Integrating Hamiltonian systems defined on the Lie groups $SO(4)$ and $SO(1,3)$

James Biggs and William Holderbaum

Abstract— **In this paper we study constrained optimal control problems on semi-simple Lie groups. These constrained optimal control problems include Riemannian, sub-Riemannian, elastic and mechanical problems. We begin by lifting these problems, through the Maximum Principle, to their associated Hamiltonian formalism. As the base manifold is a Lie group G the cotangent bundle is realized as the direct product $G\times\mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual of the Lie algebra $\mathfrak{g}$ of G. The solutions to these Hamiltonian vector fields $l\in\mathfrak{g}^*$, are called extremal curves and the projections $g(t)\in G$ are the corresponding optimal solutions. The main contribution of this paper is a method for deriving explicit expressions relating the extremal curves $l\in\mathfrak{g}^*$ to the optimal solutions $g(t)\in G$ for the special cases of the Lie groups $SO(4)$ and $SO(1,3)$. This method uses the double cover property of these Lie groups to decouple them into lower dimensional systems. These lower dimensional systems are then solved in terms of the extremals using a coordinate representation and the systems dynamic constraints. This illustrates that the optimal solutions $g(t)\in G$ are explicitly dependent on the extremal curves.**

I. Introduction

Affine control systems defined on finite-dimensional Lie groups form an important class of nonholonomic system and provide a mathematically rich setting for studying kinematic control systems [1], [2], [3], quantum control systems [4],[5] and relativistic systems [6]. The motion planning problem for such systems can be solved using optimal control theory and it follows that such problems are inseparable from problems in geometry, including the sub-Riemannian and elastic problems on the frame bundles of the planar forms [7], [8] and [9] and on the frame bundles of the space forms [10] and [11]. Each of these problems can formulated as a constrained optimal control problem. We begin here by giving a general statement of the motion planning problem:

Problem Statement 1: The motion planning problem concerns the solutions $g(t)\in G$ of the left-invariant differential system:

$$\frac{dg(t)}{dt}=g(t)(\sum_1^s u_iA_i) \tag{1}$$

that minimize the expression:

$$f_0=\frac{1}{2}\int_0^T\langle u(t),Qu(t)\rangle dt \tag{2}$$

subject to the given boundary conditions $g(0)=g_0$ and $g(T)=g_T$, $A_1,...,A_s$ are given elements of the n-dimensional Lie algebra $\mathfrak{g}$ of G and where Q is a positive definite $(s\times s)$ matrix.

The motion planning problem can be naturally viewed as an optimal control problem with $u(t)=(u_1,...,u_s)$ playing the role of control functions, which are assumed to be measurable and bounded throughout this paper. s is the number of controls which can be less than or equal to n. When s is equal to n, *Problem Statement* 1, is known as the Riemannian problem. In the case where s is less than n the kinematic system is said to be underactuated. It follows that when $n>s$, *Problem Statement* 1 describes the motion planning problem for underactuated kinematic systems on Riemannian manifolds, known as the Sub-Riemannian problem, which has been studied in [3], [12]. Additionally, such a problem statement can be slightly modified to include elastic and mechanical problems as will be shown. The Maximum Principle of optimal control (see [13],[11]) then identifies the appropriate Hamiltonian H on the dual of the Lie algebra $\mathfrak{g}^*$ of the Lie algebra G. For such problems, the solutions to the Hamiltonian vector fields called extremals are elements of $\mathfrak{g}^*$. It follows that each optimal solution $g(t)\in G$ is the projection of the extremal curves, confined to elements of the dual of the Lie algebra. Although, in this paper we do not explicitly solve the extremal solutions as these will be problem specific, we provide a method relating the extremal solutions to the corresponding solution curves $g(t)\in G$ for all systems of this form on $SO(4)$ and $SO(1,3)$. Applications motivating the study of Hamiltonian systems on Lie groups is the motions of relativistic particles [6] and applications of quantum control on $SO(4)$ [5].

$SO(4)$ and $SO(1,3)$ are semi-simple Lie groups denoted G (see [14] for general definitions) and are the orthonormal frame bundles of the 3-dimensional non-Euclidean space forms, the sphere $\mathbb{S}^3$ and Hyperboloid $\mathbb{H}^3$. The sphere $\mathbb{S}^3$ is a Riemannian manifold with its Riemannian metric inherited from the standard Euclidean metric in $\mathbb{R}^4$. The Riemannian metric on $\mathbb{H}^3$ is inherited from the Lorentzian inner product:

$$\langle x,y\rangle=-x_1y_1+x_2y_2+x_3y_3+x_4y_4 \tag{3}$$

In [15] a method for integrating these systems where the controls are time-independent is given. This paper extends this integration procedure to systems with time-dependent controls by additionally considering the lift of the control system to its appropriate Hamiltonian vector fields. The paper is divided into 3 sections which are summarized as follows:

- Section II - the optimal control problem of minimizing (2) subject to the kinematic constraint (1) is lifted to its corresponding Hamiltonian vector fields, which for

J. Biggs is with School of Systems Engineering, University of Reading, Reading, The United Kingdom j.biggs@rdg.ac.uk

W. Holderbaum is with School of Systems Engineering, University of Reading, Reading, The United Kingdom w.holderbaum@rdg.ac.uk

semi-simple Lie groups can be expressed in Lax Pair form.

- Section III- we specialize to the case for problems defined on $SO(4)$ and $SO(1,3)$, which have unique double cover properties that allow them to be mapped isomorphically to lower dimensional decoupled systems. These decoupled systems can then be solved explicitly.
- Section IV - the decoupled systems are solved using a coordinate representation and exploiting a geometric constraint of the system. The solutions to the decoupled system are then projected back onto $SO(4)$ and $SO(1,3)$, to obtain the solutions $g(t)\in G$ in terms of the extremal curves.

we complete this introduction by extending the motion planning problem (*Problem Statement* 1), to include elastic and mechanical problems.

