{F}_L(t) - \rho q(t)\lambda(t)\exp\{-\left(\Lambda(t) - \Lambda(t-\delta)\right)\}\bar{F}_L(t-\delta), \end{aligned}$$

and hence

$$r_L(t) = \lambda(t)\left[1 - \rho q(t)\exp\left\{-\int_{t-\delta}^{t} \lambda(x)dx\right\} \frac{\bar{F}_L(t-\delta)}{\bar{F}_L(t)}\right].$$

Let 2 3 . Then

$$( ) = \exp \quad ( ) + \exp \quad ( ) \quad ( ) \; ( ) \quad + \exp \quad ( ) \quad ( ) \; ( ) \; ( ) \; ( )$$

which gives

$$( ) \; = \quad ( ) \exp \quad ( ) \quad ( ) \exp \quad ( ) \quad ( ) \; ( ) \quad + \exp \quad ( ) \quad ( ) \; ( )$$

$$( ) \exp \quad ( ) \quad ( ) \; ( ) \; ( ) \; ( )$$

$$+ \exp \quad ( ) \quad ( ) \; ( ) \quad ( ) \; ( )$$

$$= \; ( ) \; ( ) \quad \exp \quad ( ) \quad ( ) \; ( ) \quad \exp \quad ( ) \quad ( ) \; ( ) \quad ( ) \; ( )$$

$$= \; ( ) \; ( ) \quad ( ) \; ( ) \exp \quad (\; ( ) \quad (\quad)) \quad (\quad)$$

and hence

$$( ) = \; ( ) \;\; 1 \quad ( ) \exp \quad ( ) \quad \frac{(\quad)}{( )}$$

By proceeding in a similar manner, we arrive at the required result.

In the following theorem, we show that the failure rate of the system asymptotically converges to the limiting intensity of the NHPP.

$$\lim \quad ( ) \quad ( ) \quad 0 \quad \overset{( )}{=} \quad 0$$

$( ) =$

$$\lim \quad ( ) = \lim \quad ( )$$

From Theorem 3.3, we have

$$\lim \quad ( ) = \lim \quad ( ) \;\; 1 \quad ( ) \exp \quad ( ) \quad \frac{(\quad)}{( )}$$

Now, we can write

$$\exp \quad ( ) \quad \frac{(\quad)}{( )} = \frac{1 + \quad ^{-} \quad (\quad ( ) \; ( ))}{1 + \quad ^{-} \quad (\quad ( ) \; ( ))} \tag{3.6}$$

which gives

$$0 \quad \exp \quad ( ) \quad \frac{(\quad)}{( )} \quad 1$$

Again, 0 ( ) 1 for all . Thus, the result trivially holds when lim ( ) = 0 and/or lim ( ) = 0. Now, consider the case when 0 lim ( ) and 0 lim ( ) 1. From the hypothesis, we have that ( ) is decreasing in 0. Then there exists a ( ) such that 1 ( ) 0 and ( ) 0, for all [ ] and for some constants , and . On using these bounds of ( ) and ( ), we get

$$(\quad) \frac{(\quad (\; +1) \;)}{!} \qquad ( ) \; ( ) \qquad ( ) \frac{(\quad (\; +1) \;)}{!}$$

for all – 1 and for all [ ). Similarly,

$$(\quad) \frac{(\quad)}{!} \qquad ( ) \; ( ) \qquad ( ) \frac{(\quad)}{!}$$

for all – and for all [ ). Again, these imply that, for all [ ),

$$1 + \; ^{-} \qquad ( ) \; ( ) \qquad ^{-} \; (\quad) \frac{(\quad (\; +1) \;)}{!}$$

and

$$1 + \; ^{-} \qquad ( ) \; ( ) \qquad ^{-} \; (\quad) \frac{(\quad)}{!}$$

On using the above two inequalities in (3.6), we get

$$0 \quad \exp \quad ( ) \quad \frac{(\quad)}{( )} \quad \frac{^{-} \; (\quad) \; \overline{\qquad}}{^{-} \; (\quad) \; \overline{\qquad}} \quad \text{for all} \quad [\quad)$$

Note that the right hand side expression approaches zero as . This holds because it is a ratio of two polynomials and the degree of the polynomial in the numerator is smaller than that in the denominator. Thus, we get

$$\lim \exp \quad ( ) \quad \frac{(\quad)}{( )} = 0$$

and hence, the result follows.

