SMALL BALL PROBABILITY ESTIMATES FOR THE HÖLDER SEMI-NORM OF THE STOCHASTIC HEAT EQUATION

MOHAMMUD FOONDUN, MATHEW JOSEPH, AND KUNWOO KIM

Abstract. We consider the stochastic heat equation on [0, 1] with periodic boundary conditions and driven by space-time white noise. Under various natural conditions, we study small ball probabilities for the Hölder semi-norms of the solutions, and provide near optimal bounds on these probabilities. As an application, we prove a support theorem in these Hölder semi-norms.

1. Introduction and Main results

We consider the stochastic heat equation (SHE) on $\mathbf{T} := [0, 1]$ with periodic boundary condition and driven by space-time white noise (we identify $\mathbf{T}$ as the one-dimensional torus, i.e., $\mathbf{T} := \mathbf{R}/\mathbf{Z}$). This is the real-valued random field $u(t, x)$, $t \in \mathbf{R}_+$, $x \in \mathbf{T}$ which solves

$$\partial_t u(t,x) = \frac{1}{2}\,\partial_x^2 u(t,x) + \sigma\big(t,x,u(t,x)\big)\cdot \dot{W}(t,x), \quad t\in\mathbf{R}_+,\, x\in\mathbf{T}, \tag{1.1}$$

with given initial profile $u(0, \cdot) = u_0 : \mathbf{T} \to \mathbf{R}$ and satisfying $u(t, 0) = u(t, 1)$ for all $t \in \mathbf{R}_+$. The space-time white noise $\dot{W}$ is a centered generalized Gaussian random field with $\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(x-y)\delta_0(t-s)$. We will make the following two assumptions on the function $\sigma : \mathbf{R}_+ \times \mathbf{T} \times \mathbf{R} \to \mathbf{R}$.

Assumption 1.1. *The function σ is uniformly elliptic, that is, there are constants $\mathscr{C}_1 > 0$ and $\mathscr{C}_2 > 0$ such that*

$$\mathscr{C}_1 \le \sigma(t,x,u) \le \mathscr{C}_2 \quad \textit{for all } t,\, x,\, u. \tag{1.2}$$

Assumption 1.2. *The function σ is Lipschitz continuous in the third variable, that is there is a constant $\mathscr{D} > 0$ such that*

$$|\sigma(t,x,u) - \sigma(t,x,v)| \le \mathscr{D}|u-v| \quad \textit{for all } t,\, x,\, u,\, v. \tag{1.3}$$

The existence and uniqueness of solutions to (1.1) under the above assumptions are well known. See for example [DKM+09] or [Wal86] for the proofs and various other properties. It is also known that the solutions are Hölder($\frac{1}{2}-$) in space and Hölder($\frac{1}{4}-$) in time.

In this paper we study the probabilities of the events that the SHE (1.1) is unusually regular, as measured in certain Hölder semi-norms, up to a fixed time. To the best of our knowledge, our paper is the first to carry out such a study even though regularity properties of SPDEs

2010 *Mathematics Subject Classification.* Primary, 60H15; Secondary, 60G17, 60G60.

Key words and phrases. heat equation, white noise, stochastic partial differential equations, small ball.

have been very well studied. See for instance [HSWX20] and [TX17] where very precise information about the modulus of continuity is given.

Our study will be framed as small ball probabilities of these semi-norms. Even though small ball probabilities have been very well studied in many different settings, not much has been done in the context of SPDEs. Only a handful of papers have looked at these types of questions for SPDEs. The paper closest to ours is that of the very recent [AJM] where the sup norm is considered. In another recent paper [Lot17], heat equations with additive noise are considered under different norms and in [Mar04], the stochastic wave equation is studied.

A lot of important developments on small ball probabilities for Gaussian processes can be found in the survey paper [LS01] and the references therein. Regarding similar questions for Hölder norms and various Sobolev norms, see for example [KLS95], [KL93] and the references in [LS01]. To phrase our results precisely we need some notations.

Fix $0 < \theta < \frac{1}{2}$ and a terminal time $T > 0$. For a function $f : [0,T] \times \mathbf{T} \to \mathbf{R}$ and for every $t \in [0,T]$ and $x \in \mathbf{T}$, let

$$\mathcal{H}_t^{(\theta)}(f) := \sup_{x \neq y} \frac{|f(t,x) - f(t,y)|}{|x-y|^{\frac{1}{2}-\theta}}$$

be the *spatial* Hölder semi-norm and let

$$\mathscr{H}_x^{(\theta)}(f) := \sup_{0 \leq s \neq t \leq T} \frac{|f(t,x) - f(s,x)|}{|t-s|^{\frac{1}{4}-\frac{\theta}{2}}}$$

be the *temporal* Hölder semi-norm. The Hölder($\frac{1}{2}-$) regularity in space and the Hölder($\frac{1}{4}-$) regularity in time of u imply that $\sup_{t\in[0,T]} \mathcal{H}_t^{(\theta)}(u)$ and $\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u)$ are finite a.s. The above quantities provide a quantitative way of measuring regularity of functions. We investigate the probability that they are exceptionally small for solutions to the stochastic heat equations.

Before we state the main result, we introduce one more notation. For each $\theta \in (0,\, 1/2)$, we let $\Lambda = \Lambda(\theta)$ be given by

$$\Lambda(\theta) := \int_{\mathbf{R}} p(1,w)|w|^{\frac{1}{2}-\theta}\, dw = \frac{2^{\frac{1}{2}-\theta}}{\sqrt{\pi}}\Gamma(1-\theta), \tag{1.4}$$

where $p(t,x)$ is the Gaussian density (2.1) and $\Gamma(t)$ is the Gamma function.

Theorem 1.1. *Let $0 < \theta < \frac{1}{2}$ and $0 < \epsilon < 1$. Suppose that the initial profile satisfies $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2}\left(1 \wedge \frac{1}{\Lambda}\right)$. Then for any $\eta > 0$ there exist positive constants $C_1, C_2, C_3, C_4 > 0$ dependent on $\mathscr{C}_1, \mathscr{C}_2, \mathscr{D}, \theta, \eta$ such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}+\eta}}\right) \leq P\left(\sup_{\substack{0\leq s,t\leq T\\ x,y\in\mathbf{T}\\ (t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|x-y|^{\frac{1}{2}-\theta} + |t-s|^{\frac{1}{4}-\frac{\theta}{2}}} \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right). \tag{1.5}$$

One can improve the lower and upper bounds in (1.5) by imposing more restrictions on σ; see Theorems 1.2, 1.3 and 1.4 below, and Remark 6.1. In Section 8 below we provide several *support* theorems where bounds on the probability that u is close (in the above Hölder semi-norm) to a function h in certain classes (such as Hölder spaces) are provided.

The reader might wonder whether the upper and lower bounds in (1.5) hold when we consider the Hölder norm

$$\|u\|_{\theta,T} := \|u\|_{\infty,T} + \sup_{\substack{0\le s,t\le T\\ x,y\in\mathbf{T}\\ (t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|t-s|^{\frac14-\frac{\theta}{2}}+|x-y|^{\frac12-\frac{\theta}{2}}}, \tag{1.6}$$

instead of semi-norm considered in (1.5), where $\|u\|_{\infty,T} := \sup_{(t,x)\in[0,T]\times\mathbf{T}}|u(t,x)|$. This is *not* the case. Indeed, due to the $1:2:4$ scaling of fluctuations, space and time for the SHE, the Hölder semi-norm of the SHE in time-space regions of the form $[t,t+\epsilon^{\frac{2}{\theta}}]\times[x,x+\epsilon^{\frac{1}{\theta}}]$ fluctuates by order ϵ. However in these regions, the solution u itself fluctuates by order $\epsilon^{\frac{1}{2\theta}}$. Intuitively, what we try to show in this article is that the $T\epsilon^{-\frac{3}{\theta}}$ time-space boxes obtained by dividing $[0,T]\times\mathbf{T}$ into subintervals of the form $[t,t+\epsilon^{\frac{2}{\theta}}]\times[x,x+\epsilon^{\frac{1}{\theta}}]$ can be *somewhat* viewed as independent regions. This explains the $T\epsilon^{-\frac{3}{\theta}}$ that we obtain in the exponents in (1.5). Moreover, by this reasoning, one should expect similar bounds on the probability $P(\|u\|_{\infty,T}\le\epsilon^{\frac{1}{2\theta}})$ if we start at $u_0\equiv 0$, for example. In fact this is what was proved in [AJM]. Therefore, while it is *not* true that we have the same bounds as (1.5) for the Hölder norm (1.6), we do have the same bounds for

$$P\left(\|u\|_{\infty,T}\le\epsilon^{\frac{1}{2\theta}},\ \sup_{\substack{0\le s,t\le T\\ x,y\in\mathbf{T}\\ (t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|x-y|^{\frac12-\theta}+|t-s|^{\frac14-\frac{\theta}{2}}}\le\epsilon\right),$$

if we start with u_0 such that $\|u_0\|_\infty\le\frac{\epsilon^{\frac{1}{2\theta}}}{2}$ and $\mathcal{H}_0^{(\theta)}(u)\le\frac{\epsilon}{2}\left(1\wedge\frac{1}{\Lambda}\right)$.

1.1. **Results.** Instead of looking at the probability of the event in (1.5) directly, we consider the probabilities of the events $\left\{\sup_{t\in[0,T]}\mathcal{H}_t^{(\theta)}(u)\le\epsilon\right\}$ and $\left\{\sup_{x\in\mathbf{T}}\mathscr{H}_x^{(\theta)}(u)\le\epsilon\right\}$ separately. The bounds in (1.5) will then follow from the bounds on the probabilities of these two events (see Section 6). It turns out that the regularity in time (as measured by the smallness of $\mathscr{H}_x^{(\theta)}(u)$) is intimately connected to the regularity in space (as measured by the smallness of $\mathcal{H}_t^{(\theta)}(u)$). Our arguments indicate that for the solution to be *regular* in time it is necessary for it to be *regular* in space. We now state the small ball probability estimates for $\sup_{t\le T}\mathcal{H}_t^{(\theta)}(u)$ and $\sup_{x\in\mathbf{T}}\mathscr{H}^{(\theta)}(u)$ with varying assumptions on the nonlinearity σ.

Theorem 1.2. *Assume that the function $\sigma(t,x,u)$ is independent of u and satisfies Assumption 1.1. Let $0<\theta<\frac12$ and $0<\epsilon<1$.*

(a) *Suppose that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2}$. *Then there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent only on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}}\right). \tag{1.7}$$

(b) *Suppose that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2\Lambda}$. *Then there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent only on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \leq P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}}\right). \tag{1.8}$$

It can be shown that u is a Gaussian random field when σ does not depend on u. The proof of the above theorem takes up a significant part of this paper and hinges on well known results specific to Gaussian processes, as well as the recently proved Gaussian correlation inequality ([Roy14]). We next consider the case when σ can also depend on u.

Theorem 1.3. *Suppose that* $\sigma(t, x, u)$ *satisfies both Assumptions 1.1 and 1.2. Let* $0 < \theta < \frac{1}{2}$ *and* $0 < \epsilon < 1$.

(a) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2}$. *Then for any* $\eta > 0$ *there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent on* $\mathscr{C}_1, \mathscr{C}_2, \mathscr{D}, \theta, \eta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}+\eta}}\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right). \tag{1.9}$$

(b) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2\Lambda}$. *Then for any* $\eta > 0$ *there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent on* $\mathscr{C}_1, \mathscr{C}_2, \mathscr{D}, \theta, \eta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}+\eta}}\right) \leq P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right). \tag{1.10}$$

Our next result sharpens the lower bound of the above theorem by imposing a further restriction on the Lipschitz coefficient of σ.

Theorem 1.4. *Suppose that* $\sigma(t, x, u)$ *satisfies both Assumptions 1.1 and 1.2. Let* $0 < \theta < \frac{1}{2}$ *and* $0 < \epsilon < 1$.

(a) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2}$. *Then there is a* $\mathscr{D}_0 > 0$ *such that for all* $\mathscr{D} < \mathscr{D}_0$, *there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent only on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right). \tag{1.11}$$

(b) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2\Lambda}$. *Then there is a* $\mathscr{D}_1 > 0$ *such that for all* $\mathscr{D} < \mathscr{D}_1$, *there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent only on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \leq P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right). \tag{1.12}$$

We now say a few words about the proofs of our theorems. As mentioned above, Theorem 1.2 relies heavily on the fact that when σ is independent of u, the solution $u(t,x)$ is a Gaussian random field. The proof of the upper bound is essentially contained in the proof of Lemma 3.2 which among other things, relies on the Gaussianity of u. Another crucial element is the sharp bound given by Lemma 2.5 whose proof uses some well known ideas presented in [AJM]. It is also interesting to note that when σ is a constant, one can further simplify the proof of the upper bound by resorting to Slepian's lemma; see Remark 3.2 for more details. The lower bounds rely even more heavily on Gaussianity of the solution in that we use the Gaussian correlation inequality in an essential way. This is done in Lemma 4.2. Another key ingredient is the use of a change of measure argument similar to [AJM]. Intuitively, this allows us to keep the solution small which gives us a better handle on the estimates required.

Under the conditions of Theorems 1.3 and 1.4, the solutions are no longer Gaussian, so the corresponding proofs require different strategies. For the lower bounds, we use a *perturbation* argument together with the proof of the lower bound in Theorem 1.2. We note that the sharper lower bound in 1.4 is also a consequence of the very same perturbation argument.

The proofs of the upper bounds in Theorems 1.3 and 1.4 are entirely different and make use of certain auxiliary random variables which have nice independence properties. These random variables are indexed by the spatial variables. Their construction is inspired by [CJK13].

In the final section of this paper, we present some extensions and prove a support theorem in the Hölder semi-norm. It will be clear later that our paper raises several questions. One such open question is whether the bounds (1.7) and (1.8) continue to hold in the general case, that is for any σ satisfying Assumptions 1.1 and 1.2. We have assumed that σ is bounded below and above by positive constants. Another avenue of investigation is to replace these assumptions by less stringent ones. Let us point that here, when $\theta = \frac{1}{2}$ the above theorems match the results recently obtained in [AJM] for the small ball probabilities of the sup norm of u.

We have studied the small ball probability estimates of $\sup_{t\in[0,T]} \mathcal{H}_t^{(\theta)}(u)$ and $\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u)$. We next consider the small ball probability estimates of $\mathcal{H}_T^{(\theta)}(u)$ for a fixed time T, and $\mathscr{H}_X^{(\theta)}(u)$ for a fixed spatial point X. We start with the Gaussian case.

Theorem 1.5. *Assume that the function $\sigma(t,x,u)$ is independent of u and satisfies Assumption 1.1, and fix a time $T>0$. Let $0<\theta<\frac{1}{2}$ and $0<\epsilon<1$.*

(a) Assume that the initial profile satisfies $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2}$. Then there exist positive constants $C_1, C_2, C_3, C_4 > 0$ dependent only on $\mathscr{C}_1, \mathscr{C}_2, \theta, T$ such that

$$C_1 \exp\left(-\frac{C_2}{\epsilon^{\frac{1}{\theta}}}\right) \leq P\left(\mathcal{H}_T^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4}{\epsilon^{\frac{1}{\theta}}}\right). \tag{1.13}$$

(b) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2\Lambda}$. *Then there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *dependent only on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{2}{\theta}}}\right) \leq P\left(\mathscr{H}_X^{(\theta)}(u) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{2}{\theta}}}\right). \tag{1.14}$$

The upper bounds are in fact an immediate consequence of the proof of Theorem 1.2, and we will see that the constants C_3 and C_4 can be chosen independently of T. The lower bounds follow from the arguments in the proof of Theorem 2.2. of [KLS95]. The proof of the lower bound above is specific to Gaussian processes and cannot be directly extended to the general case.

Theorem 1.6. *Suppose that* $\sigma(t,x,u)$ *satisfies both Assumptions 1.1 and 1.2, and fix a time* $T > 0$. *Let* $0 < \theta < \frac{1}{2}$ *and* $0 < \epsilon < 1$.

(a) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2}$. *Then there exist positive constants* $C_1, C_2 > 0$ *dependent on* $\mathscr{C}_1, \mathscr{C}_2, \theta, T$ *such that*

$$P\left(\mathcal{H}_T^{(\theta)}(u) \leq \epsilon\right) \leq C_1 \exp\left(-\frac{C_2}{\epsilon^{\frac{1}{\theta}} |\log \epsilon|^{\frac{3}{2}}}\right). \tag{1.15}$$

(b) *Assume that the initial profile satisfies* $\mathcal{H}_0^{(\theta)}(u) \leq \frac{\epsilon}{2\Lambda}$. *Then there exist positive constants* $C_1, C_2 > 0$ *dependent on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *such that*

$$P\left(\mathscr{H}_X^{(\theta)}(u) \leq \epsilon\right) \leq C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{2}{\theta}}}\right). \tag{1.16}$$

A trivial lower bound is obtained from either Theorem 1.3 or Theorem 1.4 depending on whether σ satisfies the assumptions of Theorem 1.3 or Theorem 1.4. However this is very far from the lower bound obtained in Theorem 1.5.

Remark 1.1. *Note that it is sufficient to prove the above theorems for all sufficiently small* $\epsilon < \epsilon_0$, *where* ϵ_0 *is dependent on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *(and maybe additionally on* η *in the case of Theorem 1.3 and* T *in the case of Theorems 1.5 (a) and 1.6 (a)). The conclusion for any* $0 < \epsilon < 1$ *follows from the fact that the probabilities* $P\left(\mathcal{H}_T^{(\theta)}(u) \leq \epsilon\right)$, $P\left(\mathscr{H}_X^{(\theta)}(u) \leq \epsilon\right)$, $P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \epsilon\right)$ *and* $P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \leq \epsilon\right)$ *are nondecreasing in* ϵ.

Plan: Section 2 contains some preliminary estimates. The proofs of the upper bounds in Theorems 1.2, 1.3 are given in Section 3, while the correponding lower bounds are given in Section 4. The proof of Theorem 1.4 is given in Section 5. After this, we give the proof of Theorem 1.1 in Section 6. The proofs of Theorems 1.5 and 1.6 are presented in Section 7. Finally in Section 8, we give some extensions and prove a support theorem as a corollary of our results.