To incorporate systems with drift any of the controls u_i can be set to a constant in *Problem Statement* 1. Such problems are common in applications, for example gravity induces a drift effect. Such problems are inseparable from problems in geometry, for example, a particular optimal control problem subject to a kinematic constraint with drift is known as the elastic problem, as highlighted in [1]. Following the Cartan decomposition the Lie algebra can be split into the factors $\mathfrak{p}$ and $\mathfrak{k}$ satisfying the classic relations

$$\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k},[\mathfrak{k},\mathfrak{k}]\subseteq\mathfrak{k},[\mathfrak{p},\mathfrak{k}]\subseteq\mathfrak{p},[\mathfrak{p},\mathfrak{p}]\subseteq\mathfrak{k} \tag{4}$$

the elastic problem is concerned with a left-invariant metric defined on $\mathfrak{k}$. More explicitly following the Cartan decomposition, the constraint (1) can be expressed as:

$$g(t)^{-1}\frac{dg(t)}{dt}=\sum_{i=1}^{m}A_iu_i+\sum_{m+1}^{n}A_iu_i \tag{5}$$

where n is the dimension of $\mathfrak{g}$, m is the dimension of $\mathfrak{p}$ and $(n-m+1)$ is the dimension of $\mathfrak{k}$ and

$$\begin{aligned}&\sum_{i=1}^{m}A_iu_i\in\mathfrak{p}\\&\sum_{m+1}^{n}A_iu_i\in\mathfrak{k}\end{aligned} \tag{6}$$

Following, the definition in [9], Riemannian problems are defined by a metric on the Lie algebra $\mathfrak{g}$ and the elastic problem is defined by a partial metric on $\mathfrak{k}$. For this class of problem take a fixed element $A\in\mathfrak{p}$ i.e. $u_1,...,u_m$ are constant and consider all absolutely continuous curves $g(t)\in G$ that satisfy $g^{-1}\frac{dg}{dt}-A\in\mathfrak{k}$ for almost all t in an interval $[0,T]$ and which satisfy the boundary conditions $g(0)=g_0$ and $g(T)=g_1$. Then the elastic problem in [9] considers the problem of minimizing the integral $\frac{1}{2}\int_0^T\left\langle g^{-1}\frac{dg}{dt}-A,Dg^{-1}\frac{dg}{dt}-A\right\rangle dt$, where D is a positive definite matrix. Therefore, *Problem Statement* 1 can be specified as the elastic problem:

Problem Statement 2: The elastic problem is concerned with the solutions $g(t)\in G$ of the left-invariant differential system:

$$g(t)^{-1}\frac{dg(t)}{dt}=A+\sum_{m+1}^{n}A_iu_i \tag{7}$$

that minimize the expression:

$$f_0=\frac{1}{2}\int_0^T\left\langle\sum_{m+1}^{n}A_iu_i,D\sum_{m+1}^{n}A_iu_i\right\rangle dt \tag{8}$$

subject to the given boundary conditions $g(0)=g_0$ and $g(T)=g_T$, where $A_{m+1},...,A_n$ is the standard basis of $\mathfrak{k}$ and $A\in\mathfrak{p}$ is a constant element and where D is a positive definite $(n-m+1)\times(n-m+1)$ matrix.

The constrained optimal control problem also includes all mechanical problems through the Lagrange Principle of least action where we minimize the function $f_0=\int_0^T\mathfrak{L}(t)dt$ subject to the kinematic constraint (1), where $\mathfrak{L}(t)$ is the Lagrangian of the system. In the case where the controls take the form of components of translational and angular velocities, the Hamiltonian lift will yield the dynamic equations of motion. This method of formulating the dynamics as a constrained optimal control problem is used to derive the dynamic equations of a rigid body in [16].

II. THE LAX PAIR EQUATIONS

A. Lax Pair equations on semi-simple Lie groups

The Maximum Principle of optimal control identifies the appropriate left-invariant Hamiltonian on the dual of the Lie algebra. The solutions to these integrable Hamiltonian vector fields are called extremals. The projected extremal solutions down to the level of the group are called optimal solutions. The solutions $g(t)\in G$ of (1) while minimizing the expression (2) are locally optimal, that is optimal for small terminal time T [11], however as the terminal time grows they may stop being optimal. For simplicity of terminology we will refer to all projections as the optimal solutions even though the nature of cut-locus and conjugate points have not been considered, see [11]. The parameterized control Hamiltonian corresponding to the state space (1) while minimizing the function (2) is written as (see [3]):

$$H(\xi,u,g)=\sum_{i=1}^{n}u_i\xi(gA_i)-\rho_0\sum_{i=1}^{n}c_iu_i^2 \tag{9}$$

where $\xi\in T_g^*G$ and $\rho_0=1$ for regular extremals and $\rho_0=0$ for abnormal extremals. The c_i's are constants dependent on the positive definite matrix Q or D in the elastic case. In this paper we shall only consider the regular extremals. As the vector fields are left invariant they can be pulled back by the left group action. The pull-back in this case is explicitly stated as $\xi(\cdot)=\hat{p}(g^{-1}(\cdot))$. i.e $\xi\in T^*G$ is pulled back to give a function $\hat{p}\in\mathfrak{g}^*$. The control Hamiltonian can then be written as

$$H(\hat{p},u)=\sum_{i=1}^{3}u_i\hat{p}(A_i)-\sum_{i=1}^{3}c_iu_i^2 \tag{10}$$

Through the Maximum principle of optimal control and the fact that the control Hamiltonian is a concave function of the control functions u_i, it follows by calculating $\frac{\partial H}{\partial u_i}=0$ that the optimal controls are given in feedback form:

$$u_i^*=\frac{1}{c_i}\hat{p}(A_i) \tag{11}$$

where $i=1,2,3$. Then substituting (11) back into (10) gives the optimal Hamiltonian $H(\hat{p},u^*)$ which will be denoted as H for simplicity. Define the extremal solutions $M_i=\hat{p}(A_i)$. From this the Hamiltonian vector fields can be calculated using the Poisson bracket:

$$\{M_i,M_j\}=-\hat{p}([A_i,A_j]) \tag{12}$$

Let $l(t)\in\mathfrak{g}^*$ where the coordinates of l are $M_1,...,M_i$ then the Hamiltonian vector fields can be written in compact form as:

$$\frac{dl(t)}{dt}=\{l(t),H\} \tag{13}$$

on semi-simple Lie groups each element in $\mathfrak{g}^*$ can be uniquely identified with an element in $\mathfrak{g}$ via the non-degenerate trace form, called the Killing form, which implies that the element $l(t)\in\mathfrak{g}^*$ can be identified with an element $L(t)\in\mathfrak{g}$ i.e. following the notation of [3], $L(t)=l(t)^\sharp$ where $\sharp$ is the sharp operator. It follows that the equation (13) can be expressed in Lax pair form as:

$$\dot{L}(t)=[L(t),\nabla H] \tag{14}$$

where $\nabla H\in\mathfrak{g}$ is the gradient of the function H. In addition to this equation, substituting the optimal controls (11) into (1) gives

$$\frac{dg(t)}{dt}=g(t)\nabla H \tag{15}$$

The equations (14) and (15) are the equations of motion for problems including Riemannian, sub-Riemannian, elastic and mechanical problems, where each problem differs by the appropriate left-invariant Hamiltonian H. This paper is concerned with the solutions $g(t)\in G$ of (14) and (15) for particular semi-simple Lie groups. The equations (14) and (15) can represent the equations for the Riemannian, sub-Riemannian, elastic and mechanical problems, where the equations differ by their appropriate Hamiltonian H.

B. The Lax Pair Equations on $SO(4)$ and $SO(1,3)$

Here we proceed to study two particular systems of the form (14) and (15) whose solutions are curves in the semi-simple Lie groups $g(t)\in SO(4)$ and $g(t)\in SO(1,3)$. Firstly, we define a basis for the Lie algebras of $SO(4)$ and $SO(1,3)$ as:

$$A_1=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},A_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{pmatrix}$$
$$A_3=\begin{pmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{pmatrix},B_1=\begin{pmatrix}0&-\varepsilon&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$
$$B_2=\begin{pmatrix}0&0&-\varepsilon&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},B_3=\begin{pmatrix}0&0&0&-\varepsilon\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix} \tag{16}$$

such that when $\varepsilon=1$, the basis elements $A_i,B_i\in\mathfrak{so}(4)$ and when $\varepsilon=-1$, $A_i,B_i\in\mathfrak{so}(1,3)$. Then on defining the extremals as $p_i=\hat{p}(B_i)$ and $M_i=\hat{p}(A_i)$, the corresponding Lax Pair elements in equations (14) and (15) are explicitly:

$$\nabla H=\begin{pmatrix}0&-\varepsilon\frac{\partial H}{\partial p_1}&-\varepsilon\frac{\partial H}{\partial p_2}&-\varepsilon\frac{\partial H}{\partial p_3}\\ \frac{\partial H}{\partial p_1}&0&-\frac{\partial H}{\partial M_3}&\frac{\partial H}{\partial M_2}\\ \frac{\partial H}{\partial p_2}&\frac{\partial H}{\partial M_3}&0&-\frac{\partial H}{\partial M_1}\\ \frac{\partial H}{\partial p_3}&-\frac{\partial H}{\partial M_2}&\frac{\partial H}{\partial M_1}&0\end{pmatrix} \tag{17}$$

and

$$L(t)=\begin{pmatrix}0&-\varepsilon p_1&-\varepsilon p_2&-\varepsilon p_3\\p_1&0&-M_3&M_2\\p_2&M_3&0&-M_1\\p_3&-M_2&M_1&0\end{pmatrix} \tag{18}$$

Then $g(t)\in G$ is the solution to equations (14) and (15) where $G=SO(4)$ for $\varepsilon=1$ and $G=SO(1,3)$ for $\varepsilon=-1$. These equations include Riemannian, sub-Riemannian, elastic and mechanical problems, where the equations only differ by their appropriate Hamiltonian. For example the Hamiltonian for the elastic problem in [17] i.e the Hamiltonian corresponding to *Problem Statement* 2 on $SO(4)$ and $SO(1,3)$ is:

$$H=p_1+\left(\frac{M_1^2}{c_1}+\frac{M_2^2}{c_2}+\frac{M_3^2}{c_3}\right) \tag{19}$$

where c_i are constant entries of the $(n-m+1)\times(n-m+1)$ matrix D and the Hamiltonian for a particular sub-Riemannian problem (i.e. when the metric is defined on $\mathfrak{p}$), would for example be:

$$H=\frac{1}{2}\left(\frac{p_1^2}{m_1}+\frac{p_2^2}{m_2}+\frac{p_3^2}{m_3}\right) \tag{20}$$

where m_i are the constant entries of the positive definite $(s\times s)$ matrix Q, finally the Hamiltonian for a rigid body defined on these Lie Groups, whose Lagrangian is defined entirely by it's kinetic energy i.e.

$$\mathfrak{L}(t)=\sum_{i=1}^{3}m_iv_i^2+\sum_{i=1}^{3}c_i\Omega_i^2 \tag{21}$$

where v_i are the components of translational velocity, Ω_i are the components of angular velocity, m_i the components of mass and c_i the components of inertia. Taking the angular and translational velocities to be the control functions, it follows from an application of the Maximum Principle that the appropriate energy Hamiltonian is:

$$H=\sum_{i=1}^{3}\frac{p_i^2}{m_i}+\sum_{i=1}^{3}\frac{M_i^2}{c_i} \tag{22}$$

where p_i are analogous to the components of translational momentum, M_i to the components of angular momentum.

III. Decoupling the system

In this section the system described by equations (14) and (15) defined on $SO(4)$ and $SO(1,3)$ are decoupled into two lower dimensional systems. This decoupling then allows us to compute the solutions of the decoupled systems using a simple technique. The solutions of the decoupled systems can then be projected back onto the original manifold to yield the solution to the original system (14) and (15) on $SO(4)$ and $SO(1,3)$. We begin here by describing the decoupling of the system defined on $SO(4)$.