In the following theorem, we show that the failure rate of a system for the constant -shock model has nonmonotone behaviour.

$$0$$

$$= \qquad 0$$

$(\ )\quad 1$

From Theorem 3.3, we have

$$(\ ) = \begin{cases} & \text{if } 0 \\ 1 \quad \exp \qquad \overline{} & \text{if} \end{cases}$$

where

$$(\ ) = \exp \qquad \frac{(\qquad)}{!}$$

Clearly, $(\ )$ is constant in $(0\quad)$, and $(\ ) = \quad (\qquad)\ (1 + \ (\qquad)$ for $(\ \ 2\ )$ and hence, it is increasing in $(\ \ 2\ )$. To prove the result, it su ces to show that there is an extrema point in $[2\qquad)$. We will prove it by showing that there is a local minima point in $(2\quad 3\ )$. Note that both $(\ )$ and $(\qquad)$ are di erentiable on $(2\qquad)$. This implies that $(\ )$ is di erentiable on $(2\qquad)$ and hence, we have

$$(\ ) = \quad \exp \qquad \frac{(\ )\quad(\qquad) + \quad (\qquad)\quad(\ )}{(\quad(\ ))} \qquad 2$$

If a local extrema exists in $(2\quad 3\ )$, then $(\ )$ has to be zero at this point. Now, $(\ ) = 0$ holds if and only if $(\qquad) = \quad (\ )$, which can equivalently be written as

$$\frac{(\quad 2\ )}{(\qquad)} = \frac{(\qquad)}{(\ )} \tag{3.7}$$

Further, this is equivalent to mean that

$$\frac{\exp \quad (\quad 2\ )}{\exp \quad (\qquad) + \exp \quad (\qquad)\ (\quad 2\ )} = \frac{\exp \quad (\qquad) + \exp \quad (\qquad)\ (\quad 2\ )}{\exp \qquad + \exp \qquad (\qquad) + \frac{}{}\exp \qquad (\quad 2\ )}$$

or equivalently,

$$\frac{1}{1 + \ (\quad 2\ )} = \frac{1 + \ (\quad 2\ )}{1 + \ (\qquad) + \frac{}{}(\quad 2\ )}$$

or equivalently,

$$(\quad 2\ ) + \frac{2}{}(\quad 2\ ) \quad 2 \quad - \quad = 0$$

Note that $= 2 + \quad \sqrt{\frac{}{} + \frac{}{}} \quad -$ is a solution of the above equation, and it lies in $(2\quad 3\ )$. Thus, is a local extrema of $(\ )$. Further, note that $(\ ) \quad 0$, for $(\quad 3\ )$, and $(\ ) \quad 0$,

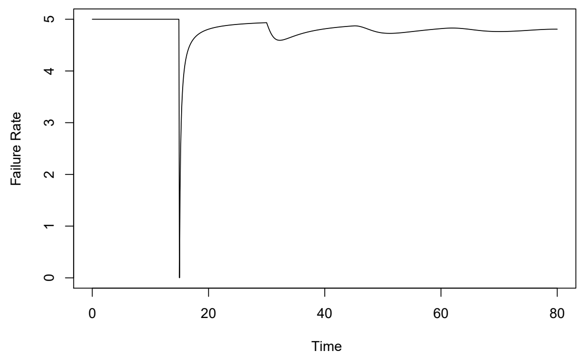

Figure 4: Plot of failure rate against (0 80], for = 5, = 15 and ( ) 1.

for (2). This implies that is a local minima point in (2 3). This completes the proof.

In what follows, we illustrate the result given in Theorem 3.5. We plot ( ) against (0 80], for xed = 5, = 15 and ( ) 1. Figure 4 shows the non-monotonic shape of the failure rate. It also equals 0 at = 15 as clearly follows from this theorem.

In this section, we discuss two applications of the proposed model, namely, the optimal replacement policy and the optimal mission duration.

In this subsection, we study the optimal replacement policy for a system under the de ned mixed shock model. This optimal replacement policy for the constant -shock model based on the HPP was rst introduced by Lam and Zhang [16]. It was further considered in Tang and Lam [28], and Eryilmaz [9] for the renewal process and the Polya process of shocks, respectively. We assume that ( ) = , 0 1 and study the problem for three di erent types of recovery functions given by ( ) = , ( ) = and ( ) = + , 1, 0, 0 . We denote these recovery functions by ( ), ( ) and ( ), respectively.

Below we give a list of assumptions and descriptions that are similar to those considered in Lam and Zhang [16], Tang and Lam [28], and Eryilmaz [9]. Note that, as in the listed papers, we are also considering the non-negligible repair times, whereas for the sake of numerical illustration, the shock process is the HPP.