Notation: Throughout this paper, C with or without subscripts will denote positive constants whose value might change from line to line. We will sometimes emphasize that the dependence of the constants on specific parameters will be denoted by specifying the parameters in brackets, e.g. $C(\delta)$. For a random variable X we denote $\|X\|_p := E[|X|^p]^{1/p}$.

2. Preliminaries

We define the heat kernel $G(t,x)$ as the fundamental solution of the heat equation on the torus $\mathbf{T}$

$$\partial_t G(t,x) = \frac{1}{2}\partial_x^2 G(t,x),$$
$$G(0,x) = \delta_0(x).$$

Let

$$p(t,x) = (2\pi t)^{-1/2} \exp\left(-\frac{x^2}{2t}\right) \tag{2.1}$$

be the fundamental solution of the heat equation on $\mathbf{R}$. It is known that the heat kernel on $\mathbf{T}$ is given explicitly by

$$G(t,x) = \sum_{k\in\mathbf{Z}} p(t,x+k). \tag{2.2}$$

We interpret the solution to (1.1) in the sense of Walsh ([Wal86]) as a random field which satisfies

$$u(t,x) = \big(G_t * u_0\big)(x) + N(t,x), \quad \text{a.s.} \tag{2.3}$$

for each t and x, where the first term on the right is the space convolution of the heat kernel with the initial profile $u_0(x)$, i.e.,

$$\big(G_t * u_0\big)(x) = \int_{\mathbf{T}} G(t,x-y)\cdot u_0(y)\,dy,$$

and the second term which we call the *noise term* is the space-time convolution of the heat kernel with the product of $\sigma\,(s,y,u(s,y))$ and white noise:

$$N(t,x) = \int_0^t \int_{\mathbf{T}} G(t-s,x-y)\cdot\sigma\big(s,y,u(s,y)\big)W(dsdy). \tag{2.4}$$

We are working on the torus $\mathbf{T} := \mathbf{R}/\mathbf{Z}$, so in the above two expressions $x-y$ should be interpreted as the unique point z in $[0,1)$ such that $x-y=z+k$ for some $k\in\mathbf{Z}$.

We now show that it is enough to prove our main results under the assumption that $u_0\equiv 0$. For a function $g:\mathbf{T}\to\mathbf{R}$ dependent only on the spatial variable x, define

$$\mathcal{H}^{(\theta)}(g) := \sup_{x\neq y\in\mathbf{T}} \frac{|g(x)-g(y)|}{|x-y|^{\frac{1}{2}-\theta}}. \tag{2.5}$$

(Note the absence of subscript t in $\mathcal{H}^{(\theta)}$). The first lemma is a simple observation about the spatial Hölder regularity of $G_t * u_0$.

Lemma 2.1. *If for some $a>0$ one has $\mathcal{H}^{(\theta)}(u_0)\le a$ then $\mathcal{H}^{(\theta)}\big(G_t * u_0\big)\le a$ for all $t>0$.*

Proof. Let $\tilde{u}_0 : \mathbf{R} \to \mathbf{R}$ be the periodization of u_0, that is $\tilde{u}_0(x+k) = u_0(x)$ for all $k \in \mathbf{Z}$ and $x \in \mathbf{T}$. We have

$$\begin{aligned}
&\left|\big(G_t * u_0\big)(x) - \big(G_t * u_0\big)(y)\right| \\
&= \left|\sum_{k\in\mathbf{Z}} \int_{\mathbf{T}} \Big[p(t, x-z+k) - p(t, y-z+k)\Big] \cdot u_0(z)\, dz\right| \\
&= \left|\sum_{k\in\mathbf{Z}} \int_{\mathbf{T}} \Big[p(t, x-z+k) - p(t, y-z+k)\Big] \cdot \tilde{u}_0(z)\, dz\right| \\
&= \left|\sum_{k\in\mathbf{Z}} \int_{-k}^{-k+1} \Big[p(t, x-w) - p(t, y-w)\Big] \cdot \tilde{u}_0(w+k)\, dw\right| \\
&= \left|\int_{\mathbf{R}} p(t,w) \cdot \Big[\tilde{u}_0(x-w) - \tilde{u}_0(y-w)\Big] dw\right|
\end{aligned}$$

We have $|\tilde{u}_0(x-w) - \tilde{u}_0(y-w)| \le A|x-y|^{\frac{1}{2}-\theta}$ by assumption and $p(t,\cdot)$ integrates to 1, therefore the result follows. ◻

We now prove a similar result for the temporal Hölder regularity of $(G_\cdot * u_0)(x)$. Recall the constant Λ introduced in (1.4).

Lemma 2.2. *If $\mathcal{H}^{(\theta)}(u_0) \le a$ then*

$$\frac{\left|\left(G_t * u_0\right)(x) - \left(G_s * u_0\right)(x)\right|}{|t-s|^{\frac{1}{4}-\frac{\theta}{2}}} \le \Lambda a$$

Proof. Without loss of generality assume that $s < t$. Let $g(x) = \left(G_s * u_0\right)(x)$ and $\tilde{g}$ be the periodization of g. Then by arguments similar to Lemma 2.1 we have

$$\begin{aligned}
\left|\left(G_t * u_0\right)(x) - \left(G_s * u_0\right)(x)\right| &= \left|\left(G_{t-s} * g\right)(x) - g(x)\right| \\
&= \int_{\mathbf{R}} p(t-s, w) \cdot |\tilde{g}(x-w) - \tilde{g}(x)| \\
&\le a \int_{\mathbf{R}} p(t-s, w)|w|^{\frac{1}{2}-\theta}\, dw \\
&\le \Lambda a |t-s|^{\frac{1}{4}-\frac{\theta}{2}},
\end{aligned}$$

by a simple change of variables. ◻

Now consider the random field

$$v(t,x) = u(t,x) - (G_t * u_0)(x), \tag{2.6}$$

where $u(t,x)$ solves (1.1) with the initial profile u_0. One can easily check that

$$\partial_t v(t,x) = \frac{1}{2}\partial_x^2 v(t,x) + \widetilde{\sigma}\big(t, x, v(t,x)\big) \cdot \dot{W}(t,x),$$

with initial profile $v_0 \equiv 0$, where

$$\widetilde{\sigma}(t,x,v) = \sigma\Big(t, x, v + (G_t * u_0)(x)\Big).$$

Furthermore $|\widetilde{\sigma}(t,x,v) - \widetilde{\sigma}(t,x,w)| \le \mathscr{D}|v-w|$ and $\widetilde{\sigma}$ is bounded below and above by $\mathscr{C}_1, \mathscr{C}_2$.

Assume $\mathcal{H}^{(\theta)}(u_0) \leq \frac{\epsilon}{2}$. Then from Lemma 2.1 and (2.6), we have the following implications:

$$\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(v) \leq \frac{\epsilon}{2} \qquad \text{implies} \qquad \sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \epsilon,$$
$$\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \frac{\epsilon}{2} \qquad \text{implies} \qquad \sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(v) \leq \epsilon.$$

Similar implications hold when we just consider $\mathcal{H}_T^{(\theta)}(u)$ and $\mathcal{H}_T^{(\theta)}(v)$ (without the supremum in t).

Similary if $\mathcal{H}^{(\theta)}(u_0) \leq \frac{\epsilon}{2\Lambda}$, then from Lemma 2.2,

$$\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(v) \leq \frac{\epsilon}{2} \qquad \text{implies} \qquad \sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \leq \epsilon,$$
$$\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \leq \frac{\epsilon}{2} \qquad \text{implies} \qquad \sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(v) \leq \epsilon.$$

Similar implications hold when we just consider $\mathscr{H}_X^{(\theta)}(u)$ and $\mathscr{H}_X^{(\theta)}(v)$ (without the supremum in x).

Remark 2.1. (Important) *From the above discussion we observe that it is sufficient to prove the main theorems stated in the introduction with $u_0 \equiv 0$. This is of course not surprising since the Laplacian is known to have smoothing effects. The above argument is merely a weak manifestation of this.* **We will assume that the initial profile $u_0 \equiv 0$ for the rest of the article.**

We will need the following lemmas which were proved in [AJM].

Lemma 2.3 (Lemma 3.3 in [AJM])**.** *There exist constants C_1, C_2 such that for all time points $0 < s < t < 1$, spatial points $x, y \in \mathbf{T}$, and $\lambda > 0$,*

$$P\left(\left|N(t,x) - N(t,y)\right| > \lambda\right) \leq C_1 \exp\left(-\frac{C_2\lambda^2}{\mathscr{C}_2^2|x-y|}\right)$$
$$P\left(\left|N(t,x) - N(s,x)\right| > \lambda\right) \leq C_1 \exp\left(-\frac{C_2\lambda^2}{\mathscr{C}_2^2|t-s|^{1/2}}\right).$$

Lemma 2.4 (Lemma 3.4 in [AJM])**.** *There exist universal constants $\mathbf{K}_1$, $\mathbf{K}_2 > 0$ such that for all $\alpha, \lambda, \epsilon > 0$, $\theta > 0$, and for all $a \in [0,1)$ with $a + \epsilon^{1/\theta} < 1$ we have*

$$P\left(\sup_{\substack{0\leq t\leq \alpha\epsilon^{\frac{2}{\theta}} \\ x\in[a,a+\epsilon^{\frac{1}{\theta}}]}} |N(t,x)| > \lambda\epsilon^{\frac{1}{2\theta}}\right) \leq \frac{\mathbf{K}_1}{1\wedge\sqrt{\alpha}} \exp\left(-\mathbf{K}_2 \frac{\lambda^2}{\mathscr{C}_2^2\sqrt{\alpha}}\right). \tag{2.7}$$

Remark 2.2. *Note that Lemma 3.4 in* [AJM] *provides* (2.7) *when $a = 0$, however, one can follow exactly the same proof to get* (2.7) *for any $a \in (0,1)$ with $a + \epsilon^{1/\theta} < 1$. It was also pointed out in* [AJM, Remark 3.1] *that if $\left|\sigma\big(s,y,u(s,y)\big)\right| \leq C_1\epsilon^{\frac{1}{2\theta}}$ then one can bound the right hand side of* (2.7) *by $\frac{\mathbf{K}_1}{1\wedge\sqrt{\alpha}} \exp\left(-\mathbf{K}_2 \frac{\lambda^2}{C_1^2\epsilon^{\frac{1}{\theta}}\sqrt{\alpha}}\right)$.*

The analysis of the following function will play a crucial role in this paper.

$$\begin{aligned}\widetilde{N}(t,x,y) &:= \int_0^t \int_{\mathbf{T}} \frac{G(t-r,x-z)-G(t-r,y-z)}{|x-y|^{\frac{1}{2}-\theta}}\, \sigma\left(r,z,u(r,z)\right) \dot{W}(drdz) \\ &= \frac{N(t,x)-N(t,y)}{|x-y|^{\frac{1}{2}-\theta}}.\end{aligned} \tag{2.8}$$

Although we have not made it explicit, the function $\widetilde{N}(t,x,y)$ clearly depends also on θ. The following lemma is used several times in the paper.

Lemma 2.5. *Let $\theta \in (0,\frac{1}{2})$. There exist constants $\mathbf{K}_3, \mathbf{K}_4$ dependent only on θ such that for all $\alpha, \lambda, \epsilon > 0$ and for all $a \in [0,1)$ with $a + \epsilon^{1/\theta} < 1$, we have*

$$P\left(\sup_{\substack{0\le t\le \alpha\epsilon^{\frac{2}{\theta}} \\ x,y\in[a,a+\epsilon^{\frac{1}{\theta}}],\, x\neq y}} \left|\widetilde{N}(t,x,y)\right| > \lambda\epsilon \right) \le \frac{\mathbf{K}_3}{1\wedge\sqrt{\alpha}} \exp\left(-\mathbf{K}_4 \frac{\lambda^2}{\mathscr{C}_2^2 \alpha^\theta}\right). \tag{2.9}$$

Proof. We will show (2.9) when $a = 0$ and the same proof works for the general $a \in (0,1)$, which is similar to Lemma 2.4 and Remark 2.2. Let us first consider the case when $\alpha \ge 1$. Consider the following grid on $[0, \alpha\epsilon^{\frac{2}{\theta}}] \times [0, \epsilon^{\frac{1}{\theta}}]$, where the first coordinate is time and the second is space:

$$\mathbb{G}_n = \left\{ \left(\frac{j}{2^{2n}}, \frac{k}{2^n}\right) : 0 \le j \le \alpha\epsilon^{\frac{2}{\theta}} 2^{2n},\ 0 \le k \le \epsilon^{\frac{1}{\theta}} 2^n \right\}.$$

The grid $\mathbb{G}_n$ will consist of only the point $(0,0)$ if $n < n_0$ where

$$n_0 := \lceil \log_2(\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}) \rceil \tag{2.10}$$

Any $p \in \mathbb{G}_n$ will be of the form $\left(\frac{j}{2^{2n}}, \frac{k}{2^n}\right)$, so for notational convenience, we set

$$N(p) := N\left(\frac{j}{2^{2n}}, \frac{k}{2^n}\right).$$

We will choose two parameters $0 < \gamma_0(\theta) < \gamma_1(\theta) < \frac{1}{2}$ which depend only on θ which satisfy the following constraint

$$\frac{1}{2} - \theta = \gamma_1 - \gamma_0. \tag{2.11}$$

We will fix the constant

$$K = \frac{1-2^{-\gamma_1}}{2^{1+\gamma_0 n_0}}, \tag{2.12}$$

and consider the events

$$A(n,\lambda) = \left\{ |N(p)-N(q)| \le \lambda K \epsilon 2^{-\gamma_1 n} 2^{\gamma_0 n_0}, \text{ for all } p,q \in \mathbb{G}_n \text{ spatial neighbors} \right\}.$$

By p,q being spatial neighbors in the grid $\mathbb{G}_n$, we mean that p,q have the same time coordinate but their spatial coordinates are adjacent in $\mathbb{G}_n$. For instance $\left(\frac{j}{2^{2n}}, \frac{k-1}{2^n}\right)$ and $\left(\frac{j}{2^{2n}}, \frac{k+1}{2^n}\right)$ are spatial neighbors of $\left(\frac{j}{2^{2n}}, \frac{k}{2^n}\right)$.

The number of such pairs of points p, q is bounded by $2 \cdot \alpha\epsilon^{\frac{2}{\theta}} 2^{2n} \cdot \epsilon^{\frac{1}{\theta}} 2^n \leq 2^4 \cdot 2^{3(n-n_0)}$, where we have used (2.10). Therefore a union bound along with the first tail bound in Lemma 2.3 gives

$$\begin{aligned} P\big(A(n,\lambda)^c\big) &\leq C_1 2^{3(n-n_0)} \exp\left(-\frac{C_2\lambda^2 K^2 \epsilon^2 2^{-2\gamma_1 n} 2^{2\gamma_0 n_0}}{\mathscr{C}_2^2 2^{-n}}\right) \\ &\leq C_1 2^{3(n-n_0)} \exp\left(-\frac{C_2\lambda^2 K^2 \alpha^{-\theta} 2^{-2n_0\theta} 2^{-2\gamma_1 n} 2^{2\gamma_0 n_0}}{\mathscr{C}_2^2 2^{-n}}\right) \\ &\leq C_1 2^{3(n-n_0)} \exp\left(-\frac{C_2\lambda^2 K^2 2^{(1-2\gamma_1)(n-n_0)}}{\mathscr{C}_2^2 \alpha^{\theta}}\right), \end{aligned}$$

where the second inequality follows since $\alpha^\theta \epsilon^2 2^{2n_0\theta} \geq 1$ by our choice of n_0 and the final inequality is obtained using the choice γ_0, γ_1 in (2.11).

We now let $A(\lambda) := \cap_{n \geq n_0} A(n, \lambda)$ and use a union bound once again to obtain

$$\begin{aligned} P\big(A(\lambda)^c\big) &\leq \sum_{n \geq n_0} P\big(A(n,\lambda)^c\big) \\ &\leq C_1 \sum_{n \geq n_0} 2^{3(n-n_0)} \exp\left(-\frac{C_2\lambda^2 K^2 2^{(1-2\gamma_1)(n-n_0)}}{\mathscr{C}_2^2 \alpha^{\theta}}\right) \\ &\leq C_3 \exp\left(-\frac{C_4 \lambda^2 K^2}{\mathscr{C}_2^2 \alpha^\theta}\right). \end{aligned}$$

Now on the event $A(\lambda)$, one has for p, q spatial neighbors in $\mathbb{G}_n$

$$\frac{|N(p) - N(q)|}{|p-q|^{\frac{1}{2}-\theta}} \leq \frac{\lambda K \epsilon 2^{-\gamma_1 n} 2^{\gamma_0 n_0}}{2^{-n(\frac{1}{2}-\theta)}} \leq \lambda\epsilon,$$

by our choice of γ_0, γ_1 and K in (2.11) and (2.12).

We now show that the above bound continues to hold when $p, q \in \mathbb{G}_n$ are no longer spatial neighbors but have the same time coordinate. Let the spatial coordinate of p be $k2^{-n}$ and let the spatial coordinate of q be $l2^{-n}$, and without loss of generality assume $k < l$. Find the smallest positive integer n_1 with $n_0 \leq n_1 \leq n$ such that

$$\frac{k}{2^n} \leq \frac{k_1}{2^{n_1}} < \frac{k_1+1}{2^{n_1}} \leq \frac{l}{2^n} \tag{2.13}$$

for some nonnegative integer k_1. First note that we must have

$$\frac{1}{2^{n_1}} \leq \left|\frac{k}{2^n} - \frac{l}{2^n}\right| \leq \frac{4}{2^{n_1}}. \tag{2.14}$$

The lower bound is clear by (2.13) and the upper bound follows from the minimality of n_1, for if the difference between $k2^{-n}$ and $l2^{-n}$ was larger than 2^{2-n_1} then there would be two spatial neighbors in $\mathbb{G}_{n_1-1}$ between them.