A. Decoupling the system on $SO(4)$

The system defined by the differential equations (14) and (15) on $SO(4)$ can be decoupled into two lower dimensional systems. The decoupling is possible as the Lie algebra $\mathfrak{so}(4)$ is isomorphic to $\mathfrak{su}(2)\times\mathfrak{su}(2)$ and an element $A\in\mathfrak{so}(4)$ can be identified with the elements $(V_1,V_2)\in\mathfrak{su}(2)\times\mathfrak{su}(2)$ using the theorem from [15] which is stated below:

Theorem 1: $\mathfrak{so}(4)$ is isomorphic to $\mathfrak{su}(2)\times\mathfrak{su}(2)$ where an element $A\in\mathfrak{so}(4)$ is associated with the elements $(V_1,V_2)\in\mathfrak{su}(2)\times\mathfrak{su}(2)$ via the following mapping:

$$\begin{aligned}&A\mapsto(V_1,V_2)=\\&\begin{pmatrix}0&-b_1&-b_2&-b_3\\b_1&0&-a_3&a_2\\b_2&a_3&0&-a_1\\b_3&-a_2&a_1&0\end{pmatrix}\\&\mapsto\frac{1}{2}\begin{pmatrix}(a_1+b_1)i&(a_2+b_2)+(a_3+b_3)i\\-(a_2+b_2)+(a_3+b_3)i&-(a_1+b_1)i\end{pmatrix}\\&,\frac{1}{2}\begin{pmatrix}(a_1-b_1)i&(a_2-b_2)+(a_3-b_3)i\\-(a_2-b_2)+(a_3-b_3)i&-(a_1-b_1)i\end{pmatrix}\end{aligned}\tag{23}$$

where $a_1,a_2,a_3,b_1,b_2,b_3\in\mathbb{R}$

For simplicity of exposition define the basis:

$$E_1=\begin{pmatrix}i&0\\0&-i\end{pmatrix},E_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix},E_3=\begin{pmatrix}0&i\\i&0\end{pmatrix}\tag{24}$$

then the Lax pair elements defined on $\mathfrak{su}(2)$ can be expressed in the general form:

$$L(t)=\frac{1}{2}\begin{pmatrix}l_1i&l_2+l_3i\\-l_2+l_3i&-l_1i\end{pmatrix}\tag{25}$$

where $l_1,l_2,l_3\in\mathbb{R}$ and

$$\nabla H=\frac{1}{2}\begin{pmatrix}x_1i&x_2+x_3i\\-x_2+x_3i&-x_1i\end{pmatrix}\tag{26}$$

and $x_1,x_2,x_3\in\mathbb{R}$ then it follows from *Theorem* 2, that the system defined by (14) and (15) on $SO(4)$ can be decoupled into a system on $SU(2)\times SU(2)$ where $(g_1(t),g_2(t))\in SU(2)\times SU(2)$ are the solutions of the following differential equations:

$$\begin{aligned}\frac{dg_1(t)}{dt}&=g_1(t)\nabla H_1\\\frac{dL_1(t)}{dt}&=[L_1(t),\nabla H_1]\\\frac{dg_2(t)}{dt}&=g_2(t)\nabla H_2\\\frac{dL_2(t)}{dt}&=[L_2(t),\nabla H_2]\end{aligned}\tag{27}$$

where

$$L_1(t)=\frac{1}{2}\begin{pmatrix}l_1i&l_2+l_3i\\-l_2+l_3i&-l_1i\end{pmatrix}\tag{28}$$

with:

$$\begin{aligned}l_1&=(M_1+p_1)\\l_2&=(M_2+p_2)\\l_3&=(M_3+p_3)\end{aligned}\tag{29}$$

and

$$L_2(t)=\frac{1}{2}\begin{pmatrix}l_1i&l_2+l_3i\\-l_2+l_3i&-l_1i\end{pmatrix}\tag{30}$$

with:

$$\begin{aligned}l_1&=(M_1-p_1)\\l_2&=(M_2-p_2)\\l_3&=(M_3-p_3)\end{aligned}\tag{31}$$

and

$$\nabla H_1=\frac{1}{2}\begin{pmatrix}x_1i&x_2+x_3i\\-x_2+x_3i&-x_1i\end{pmatrix}\tag{32}$$

with:

$$\begin{aligned}x_1&=\frac{\partial H}{\partial M_1}+\frac{\partial H}{\partial p_1}\\x_2&=\frac{\partial H}{\partial M_2}+\frac{\partial H}{\partial p_2}\\x_3&=\frac{\partial H}{\partial M_3}+\frac{\partial H}{\partial p_3}\end{aligned}\tag{33}$$

and finally

$$\nabla H_2=\frac{1}{2}\begin{pmatrix}x_1i&x_2+x_3i\\-x_2+x_3i&-x_1i\end{pmatrix}\tag{34}$$

with:

$$\begin{aligned}x_1&=\frac{\partial H}{\partial M_1}-\frac{\partial H}{\partial p_1}\\x_2&=\frac{\partial H}{\partial M_2}-\frac{\partial H}{\partial p_2}\\x_3&=\frac{\partial H}{\partial M_3}-\frac{\partial H}{\partial p_3}\end{aligned}\tag{35}$$

The decoupled systems can then be integrated using a simple technique which will be described in the following section.

B. Decoupling the system on $SO(1,3)$

The system described by equation (14) and (15) on $SO(1,3)$ is decoupled into two lower dimensional systems. This decoupling is performed by using the theorem outlined in [15] which is stated below

Theorem 2: $\mathfrak{so}(1,3)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ where an element $A \in \mathfrak{so}(1,3)$ is identified with the elements (U,U^*) where $U,U^* \in \mathfrak{sl}_2(\mathbb{C})$ via the following mapping:

$$\begin{aligned}
&A \mapsto (U,U^*) = \\
&\begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{pmatrix} \\
&\mapsto \frac{1}{2}\begin{pmatrix} (ia_1+b_1) & (a_2+b_3)+i(a_3-b_2) \\ (b_3-a_2)+i(a_3+b_2) & -(ia_1+b_1) \end{pmatrix} \\
&, \frac{1}{2}\begin{pmatrix} (b_1-ia_1) & (b_3-a_2)-i(a_3+b_2) \\ (b_3+a_2)-i(a_3-b_2) & -(b_1-ia_1) \end{pmatrix}
\end{aligned} \tag{36}$$

where $a_1,a_2,a_3,b_1,b_2,b_3 \in \mathbb{R}$ and the $*$ notation denotes the conjugate transpose.