1. A new system is incepted into operation at t = 0 and it is repaired immediately once it is failed. The system is replaced by a new identical one after the -th failure is observed.
2. The system is subject to external shocks that occur according to the HPP with intensity 0.
3. After the -th repair, the new recovery function is given by : 0 [0) such that ( ) = , for 1.
4. Let be the repair time of the system after the -th failure, = 1 2 . Then the sequence forms an increasing geometric process such that ( ) = — = 1 2 .
5. The repair cost is ; the reward rate is when the system is operating. The replacement cost has two parts: the basic replacement cost is , whereas the other one is proportional to the replacement time with rate . Further, we assume that ( ) = .
6. The HPP, the geometric process and the replacement time are independent.

Let denote the random operating time of the system to the rst failure. Further, let denote the operating time of the system after the (1)-th repair to the -th failure, = 2 3 . Let be a random length of a cycle under the replacement policy . Then

$$= \quad + \quad +$$

From Corollary 3.3, we have

$$(\quad) = \frac{1}{\ } \quad 1 + \quad (\ ) \quad - \quad - \quad \exp$$

On using the above expression, we get

$$(\quad) = \frac{1}{\ } \quad 1 + \quad (\ ) \quad - \quad - \quad \exp \quad + \quad \frac{\ }{\ } +$$

$$= \frac{\ }{\ } + \frac{1}{\ } \quad (\ ) \quad - \quad - \quad \exp \quad + \quad \frac{\ }{\ } +$$

Further, the expected cost on a cycle is given by

$$+ \quad +$$

$$= \quad —— \quad —— \quad - \quad (\ ) \quad ^{-} \quad ^{-} \quad \exp \quad + \quad +$$

Then the average (long-run) replacement cost rate of the system, denoted by ( ), can be calculated as

$$(\ ) = \frac{\text{Expected cost incurred in a cycle}}{\text{Expected length of a cycle}}$$

$$= \frac{—— \quad —— \quad - \quad (\ ) \quad ^{-} \quad ^{-} \quad \exp \quad + \quad +}{- + - \quad (\ ) \quad ^{-} \quad ^{-} \quad \exp \quad + \quad —— +}$$

Let ( ) be the average replacement cost of the system when the recovery function is ( ), = 1 2 3. In Table 1, we calculate ( ), = 1 2 3, for di erent values of . We assume the model parameter values as follows: = 1 05 = 0 8 = 3 = 2 = 100 = 30 = 5 = 10 = 0 8 = 0 3 = 0 99 = 1 = 1 2 = 0 7. Table 1 indicates the minimum average replacement costs, for di erent recovery functions, as (9) = 2 495398, (9) = 2 545428 and (10) = 2 598365. Thus, the system should be replaced immediately after the = 9-th failure when the recovery functions are ( ) and ( ). On the other hand, it should be done immediately after the = 10-th failure when the recovery function is ( ). Moreover, the graphical representation of ( ), = 1 2 3 against = 1 2 30, is given in Figure 5.

Further, we illustrate the e ect of on ( ) and , = 1 2 3. Figure 6 shows that the optimal values of ( ) and decrease as increases.

The notion of the was introduced by Finkelstein and Levitin [12] for a non-repairable system subject to shocks and internal failures. In this paper, they showed that the mission completion is not always bene cial in terms of cost for a degrading system. In many real life scenarios, it may be a good strategy to abort the mission before its completion. The mission abort usually results in a reward that depends on the system s operation time and a penalty. On the other hand, the mission completion results in an additional reward. Moreover,

Table 1: Values of ( ) for = 1 2 3

N	( )	( )	( )	N	( )	( )	( )
1	6.310475	6.403005	6.511520	14	2.635177	2.660795	2.690814
2	4.608003	4.707498	4.825231	15	2.675697	2.697639	2.723330
3	3.707055	3.800422	3.911283	16	2.714776	2.733445	2.755292
4	3.183374	3.268529	3.369722	20	2.842405	2.851637	2.862433
5	2.867401	2.944476	3.036020	24	2.920088	2.924337	2.929310
6	2.677378	2.746925	2.829422	28	2.961543	2.963408	2.965595
7	2.568473	2.631047	2.705151	32	2.982100	2.982895	2.983828
8	2.513652	2.569741	2.636042	36	2.991852	2.992185	2.992575
9			2.604458	40	2.996348	2.996486	2.996647
10	2.501744	2.546109		55	2.999833	2.999838	2.999844
11	2.524219	2.563305	2.609268	70	2.999993	2.999993	2.999993
12	2.556706	2.590900	2.631055	85	3.000000	3.000000	3.000000
13	2.594759	2.624462	2.659301	100	3.000000	3.000000	3.000000