One next observes that we can find a sequence of points p_i, $n_1 \leq i \leq n$ and q_i, $n_1 \leq i \leq n$ with the same time coordinates as p, q, such that p_i, p_{i+1} (resp. q_i, q_{i+1}) are either equal or adjacent spatial points in $\mathbb{G}_i$. In addition at most one such adjacent spatial pair (p_i, p_{i+1}) (resp. (q_i, q_{i+1})) is in each $\mathbb{G}_j$, $n_1 \le j \le n$, and $p_n = p, q_n = q$. Therefore

$$\begin{aligned}|N(p) - N(q)| &\le \sum_{i=n_1}^{n} \left|N(p_i) - N(p_{i+1})\right| + \sum_{i=n_1}^{n} \left|N(q_i) - N(q_{i+1})\right| \\ &\le 2 \sum_{i=n_1}^{n} \lambda K \epsilon 2^{-\gamma_1 i} 2^{\gamma_0 n_0}\end{aligned}$$

on the event $A(\lambda)$. As a consequence, on this event

$$\frac{|N(p) - N(q)|}{|p-q|^{\frac{1}{2}-\theta}} \le \frac{2\lambda K\epsilon}{1-2^{-\gamma_1}} \cdot \frac{2^{\gamma_0 n_0} 2^{-\gamma_1 n_1}}{2^{-n_1(\frac{1}{2}-\theta)}} \le \lambda\epsilon$$

by (2.14) and our choice of γ_0, γ_1 and K in (2.11) and (2.12). This completes the proof in the case $\alpha \ge 1$.

In the case $0 < \alpha < 1$, we divide the spatial interval into smaller intervals of length $\sqrt{\alpha}\epsilon^{\frac{1}{\theta}}$. A simple union bound along with stationarity gives

$$P\left(\sup_{\substack{0\le t\le \alpha\epsilon^{\frac{2}{\theta}} \\ x,y\in[0,\epsilon^{\frac{1}{\theta}}],\, x\neq y}} \left|\widetilde{N}(t,x,y)\right| > \lambda\epsilon\right) \le \frac{1}{\sqrt{\alpha}} P\left(\sup_{\substack{0\le t\le \alpha\epsilon^{\frac{2}{\theta}} \\ x,y\in[0,\sqrt{\alpha}\epsilon^{\frac{1}{\theta}}],\, x\neq y}} \left|\widetilde{N}(t,x,y)\right| > \frac{\lambda(\alpha^{\frac{\theta}{2}}\epsilon)}{\alpha^{\frac{\theta}{2}}}\right),$$

and now we apply the previous argument. □

Remark 2.3. *From Remark 2.2, it also follows from the proof that if* $\left|\sigma\big(s,y,u(s,y)\big)\right| \le C_1\epsilon^{\frac{1}{2\theta}}$ *then one can bound the right hand side of* (2.9) *by* $\frac{\mathbf{K}_3}{1\wedge\sqrt{\alpha}} \exp\left(-\mathbf{K}_4 \frac{\lambda^2}{C_1^2 \epsilon^{\frac{1}{\theta}} \alpha^\theta}\right)$.

Define

$$N^{\#}(s,t,x) := \frac{N(t,x) - N(s,x)}{|t-s|^{\frac{1}{4}-\frac{\theta}{2}}}. \tag{2.15}$$

The proof of the following lemma is similar to that of Lemma 2.5 and is therefore omitted.

Lemma 2.6. *Let* $\theta \in (0, \frac{1}{2})$. *There exist constants* $\mathbf{K}_7, \mathbf{K}_8$ *dependent only on* θ *such that for all* $\alpha, \lambda, \epsilon > 0$ *and for all* $a \in [0,1)$ *with* $a + \epsilon^{1/\theta} < 1$, *we have*

$$P\left(\sup_{\substack{0\le s,t\le \alpha\epsilon^{\frac{2}{\theta}},\, s\neq t \\ x\in[a,a+\epsilon^{\frac{1}{\theta}}]}} \left|N^{\#}(s,t,x)\right| > \lambda\epsilon\right) \le \frac{\mathbf{K}_7}{1\wedge\sqrt{\alpha}} \exp\left(-\mathbf{K}_8 \frac{\lambda^2}{\mathscr{C}_2^2 \alpha^\theta}\right). \tag{2.16}$$

Remark 2.4. *Note also here that a similar statement to that of Remark 2.3 also holds in this case. That is, if* $\left|\sigma\big(s,y,u(s,y)\big)\right| \le C_1\epsilon^{\frac{1}{2\theta}}$, *then one can bound the right hand side of* (2.7) *by* $\frac{\mathbf{K}_7}{1\wedge\sqrt{\alpha}} \exp\left(-\mathbf{K}_8 \frac{\lambda^2}{C_1^2\epsilon^{\frac{1}{\theta}} \alpha^\theta}\right)$.

We will also need some estimates concerning $G(t,\, x)$, which come from Lemmas 3.1 and 3.2 of [AJM]. For the lemma below let

$$x_* = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ x-1, & \frac{1}{2} < x \le 1. \end{cases}$$

We have

Lemma 2.7. *There exist positive constants C_0, C_1, C_2, C_3 such that*

$$G(t,\, x) \le C_0 p(t,\, x_*) \quad \text{for all } x \in [0,\, 1],\, t \in \mathbf{T}, \tag{2.17}$$

$$\int_0^t \int_{\mathbf{T}} |G(s, x-z) - G(s, y-z)|^2 dz\, ds \le C_0|x-y| \quad \text{for all } x \in [0,\, 1] \text{ and } t \ge 0, \tag{2.18}$$

$$\int_s^t \int_{\mathbf{T}} G^2(r, x) dx dr \le C_1\sqrt{t-s} \quad \text{for } 0 < s \le t \le s+1, \tag{2.19}$$

(2.20)
$$C_2\sqrt{t-s} \le \int_0^s \int_{\mathbf{T}} \left[G(t-r, z) - G(s-r, z)\right]^2 dz dr \le C_3\sqrt{t-s} \quad \text{for all } 0 < s \le t < \infty.$$

With the preliminaries in place we can now move on to proving the theorems stated in the introduction.

3. Upper bounds

3.1. **Upper bound in Theorem 1.2 (a).** We are assuming that the function $\sigma(t, x, u) = \sigma(t, x)$ does not depend on the third variable so the random field $u(t, x)$ is Gaussian. Before proving the required estimates, we describe the main strategy behind the proof.

Fix parameters $c_0 > 0, c_1 \ge 4$ to be specified later, and let

$$\delta := \epsilon^{\frac{1}{\theta}}.$$

We consider discrete time-space points (t_i, x_j), where the time points t_i are uniformly spaced in $[0, T]$ and space points x_j are uniformly spaced in $\mathbf{T}$:

$$\begin{aligned} t_i &= ic_0\delta^2, \qquad i = 0,\, 1,\, \cdots, I := \left[\frac{T}{c_0\delta^2}\right] \\ x_j &= jc_1\delta, \qquad j = 0,\, 1,\, \cdots, J := \left[\frac{1}{c_1\delta}\right]. \end{aligned} \tag{3.1}$$

We clearly have

$$\begin{aligned} P\left(\sup_{0\le t\le T} \mathcal{H}_t^{(\theta)}(u) \le \epsilon\right) &\le P\left(\max_{\substack{i=0,1,\cdots,I \\ j=0,1\cdots J}} \frac{\left|u(t_i, x_j+\delta) - u(t_i, x_j)\right|}{\delta^{\frac{1}{2}-\theta}} \le \epsilon\right) \\ &\le P\left(\max_{\substack{i=0,1,\cdots,I \\ j=0,1\cdots J}} \left|u(t_i, x_j+\delta) - u(t_i, x_j)\right| \le \epsilon^{\frac{1}{2\theta}}\right), \end{aligned} \tag{3.2}$$

Consider the events

$$A_i := \left\{ \max_{j=0,1,\cdots,J} \left| u(t_i, x_j + \delta) - u(t_i, x_j) \right| \leq \epsilon^{\frac{1}{2\theta}} \right\}. \tag{3.3}$$

From the above

$$\begin{aligned} P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u) \leq \epsilon \right) &\leq P\left(\bigcap_{i=0}^{I} A_i \right) \\ &= \prod_{i=0}^{I} P\left(A_i \middle| A_0, A_1 \cdots A_{i-1} \right). \end{aligned} \tag{3.4}$$

We will show in Lemma 3.2 below that for some $0 < \eta < 1$,

$$P\left(A_i \,\middle|\, u(s,x),\, s \leq t_{i-1},\, x \in \mathbf{T} \right) \leq \eta^J \tag{3.5}$$

uniformly in i. Since the above bound holds regardless of the profile up to time t_{i-1} one can conclude that the right hand side of (3.4) is bounded by $\eta^{J(I+1)}$, which gives us the required upper bound in Theorem 1.2.

We have the following lemma which plays an important role along with the fact the solution is Gaussian. For $k \in \mathbf{N}^+$ and $\delta > 0$, we define

$$\tilde{\Delta}_k := N(t_1, x_k + \delta) - N(t_1, x_k). \tag{3.6}$$

Lemma 3.1. *Fix* $c_0 > 0$ *and* $c_1 \geq 4$. *Then there exist positive constants* $C_0(c_0, \mathscr{C}_1)$, $C_1(c_0, \mathscr{C}_2)$, $C_2(\mathscr{C}_2)$ *such that for all* δ *small enough,*

$$C_0\delta \leq \mathrm{Var}\big(\tilde{\Delta}_k\big) \leq C_1\delta \tag{3.7}$$

uniformly in k. *If* $0 < |x_k - x_l| < \frac{1}{2}$ *then*

$$\left| \mathrm{Cov}\big(\tilde{\Delta}_k,\, \tilde{\Delta}_l\big) \right| \leq C_2\delta \exp\left(-\frac{|x_k - x_l|^2}{64 t_1} \right). \tag{3.8}$$

Proof. Since $\tilde{\Delta}_k$ is a mean zero random variable, we can use Itô's isometry along with the bound on σ given by (1.2) to obtain

$$\begin{aligned} \mathrm{Var}\big(\tilde{\Delta}_k\big) &\leq \mathscr{C}_2^2 \int_0^{t_1} \int_{\mathbf{T}} \big[G(s, y+\delta) - G(s,y) \big]^2 dyds \\ &\leq \mathscr{C}_2^2 C\delta, \end{aligned}$$

where the last inequality comes from Lemma 2.7. This gives the required upper bound in (3.7)

Next, we have

$$\begin{aligned} \mathrm{Var}\big(\tilde{\Delta}_k\big) &\geq \mathscr{C}_1^2 \int_0^{t_1} \int_{\mathbf{T}} \big[G(s, y+\delta) - G(s,y) \big]^2 dyds \\ &= 2\mathscr{C}_1^2 \int_0^{t_1} [G(2s, 0) - G(2s, \delta)] ds \\ &\geq C\mathscr{C}_1^2 \int_0^{t_1} \frac{1}{\sqrt{s}} ds, \end{aligned}$$

which gives the lower bound in (3.7). Let us explain how we obtain the last inequality above. For $k \geq 1$ and $\delta < \frac{1}{2}$ one has $(k-\delta)^2 \geq \frac{1}{10}(k^2+\delta^2)$. Thus for all $0 \leq s \leq 2t_1$,

$$\begin{aligned} G(s,0) - G(s,\delta) &= \frac{1}{\sqrt{2\pi s}} \left\{ \sum_{k\in\mathbf{Z}} e^{-k^2/2s} - \sum_{k\in\mathbf{Z}} e^{-(k+\delta)^2/2s} \right\} \\ &\geq \frac{1}{\sqrt{2\pi s}} \left\{ 1 - e^{-\delta^2/2s} - \sum_{k=1}^{\infty} e^{-(k-\delta)^2/2s} \right\} \\ &\geq \frac{1}{\sqrt{2\pi s}} \left\{ 1 - e^{-1/4c_0} - e^{-\delta^2/20s} \sum_{k=1}^{\infty} e^{-k^2/20s} \right\} \\ &\geq \frac{C}{\sqrt{2\pi s}} \left\{ 1 - e^{-1/4c_0} - \frac{\sqrt{40\pi t_1}}{2} \cdot e^{-1/(40c_0)} \right\} \\ &\geq \frac{C}{\sqrt{s}}. \end{aligned}$$

The second last inequality is a consequence of bounding the sum from above by an appropriate integral from a Riemann sum approximation.

We next turn to the bound on the covariance. Observe that if we assume that $k > l$, we have $a = x_k - x_l = (k-l)c_1\delta$ and therefore $a + \delta > a - \delta > a/4$. Using this, the semigroup property of the heat kernel and (2.17) we have

$$\begin{aligned} \left| \mathrm{Cov}\big(\tilde{\Delta}_k, \tilde{\Delta}_l\big) \right| &\leq \mathscr{C}_2^2 \int_0^{t_1} \int_{\mathbf{T}} |G(s,y+\delta) - G(s,y)| \cdot |G(s,y+a+\delta) - G(s,y+a)| \, dy \, ds \\ &\leq \mathscr{C}_2^2 \int_0^{t_1} (2G(2s,a) + G(2s,a-\delta) + G(2s,a+\delta)) \, ds \\ &\leq C\mathscr{C}_2^2 \int_0^{t_1} (2p(2s,a) + p(2s,a-\delta) + p(2s,a+\delta)) \, ds \\ &\leq C\mathscr{C}_2^2 \sqrt{t_1} \exp\left(-\frac{|a|^2}{64t_1} \right), \end{aligned}$$

which completes the proof since $t_1 = c_0\delta^2$ and $|a| = |x_k - x_l|$. □

For the next lemma recall the events A_i defined in (3.3)

Lemma 3.2. *Let $c_0 = 1$. We can find $c_1 \geq 4$ large enough and $0 < \eta < 1$ such that for an arbitrary initial profile u_0,*

$$P(A_1 | u_0) \leq \eta^J.$$

Remark 3.1. *The above lemma implies* (3.5) *because the initial profile is allowed to be arbitrary, and by the Markov property, A_i depends only on the profile $u(t_{i-1}, \cdot)$.*

Proof. For an arbitrary initial profile u_0

$$P(A_1) = \prod_{j=0}^{J-1} P\left(B_j \,|\, B_1, B_2, \cdots, B_{j-1}\right), \tag{3.9}$$

where

$$B_j = \left\{ |u(t_1, x_j + \delta) - u(t_1, x_j)| \le \epsilon^{\frac{1}{2\theta}} \right\}.$$

We will show that each of the terms inside the product sign in (3.9) is uniformly (in j) bounded away from 1, which will imply the lemma. We will in fact prove a stronger statement that $P\left(B_j \,|\, \mathcal{G}_{j-1}\right)$ is uniformly (in j) bounded away from 1, where $\mathcal{G}_{j-1}$ is the σ algebra generated by the random variables $\tilde{\Delta}_k = N(t_1, x_k + \delta) - N(t_1, x_k)$, $k \le j - 1$. We thus need to show the existence of some $0 < \eta < 1$ such that

$$P\left(|\Delta_j| \le \epsilon^{\frac{1}{2\theta}} \,\middle|\, \mathcal{G}_{j-1} \right) \le \eta,$$

where $\Delta_k := u(t_1, x_k + \delta) - u(t_1, x_k)$. We will obtain this by showing

$$\operatorname{Var}\left(\Delta_j \,\middle|\, \mathcal{G}_{j-1} \right) \ge C\epsilon^{\frac{1}{\theta}}, \tag{3.10}$$

for some constant C independent of j. We can use general properties of Gaussian random vectors to write

$$\begin{aligned} \Delta_j &= \Big[\left(G_{t_1} * u_0\right)(x_j + \delta) - \left(G_{t_1} * u_0\right)(x_j) \Big] + \tilde{\Delta}_j \\ &= \Big[\left(G_{t_1} * u_0\right)(x + j + \delta) - \left(G_{t_1} * u_0\right)(x_j) \Big] + X + Y, \end{aligned} \tag{3.11}$$

where

$$X = \sum_{k=0}^{j-1} \beta_k \tilde{\Delta}_k$$

is the conditional expectation of $\tilde{\Delta}_j$ given $\mathcal{G}_{j-1}$. The variance of Y is the conditional variance of $\tilde{\Delta}_j$ given $\mathcal{G}_{j-1}$, which is also the conditional variance in (3.10). Moreover Y is independent of $\mathcal{G}_{j-1}$ and thus

$$\operatorname{Cov}\left(Y, \tilde{\Delta}_l\right) = 0, \quad l = 0, 1, \cdots, j-1.$$

Therefore for all $l = 0, 1, \cdots, j-1$ we have

$$\operatorname{Cov}\left(\tilde{\Delta}_j, \tilde{\Delta}_l\right) = \sum_{k=0}^{j-1} \beta_k \operatorname{Cov}\left(\tilde{\Delta}_k, \tilde{\Delta}_l\right) \tag{3.12}$$

Let $\mathbf{y} = (y_0, y_1, \cdots, y_{j-1})^T$ where y_l represents the entry on the left hand side above, $\boldsymbol{\beta}$ be the vector of the β_l's, and let

$$\mathbf{S} = \left(\left(\operatorname{Cov}(\tilde{\Delta}_k, \tilde{\Delta}_l) \right) \right)_{0 \le k, l \le j-1}$$

be the covariance matrix of the increments $\tilde{\Delta}_l$. We can thus rewrite (3.12) in matrix form

$$\mathbf{y} = \mathbf{S}\boldsymbol{\beta}. \tag{3.13}$$

Let us next show that $\mathbf{S}$ is invertible. Write $\mathbf{S} = \mathbf{D}(\mathbf{I} - \mathbf{A})\mathbf{D}$, where $\mathbf{D} \in \mathbf{R}^{j \times j}$ is the diagonal matrix with diagonal entries

$$\operatorname{Std}\left(\tilde{\Delta}_k\right), \quad k = 0, 1, \cdots, j-1,$$

and $\mathbf{I} - \mathbf{A}$ is the correlation matrix of $\tilde{\Delta}_l$. Above Std denotes the standard deviation of the random variable in parentheses. Denote by $\|\cdot\|_{1,1}$ the norm on matrices in $\mathbf{R}^{j \times j}$ induced by the ℓ_1 norm $\|\cdot\|_1$ on $\mathbf{R}^j$. Now $\|\mathbf{A}\|_{1,1} = \max_j \sum_{i=1}^n |a_{i,j}|$ (see page 259 in [RB00]), we can use (3.8) to obtain $\|\mathbf{A}\|_{1,1} \le 1/3$ by choosing c_1 large enough. In that case the inverse of $\mathbf{I} - \mathbf{A}$ exists and moreover

$$\|(\mathbf{I} - \mathbf{A})^{-1}\|_{1,1} \le \frac{1}{1 - \|\mathbf{A}\|_{1,1}} \le \frac{3}{2}.$$

Using this along with the lower bound in (3.7) we obtain that $\mathbf{S}$ is invertible and moreover

$$\|\mathbf{S}^{-1}\|_{1,1} \le \|\mathbf{D}^{-1}\|_{1,1} \cdot \|(\mathbf{I} - \mathbf{A})^{-1}\|_{1,1} \cdot \|\mathbf{D}^{-1}\|_{1,1} \le \frac{C_0(1, \mathscr{C}_1)}{\delta}. \tag{3.14}$$

Note also from (3.8)

$$\|\mathbf{y}\|_1 \le C_2(\mathscr{C}_2)\delta \sum_{k=1}^{\infty} \exp\left(-\frac{c_1^2 k^2}{64}\right) \le \frac{C_2(\mathscr{C}_2)\delta}{c_1}. \tag{3.15}$$

By choosing c_1 large the above can be made very small.