Then the equations (14) and (15) can be decoupled into two systems in a similar manner to the system on $SO(4)$. The solutions of these systems are called $g_3(t)$ and $g_3^*(t)$. In this case $g_3(t) = g_3(0)\exp(Ut)$ and $g_3^*(t) = g_3^*(0)\exp(U^*t)$ are explicitly related by the conjugate transpose and therefore, reduces computation since we only need to solve for $g_3(t)$. In addition we can express the equations on $\mathfrak{sl}_2(\mathbb{C})$ in the basis of (24) and then it is a matter of solving the differential equations:

$$\begin{aligned}
\frac{dg_3(t)}{dt} &= g_3(t)\nabla H_3 \\
\frac{dL_3(t)}{dt} &= [L_3(t), \nabla H_3]
\end{aligned} \tag{37}$$

where

$$L_3(t) = \frac{1}{2}\begin{pmatrix} l_1 i & l_2 + l_3 i \\ -l_2 + l_3 i & -l_1 i \end{pmatrix} \tag{38}$$

with:

$$\begin{aligned}
l_1 &= M_1 - ip_1 \\
l_2 &= M_2 - ip_2 \\
l_3 &= M_3 - ip_3
\end{aligned} \tag{39}$$

and

$$\nabla H_3 = \frac{1}{2}\begin{pmatrix} x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & -x_1 i \end{pmatrix} \tag{40}$$

with

$$\begin{aligned}
x_1 &= \frac{\partial H}{\partial M_1} - i\frac{\partial H}{\partial p_1} \\
x_2 &= \frac{\partial H}{\partial M_2} - i\frac{\partial H}{\partial p_2} \\
x_3 &= \frac{\partial H}{\partial M_3} - i\frac{\partial H}{\partial p_3}
\end{aligned} \tag{41}$$

This decoupling has therefore greatly simplified the integration procedure.

IV. The integration procedure

We assume that the extremal curves $p_i, M_i \in \mathfrak{g}^*$ have been solved (either explicitly e.g for the elastic problem in [17] or numerically for Kirchhoff's equations in [**?**]). This section derives equations relating the extremal curves $l \in \mathfrak{g}^*$ to the optimal solutions of the decoupled systems $g_1(t), g_2(t), g_3(t), g_3^*(t)$. Each of the decoupled equations in (27) and (37) can be expressed in the form (14) and (15), where the Lax pair $L(t)$ and ∇H are defined by (25) and (26) respectively. To solve for $g(t) \in G$, we make use of the following theorem:

Theorem 3: The general solution to the differential equation (14) can be expressed as

$$L(t) = g(t)^{-1} L(0) g(t) \tag{42}$$

where $L(0)$ is the $L(t)$ matrix at $t=0$ and is therefore a matrix with constant entries

Proof. Firstly, recall that if $g(t) \in G$ is a solution to the differential equation (15), then $g(t)^{-1} \in G$ is a solution to (see [11]):

$$\frac{dg(t)^{-1}}{dt} = -\nabla H g(t)^{-1} \tag{43}$$

then it follows on differentiating (42) that:

$$\frac{dL(t)}{dt} = \frac{dg(t)^{-1}}{dt} L(0) g(t) + g(t)^{-1} L(0) \frac{dg(t)}{dt} \tag{44}$$

and on substituting (15) and (43) into (44) yields:

$$\begin{aligned}
\frac{dL(t)}{dt} &= -\nabla H g(t)^{-1} L(0) g(t) + g(t)^{-1} L(0) g(t) \nabla H \\
&= L(t)\nabla H - \nabla H L(t) \\
&= [L(t), \nabla H]
\end{aligned} \tag{45}$$

$\square$.

It follows from (42) that as $g(t)$ varies, $g(t)L(t)g(t)^{-1}$ describes the conjugacy class of $L(t)$ which is equal to the constant matrix $L(0)$. As a consequence of this the trace power of $L^j(t)$ must be a constant of motion or equivalently, the coefficients of the characteristic polynomial of $L(t)$ must be a constant of motion. From here on we specialize to the case where $L(t)$ is defined by equation (25). Therefore, we can define a constant K by the following formula:

$$K^2 = -2trace(L(t)^2) = l_1^2 + l_2^2 + l_3^2 \tag{46}$$

it follows that $g(t)L(t)g(t)^{-1}$ can be diagonalized under suitable conjugation such that

$$g(t)L(t)g(t)^{-1} = \frac{K}{2} E_1 \tag{47}$$

Integrating the system with respect to the particular solution (47) greatly simplifies the integration procedure as is now shown.

A. Explicit Solutions

To integrate the system coordinates are introduced for $g(t) \in G$. We shall use $\varphi_1, \varphi_2, \varphi_3$ to denote the coordinates of a point $g(t)$ subject to the equation:

$$g(t) = \exp(\frac{1}{2}\varphi_1 E_1)\exp(\frac{1}{2}\varphi_2 E_2)\exp(\frac{1}{2}\varphi_3 E_1) \tag{48}$$

where E_1 and E_2 are as in (24). Assume now that K is non-zero. It follows from (47) that:

$$L(t) = \frac{K}{2} g^{-1}(t) E_1 g(t) \tag{49}$$

and substituting (48) into (49) yields:

$$L(t) = \frac{K}{2} e^{-\frac{1}{2}E_1\varphi_3} e^{-\frac{1}{2}E_2\varphi_2} E_1 e^{\frac{1}{2}E_2\varphi_2} e^{\frac{1}{2}E_1\varphi_3} \tag{50}$$