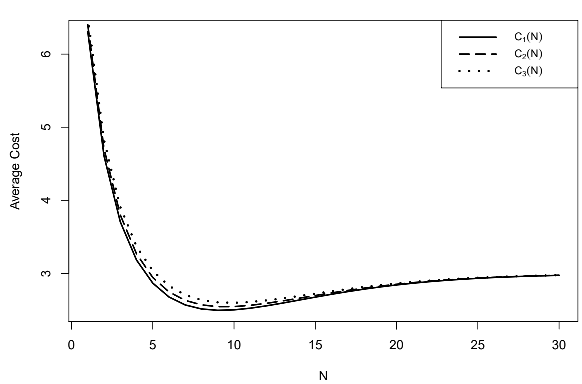

Figure 5: Plot of ( ) for = 1 2 3

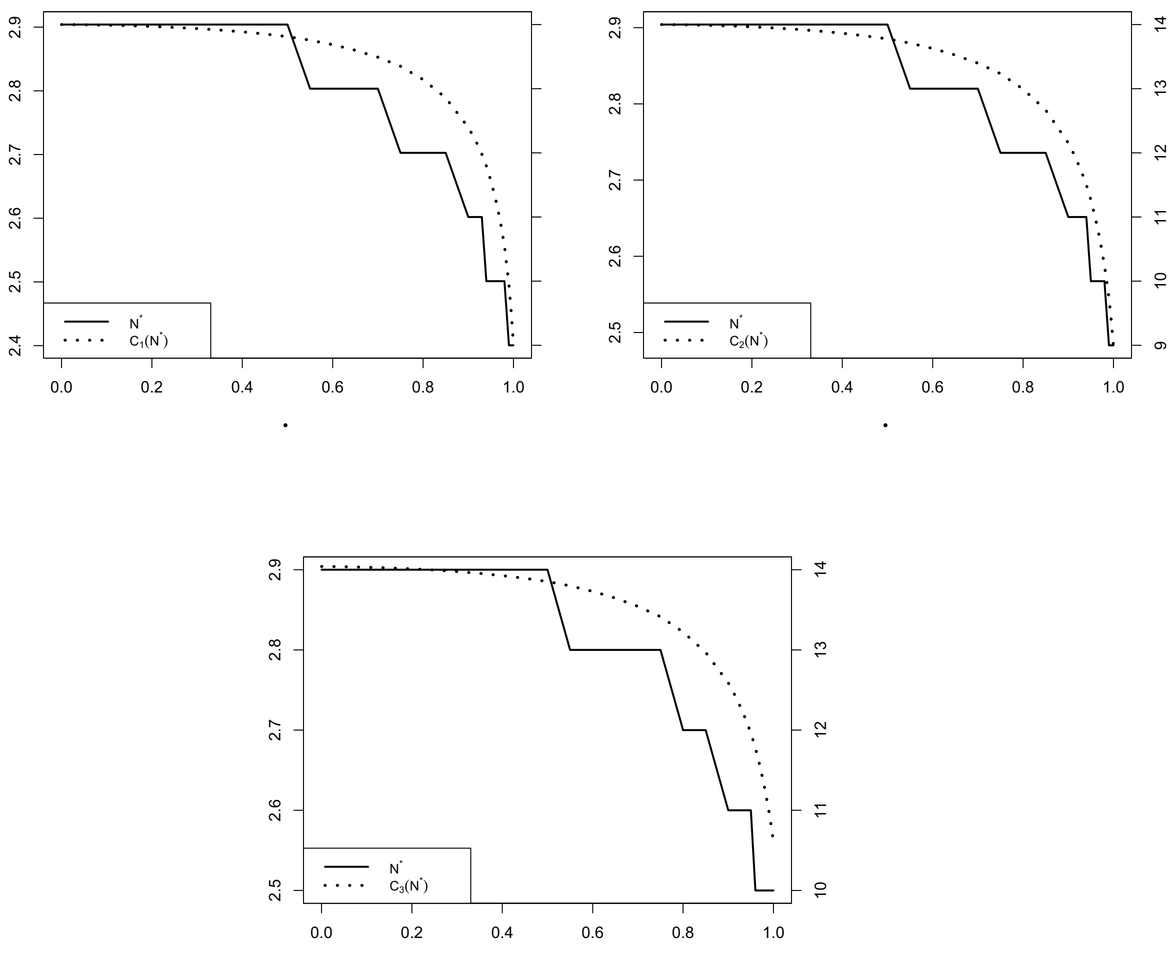

Figure 6: Plot of optimal average cost and N^* against $\sigma \in (0, 1)$

the failure of the system during the mission also results in a penalty because it incurs additional costs due to failure of a mission. In this subsection, we discuss the optimal mission duration for a system under the de ned mixed shock model. Below we give a list of assumptions.