Let us now return to (3.13) which we write as $\boldsymbol{\beta} = \mathbf{S}^{-1}\mathbf{y}$. From this we obtain

$$\|\boldsymbol{\beta}\|_1 \le \|\mathbf{S}^{-1}\|_{1,1} \cdot \|\mathbf{y}\|_1.$$

We have shown in (3.14) and (3.15) that by choosing c_1 large enough we can make $\|\boldsymbol{\beta}\|_1$ arbitrary small so that

$$\mathrm{Std}(X) \le \|\boldsymbol{\beta}\|_1 \cdot \sup_k \mathrm{Std}\big(\tilde{\Delta}_k\big) \le (C_1(1, \mathscr{C}_2) \cdot \delta)^{\frac{1}{2}} \|\boldsymbol{\beta}\|_1$$

can be made a small multiple of $\sqrt{\delta}$. We have used the upper bound in (3.7) above. Using this along with the lower bound in (3.7) once again we obtain

$$\mathrm{Std}(Y) \ge \mathrm{Std}\big(\tilde{\Delta}_j\big) - \mathrm{Std}(X) \ge C(c_1, \mathscr{C}_1, \mathscr{C}_2) \cdot \sqrt{\delta}$$

uniformly in j for some positive constant $C(c_1, \mathscr{C}_1, \mathscr{C}_2)$, and this proves (3.10), completing the proof of the lemma. □

Remark 3.2. *We note that if $\sigma(s,y)$ is a constant function, then Lemma 3.2 can be easily proved by using Slepian's inequality. For instance, if $\sigma(s,y) = 1$, then it is easy to see that there exist $c_0 > 0$ and $c_1 > 0$ such that $G(t_1, z)$ is convex for all $|z| \ge c_1\delta$, and then the convexity implies that*

$$Cov(\tilde{\Delta}_k, \tilde{\Delta}_l) \le 0.$$

Using now Slepian's inequality, we get

$$P\left(\max_k |\tilde{\Delta}_k| \le \epsilon^{1/2\theta}\right) \le P\left(\max_k \tilde{\Delta}_k \le \epsilon^{1/2\theta}\right) \le \prod_k P\left(\tilde{\Delta}_k \le \epsilon^{1/2\theta}\right),$$

which provides an upper bound on $P(A_1)$ as in (3.9) since $\tilde{\Delta}_k$ is mean-zero Gaussian with variance estimated in (3.7).

3.2. **Upper bound in Theorem 1.2 (b).** We provide the outline of the proof of the upper bound. The details are quite similar to that of the proof of the upper bound of Theorem 1.2 (a) and are left to the reader. Using the same discrete time-space points as in (3.1) we obtain

$$P\left(\sup_{x \in \mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \le \epsilon\right) \le P\left(\max_{\substack{i=0,1,\cdots,I \\ j=0,1\cdots J}} \big|u(t_i + \delta^2, x_j) - u(t_i, x_j)\big| \le \epsilon^{\frac{1}{2\theta}}\right). \tag{3.16}$$

Defining

$$A_i^{\#} := \left\{ \max_{j=0,1,\cdots,J} \left| u(t_i+\delta^2, x_j) - u(t_i, x_j) \right| \le \epsilon^{\frac{1}{2\theta}} \right\},$$

the upper bound will follow once we show the existence of a $0 < \tilde{\eta} < 1$ such that

$$P\left(A_i^{\#} \,\middle|\, u(s,x),\, s \le t_i,\, x \in \mathbf{T} \right) \le \tilde{\eta}^J.$$

Note here the slight change from (3.5); here we condition on the profile up to t_i. One could have conditioned up to time t_{i-1} but conditioning up to time t_i makes the argument simpler. Define

$$\tilde{\Delta}_k^{\#} := N(\delta^2, x_k).$$

Using (2.19), one obtains a similar result to Lemma 3.1 with the random variables $\tilde{\Delta}_k$ replaced by $\tilde{\Delta}_k^{\#}$. Thus, by the Markov property again, we only need to show

$$P(A_0^{\#} | u_0) \le \tilde{\eta}^J$$

for some $0 < \tilde{\eta} < 1$. For this we note

$$P(A_0^{\#}) = \prod_{j=0}^{J-1} P\left(B_j^{\#} \,|\, B_1^{\#}, B_2^{\#}, \cdots, B_{j-1}^{\#} \right),$$

where

$$B_j^{\#} := \left\{ |u(\delta^2, x_j) - u(0, x_j)| \le \epsilon^{\frac{1}{2\theta}} \right\}.$$

Define $\Delta_k^{\#} = u(\delta^2, x_k) - u(0, x_k)$ and then show

$$P\left(|\Delta_j^{\#}| \le \epsilon^{\frac{1}{2\theta}} \,\middle|\, \mathcal{G}_{j-1} \right) \le \tilde{\eta}$$

for some $0 < \tilde{\eta} < 1$, where $\mathcal{G}_{j-1}$ is the σ algebra generated by the random variables $\tilde{\Delta}_k^{\#}$, $k \le j-1$. Note that although

$$\Delta_j^{\#} = \left[\left(G_{\delta^2} * u_0\right)(x_j) - u_0(x_j) \right] + \tilde{\Delta}_j^{\#}$$

is of a slightly different form than that of (3.11), the term in the square brackets does not play any role in the argument of Lemma 3.2.

3.3. **Upper bound in Theorem 1.3 (a).** The function $\sigma(t,x,u)$ now depends on the third variable, so the resulting random field is no longer Gaussian. Therefore, we will need an alternative argument based on an approximation procedure. For $\beta > 0$, we define the following equation,

(3.17)

$$V^{(\beta)}(t,\, x) = (G_t * u_0)(x) + \int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} G(t-s, x-y) \sigma\left(s, y, V^{(\beta)}(s,y)\right) W(dsdy).$$

Of course, here, we treat $x \pm \sqrt{\beta t} \in \mathbf{T}$.

Existence and uniqueness of a solution to the above equation is not an issue. In fact, this can be easily proved by considering the following Picard iterates.

(3.18)

$$V^{(\beta),l}(t,\, x) = (G_t * u_0)(x) + \int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} G(t-s, x-y) \sigma\left(s, y, V^{(\beta),l-1}(s,y)\right) W(dsdy).$$

with $V^{(\beta),0}(t,\, x) := (G_t * u_0)(x)$. We will need the following result.

Proposition 3.1. *Assume $\beta t < \frac{1}{4}$. There exist positive constants C_1 and C_2 dependent on $\mathscr{D}$ such that*

$$\sup_{x\in\mathbf{T}} E\left[|V^{(\beta),l}(t,\, x) - V^{(\beta)}(t,\, x)|^p\right] \le C_1^p e^{C_2 p^3 t - \frac{pl}{2}}.$$

Proof. We use (3.17) and (3.18) to write

$$\begin{aligned}&V^{(\beta),l}(t,\, x) - V^{(\beta)}(t,\, x)\\ &:= \int_0^t \int_{[x-\sqrt{\beta t},x+\sqrt{\beta t}]} G(t-s, x-y)\left[\sigma\left(s, y, V^{(\beta),l-1}(s,y)\right) - \sigma\left(s,y,V^{(\beta)}(s,y)\right)\right] W(dsdy).\end{aligned}$$

For notational convenience, we set $f(l,t) := \sup_{x\in\mathbf{T}} \|V^{(\beta),l}(t,\, x) - V^{(\beta)}(t,\, x)\|_p^2$. We now use Burkholder's inequality and the fact that σ is globally Lipschitz to obtain

$$\begin{aligned}f(l,t) &\le Cp\int_0^t f(l-1,s)\int_{[x-\sqrt{\beta t},x+\sqrt{\beta t}]} G(t-s,x-y)^2 dyds\\ &\le 2Cp\int_0^t \frac{f(l-1,s)}{\sqrt{t-s}}ds,\end{aligned}$$

where we have used the heat kernel estimate (2.17) to get the last bound in the above. Upon setting $F(l) := \sup_{s>0} e^{-ks} f(l,s)$, the above immediately yields

$$F(l) \le \frac{2Cp}{\sqrt{k}} F(l-1).$$

Upon choosing $k = Cp^2$ with some large constant C and iterating, we obtain $F(l) \le C\left(\frac{1}{2}\right)^l$. This gives the result. $\square$

We also have the following convergence result.

Proposition 3.2. *Assume $\beta t < \frac{1}{4}$. Then there exist positive constants C_1 and C_2 dependent on $\mathscr{C}_2, \mathscr{D}$ such that*

$$\sup_{x\in\mathbf{T}} E\left[|u(t,\, x) - V^{(\beta)}(t,\, x)|^p\right] \le C_1 e^{C_2 p^3 t - \frac{\beta p}{4}}.$$

Proof. We use (3.17) and the mild formulation of the $u(t,x)$ to write

$$\begin{aligned}&u(t,x) - V^{(\beta)}(t,\, x)\\ &= \int_0^t \int_{[x-\sqrt{\beta t},x+\sqrt{\beta t}]} G(t-s,x-y)\left[\sigma\left(s,y,u(s,y)\right) - \sigma\left(s,y,V^{(\beta)}(s,y)\right)\right] W(dsdy)\\ &\quad + \int_0^t \int_{[x-\sqrt{\beta t},x+\sqrt{\beta t}]^c} G(t-s,x-y)\cdot\sigma\left(s,y,u(s,y)\right) W(dsdy).\end{aligned}$$

We now use Burkholder's inequality together with the Lipschitz continuity of σ to write

$$\begin{aligned}
&\|u(t,x) - V^{(\beta)}(t,\, x)\|_p^2 \\
&\quad \leq Cp\int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]} G^2(t-s, x-y)\|u(s,y) - V^{(\beta)}(s,y)\|_p^2 ds dy \\
&\qquad + Cp\int_0^t \int_{[x-\sqrt{\beta t}, x+\sqrt{\beta t}]^c} G^2(t-s, x-y)\|\sigma\left(s, y, u(s,y)\right)\|_p^2 ds dy \\
&\quad := I_1 + I_2.
\end{aligned}$$

We bound I_2 first. We now use the heat kernel estimate (2.17) and the fact that σ is bounded above to get that

$$\begin{aligned}
I_2 &\leq Cpe^{-\beta/2}\int_0^t \frac{1}{\sqrt{t-s}} ds \\
&\leq Cpe^{-\beta/2}\sqrt{t}.
\end{aligned}$$

We now set $F(k) := \sup_{t>0,\, x\in \mathbf{T}} e^{-kt}\|u(t,x) - V^{(\beta)}(t,\, x)\|_p^2$ and bound I_1 as in the Proposition above to obtain

$$F(k) \leq \frac{Cp}{\sqrt{k}} F(k) + \frac{Cp}{\sqrt{k}} e^{-\beta/2}.$$

This finishes the proof upon choosing the k to be an appropriate multiple of p^2. $\square$

We will use the following straightforward consequence of the above:

$$\sup_{x\in\mathbf{T}} E\left[|u(t,\, x) - V^{(\beta),l}(t,\, x)|^p\right] \leq D_1 e^{D_2 p^3 t}\left(e^{-\frac{\beta p}{4}} + e^{-\frac{pl}{2}}\right), \tag{3.19}$$

where D_1 and D_2 are some positive constants depending on $\mathscr{C}_2, \mathscr{D}$. The following lemma along with (3.19) suggests that we can construct independent random variables that are close to $u(t,x)$. The proof of Lemma 3.3 is essentially the same as that of Lemma 4.4 of [CJK13].

Lemma 3.3. *Let $\beta, t > 0$ and $l \geq 0$. Fix a collection of points $x_1, x_2, \cdots \in \mathbf{T}$ such that the distance between x_i and x_j is greater than $2l\sqrt{\beta t}$ whenever $i \neq j$. Then $\{V^{(\beta),l}(t,\, x_j)\}$ forms a collection of independent random variables.*

We can now prove the upper bound. Recall $\delta = \epsilon^{\frac{1}{\theta}}$ and the time points $t_i := ic_0\delta^2$ as in (3.1); the parameter c_0 will be chosen suitably later in the proof. We shall consider now the spatial points $x_{2j} := j(\delta+\rho)$ and $x_{2j-1} := j(\delta+\rho) - \rho$ for $j = 1, \ldots, J$ where $J := [1/2(\delta+\rho)]$ and $\rho := |\alpha \log \epsilon|^{\frac{3}{2}}\delta$. Here $\alpha > 4(D_2+1)\sqrt{c_0} + 16\theta$ is a constant which is independent of ϵ and i, j where D_2 is in (3.19). From this definition, we have $|x_{2j+1} - x_{2j}| = \delta$ and $|x_{2j+2} - x_{2j+1}| = \rho$ for $j = 0, \ldots, J$. As in the proof of the upper bound for the Gaussian case, we have

$$\begin{aligned}P\left(\sup_{\substack{x\neq y\in\mathbf{T}\\0\le t\le T}}\frac{|u(t,x)-u(t,y)|}{|x-y|^{\frac12-\theta}}\le\epsilon\right)&\le P\left(\max_{\substack{i=0,1,\cdots,I\\j=0,1\cdots J-1}}\frac{\left|u(t_i,x_{2j+1})-u(t_i,x_{2j})\right|}{\delta^{\frac12-\theta}}\le\epsilon\right)\\&\le P\left(\max_{\substack{i=0,1,\cdots,I\\j=0,1\cdots J}}\left|u(t_i,x_{2j}+\delta)-u(t_i,x_{2j})\right|\le\epsilon^{\frac{1}{2\theta}}\right).\end{aligned}$$

We will show below that *uniformly* over initial profiles u_0

$$P\left(\max_{j=0,1,\cdots,J}\left|u(t_1,x_{2j}+\delta)-u(t_1,x_{2j})\right|\le\epsilon^{\frac{1}{2\theta}}\right)\le\exp\left(-\frac{C}{|\log\epsilon|^{\frac32}\epsilon^{\frac1\theta}}\right)\tag{3.20}$$

for some positive constant C. One then uses (3.4) and notes that the number of time intervals $I=\left[\frac{T}{c_0\delta^2}\right]=\left[\frac{T}{c_0\epsilon^{2/\theta}}\right]$ to get the required upper bound.