It follows after simplification that

$$L(t) = \frac{iK}{2}\begin{pmatrix} \cos\varphi_2 & e^{-i\varphi_3}\sin\varphi_2 \\ e^{i\varphi_3}\sin\varphi_2 & -\cos\varphi_2 \end{pmatrix} \tag{51}$$

Then equating this to (25) gives

$$l_1 = K\cos\varphi_2 \tag{52}$$

and furthermore

$$\begin{aligned} l_2 + il_3 &= iKe^{-i\varphi_3}\sin\varphi_2 \\ -l_2 + il_3 &= iKe^{i\varphi_3}\sin\varphi_2 \end{aligned} \tag{53}$$

from (52) it is easily shown that:

$$\sin\varphi_2 = \frac{\sqrt{K^2 - {l_1}^2}}{K} \tag{54}$$

substituting equation (54) into the equations (53) then adding the two equations and simplifying gives:

$$\cos\varphi_3 = \frac{l_3}{\sqrt{K^2 - {l_1}^2}} \tag{55}$$

following the same procedure but subtracting one equation from another in (53) yields:

$$\sin\varphi_3 = \frac{l_2}{\sqrt{K^2 - {l_1}^2}} \tag{56}$$

It remains to solve for φ_1. Using the coordinate representation of $g(t)$ as (48) and substituting into $g(t)^{-1}\frac{dg(t)}{dt}$ to obtain a coordinate representation of the equation (14) yields:

$$\begin{aligned} g(t)^{-1}\frac{dg(t)}{dt} &= \frac{\dot{\varphi}_1}{2}\begin{pmatrix} i\cos\varphi_2 & ie^{-i\varphi_3}\sin\varphi_2 \\ ie^{i\varphi_3}\sin\varphi_2 & -i\cos\varphi_2 \end{pmatrix} \\ &+ \frac{\dot{\varphi}_2}{2}\begin{pmatrix} 0 & e^{-i\varphi_3} \\ -e^{i\varphi_3} & 0 \end{pmatrix} + \frac{\dot{\varphi}_3}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \nabla H \end{aligned} \tag{57}$$

then equating (57) to ∇H in (26) yields:

$$x_1 = \dot{\varphi}_1\cos\varphi_2 + \dot{\varphi}_3 \tag{58}$$

and

$$\begin{aligned} x_2 + ix_3 &= \dot{\varphi}_1 ie^{-i\varphi_3}\sin\varphi_2 + \dot{\varphi}_2 e^{-i\varphi_3} \\ -x_2 + ix_3 &= \dot{\varphi}_1 ie^{i\varphi_3}\sin\varphi_2 - \dot{\varphi}_2 e^{i\varphi_3} \end{aligned} \tag{59}$$

the two equations in (59) can be rearranged to give:

$$\begin{aligned} \frac{x_2}{e^{-i\varphi_3}} + \frac{ix_3}{e^{-i\varphi_3}} &= \dot{\varphi}_1 i\sin\varphi + \dot{\varphi}_2 \\ -\frac{x_2}{e^{i\varphi_3}} + \frac{ix_3}{e^{i\varphi_3}} &= \dot{\varphi}_1 i\sin\varphi - \dot{\varphi}_2 \end{aligned} \tag{60}$$

then adding the two equations in (60) yields:

$$\frac{x_2}{e^{-i\varphi_3}} - \frac{x_2}{e^{i\varphi_3}} + \frac{ix_3}{e^{i\varphi_3}} + \frac{ix_3}{e^{-i\varphi_3}} = 2\dot{\varphi}_1 i\sin\varphi_2 \tag{61}$$

on substituting the expressions (53) into (61) and simplifying we obtain:

$$\dot{\varphi}_1 = K\left(\frac{x_2 l_2 + x_3 l_3}{l_2^2 + l_3^2}\right) \tag{62}$$

Therefore, all the coordinates $\varphi_1, \varphi_2, \varphi_3$ have been solved. To write $g(t)$ in a compact form we calculate (48) explicitly, which yields:

$$g(t) = \begin{pmatrix} e^{\frac{1}{2}i\varphi_1}e^{\frac{1}{2}i\varphi_3}\cos\frac{\varphi_2}{2} & e^{\frac{1}{2}i\varphi_1}e^{-\frac{1}{2}i\varphi_3}\sin\frac{\varphi_2}{2} \\ -e^{-\frac{1}{2}i\varphi_1}e^{\frac{1}{2}i\varphi_3}\sin\frac{\varphi_2}{2} & e^{-\frac{1}{2}i\varphi_1}e^{-\frac{1}{2}i\varphi_3}\cos\frac{\varphi_2}{2} \end{pmatrix} \tag{63}$$

then using the identities:

$$\begin{aligned} \cos\frac{\varphi_2}{2} &= \sqrt{\frac{1+\cos\varphi_2}{2}} \\ \sin\frac{\varphi_2}{2} &= \sqrt{\frac{1-\cos\varphi_2}{2}} \\ e^{\pm\frac{1}{2}i\varphi_3} &= (\cos\varphi_3 \pm \sin\varphi_3)^{1/2} \end{aligned} \tag{64}$$

it follows from substituting (52), (55) and (56) into (64) that:

$$\begin{aligned} \cos\frac{\varphi_2}{2} &= \sqrt{\frac{K+l_1}{2K}} \\ \sin\frac{\varphi_2}{2} &= \sqrt{\frac{K-l_1}{2K}} \\ e^{\pm\frac{1}{2}i\varphi_3} &= \frac{(l_3 \pm l_2)^{1/2}}{\left(K^2 - l_1^2\right)^{1/4}} \end{aligned} \tag{65}$$

additionally $\dot{\varphi}_1$ defined by (62) can be integrated and $e^{\frac{1}{2}i\varphi_1}$ and $e^{-\frac{1}{2}i\varphi_1}$ can be computed and substituted along with (65) into (63) which yields a simple expression for $g(t)$:

$$\begin{aligned} g(t) = &\frac{1}{(K^2 - l_1^2)^{1/4}(2K)^{1/2}} \times \\ &\begin{pmatrix} e^{\frac{1}{2}i\varphi_1}\sqrt{(l_3+l_2)(K+l_1)} & e^{\frac{1}{2}i\varphi_1}\sqrt{(l_3-l_2)(K-l_1)} \\ -e^{-\frac{1}{2}i\varphi_1}\sqrt{(l_3+l_2)(K-l_1)} & e^{-\frac{1}{2}i\varphi_1}\sqrt{(l_3-l_2)(K+l_1)} \end{pmatrix} \end{aligned} \tag{66}$$

Therefore, all the solutions $g_1(t), g_2(t) \in SU(2)$ and $g_3(t), g_3^*(t) \in SL_2(\mathbb{C})$ of the decoupled systems can be expressed explicitly in terms of the extremal curves.