1. A new system with lifetime starts a mission at time $= 0$. The mission duration is . The mission can be terminated at any time $(0 \quad]$.
2. The system is subject to external shocks that occur according to the HPP with intensity 0 under the de ned model with $( ) =$ for 0 1.
3. The system gets pro t ( ) when the mission is completed (i.e., the system does not fail during the mission or the mission is not aborted in $[0 \quad])$. The per time unit reward, when the system is working, is and the per time unit operational cost is , where .
4. A penalty is imposed if the system fails during the mission. In case of premature termination, the xed penalty ( ) is administrated. Further, is an additional reward for the mission completion.
5. Reward after the failure is discarded.

Based on aforementioned assumptions, the pro t ( ) upon mission completion can be expressed as

$$( ) = (\qquad) \quad +$$

Note that the mission is aborted at time if the total pro t at termination exceeds the expected pro t in case of mission continuation. The pro t at termination at time is equal to ( ) . On the other hand, the expected pro t in the case of mission continuation is

$$\frac{( )}{( )}((\qquad) \; + \quad) \qquad 1 \quad \frac{( )}{( )}$$

where ( ) ( ) is the probability that a system will not fail in the remaining mission time given that it is operable at time ; here ( ) is the same as in (3.1) with 0 $= 1$ $= 1$ and $=$ for all 0 . Thus, if for some , the expression

$$( ) = \frac{( )}{( )}((\qquad) \; + \quad) \qquad 1 \quad \frac{( )}{( )} \qquad ((\qquad) \qquad)$$

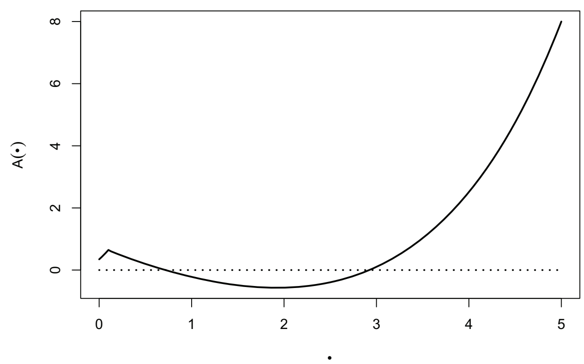

Figure 7: Profit comparison function $A(\tau)$ against $\tau \in [0, 5]$

is non-negative, then the mission should not be terminated at time τ. Clearly, $A(0) \geq 0$, as there is no need to terminate the mission that had just started. Since the expression of $A(\tau)$ is complicated, it is not analytically possible to find out the values of τ for which $A(\tau) \geq 0$. Thus, we consider the following numerical example.

Let us assume $T = 5, c_p = 2.5, c_0 = 0.5, C_R = 3, C_f = 8, C_t = 5, \rho = 0.95, \delta = 0.1, \sigma = 0.95$ and $\lambda = 1.4$. Based on these parameter values, we plot the profit comparison function $A(\tau)$ against $\tau \in [0, 5]$. Figure 7 shows that $A(\tau)$ is increasing in $\tau \in [0, 0.1]$ and is in U-shaped in $\tau \in (0.1, 5]$. Further, note that it takes negative values in $\tau \in [0.72, 2.92]$. This implies that the mission should not be terminated in the interval $[0, 0.72)$ and $(2.92, 5]$, whereas it should be aborted just at $\tau = 0.72$ as it is the optimal solution. In case the mission is not terminated at time $\tau = 0.72$, it may be terminated at any time in the interval $[0.72, 2.92]$. Further, if this is not done, then the mission should not be terminated at all because its termination in the interval $(2.92, 5]$ is not beneficial.

5 Concluding remarks

A combination of the history-dependent extreme shock model and the history-dependent δ-shock model is considered in this paper. This model is a generalization of some of the existing models in the literature, namely, the classical extreme shock model, the history-dependent extreme shock model and the constant δ-shock model. Further, we assume that shocks occur according to the GPP which is a generalization of some of the commonly used counting processes, namely, the HPP, the NHPP, and the Polya process. For the de ned model, we derive the survival function, the mean lifetime and the failure rate of a system. Further, we study the long run behaviour and the non-monotone behaviour of the failure rate. As applications of the proposed model, we consider the optimal replacement policy and the optimal mission duration.

In our study, we have considered binary systems, when a system can be only in two states (operable or failed). In the future research, we plan to discuss the multistate systems under the history dependent mixed shock models. These models could provide a better stochastic description at may practical instances.

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