Let us therefore turn to the proof of (3.20). Using the triangle inequality, the right hand side of (3.20) is bounded above by

$$\begin{aligned}&2P\left(\max_{j=0,1,\cdots,2J}\left|u(t_1,x_j)-V^{(\beta),l}(t_1,x_j)\right|>\epsilon^{1/2\theta}\right)\\&\qquad+P\left(\max_{j=0,1,\cdots,J}\left|V^{(\beta),l}(t_1,x_{2j+1})-V^{(\beta),l}(t_1,x_{2j})\right|\le3\epsilon^{1/2\theta}\right)\\&:=L_1+L_2.\end{aligned}$$

Before we consider L_1 and L_2, we define

$$\beta=l:=\lfloor\alpha|\log\epsilon|\rfloor\quad\text{ and }\quad p:=\left\lfloor\sqrt{|\log\epsilon|/\delta^2}\right\rfloor.\tag{3.21}$$

Let us now consider L_1 first. By Chebyshev's inequality and (3.19) there exist constants $C_1>0$ and $C_2>0$ which are independent of ϵ such that

$$\sup_{x\in\mathbf{T}}P\left(\left|u(t_1,x)-V^{(\beta),l}(t_1,x)\right|\ge\epsilon^{1/2\theta}\right)\le C_1\exp\left(-\frac{C_2|\log\epsilon|^{3/2}}{\epsilon^{1/\theta}}\right).\tag{3.22}$$

Since $2J\le1/(\delta+\rho)\le1/\delta=\epsilon^{-1/\theta}$, we have for some other positive constants $\tilde C_1$ and $\tilde C_2$ independent of ϵ

$$L_1\le2J\sup_{x\in\mathbf{T}}P\left(\left|u(t_1,x)-V^{(\beta),l}(t_1,x)\right|\ge\epsilon^{1/2\theta}\right)\le\tilde C_1\exp\left(-\frac{\tilde C_2|\log\epsilon|^{3/2}}{\epsilon^{1/\theta}}\right).$$

Let us now consider L_2. First observe that $W_j:=\left(V^{(\beta),l}(t_1,x_{2j}),V^{(\beta),l}(t_1,x_{2j+1})\right)$ are independent for $j=0,1,\ldots,J-1$ by Lemma 3.3 since the distance between x_{2j+1} and x_{2j+2} is greater than $2l^{3/2}\sqrt{t_1}$. Thus, we have

$$\begin{aligned}L_2=&P\left(\max_{j=0,1,\cdots,J}\left|V^{(\beta),l}(t_1,x_{2j+1})-V^{(\beta),l}(t_1,x_{2j})\right|\le3\epsilon^{1/2\theta}\right)\\=&\prod_{j=1}^{J}P\left(\left|V^{(\beta),l}(t_1,x_{2j+1})-V^{(\beta),l}(t_1,x_{2j})\right|\le3\epsilon^{1/2\theta}\right).\end{aligned}$$

Using the triangle inequality, we have

$$
\begin{aligned}
&P\left(\left|V^{(\beta),l}(t_1,x_{2j+1})-V^{(\beta),l}(t_1,x_{2j})\right|\le 3\epsilon^{1/2\theta}\right)\\
&\le 2\max_{0\le j\le 2J} P\left(\left|u(t_1,x_j)-V^{(\beta),l}(t_1,x_j)\right|>\epsilon^{1/2\theta}\right)+P\left(|u(t_1,x_{2j+1})-u(t_1,x_{2j})|\le 5\epsilon^{1/2\theta}\right)\\
&=: L_{21}+L_{22}.
\end{aligned}
\tag{3.23}
$$

Let us first consider L_{22}. Consider the following martingale M_s for $0\le s\le t_1$:

$$
\begin{aligned}
M_s=&\left[\left(G_{t_1}*u_0\right)(x_{2j+1})-\left(G_{t_1}*u_0\right)(x_{2j})\right]\\
&+\int_0^s\int_{\mathbf{T}}\left[G(t_1-r,x_{2j+1}-y)-G(t_1-r,x_{2j}-y)\right]\cdot\sigma\left(r,y,u(r,y)\right)W(dydr).
\end{aligned}
$$

Note that M_{t_1} is $u(t_1,x_{2j+1})-u(t_1,x_{2j})$. The quadratic variation of the martingale is given by

$$
\langle M\rangle_s=\int_0^s\int_{\mathbf{T}}\left[G(t_1-r,\,x_{2j+1}-y)-G(t_1-r,x_{2j}-y)\right]^2\sigma\left(r,y,u(r,y)\right)^2dydr.
$$

We use (3.7) to obtain

$$
C_0\delta\le\langle M\rangle_{t_1}\le C_1\delta.
$$

Since M_t is a continuous martingale, it is a time change of a Brownian motion B, i.e., $M_t=M_0+B_{\langle M\rangle_t}$. Hence, recalling $\delta=\epsilon^{1/\theta}$, we have

$$
\begin{aligned}
P\left(|u(t_1,x_{2j+1})-u(t_1,x_{2j})|\le 5\epsilon^{1/2\theta}\right)&\le P\left(\left|M_0+B_{\langle M\rangle_{t_1}}\right|\le 5\epsilon^{1/2\theta}\right)\\
&\le P\left(\inf_{C_0\delta\le t\le C_1\delta}|M_0+B_t|\le 5\sqrt{\delta}\right)\\
&\le P\left(\inf_{C_0\delta\le t\le C_1\delta}|B_t|\le 5\sqrt{\delta}\right)=:\gamma,
\end{aligned}
\tag{3.24}
$$

for some $\gamma<1$ independent of δ. The last inequality can be obtained as follows: We consider a countable dense subset $D:=\{t_0,t_1,\dots\}$ of $[C_0\delta,C_1\delta]$ ($t_i\neq t_j$ whenever $i\neq j$) and $D_m:=\{t_0,\dots,t_m\}$. Since $\{B_{t_0},\dots B_{t_m}\}$ is a non-degenerate Gaussian vector, for any non-random number $A\in\mathbf{R}$ and $\eta>0$, we have

$$
P\left(\min_{0\le i\le m}|A+B_{t_i}|\le\eta\right)\le P\left(\min_{0\le i\le m}|B_{t_i}|\le\eta\right).
$$

Since this holds for all $m\ge 1$, we get the last inequality above.

Let us now consider L_{21}. Here, (3.22) implies L_{21} can be made arbitrarily small by choosing ϵ small enough. Therefore, there exists a constant $\eta<1$ independent of ϵ such that

$$
L_{21}+L_{22}\le\eta<1,
$$

which implies from (3.23)

$$
L_2\le\eta^J\le\exp\left(-\frac{C}{\epsilon^{1/\theta}|\log\epsilon|^{3/2}}\right).
$$

Combining our bounds on I_1 and I_2, we finish the proof. □

3.4. **Upper bound in Theorem 1.3 (b).** The proof follows a similar strategy to that of the upper bound proved above and we will sketch the proof focusing on the main differences. Note that we use the same choice of β and l as in part (a) (see (3.21)). Then, we have

$$P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \le \epsilon\right) \le P\left(\max_{\substack{i=0,1,\cdots,I\\ j=0,1\cdots J}} \left|u(t_i+\delta^2, x_j) - u(t_i, x_j)\right| \le \epsilon^{\frac{1}{2\theta}}\right),$$

where the points t_i are given by (3.1) while $x_j = jc_1|\log\epsilon|^{3/2}\delta,\ j = 0, 1, \cdots, J := \left[\frac{1}{c_1|\log\epsilon|\delta}\right]$. Here, we choose c_1 as a fixed constant that only depends on α (see part (a) for the definition of α) and c_0 such that $|x_i - x_j| \ge 2\ell\sqrt{\beta(t_1+\delta^2)}$. In other words, by our choices of x_j, $\left\{V^{(\beta,l)}(t_1+\delta^2, x_j) - V^{(\beta),l}(t_1, x_j)\right\}_{j=0}^{J}$ is a collection of independent random variables. Now we have

$$\begin{aligned}
P\left(\left|u(t_1+\delta^2, x_j) - u(t_1, x_j)\right| \le \epsilon^{\frac{1}{2\theta}}\right) &\le P\left(\left|u(t_1+\delta^2, x_j) - V^{(\beta),l}(t_1+\delta^2, x_j)\right| \ge \epsilon^{\frac{1}{2\theta}}\right)\\
&\quad + P\left(\left|u(t_1, x_j) - V^{(\beta),l}(t_1, x_j)\right| \ge \epsilon^{\frac{1}{2\theta}}\right)\\
&\quad + P\left(\left|V^{(\beta,l)}(t_1+\delta^2, x_j) - V^{(\beta),l}(t_1, x_j)\right| \le 3\epsilon^{\frac{1}{2\theta}}\right).
\end{aligned}$$

For the first two terms, we have similar upper bounds as the one given by (3.22);

$$\begin{aligned}
P\left(\left|u(t_1+\delta^2, x_j) - V^{(\beta),l}(t_1+\delta^2, x_j)\right| \ge \epsilon^{\frac{1}{2\theta}}\right) + P\left(\left|u(t_1, x_j) - V^{(\beta),l}(t_1, x_j)\right| \ge \epsilon^{\frac{1}{2\theta}}\right)&\\
\le C_1 \exp\left(-\frac{C_2|\log\epsilon|^{3/2}}{\epsilon^{1/\theta}}\right)&,
\end{aligned}$$

for some positive constants C_1 and C_2. For the final term, we have

$$\begin{aligned}
P\left(\left|V^{(\beta,l)}(t_1+\delta^2, x_j) - V^{(\beta),l}(t_1, x_j)\right| \le 3\epsilon^{\frac{1}{2\theta}}\right) &\le P\left(\left|V^{(\beta,l)}(t_1+\delta^2, x_j) - u(t_1+\delta^2, x_j)\right| \ge \epsilon^{\frac{1}{2\theta}}\right)\\
&\quad + P\left(\left|u(t_1, x_j) - V^{(\beta,l)}(t_1, x_j)\right| \ge \epsilon^{\frac{1}{2\theta}}\right)\\
&\quad + P\left(\left|u(t_1+\delta^2, x_j) - u(t_1, x_j)\right| \le 5\epsilon^{\frac{1}{2\theta}}\right).
\end{aligned}$$

The bound for the last term is similar to the bound given by (3.24). The martingale term is slightly different. For $0 \le s \le t_1 + \delta^2$

$$\begin{aligned}
M_s &= \left[\left(G_{t_1+\delta^2} * u_0\right)(x_j) - \left(G_{t_1} * u_0\right)(x_j)\right]\\
&\quad + \int_0^s \int_{\mathbf{T}} \left[G(t_1+\delta^2-r, x_j-y) - G(t_1-r, x_j-y)1_{r\le t_1}\right] \cdot \sigma\left(r, y, u(r,y)\right) W(dydr).
\end{aligned}$$

We now use (2.20) to show that there exist constants C_3 and C_4 such that

$$C_3\delta \le \langle M\rangle_{t_1+\delta^2} \le C_4\delta.$$

A similar argument to that of (3.24) shows that

$$P\left(\left|u(t_1+\delta^2, x_j) - u(t_1, x_j)\right| \le 5\epsilon^{\frac{1}{2\theta}}\right) \le \gamma,$$

where $\gamma < 1$. The proof now follows from part (a).

4. Lower bounds

4.1. **Lower bound in Theorem 1.2 (a).** Recall our time discretizations from (3.1): $t_i = ic_0\delta^2 = ic_0\epsilon^{\frac{2}{\theta}}$, $i = 0, 1, \cdots, I$, and consider now the events

$$B_i = U_i \cap H_i, \tag{4.1}$$

where the event U_i puts restriction on the supremum norm of $u(t, \cdot)$ in the time interval $[t_i, t_{i+1}]$:

$$U_i = \left\{ \sup_{x\in\mathbf{T}} |u(t_{i+1}, x)| \le \frac{\epsilon^{\frac{1}{2\theta}}}{6}, \text{ and } \sup_{x\in\mathbf{T}} |u(t,x)| \le \frac{2\epsilon^{\frac{1}{2\theta}}}{3} \text{ for all } t \in [t_i, t_{i+1}] \right\}, \tag{4.2}$$

and the event H_i puts restriction on the Hölder norm of u in the time interval $[t_i, t_{i+1}]$:

$$H_i = \left\{ \mathcal{H}^{(\theta)}_{t_{i+1}}(u) \le \frac{\epsilon}{6}, \text{ and } \mathcal{H}^{(\theta)}_{t}(u) \le \frac{2\epsilon}{3} \text{ for all } t \in [t_i, t_{i+1}] \right\}. \tag{4.3}$$

It is clear that

$$\begin{aligned} P\left(\sup_{0\le t\le T} \mathcal{H}^{(\theta)}_t(u) \le \epsilon \right) &\ge P\left(\sup_{0\le t\le T} \mathcal{H}^{(\theta)}_t(u) \le \epsilon,\ \sup_{\substack{0\le t\le T\\ x\in[0,1]}} |u(t,\,x)| \le \epsilon^{\frac{1}{2\theta}} \right) \\ &\ge P\left(\cap_{i=0}^{I-1} B_i \right) \\ &= \prod_{i=0}^{I-1} P\left(B_i \middle| B_0, B_1 \cdots B_{i-1} \right). \end{aligned} \tag{4.4}$$

Similar to the method of the upper bound, our main task will be to obtain a uniform lower bound on $P\left(B_i \middle| B_0, B_1 \cdots B_{i-1}\right)$. It turns out that with an appropriate choice of c_0 one can in fact obtain such a uniform lower bound. We do this in Lemma 4.3 below (see also Remark 4.1), and then the lower bound in Theorem 1.2 (a) follows immediately. We first need a couple of lemmas which we turn to next.

Lemma 4.1. *There exists a constant $\mathbf{K}_5$ dependent only on θ and $\alpha_0(\theta, \mathscr{C}_2) > 0$ such that for $\alpha < \alpha_0(\theta, \mathscr{C}_2)$ we have*

$$P\left(\sup_{t\le\alpha\epsilon^{\frac{2}{\theta}},\, x\ne y\in\mathbf{T}} \left|\widetilde{N}(t,x,y)\right| \le \epsilon \right) \ge \exp\left(-\frac{2}{\alpha^{\frac12}\epsilon^{\frac1\theta}} \exp\left(-\frac{\mathbf{K}_5}{\mathscr{C}_2^2\alpha^\theta} \right) \right). \tag{4.5}$$

Proof. We first split $\mathbf{T}^2$ into squares S of side length $\alpha^{\frac12}\epsilon^{\frac1\theta}$. By the Gaussian correlation inequality ([LaM17], [Roy14]) we have

$$P\left(\sup_{t\le\alpha\epsilon^{\frac{2}{\theta}},\, x\ne y\in\mathbf{T}} \left|\widetilde{N}(t,x,y)\right| \le \epsilon \right) \ge \prod_S P\left(\sup_{\substack{t\le\alpha\epsilon^{\frac{2}{\theta}},\\ (x,y)\in S,\, x\ne y}} \left|\widetilde{N}(t,x,y)\right| \le \epsilon \right).$$

For $k = 0, 1, \cdots, \alpha^{-\frac12}\epsilon^{-\frac1\theta} - 1$, let S_k be a square in $\mathbf{T}^2$ whose center is $k2^{-\frac12}\alpha^{\frac12}\epsilon^{\frac1\theta}$ from the diagonal $x = y$. There are at most $2\alpha^{-\frac12}\epsilon^{-\frac1\theta}$ of such squares. Therefore the above probability is bounded below by

$$\prod_{k=0}^{\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}-1}\left[P\left(\sup_{\substack{t\le\alpha\epsilon^{\frac{2}{\theta}},\\ (x,y)\in S_k,\, x\neq y}}\left|\widetilde{N}(t,x,y)\right|\le\epsilon\right)\right]^{2\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}}. \tag{4.6}$$

Let us now give a lower bound of the expression inside the square brackets. We first consider the case when $k\ge 1$. For any $(x,y)\in S_k$ one has a lower bound $|x-y|\ge\frac{1}{4}\left(k\alpha^{\frac{1}{\theta}}\epsilon^{\frac{1}{\theta}}\right)$ and therefore

$$\begin{aligned}
&P\left(\sup_{t\le\alpha\epsilon^{\frac{2}{\theta}},\,(x,y)\in S_k}\left|\widetilde{N}(t,x,y)\right|\le\epsilon\right)\\
&\ge P\left(\sup_{t\le\alpha\epsilon^{\frac{2}{\theta}},\,(x,y)\in S_k}\left|N(t,x)-N(t,y)\right|\le\frac{\epsilon}{4}\left(k\alpha^{\frac{1}{2}}\epsilon^{\frac{1}{\theta}}\right)^{\frac{1}{2}-\theta}\right)\\
&\ge P\left(\sup_{t\le\alpha\epsilon^{\frac{2}{\theta}},\,(x,y)\in S_k}\max\left\{\left|N(t,x)\right|,\,\left|N(t,y)\right|\right\}\le\frac{\epsilon}{8}\left(k\alpha^{\frac{1}{2}}\epsilon^{\frac{1}{\theta}}\right)^{\frac{1}{2}-\theta}\right)\\
&\ge 1-2\mathbf{K}_1\exp\left(-\mathbf{K}_2\frac{k^{1-2\theta}}{64\mathscr{C}_2^2\alpha^{\theta}}\right),
\end{aligned} \tag{4.7}$$

the last inequality follows from (2.7). Therefore there exists an $\alpha_1(\theta,\mathscr{C}_2)>0$ small enough such that for all positive $\alpha<\alpha_1(\theta,\mathscr{C}_2)$ one has

$$P\left(\sup_{t\le\alpha\epsilon^{\frac{2}{\theta}},\,(x,y)\in S_k}\left|\widetilde{N}(t,x,y)\right|\le\epsilon\right)\ge 1-\exp\left(-\frac{\mathbf{K}_2k^{1-2\theta}}{128\mathscr{C}_2^2\alpha^{\theta}}\right).$$

Returning to (4.6) we can obtain a lower bound on the product of terms for which $k\neq 0$ by choosing an $\alpha_2(\theta,\mathscr{C}_2)>0$ small enough such that for $\alpha<\alpha_2(\theta,\mathscr{C}_2)$ we have

$$\begin{aligned}
&\prod_{k=1}^{\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}-1}\left[P\left(\sup_{\substack{t\le\alpha\epsilon^{\frac{2}{\theta}},\\ (x,y)\in S_k,\, x\neq y}}\left|\widetilde{N}(t,x,y)\right|\le\epsilon\right)\right]^{2\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}}\\
&\ge\exp\left(2\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}\sum_{k=1}^{\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}-1}\log\left\{1-\exp\left(-\frac{\mathbf{K}_2}{2}\frac{k^{1-2\theta}}{64\mathscr{C}_2^2\alpha^{\theta}}\right)\right\}\right)\\
&\ge\exp\left(-2\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}\sum_{k=1}^{\alpha^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}-1}\exp\left(-\frac{\mathbf{K}_2k^{1-2\theta}}{128\mathscr{C}_2^2\alpha^{\theta}}\right)\right)\\
&\ge\exp\left(-2\alpha^{-\frac{1}{2}}\exp\left(-\frac{\mathbf{K}_2}{256\mathscr{C}_2^2\alpha^{\theta}}\right)\cdot\epsilon^{-\frac{1}{\theta}}\right).
\end{aligned} \tag{4.8}$$

Finally we consider the $k = 0$ term in (4.6). For a small $\alpha_3(\theta, \mathscr{C}_2) > 0$ one has for $\alpha < \alpha_3(\theta, \mathscr{C}_2)$

$$
\begin{aligned}
&\left[P\left(\sup_{\substack{t \le \alpha\epsilon^{\frac{2}{\theta}},\\ (x,y)\in S_0,\, x\neq y}} \left|\widetilde{N}(t,x,y)\right| \le \frac{(\alpha^{\frac12}\epsilon^{\frac1\theta})^\theta}{\alpha^{\frac{\theta}{2}}} \right)\right]^{2\alpha^{-\frac12}\epsilon^{-\frac1\theta}} \\
&\ge \left[1 - \exp\left(-\frac{\mathbf{K}_4}{2\mathscr{C}_2^2\alpha^\theta}\right)\right]^{2\alpha^{-\frac12}\epsilon^{-\frac1\theta}} \\
&\ge \exp\left(-2\alpha^{-\frac12}\exp\left(-\frac{\mathbf{K}_4}{4\mathscr{C}_2^2\alpha^\theta}\right)\cdot \epsilon^{-\frac1\theta}\right),
\end{aligned}
\tag{4.9}
$$

where the second inequality follows by Lemma 2.5. We now use the bounds (4.9) and (4.8) in (4.6). The statement (4.5) follows immediately from this by choosing $\alpha_0 \le \alpha_1 \wedge \alpha_2 \wedge \alpha_3$ small enough. □