B. Projecting the decoupled system back onto the original system

The preceding argument illustrates that a control system defined on $SO(4)$ and $SO(1,3)$ and its Hamiltonian lift can be decoupled and solved. However, it is necessary to reconstruct the solutions on the original Lie groups from the solutions of the decoupled systems. This reconstruction is performed in the form of a projection detailed in the paper [15]. For completeness we illustrate the main results:

1) Projecting back onto $SO(4)$: Let us define the set:

$$X=\left\{\left(\begin{array}{cc} x_0+ix_1 & x_2+ix_3 \\ -x_2+ix_3 & x_0-ix_1 \end{array}\right): x_0,x_1,x_2,x_3\in\mathbb{R}\right\} \quad (67)$$

then for any element $\hat{z}\in\mathbb{R}^4$ associate an element $Z\in X$ via the mapping:

$$\hat{z}=\left[\begin{array}{c} z_0 \\ z_1 \\ z_2 \\ z_3 \end{array}\right]\rightarrow Z=\left(\begin{array}{cc} z_0+iz_1 & z_2+iz_3 \\ -z_2+iz_3 & z_0-iz_1 \end{array}\right) \quad (68)$$

where $z_0,z_1,z_2,z_3\in\mathbb{R}$ and define a second element, for simplicity of exposition, as $\hat{w}\in\mathbb{R}^4$ associated to $W\in X$ in the same way as equation (68):

$$\hat{w}=\left[\begin{array}{c} w_0 \\ w_1 \\ w_2 \\ w_3 \end{array}\right]\rightarrow W=\left(\begin{array}{cc} w_0+iw_1 & w_2+iw_3 \\ -w_2+iw_3 & w_0-iw_1 \end{array}\right) \quad (69)$$

where $w_0,w_1,w_2,w_3\in\mathbb{R}$ then define the homomorphism $\Phi: SU(2)\times SU(2)\rightarrow SO(4)$ by:

Theorem 4: The homomorphism $\Phi: SU(2)\times SU(2)\rightarrow SO(4)$ is defined through the following equivalent group actions:

$$g(t)\hat{z}=\hat{w} \quad (70)$$

for $g(t)\in SO(4)$ if and only if

$$g_1(t)Zg_2^{-1}(t)=W \quad (71)$$

where $g_1(t),g_2(t)\in SU(2)$.

Proof. see [10] and [14]. Using this Theorem we can construct a closed form solution $g(t)\in SO(4)$ from the closed form solutions $g_1(t),g_2(t)\in SU(2)$, firstly note that $g_1(t),g_2(t)\in SU(2)$ can be projected onto $\mathbb{R}^4$ following the equations (71) and (69). Expressing these two equations as one projection yields:

$$g_1(t)Zg_2^{-1}(t)=W\mapsto\hat{w}\in\mathbb{R}^4 \quad (72)$$

using the projection (72) and the equivalence of the group actions (70) and (71) implies that the solution $g(t)\in SO(4)$ can be constructed by associating the first column of $g(t)\in SO(4)$ which we call $\hat{w}_1$ with the first basis element of the orthonormal frame $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]^T\in\mathbb{R}^4$ via the projection:

$$\hat{w}_1=g(t)\cdot\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \end{array}\right]^T$$

with the first basis element of the orthonormal frame in X (67):

$$g_1(t)\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)g_2^{-1}(t)=W_1\rightarrow\hat{w}_1$$

in the same manner it follows that the remaining columns of $SO(4)$ are identified with:

$$\begin{aligned} g_1(t)\left(\begin{array}{cc} i & 0 \\ 0 & -i \end{array}\right)g_2^{-1}(t)=W_2\rightarrow\hat{w}_2 \\ g_1(t)\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)g_2^{-1}(t)=W_3\rightarrow\hat{w}_3 \\ g_1(t)\left(\begin{array}{cc} 0 & i \\ i & 0 \end{array}\right)g_2^{-1}(t)=W_3\rightarrow\hat{w}_4 \end{aligned} \quad (73)$$

where

$$g(t)=\left(\begin{array}{cccc} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 \end{array}\right) \quad (74)$$

This can be expressed explicitly but due to space constraints cannot be written in its most complete form here. This section has provided a method for integrating control systems defined on the Lie group SO(4) where the controls are time-dependent.

2) Projecting back onto $SO(1,3)$: In a similar manner to (IV-B.1) define a set of matrices X such that:

$$X=\left\{\left(\begin{array}{cc} x_0+x_1 & x_3-ix_2 \\ x_3+ix_2 & x_0-x_1 \end{array}\right): x_0,x_1,x_2,x_3\in\mathbb{C}\right\} \quad (75)$$

this is the real vector space of Hermitian 2×2 matrices, in addition let V_{-1} denote the real linear space spanned by the basis ie_0,e_1,e_2,e_3. For any element $\hat{z}=z_0ie_0+z_1e_1+z_2e_2+z_3e_3$ in V_{-1}, with $z_0,z_1,z_2,z_3\in\mathbb{C}$ is associated to $Z\in X$ via the mapping:

$$\hat{z}=\left[\begin{array}{c} z_0 \\ z_1 \\ z_2 \\ z_3 \end{array}\right]\rightarrow Z=\left(\begin{array}{cc} z_0+z_1 & z_3-iz_2 \\ z_3+iz_2 & z_0-z_1 \end{array}\right) \quad (76)$$

and for simplicity of exposition define a second element $\hat{w}\in V_{-1}$ associated to $W\in X$ via the mapping:

$$\hat{w}=\left[\begin{array}{c} w_0 \\ w_1 \\ w_2 \\ w_3 \end{array}\right]\rightarrow W=\left(\begin{array}{cc} w_0+w_1 & w_3-iw_2 \\ w_3+iw_2 & w_0-w_1 \end{array}\right) \quad (77)$$

then the mapping $\Phi: SL_2(\mathbb{C})\rightarrow SO(1,3)$ is defined as:

Theorem 5: The homomorphism $\Phi: SL_2(\mathbb{C})\rightarrow SO(1,3)$ is defined through the following equivalent group actions:

$$g(t)\hat{z}=\hat{w} \quad (78)$$

for $g(t)\in SO(1,3)$ whenever

$$g_3(t)Zg_3^*(t)=W \quad (79)$$

for $g_3(t)\in SL_2(\mathbb{C})$ and where $g_3^*(t)$ is the conjugate transpose of $g_3(t)$.

Proof. see [10] and [14]. Therefore, we can obtain the solution $g(t)\in SO(1,3)$ by using this homomorphism defined by equations (78) and (79). Then each column of $SO(1,3)$

is identified with:

$$
\begin{aligned}
g_1(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_1{}^*(t) = W_1 \to \hat{w}_1 \\
g_1(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g_1{}^*(t) = W_2 \to \hat{w}_2 \\
g_1(t)\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} g_1{}^*(t) = W_3 \to \hat{w}_3 \\
g_1(t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_1{}^*(t) = W_3 \to \hat{w}_4
\end{aligned} \tag{80}
$$

where $g(t) \in SO(1,3)$ is defined by:

$$
g(t) = \begin{pmatrix} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 \end{pmatrix} \tag{81}
$$

This can be expressed explicitly but due to space constraints cannot be written in its most complete form here. This section has provided a method for integrating control systems defined on the Lie group SO(1,3) where the controls are time-dependent.

V. Conclusion

In this paper we formulate Riemannian, sub-Riemannian, elastic and mechanical problems as constrained optimal control problems, and lift them to their corresponding Hamiltonian vector fields through the Maximum Principle. We specialize to a particular case and illustrate a method for deriving explicit expressions, relating the extremal curves $l \in \mathfrak{g}^*$ to the optimal solutions $g(t) \in G$ for the semi-simple Lie groups $SO(4)$ and $SO(1,3)$. This method uses the double cover property of these Lie groups to decouple them into lower dimensional systems. These lower dimensional systems are solved in terms of the extremals using a coordinate representation and the systems dynamic constraints. The solutions of the decoupled system are confined to elements in the dual of the Lie algebra. Finally, the solutions to the decoupled systems are projected back onto the original systems to yield their optimal solutions.

References

[1] Walsh, G., Montgomery, R., Sastry, S., ‘Optimal Path Planning on Matrix Lie Groups’. Proceedings of IEEE Conference on Decision and Control, 1994.

[2] Leonard, N., Krishnaprasad, P. S., ‘Motion control of drift free, left-invariant systems on Lie groups’. IEEE Transactions on Automatic control, 1995.

[3] Bloch, A. M., ‘Nonholonomic Mechanics and Control’. Springer-Verlag, New York,2003.

[4] D’ Alessandro, D., Dahleh, M., ‘Optimal Control of Two-Level Quantum Systems’. IEEE Transactions on Automatic Control, Vol. 46, No. 6, p 866-876, 2001.

[5] D’ Alessandro, D., ‘The optimal control problem on SO(4) and its applications to quantum control’, IEEE Transactions on Automatic Control, 47, No.1, 2002, p. 87-92

[6] Arroyo, J., Barros, M., Garay, O., ‘Models of relativistic particles with curvature and torsion revisited’. General Relativity and Gravitation, Springer,Volume 36, Number 6, p.1441 - 1451, 2004.

[7] Jurdjevic, V., ‘Non-Euclidean Elastica’. American Journal of Mathematics, 117, p.93-124, 1995.

[8] Noakes, L., Popiel, T., ‘Quadratures and cubics in $SO(3)$ and $SO(1,2)$’. IMA Journal of Mathematical Control and Information, p.1-11, 2005.

[9] Jurdjevic, V., ‘Hamiltonian point of view of non-Euclidean geometry and elliptic functions’. Systems and Control Letters, 43, p.25-41, 2001.

[10] Jurdjevic, V., ‘Integrable Hamiltonian Systems on Complex Lie groups’. American Mathematical Society, Vol. 178, No. 838, 2005.

[11] Jurdjevic, V., ‘Geometric Control Theory’. Advanced Studies in Mathematics, Cambridge University Press, 52,1997.

[12] Bloch, A., Crouch, P., Ratui, T., ‘Sub-Riemannian optimal control problems’. Fields Institute Communications, vol. 3, pp. 35-48, 1994.

[13] Sussmann, H. J., ‘An introduction to the coordinate-free maximum principle’. In Geometry of Feedback and Optimal Control,B. Jakubczyk and W. Respondek Eds., Marcel Dekker, New York, pp. 463-557, 1997.

[14] Hall, B. C., ‘Lie Groups, Lie Algebras, and Representations: An Elementary Introduction’. Springer-Verlag, New York, 2003.

[15] Biggs, J., Holderbaum, W., ‘Integrating Control Systems defined on the frame bundles of the space forms’. Proceedings of IEEE Conference on Decision and Control, San Diego, 2006.

[16] Hussein, I. I., Bloch, A. M.,‘Constrained Optimal Trajectory on the Group of Rigid Body Motions’. Proceedings of IEEE CDC-ECC,Seville, 2005.

[17] Biggs, J., Holderbaum, W., ‘The Geometry of Optimal Control problems on some Six Dimensional Lie Groups’. Proceedings of IEEE CDC-ECC, 2005.