Lemma 4.2. *There exists a constant $\mathbf{K}_6$ dependent only on θ, and $\tilde{\alpha}_0(\theta, \mathscr{C}_2) > 0$ such that for $\alpha < \tilde{\alpha}_0(\theta, \mathscr{C}_2)$ small enough one has*

$$
P\left(\sup_{t\le\alpha\epsilon^{\frac2\theta}, x\in\mathbf{T}} |N(t,x)| \le \epsilon^{\frac{1}{2\theta}},\ \sup_{t\le\alpha\epsilon^{\frac2\theta},\, x\neq y\in\mathbf{T}} |\widetilde{N}(t,x,y)| \le \epsilon \right) \ge \exp\left(-\frac{1}{\alpha^{\frac12}\epsilon^{\frac1\theta}}\exp\left(-\frac{\mathbf{K}_6}{\mathscr{C}_2^2\alpha^\theta}\right)\right)
\tag{4.10}
$$

Proof. An application of the Gaussian correlation inequality ([Roy14], [LaM17]) gives

$$
\begin{aligned}
&P\left(\sup_{t\le\alpha\epsilon^{\frac2\theta}, x\in\mathbf{T}} |N(t,x)| \le \epsilon^{\frac{1}{2\theta}},\ \sup_{t\le\alpha\epsilon^{\frac2\theta},\, x\neq y\in\mathbf{T}} |\widetilde{N}(t,x,y)| \le \epsilon \right) \\
&\ge P\left(\sup_{t\le\alpha\epsilon^{\frac2\theta}, x\in\mathbf{T}} |N(t,x)| \le \epsilon^{\frac{1}{2\theta}} \right)\cdot P\left(\sup_{t\le\alpha\epsilon^{\frac2\theta},\, x\neq y\in\mathbf{T}} |\widetilde{N}(t,x,y)| \le \epsilon \right).
\end{aligned}
$$

We now partition $\mathbf{T}$ into disjoint intervals $[a_i, a_{i+1})$ where $a_i := i\alpha^{\frac12}\epsilon^{\frac1\theta}$ for $i = 1, \dots, \alpha^{-\frac12}\epsilon^{-\frac1\theta}$. Applying the Gaussian correlation inequality once again and (2.7), one obtains

$$
\begin{aligned}
P\left(\sup_{t\le\alpha\epsilon^{\frac2\theta}, x\in\mathbf{T}} |N(t,x)| \le \epsilon^{\frac{1}{2\theta}} \right) &\ge \prod_{i=1}^{\alpha^{-\frac12}\epsilon^{-\frac1\theta}} P\left(\sup_{t\le\alpha\epsilon^{\frac2\theta}, x\in[0,\sqrt{\alpha}\epsilon^{\frac1\theta}]} |N(t,x)| \le \epsilon^{\frac{1}{2\theta}} \right) \\
&\ge \left\{1 - \exp\left(-\frac{\mathbf{K}_2}{2\mathscr{C}_2^2\alpha^{\frac12}}\right)\right\}^{\alpha^{-\frac12}\epsilon^{-\frac1\theta}} \\
&\ge \exp\left(-\alpha^{-\frac12}\epsilon^{-\frac1\theta}\exp\left(-\frac{\mathbf{K}_2}{4\mathscr{C}_2^2\alpha^{\frac12}}\right)\right)
\end{aligned}
$$

if $\alpha < \alpha_4(\theta, \mathscr{C}_2)$ is small enough. The result now follows from this and (4.5). □

For the next lemma recall the events B_i defined in (4.1).

Lemma 4.3. *For all initial profiles u_0 with $|u_0(x)| \le \frac{\epsilon^{\frac{1}{2\theta}}}{3}$ and $\mathcal{H}^{(\theta)}(u_0) \le \frac{\epsilon}{3}$, one has*

$$P(B_0) \ge \exp\left(-\frac{2}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_6}{36\mathscr{C}_2^2 c_0^\theta}\right) - \frac{2}{9c_0\mathscr{C}_1^2\epsilon^{\frac{1}{\theta}}}\right)$$

when $c_0 6^{\frac{2}{\theta}} < \tilde{\alpha}_0$, where $\tilde{\alpha}_0$ is defined in Lemma 4.2.

Remark 4.1. *Note that we have the same lower bound for $P(B_i|B_0, B_1, \cdots, B_{i-1})$. This is because given B_{i-1} the profile $u(t_{i-1}, \cdot)$ has sup norm at most $\frac{\epsilon^{\frac{1}{2\theta}}}{3}$ and Hölder norm at most $\frac{\epsilon}{3}$. One can then use the Markov property and the above result.*

Proof. We will use a change of measure argument inspired by a technique in large deviation theory. A similar method was employed in [AJM]. Consider the measure Q defined by

$$\frac{dQ}{dP} = \exp\left(Z_{t_1}^{(1)} - \frac{1}{2}Z_{t_1}^{(2)}\right),$$

where

$$Z_{t_1}^{(1)} = -\int_0^{t_1}\int_{\mathbf{T}} \frac{1}{\sigma(r,z)}\frac{(G_r * u_0)(z)}{t_1} W(dzdr),$$
$$Z_{t_1}^{(2)} = \int_0^{t_1}\int_{\mathbf{T}} \left|\frac{1}{\sigma(r,z)}\frac{(G_r * u_0)(z)}{t_1}\right|^2 dzdr.$$

Define

$$\widetilde{\dot{W}}(r,z) := \dot{W}(r,z) + \frac{1}{\sigma(r,z)}\cdot\frac{(G_r * u_0)(z)}{t_1}.$$

It follows from Lemma 2.1 in [AJM] that $\widetilde{\dot{W}}$ is a white noise under the measure Q.

By change of measure

$$Q(B_0) = E_P\left(\frac{dQ}{dP}\cdot \mathbf{1}\{B_0\}\right),$$

and so Cauchy-Schwarz inequality gives

$$Q(B_0) \le \left[E_P\left(\frac{dQ}{dP}\right)^2\right]^{\frac{1}{2}} \cdot P(B_0)^{\frac{1}{2}},$$

from which we obtain

$$P(B_0) \ge Q(B_0)^2\left\{E_P\left(\frac{dQ}{dP}\right)^2\right\}^{-1}. \tag{4.11}$$

Now

$$E_P\left(\frac{dQ}{dP}\right)^2 = \exp\left(\int_0^{t_1}\int_{\mathbf{T}}\left|\frac{1}{\sigma(s,y)}\cdot\frac{(G_s * u_0)(y)}{t_1}\right|^2 dyds\right) \le \exp\left(\frac{1}{9c_0\mathscr{C}_1^2\epsilon^{\frac{1}{\theta}}}\right). \tag{4.12}$$

We next provide a lower bound on $Q(B_0)$. First observe that

$$u(t,x) = \left(1-\frac{t}{t_1}\right)(G_t * u_0)(x) + \int_0^t\int_{\mathbf{T}} G(t-r, x-z)\sigma(r,z)\widetilde{W}(drdz), \tag{4.13}$$

and

$$\begin{aligned}\frac{u(t,x)-u(t,y)}{|x-y|^{\frac{1}{2}-\theta}} &= \left(1-\frac{t}{t_1}\right)\cdot\left[\frac{(G_t*u_0)(x)-(G_t*u_0)(y)}{|x-y|^{\frac{1}{2}-\theta}}\right] \\ &\quad + \int_0^t\int_{\mathbf{T}}\frac{G(t-r,x-z)-G(t-r,y-z)}{|x-y|^{\frac{1}{2}-\theta}}\sigma(r,z)\dot{\widetilde{W}}(drdz).\end{aligned} \tag{4.14}$$

The deterministic term in (4.13) is bounded uniformly (in x) by $\frac{\epsilon^{\frac{1}{2\theta}}}{3}$ in the interval $[0,t_1]$ and is equal to 0 at the terminal time t_1. Similarly, due to Lemma 2.1, the first term in (4.14) is bounded uniformly (in x,y) by $\frac{\epsilon}{3}$ in the same interval and is also equal to 0 at the terminal time t_1. We define $N_1(t,x)$ and $\widetilde{N}_1(t,x,y)$ as $N(t,x)$ and $\widetilde{N}(t,x,y)$ as in (2.4) and (2.8) respectively but by replacing $\dot{W}$ by $\dot{\widetilde{W}}$. It therefore follows

$$\begin{aligned}Q(B_0) &\geq Q\left(\sup_{t\leq t_1,\,x\in\mathbf{T}}|N_1(t,x)|\leq\frac{\epsilon^{\frac{1}{2\theta}}}{6},\ \sup_{t\leq t_1,\,x\neq y\in\mathbf{T}}\left|\widetilde{N}_1(t,x,y)\right|\leq\frac{\epsilon}{6}\right) \\ &\geq Q\left(\sup_{t\leq c_0 6^{\frac{2}{\theta}}(\epsilon/6)^{\frac{2}{\theta}},\,x\in\mathbf{T}}|N_1(t,x)|\leq\left(\frac{\epsilon}{6}\right)^{\frac{1}{2\theta}},\ \sup_{t\leq c_0 6^{\frac{2}{\theta}}(\epsilon/6)^{\frac{2}{\theta}},\,x\neq y\in\mathbf{T}}\left|\widetilde{N}_1(t,x,y)\right|\leq\frac{\epsilon}{6}\right) \\ &\geq \exp\left(-\frac{1}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_6}{36\mathscr{C}_2^2c_0^\theta}\right)\right),\end{aligned} \tag{4.15}$$

as long as $c_0 6^{\frac{2}{\theta}}<\tilde{\alpha}_0$ from Lemma 4.2. If we use (4.15) and (4.12) into (4.11) we obtain

$$P(B_0)\geq\exp\left(-\frac{2}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_6}{36\mathscr{C}_2^2c_0^\theta}\right)-\frac{2}{9c_0\mathscr{C}_1^2\epsilon^{\frac{1}{\theta}}}\right)$$

as long as $c_0 6^{\frac{2}{\theta}}<\tilde{\alpha}_0$. □

4.2. **Lower bound in Theorem 1.2 (b).** The argument in Section 4.1 has to be modified at quite a few places. We first note the following lemma which follows immediately from the Gaussian correlation inequality.

Lemma 4.4. *There is $\alpha_0^{\#}(\theta,\mathscr{C}_2)>0$ such that for $\alpha<\alpha_0^{\#}(\theta,\mathscr{C}_2)$ we have*

$$P\left(\sup_{\substack{0\leq s,t\leq\alpha\epsilon^{\frac{2}{\theta}},\,s\neq t\\ x\in\mathbf{T}}}\left|N^{\#}(s,t,x)\right|\leq\epsilon\right)\geq\exp\left(-\frac{1}{\alpha^{\frac{1}{2}}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_9}{\mathscr{C}_2^2\alpha^\theta}\right)\right)$$

We shall consider time discretizations $t_i=ic_2\delta^2=ic_2\epsilon^{\frac{2}{\theta}},\ i=0,1,\cdots,I$. The constant c_2 will be appropriately chosen so as to get a uniform lower bound on $P\left(B_i^{\#}\middle|B_0^{\#},B_1^{\#}\cdots B_{i-1}^{\#}\right)$ in (4.20) below. It will only depend on θ and $\mathscr{C}_2$. In this section let

$$B_i^{\#}:=U_i^{\#}\cap H_i^{\#}\cap T_i^{\#},$$

where, similar to Section 4.1,

$$U_i^{\#} = \left\{ \sup_{x\in\mathbf{T}} |u(t_{i+1},x)| \le \frac{\epsilon^{\frac{1}{2\theta}}}{8c_2^{\frac{\theta}{2}-\frac{1}{4}}}, \text{ and } \sup_{x\in\mathbf{T}} |u(t,x)| \le \frac{\epsilon^{\frac{1}{2\theta}}}{4c_2^{\frac{\theta}{2}-\frac{1}{4}}} \text{ for all } t\in[t_i,t_{i+1}] \right\}, \tag{4.16}$$

$$H_i^{\#} = \left\{ \mathcal{H}^{(\theta)}_{t_{i+1}}(u) \le \frac{\epsilon}{8\Lambda}, \text{ and } \mathcal{H}^{(\theta)}_{t}(u) \le \frac{\epsilon}{2\Lambda} \text{ for all } t\in[t_i,t_{i+1}] \right\}, \tag{4.17}$$

$$T_i^{\#} = \left\{ \sup_{\substack{x\in\mathbf{T}\\ t_i\le s,t\le t_{i+1},\, s\neq t}} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} \le \frac{\epsilon}{2} \right\}. \tag{4.18}$$

Here, we recall the constant Λ in (1.4). Let us first consider the following lemma.

Lemma 4.5. *We have the following inclusion.*

$$\cap_{i=0}^{I} B_i^{\#} \subset \left\{ \sup_{x\in\mathbf{T}} \mathscr{H}^{(\theta)}_x(u) \le \epsilon \right\}. \tag{4.19}$$

Proof. We take a realization $u(\cdot,\cdot)$ of the left hand side. We need to show for any $s<t\in[0,T]$,

$$\frac{|u(t,x)-u(s,x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} \le \epsilon.$$

Suppose $s<t$ are both in $[t_i,t_{i+1}]$. Then, since the profile is in $T_i^{\#}$,

$$\frac{|u(t,x)-u(s,x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} \le \frac{\epsilon}{2}.$$

Next we consider the case when $s\in[0,t_{i-1}]$ and $t\in[t_i,t_{i+1}]$. In this case we have

$$\begin{aligned}\frac{|u(t,x)-u(s,x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} &\le \frac{|u(t,x)-u(t_i,x)|}{|t-t_i|^{\frac14-\frac{\theta}{2}}} + \frac{|u(t_i,x)-u(s,x)|}{|t_i-s|^{\frac14-\frac{\theta}{2}}} \\ &\le \frac{\epsilon}{2} + \frac{2\epsilon^{\frac{1}{2\theta}}}{4c_2^{\frac{\theta}{2}-\frac14}} \cdot \left(\frac{1}{c_2\epsilon^{\frac{2}{\theta}}}\right)^{\frac14-\frac{\theta}{2}} \\ &\le \epsilon,\end{aligned}$$

since $u\in\cap_{i=0}^{I-1}U_i^{\#}$. Finally consider $s\in[t_{i-1},t_i]$ and $t\in[t_i,t_{i+1}]$. In this case

$$\frac{|u(t,x)-u(s,x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} \le \frac{|u(t,x)-u(t_i,x)|}{|t-t_i|^{\frac14-\frac{\theta}{2}}} + \frac{|u(t_i,x)-u(s,x)|}{|t_i-s|^{\frac14-\frac{\theta}{2}}} \le \epsilon.$$

This shows that the realization is in $\left\{\sup_{x\in\mathbf{T}} \mathscr{H}^{(\theta)}_x(u) \le \epsilon\right\}$. □

Remark 4.2. *Observe that the events $H_i^{\#}$ play no role in the argument above, and we can in fact take the larger set $\cap_{i=0}^{I}(U_i^{\#}\cap T_i^{\#})$ in the left hand side of (4.19). However, as we will see in Proposition 4.1 below, to get a lower bound on $P(T_i^{\#})$ we will need a control on the spatial Hölder norms of u as given by the events $H_i^{\#}$.*

From the above lemma

$$
\begin{aligned}
P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u)\le \epsilon\right) &\ge P\left(\cap_{i=0}^I B_i^{\#}\right) \\
&= \prod_{i=0}^I P\left(B_i^{\#}\big| B_0^{\#}, B_1^{\#}\cdots B_{i-1}^{\#}\right).
\end{aligned} \tag{4.20}
$$

The lower bound follows from the Markov property and the following

Proposition 4.1. *Suppose the initial profile u_0 satisfies*

$$
\sup_{x\in\mathbf{T}} |u_0(x)| \le \frac{\epsilon^{\frac{1}{2\theta}} c_2^{\frac14-\frac{\theta}{2}}}{8}, \quad \mathcal{H}^{(\theta)}(u_0)\le \frac{\epsilon}{8\Lambda}.
$$

Then there exists a universal constant $\mathbf{K}_{10}>0$ and $\tilde{\alpha}_0^{\#}(\theta,\mathscr{C}_2)>0$ such that for all $c_2<\tilde{\alpha}_0^{\#}$ one has

$$
P\left(B_0^{\#}\right)\ge \exp\left(-\frac{1}{\sqrt{c_2}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_{10}}{(1+\Lambda^2)\mathscr{C}_2^2 c_2^{\theta}}\right) - \frac{1}{64\mathscr{C}_1^2 c_2^{\frac12+\theta}\epsilon^{\frac{1}{\theta}}}\right).
$$

Proof. We work with the measure Q constructed in the proof of Lemma 4.3. From (4.13) we have

$$
\begin{aligned}
&u(t,x)-u(s,x) \\
&=\left\{\left[1-\frac{t}{t_1}\right](G_t*u_0)(x)+N_1(t,x)\right\}-\left\{\left[1-\frac{s}{t_1}\right](G_s*u_0)(x)+N_1(s,x)\right\},
\end{aligned}
$$

where we recall

$$
N_1(t,x)=\int_0^t\int_{\mathbf{T}} G(t-r,x-z)\sigma(r,z)\widetilde{\dot{W}}(drdz).
$$

Define

$$
N_1^{\#}(s,t,x):=\frac{N_1(t,x)-N_1(s,x)}{|t-s|^{\frac14-\frac{\theta}{2}}}.
$$

For $s,t\in[0,t_1]$

$$
\begin{aligned}
\frac{|u(t,x)-u(s,x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} &\le \left[1+\frac{s}{t_1}\right]\frac{|(G_t*u_0)(x)-(G_s*u_0)(x)|}{|t-s|^{\frac14-\frac{\theta}{2}}} \\
&\quad +\frac{|t-s|}{t_1}\frac{|(G_t*u_0)(x)|}{|t-s|^{\frac14-\frac{\theta}{2}}}+|N_1^{\#}(s,t,x)|.
\end{aligned}
$$

The first term on the right is less than $\frac{\epsilon}{4}$ thanks to Lemma 2.2. The second term is less than $\frac{\epsilon}{8}$ by the assumption on the initial profile. Now

$$\begin{aligned}
Q(B_0^{\#}) &\geq Q\Bigg(\sup_{t\leq t_1,\, x\in\mathbf{T}} |N_1(t,x)| \leq \frac{\epsilon^{\frac{1}{2\theta}}}{8c_2^{\frac{\theta}{2}-\frac14}},\ \sup_{t\leq t_1,\, x\neq y\in\mathbf{T}} \left|\widetilde{N}_1(t,x,y)\right| \leq \frac{\epsilon}{8\Lambda}, \\
&\qquad \sup_{x\in\mathbf{T},\, s\neq t\in[0,t_1]} |N_1^{\#}(s,t,x)| \leq \frac{\epsilon}{8}\Bigg) \\
&\geq Q\Bigg(\sup_{t\leq t_1,\, x\in\mathbf{T}} |N_1(t,x)| \leq \frac{\epsilon^{\frac{1}{2\theta}}}{8c_2^{\frac{\theta}{2}-\frac14}},\ \sup_{t\leq t_1,\, x\neq y\in\mathbf{T}} \left|\widetilde{N}_1(t,x,y)\right| \leq \frac{\epsilon}{8\Lambda}\Bigg) \\
&\qquad \times Q\Bigg(\sup_{x\in\mathbf{T},\, s\neq t\in[0,t_1]} |N_1^{\#}(s,t,x)| \leq \frac{\epsilon}{8}\Bigg)
\end{aligned}$$

by the Gaussian correlation inequality. By splitting the interval $\mathbf{T}$ into smaller intervals of length $c_2^{\frac12}\epsilon^{\frac{1}{\theta}}$ and using Gaussian correlation inequality repeatedly (see also Lemma 4.4 and Lemma 4.2) one obtains

$$Q(B_0^{\#}) \geq \exp\left(-\frac{1}{c_2^{\frac12}\epsilon^{\frac1\theta}} \exp\left(-\frac{\mathbf{K}_{11}}{(1+\Lambda^2)\mathscr{C}_2^2 c_2^{\theta}} \right)\right) \tag{4.21}$$

as long as c_2 is small enough. Following the arguments of Lemma 4.3 we obtain

$$P\left(B_0^{\#}\right) \geq Q\left(B_0^{\#}\right)^2 \left\{ E_P\left(\frac{dQ}{dP}\right)^2 \right\}^{-1}$$

where Q is the measure constructed there. As in (4.12) we have

$$E_P\left(\frac{dQ}{dP}\right)^2 = \exp\left(\int_0^{t_1}\int_{\mathbf{T}} \left| \frac{1}{\sigma(s,y)}\cdot \frac{(G_s * u_0)(y)}{t_1}\right|^2 dyds\right) \leq \exp\left(\frac{1}{64\mathscr{C}_1^2 c_2^{\frac12+\theta}\epsilon^{\frac1\theta}}\right).$$

Using the above along with (4.21), the proof is complete. □

4.3. **Lower bound in Theorem 1.3 (a).** We begin by describing the idea behind the proof first. The same idea will be used for the proof of Theorem 1.4. We will consider the following modifications of the temporal discretisation given by (3.1),

$$t_i = ic_0\delta^{2+\eta}, \qquad i = 0,\, 1,\, \cdots, I := \left[\frac{T}{c_0\delta^{2+\eta}}\right]. \tag{4.22}$$

Define

$$R_i := \left\{ |u(t_{i+1},x)| \leq \frac{\epsilon^{\frac{1}{2\theta}}}{3} \text{ for all } x\in\mathbf{T}, \text{ and } |u(t,x)| \leq \epsilon^{\frac{1}{2\theta}} \text{ for all } t\in[t_i,t_{i+1}],\, x\in\mathbf{T}\right\},$$

and

$$S_i := \left\{ \mathcal{H}_{t_{i+1}}^{(\theta)}(u) \leq \frac{\epsilon}{3}, \text{ and } \mathcal{H}_t^{(\theta)}(u) \leq \epsilon \text{ for all } t\in[t_i,t_{i+1}]\right\}.$$

We consider the event

$$A_i = R_i \cap S_i.$$

Our goal is to provide a lower bound on $P(A_i)$. By the Markov property it is sufficient to obtain a lower bound on $P(A_0)$ under the assumption that the initial profile satisfies $|u_0(x)| \le \epsilon^{\frac{1}{2\theta}}/3$ and $\mathcal{H}^{(\theta)}(u_0) \le \epsilon/3$.

Consider the evolution of $u(t,\cdot)$ in $[0,t_1]$ and write

$$u(t,x) = u_g(t,x) + D(t,x),$$

where $u_g(t,x)$ solves

$$\partial_t u_g(t,x) = \frac{1}{2}\partial_x^2 u_g(t,x) + \sigma\big(t,x,u_0(x)\big)\cdot \dot{W}(t,x), \quad t\in\mathbf{R}_+,\ x\in\mathbf{T},$$

with initial profile $u_0(x)$. Note that the third coordinate in σ is now $u_0(x)$ and therefore u_g is a Gaussian random field. Therefore if we define as in (4.1)

$$B_0^{(g)} = U_0^{(g)} \cap H_0^{(g)},$$

with $U_0^{(g)}$ and $H_0^{(g)}$ defined similarly as in (4.2) and (4.3) but for the process u_g in place of u, and with the new value of $t_1 = \delta^{2+\eta} = \epsilon^{(2+\eta)/\theta}$.

Now $B_0^{(g)} \supset \widetilde{B}_0^{(g)}$, where $\widetilde{B}_0^{(g)}$ is similar to (4.1) but with u replaced by $u^{(g)}$, ϵ replaced by $\widetilde{\epsilon} = \frac{\epsilon}{8}$, and $t_1 = c_0(\widetilde{\epsilon})^{\frac{2}{\theta}}$. Therefore

$$P(B_0^{(g)}) \ge \exp\left\{-\frac{2\cdot 8^{\frac{1}{\theta}}}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_6}{36\mathscr{C}_2^2 c_0^\theta}\right) - \frac{2\cdot 8^{\frac{1}{\theta}}}{9c_0\mathscr{C}_1^2\epsilon^{\frac{1}{\theta}}}\right\} \tag{4.23}$$

when $c_0 6^{\frac{2}{\theta}} < \tilde{\alpha}_0$. The difference between u and u_g is

$$D(t,x) = \int_0^t\int_{\mathbf{T}} G(t-s,x-y)\cdot\big[\sigma\big(s,y,u(s,y)\big) - \sigma\big(s,y,u_0(y)\big)\big]\, W(dsdy).$$

Consider the set

$$V := \left\{|D(t,x)| \le \frac{\epsilon^{\frac{1}{2\theta}}}{6} \text{ for all } t\in[0,t_1],\ x\in\mathbf{T}\right\} \cap \left\{\mathcal{H}_t^{(\theta)}(D) \le \frac{\epsilon}{6} \text{ for all } t\in[0,t_1]\right\}. \tag{4.24}$$

Define now

$$\tau := \inf\left\{t\ge 0 : |u(t,x) - u_0(x)| \ge 2\epsilon^{\frac{1}{2\theta}} \text{ for some } x\in\mathbf{T}\right\},$$

and let

$$\widetilde{D}(t,x) := \int_0^t\int_{\mathbf{T}} G(t-s,x-y)\cdot\big[\sigma\big(s,y,u(s\wedge\tau,y)\big) - \sigma\big(s,y,u_0(y)\big)\big]\, W(dsdy). \tag{4.25}$$

Let the event $\widetilde{V}$ be the same as V (see (4.24)) but with D replaced by $\tilde{D}$. Now

$$\begin{aligned} P(A_0) &\ge P\left(B_0^{(g)} \cap V\right) \\ &= P\left(B_0^{(g)} \cap \widetilde{V}\right) \\ &\ge P\left(B_0^{(g)}\right) - P\left(\widetilde{V}^c\right) \qquad (4.26) \\ &\ge P\left(B_0^{(g)}\right) - P\left(\sup_{\substack{0\le t\le t_1\\ x\in\mathbf{T}}} |\widetilde{D}(t,x)| > \frac{\epsilon^{\frac{1}{2\theta}}}{6}\right) - P\left(\sup_{0\le t\le t_1} \mathcal{H}_t^{(\theta)}(\widetilde{D}) > \frac{\epsilon}{6}\right). \end{aligned}$$

The equality holds because on the event A_0 we have $\|u(t,\cdot)-u_0\|_\infty < 2\epsilon^{\frac{1}{2\theta}}$ (recall that our initial profile is everywhere less than $\epsilon^{\frac{1}{2\theta}}/3$), and so $D(t,\cdot)=\widetilde{D}(t,\cdot)$ up to time t_1 on the event A_0. Now we use Remark 2.2 together with the fact that now $t_1 = c_0\epsilon^{\frac{2}{\theta}+\frac{\eta}{\theta}}$ to obtain

(4.27)

$$P\left(\sup_{\substack{0\le t\le t_1\\ x\in\mathbf{T}}}|\widetilde{D}(t,x)| > \frac{\epsilon^{\frac{1}{2\theta}}}{6}\right) \le \sum_{i=1}^{c_0^{-\frac{1}{2}}\epsilon^{-\frac{1}{\theta}}} P\left(\sup_{\substack{0\le t\le t_1\\ x\in\left[(i-1)\sqrt{c_0}\epsilon^{\frac{1}{\theta}},\, i\sqrt{c_0}\epsilon^{\frac{1}{\theta}}\right]}}|\widetilde{D}(t,x)| > \frac{1}{6c_0^{\frac{1}{4}}}\left(c_0^{\frac{1}{4}}\epsilon^{\frac{1}{2\theta}}\right)\right)$$

$$\le \frac{\mathbf{K}_1}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}+\frac{\eta}{2\theta}}}\exp\left(-\frac{\mathbf{K}_2}{144\sqrt{c_0}\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\frac{\eta}{2\theta}}}\right).$$

Next we focus on the last term in (4.26). We divide $\mathbf{T}^2$ into squares S of side length $\sqrt{c_0}\epsilon^{\frac{1}{\theta}}$. Let

$$\widetilde{N}^{(\widetilde{D})}(t,x,y) := \frac{\widetilde{D}(t,x)-\widetilde{D}(t,y)}{|x-y|^{\frac{1}{2}-\theta}}.$$

Using Lemma 2.5 we obtain

$$P\left(\sup_{0\le t\le t_1}\mathcal{H}_t^{(\theta)}(\widetilde{D}) > \frac{\epsilon}{6}\right) \le \frac{1}{c_0\epsilon^{\frac{2}{\theta}}}\cdot\sup_S P\left(\sup_{\substack{0\le t\le c_0\epsilon^{\frac{2}{\theta}+\frac{\eta}{\theta}}\\ (x,y)\in S,\, x\ne y}}|\widetilde{N}^{(\widetilde{D})}(t,x,y)| > \frac{\epsilon}{6}\right)$$

(4.28)

$$= \frac{1}{c_0\epsilon^{\frac{2}{\theta}}}\cdot\sup_S P\left(\sup_{\substack{0\le t\le c_0\epsilon^{\frac{2}{\theta}+\frac{\eta}{\theta}}\\ (x,y)\in S,\, x\ne y}}|\widetilde{N}^{(\widetilde{D})}(t,x,y)| > \frac{1}{6c_0^{\frac{\theta}{2}}}\cdot c_0^{\frac{\theta}{2}}\epsilon\right)$$

$$\le \frac{\mathbf{K}_3}{c_0\epsilon^{\frac{2}{\theta}+\frac{\eta}{2\theta}}}\exp\left(-\frac{\mathbf{K}_4}{144c_0^{\theta}\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\eta}}\right).$$

We plug in the bounds (4.28), (4.27) and (4.23) into (4.26) to obtain

$$P(A_0) \ge \exp\left\{-\frac{2\cdot 8^{\frac{1}{\theta}}}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_6}{36\mathscr{C}_2^2c_0^{\theta}}\right) - \frac{2\cdot 8^{\frac{1}{\theta}}}{9c_0\mathscr{C}_1^2\epsilon^{\frac{1}{\theta}}}\right\}$$
$$-\frac{\mathbf{K}_1}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}+\frac{\eta}{2\theta}}}\exp\left(-\frac{\mathbf{K}_2}{144\sqrt{c_0}\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\frac{\eta}{2\theta}}}\right) - \frac{\mathbf{K}_3}{c_0\epsilon^{\frac{2}{\theta}+\frac{\eta}{2\theta}}}\exp\left(-\frac{\mathbf{K}_4}{144c_0^{\theta}\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\eta}}\right).$$

The last two terms are much smaller than the first term for small ϵ. Therefore $P(A_0)\ge P(B_0^{(g)})/2$ when ϵ is small enough. We thus have a lower bound on $P(A_i)$ for all i. As mentioned earlier, the proof of (1.11) then follows from the Markov property. $\square$

4.4. **Lower bound in Theorem 1.3 (b).** The argument follows that of Section 4.3 with some modifications. Now let $t_i = ic_2\delta^{2+\eta\theta} = \epsilon^{\frac{2}{\theta}+\eta}$. Similar to before, for $t\in[t_i,t_{i+1}]$, we write $u(t,x) = u_g^{(i)}(t,x) + D^{(i)}(t,x)$. Here

$$\begin{aligned} \partial_t u_g^{(i)} &= \frac{1}{2}\partial_x^2 u_g^{(i)} + \sigma\left(t,x,u(t_i,x)\right)\dot{W}(t,x), \qquad t\in[t_i,t_{i+1}],\\ u_g^{(i)}(t_i,\cdot) &\equiv u(t_i,\cdot), \end{aligned}$$

and

$$D^{(i)}(t,x) = \int_{t_i}^t \int_{\mathbf{T}} G(t-s,x-y)\left[\sigma(s,y,u(s,y)) - \sigma(s,y,u(t_i,y)\right] W(dyds).$$

Now define

$$B_i^{(g),\#} = U_i^{(g),\#} \cap H_i^{(g),\#} \cap T_i^{(g),\#},$$

where $H_i^{(g),\#}$, $T_i^{(g),\#}$ are as in (4.17) and (4.18) but with u_g in place of u, and with the $t_i = ic_2\delta^{2+\eta\theta} = ic_2\epsilon^{\frac{2}{\theta}+\eta}$. On the other hand we define

$$\begin{aligned} U_i^{(g),\#} := \Bigg\{ \sup_{x\in\mathbf{T}} \left|u_g^{(i)}(t_{i+1},x)\right| \le \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{8c_2^{\frac{\theta}{2}-\frac14}}, \\ \text{and } \sup_{x\in\mathbf{T}} \left|u_g^{(i)}(t,x)\right| \le \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{4c_2^{\frac{\theta}{2}-\frac14}} \text{ for all } t\in[t_i,t_{i+1}]\Bigg\}. \end{aligned}$$

Now let

$$V_i^{\#} = V_{i,1}^{\#} \cap V_{i,2}^{\#},$$

where

$$\begin{aligned} V_{i,1}^{\#} &= \left\{ \sup_{\substack{x\in\mathbf{T}\\ t\in[t_i,t_{i+1}]}} |D^{(i)}(t,x)| \le \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{4c_2^{\frac{\theta}{2}-\frac14}} \right\}\\ V_{i,2}^{\#} &= \left\{ \sup_{\substack{x\in\mathbf{T}\\ t_i\le s,t\le t_{i+1},\, s\neq t}} \frac{\left|D^{(i)}(t,x) - D^{(i)}(s,x)\right|}{|t-s|^{\frac14-\frac{\theta}{2}}} \le \frac{\epsilon}{2} \right\}. \end{aligned}$$

It follows from arguments similar to Lemma 4.5 that

$$\bigcap_{i=0}^{I} \mathscr{B}_i^{\#} \subset \left\{ \sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u) \le 2\epsilon \right\}, \tag{4.29}$$

where

$$\mathscr{B}_i^{\#} := B_i^{(g),\#} \cap V_i^{\#}.$$

By the Markov property it is enough to give a lower bound on $P(\mathscr{B}_0^{\#})$ under the assumption that $|u_0(x)| \le \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{8c_2^{\frac{\theta}{2}-\frac14}}$ and $\mathcal{H}_0^{(\theta)}(u) \le \frac{\epsilon}{8\Lambda}$. Let

$$\tau := \inf\left\{ t\ge 0 : |u(t,x) - u_0(x)| \ge \frac{2}{c_2^{\frac{\theta}{2}-\frac14}}\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})} \text{ for some } x\in\mathbf{T} \right\},$$

Let $\widetilde{V}_0^{\#}$ be defined as $V_0^{\#}$ but with $\widetilde{D}^{(0)}$ in place of $D^{(0)}$. Here $\widetilde{D}_0$ is as in (4.25) but with the above τ.

$$\begin{aligned}P(\mathscr{B}_0^{\#}) &= P(B_0^{(g),\#}\cap V_0^{\#})\\ &= P(B_0^{(g),\#}\cap \widetilde{V}_0^{\#})\\ &\geq P\left(B_0^{(g),\#}\right) - P\left((\widetilde{V}_{0,1}^{\#})^c\right) - P\left((\widetilde{V}_{0,2}^{\#})^c\right)\end{aligned}$$

Using Remark 2.2 and the argument in (4.27) we obtain

$$\begin{aligned}P\left((\widetilde{V}_{0,1}^{\#})^c\right) &= P\left(\sup_{\substack{0\leq t\leq t_1\\ x\in\mathbf{T}}}|\widetilde{D}^{(0)}(t,x)| > \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{4c_2^{\frac{\theta}{2}-\frac14}}\right)\\ &\leq \frac{\mathbf{K}_1}{\sqrt{c_2}\epsilon^{\frac{1}{\theta}+\frac{\eta}{2}}}\exp\left(-\frac{\mathbf{K}_2}{64\sqrt{c_2}\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\frac{\eta}{2}}}\right).\end{aligned}\tag{4.30}$$

Similarly using Remark 2.4 we obtain

$$P\left((\widetilde{V}_{0,2}^{\#})^c\right) \leq \frac{\mathbf{K}_7}{\sqrt{c_2}\epsilon^{\frac{1}{\theta}+\frac{\eta}{2}}}\exp\left(-\frac{\mathbf{K}_8}{16\sqrt{c_2}\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\frac{\eta}{2}}}\right).\tag{4.31}$$

Lemma 4.6. *We have when ϵ is small enough*

$$P\left(B_0^{(g),\#}\right)\geq \exp\left(-\frac{3}{c_2^{\frac12}\epsilon^{\frac{1}{\theta}+\eta(\frac12-\theta)}}\exp\left(-\frac{\mathbf{K}_2}{128c_2^{\theta}\mathscr{C}_2^2}\right)\right)$$

Proof. As in the proof of Proposition 4.1, with this new choice of $t_1 = c_2\epsilon^{\frac{2}{\theta}+\eta}$ and with

$$N_1^{(g)}(t,x) := \int_0^t\int_{\mathbf{T}} G(t-r,x-z)\sigma(r,z,u_0(z))\dot{\widetilde{W}}(drdz),$$

$\widetilde{N}_1^{(g)}$, $N_1^{\#,(g)}$ defined in terms of $N_1^{(g)}$, we have

(4.32)

$$\begin{aligned}Q(B_0^{(g),\#}) \geq Q&\Bigg(\sup_{t\leq t_1,\, x\in\mathbf{T}}|N_1^{(g)}(t,x)|\leq \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{8c_2^{\frac{\theta}{2}-\frac14}},\ \sup_{t\leq t_1,\, x\neq y\in\mathbf{T}}\left|\widetilde{N}_1^{(g)}(t,x,y)\right|\leq\frac{\epsilon}{8\Lambda}\Bigg)\\ &\times Q\Bigg(\sup_{x\in\mathbf{T},\, s\neq t\in[0,t_1]}|N_1^{\#,(g)}(s,t,x)|\leq\frac{\epsilon}{8}\Bigg)\end{aligned}$$

A lower bound on the last probability is obtained by taking the supremum of $s\neq t$ over $[0, c_2\epsilon^{\frac{2}{\theta}}]$ instead of $[0,t_1]$. This gives

$$Q\Bigg(\sup_{x\in\mathbf{T},\, s\neq t\in[0,t_1]}|N_1^{\#,(g)}(s,t,x)|\leq\frac{\epsilon}{8}\Bigg)\geq \exp\left(-\frac{1}{c_2^{\frac12}8^{\frac{1}{\theta}}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_9}{64\mathscr{C}_2^2c_2^{\theta}}\right)\right).\tag{4.33}$$

Next let

$$\tilde{t}_1 := c_2\epsilon^{\frac{2}{\theta}+\eta(1-2\theta)}.$$

Clearly $\tilde{t}_1 > t_1$ and so a lower bound of the first term on the right of (4.32) is (note the t_1 in the sup has been replaced by $\tilde{t}_1$)

$$\begin{aligned}
(4.34)\quad & Q\left(\sup_{t\le \tilde{t}_1,\, x\in\mathbf{T}} |N_1^{(g)}(t,x)| \le \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{8c_2^{\frac{\theta}{2}-\frac14}},\ \sup_{t\le\tilde{t}_1,\, x\ne y\in\mathbf{T}} \left|\widetilde{N}_1^{(g)}(t,x,y)\right| \le \frac{\epsilon}{8\Lambda}\right) \\
&\ge Q\left(\sup_{t\le \tilde{t}_1,\, x\in\mathbf{T}} |N_1^{(g)}(t,x)| \le \frac{\epsilon^{\frac{1}{2\theta}+\eta(\frac14-\frac{\theta}{2})}}{8c_2^{\frac{\theta}{2}-\frac14}}\right)\cdot Q\left(\sup_{t\le c_2\epsilon^{\frac{2}{\theta}},\, x\ne y\in\mathbf{T}} \left|\widetilde{N}_1^{(g)}(t,x,y)\right| \le \frac{\epsilon}{8\Lambda}\right) \\
&\ge \exp\left(-\frac{1}{c_2^{\frac12}\epsilon^{\frac1\theta+\eta(\frac12-\theta)}}\exp\left(-\frac{\mathbf{K}_2}{128c_2^\theta\mathscr{C}_2^2}\right)\right)\cdot\exp\left(-\frac{2}{c_2^{\frac12}\epsilon^{\frac1\theta}}\exp\left(-\frac{\mathbf{K}_5}{64\Lambda^2\mathscr{C}_2^2c_2^\theta}\right)\right)
\end{aligned}$$

when c_2 is chosen small enough, using the arguments in Lemma 4.2. We have used the Gaussian correlation inequality in the second step. Note that the sup in t in the second probabillity is over a larger time interval $[0, c_2\epsilon^{\frac{2}{\theta}}]$.

The event $B_0^{(g),\#}$ depends on the noise up to time $c_2\epsilon^{\frac{2}{\theta}}$, and so

$$P\left(B_0^{(g),\#}\right) \ge Q\left(B_0^{(g),\#}\right)^2\left\{E_P\left(\frac{dQ}{dP}\bigg|_{[0,c_2\epsilon^{\frac{2}{\theta}}]}\right)^2\right\}^{-1}. \tag{4.35}$$

We have the following upper bound (similar to (4.12):

$$E_P\left(\frac{dQ}{dP}\bigg|_{[0,c_2\epsilon^{\frac{2}{\theta}}]}\right)^2 \le \exp\left(-\frac{1}{64\mathscr{C}_1^2c_2^{\theta+\frac12}\epsilon^{\frac1\theta-\eta(\frac12-\theta)}}\right). \tag{4.36}$$

Plugging in the bounds (4.33),(4.34) and (4.36) into (4.35) we obtain the lemma. ☐

From the above lemma as well as (4.30) and (4.31) we obtain

$$P(\mathscr{B}_0^{\#}) \ge \exp\left(-\frac{4}{c_2^{\frac12}\epsilon^{\frac1\theta+\eta(\frac12-\theta)}}\exp\left(-\frac{\mathbf{K}_2}{128c_2^\theta\mathscr{C}_2^2}\right)\right)$$

when ϵ is small enough, and thus from (4.29) one gets

$$P\left(\sup_{x\in\mathbf{T}}\mathscr{H}_x^{(\theta)}(u)\le 2\epsilon\right) \ge \exp\left(-\frac{C(\theta,\mathscr{C}_2)T}{\epsilon^{\frac3\theta+\eta(\frac32-\theta)}}\right),$$

for some constant $C(\theta,\mathscr{C}_2) > 0$ dependent only on $\mathscr{C}_2$ and θ. This completes the proof of the lower bound since η is arbitrary. ☐

5. Proof of Theorem 1.4

The proofs of the upper bounds in both statements in Theorem 1.4 are the same as that of Theorem 1.3. The proof of the lower bounds follows the same ideas as in the proofs for the lower bounds of Theorem 1.3. We show this only for statement (a); the proof of statement

(b) is similar. The only difference as compared to the proof in Theorem 1.2 (a) is that we revert back to the discretisation given by (3.1). We therefore have

$$P(A_0) \geq \exp\left\{-\frac{2}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_6}{36\mathscr{C}_2^2 c_0^{\theta}}\right) - \frac{2}{9c_0\mathscr{C}_1^2\epsilon^{\frac{1}{\theta}}}\right\}$$
$$-\frac{\mathbf{K}_1}{\sqrt{c_0}\epsilon^{\frac{1}{\theta}}}\exp\left(-\frac{\mathbf{K}_2}{144\sqrt{c_0}\mathscr{D}^2\epsilon^{\frac{1}{\theta}}}\right) - \frac{\mathbf{K}_3}{c_0\epsilon^{\frac{2}{\theta}}}\exp\left(-\frac{\mathbf{K}_4}{144c_0^{\theta}\mathscr{D}^2\epsilon^{\frac{1}{\theta}}}\right).$$

For any fixed c_0, $\mathscr{C}_1$ and $\mathscr{C}_2$, we can choose $\mathscr{D}$ small enough so that as ϵ decreases, the final two term goes to zero much faster than the first term. Therefore for small ϵ a lower bound on $P(A_0)$ (and hence $P(A_i)$) is one half times the first term above. An application of the Markov property then finishes the proof. □

6. Proof of Theorem 1.1

We first prove the upper bound. This follows immediately from

$$P\left(\sup_{\substack{0\leq s,t\leq T\\ 0\leq x,y\leq 1\\ (t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|x-y|^{\frac{1}{2}-\theta}+|t-s|^{\frac{1}{4}-\frac{\theta}{2}}} \leq \epsilon\right) \leq P\left(\sup_{t\leq T}\mathcal{H}_t^{(\theta)}(u) \leq \epsilon\right),$$

and Theorem 1.3. Let us turn our attention to the lower bound. In the proof of the lower bound of Theorem 1.3 (b), we let

$$V_{i,3}^{\#} := \left\{\sup_{\substack{x\neq y\in\mathbf{T}\\ t_i\leq t\leq t_{i+1}}} \frac{\left|D^{(i)}(t,x)-D^{(i)}(t,y)\right|}{|x-y|^{\frac{1}{2}-\theta}} \leq \frac{\epsilon}{2\Lambda}\right\},$$

and redefine

$$V_i^{\#} := V_{i,1}^{\#}\cap V_{i,2}^{\#}\cap V_{i,3}^{\#} \quad \text{and} \quad \mathscr{B}_i^{\#} := B_i^{(g),\#}\cap V_i^{\#}$$

We then have

$$\bigcap_{i=0}^{I}\mathscr{B}_i^{\#} \subset \left\{\sup_{x\in\mathbf{T}}\mathscr{H}_x^{(\theta)}(u)\leq 2\epsilon\right\}\cap\left\{\sup_{t\in[0,T]}\mathcal{H}_t^{(\theta)}(u)\leq\frac{\epsilon}{\Lambda}\right\}.$$

In addition, similar to (4.30) and (4.31), Remark 2.3 says that

$$P\left((\widetilde{V}_{0,3}^{\#})^c\right) \leq \frac{\mathbf{K}_3}{\sqrt{c_2}\epsilon^{\frac{1}{\theta}+\frac{\eta}{2}}}\exp\left(-\frac{\mathbf{K}_4}{16\sqrt{c_2}\Lambda^2\mathscr{D}^2\epsilon^{\frac{1}{\theta}+\frac{\eta}{2}}}\right),$$

Now it is easy to see that

$$P\left(\sup_{\substack{0\leq s,t\leq T\\ 0\leq x,y\leq 1\\ (t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|t-s|^{\frac{1}{4}-\frac{\theta}{2}}+|x-y|^{\frac{1}{2}-\frac{\theta}{2}}} \leq \epsilon\left[2+\frac{1}{\Lambda}\right]\right)$$
$$\geq P\left(\left\{\sup_{x\in\mathbf{T}}\mathscr{H}_x^{(\theta)}(u)\leq 2\epsilon\right\}\cap\left\{\sup_{t\in[0,T]}\mathcal{H}_t^{(\theta)}(u)\leq\frac{\epsilon}{\Lambda}\right\}\right).$$

It then follows quite easily that under the same assumptions of Theorem 1.3, for any $\eta > 0$, there exist positive constants $C_1, C_2 > 0$ dependent on $\mathscr{C}_1, \mathscr{C}_2, \theta, \eta$ such that

$$P\left(\sup_{\substack{0\le s,t\le T\\ 0\le x,y\le 1\\ (t,x)\neq(s,y)}} \frac{|u(t,x)-u(s,y)|}{|t-s|^{\frac14-\frac{\theta}{2}}+|x-y|^{\frac12-\frac{\theta}{2}}} \le \epsilon\right) \ge C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}+\eta}}\right)$$

This finishes the proof of Theorem 1.1. □

Remark 6.1. *It is easy to see from the argument presented here that under the assumptions of Theorem 1.2 (resp. Theorem 1.4) we have the same bounds as in* (1.7) *(resp.* (1.11)*) for the Hölder semi-norm. We leave the verification to the reader.*

7. Proofs of Theorems 1.5 and 1.6

The proof of the lower bound of Theorem 1.5 relies heavily on Theorem 2.2 of [KLS95]. We will use some notations from its proof and indicate only the main differences. The proofs of Theorem 1.6 follow from Theorem 1.5 using the same arguments used previously to deal with the non-gaussian case.

Proof of Theorem 1.5 (a). The upper bound is a result of Lemma 3.2 (recall the event A_i defined in (3.3)); note that the initial profile in Lemma 3.2 is arbitrary. Indeed, we might condition on the profile at time $T - c_0\delta^2$ and conclude from the above lemma that

$$P\left[\max_{j=0,1,\cdots,J} \left|u(T,x_j+\delta)-u(T,x_j)\right| \le \epsilon^{\frac{1}{2\theta}} \;\middle|\; u(T-c_0\delta^2,\cdot)\right] \le \eta^J,$$

where $0<\eta<1$ and $J = \left[\frac{1}{c_1\delta}\right]$, and $c_0 = 1, c_1, \delta = \epsilon^{\frac{1}{\theta}}$ are as in Section 3.1. From this and (3.4) we obtain

$$P\left(\mathcal{H}_T^{(\theta)}(u) \le \epsilon \,\middle|\, u(T-c_0\delta^2,\cdot)\right) \le \eta^J.$$

Integrating over the profile $u(T-c_0\delta^2,\cdot)$ we obtain the upper bound.

We next turn our attention to the lower bound. As mentioned above, the proof follows along the lines of the proof of Theorem 2.2 of [KLS95], and we just sketch the necessary modifications in the proof. Recall that we assume that our initial profile $u_0 \equiv 0$, and therefore

$$E\left[\{u(T,x)-u(T,y)\}^2\right] = E\left[\{N(T,x)-N(T,y)\}^2\right].$$

Defining $\boldsymbol{\sigma}^2(\gamma) := E\left[\{N(T,x+\gamma)-N(T,x)\}^2\right]$ it follows from the proof of (3.7) that

$$C(T)\gamma \le \boldsymbol{\sigma}^2(\gamma) \le \sqrt{C_1}\gamma$$

for $\gamma > 0$ small enough, where $C(T)$ is a constant dependent on T and C_1 is the constant in (3.7). The above is the key ingredient in the proof of the lower bound. We take $\boldsymbol{\beta} = \theta$ and $f(x) = x^{\frac12-\theta}$ in Theorem 2.2 in [KLS95]. While it is not true that $\boldsymbol{\sigma}(x)/x^{\boldsymbol{\beta}}f(x)$ is nondecreasing in x as in Theorem 2.2 of [KLS95], a close examination of the proof reveals that all we require is that $\boldsymbol{\sigma}(ax)/f(ax) \le C_2 a^{\boldsymbol{\beta}}\boldsymbol{\sigma}(x)/f(x)$ for some positive constant C_2, for all $0 < a < 1$ and x small enough. This clearly holds for us. The sequences x_l and $y_{j,l}$ encountered in the proof there should be modified by multiplying by $\frac{1}{C_2}$. Similarly, while going through the arguments of the lower bounds of the terms A, B, C defined in the paper, one just gets an additional constant multiple inside the exponentials and this does not change the result. We leave this routine checking to the interested reader. The lower bound in Theorem 1.5 follows immediately from the lower bound of Theorem 2.2 in [KLS95]. □

Proof of Theorem 1.5 (b). For the upper bound, let $t_i = i\epsilon^{\frac{2}{\theta}}$, $i = 0, 1, \cdots I = T\epsilon^{-\frac{2}{\theta}}$.

$$P\left(\mathscr{H}_X^{(\theta)}(u) \le \epsilon\right) \le P\left(\frac{|u(t_{i+1}, X) - u(t_i, X)|}{(t_{i+1} - t_i)^{\frac{1}{4} - \frac{\theta}{2}}} \le \epsilon, \quad \text{for all } i = 0, 1, \cdots, I\right).$$

By considering the profile at time t_i we obtain

$$u(t_{i+1}, X) = \left(G_{t_i - t_{i+1}} * u(t_i, \cdot)\right)(X) + \mathcal{N}(t_i, t_{i+1}, X).$$

Note that $\mathcal{N}(t_i, t_{i+1}, X)$ is really the noise term from time t_i to t_{i+1}, that is thinking of time t_i as the new time *zero* . Similar to arguments used a few times in this paper we have

$$P\left(\frac{|u(t_{i+1}, X) - u(t_i, X)|}{(t_{i+1} - t_i)^{\frac{1}{4} - \frac{\theta}{2}}} \le \epsilon \;\Big|\; u(s, \cdot),\, s \le t_i\right) \le P\left(\frac{|\mathcal{N}(t_i, t_{i+1}, X)|}{(t_{i+1} - t_i)^{\frac{1}{4} - \frac{\theta}{2}}} \le \epsilon \Big|\, u(s, \cdot),\, s \le t_i\right),$$

which is bounded uniformly (in i) by a number less than 1 (note that the variance of $\mathcal{N}(t_i, t_{i+1}, X)$ is bounded above and below by constant multiples of $\epsilon^{\frac{1}{2\theta}}$). The Markov property then gives the upper bound.

Consider the process $Y_t := u(tT, X)$, $0 \le t \le 1$. As we are under the assumption $u_0 \equiv 0$ we have

$$E\left[(Y_t - Y_s)^2\right] = E\left[\{N(tT, X) - N(sT, X)\}^2\right].$$

Defining $\boldsymbol{\sigma}^2(\gamma) := E\left[(Y_{t+\gamma} - Y_t)^2\right]$ and using (2.19) and (2.20) one obtains

$$C_1\sqrt{T\gamma} \le \boldsymbol{\sigma}^2(\gamma) \le C_2\sqrt{T\gamma}$$

for constants C_1, C_2 independent of T. One can then follow the argument of the lower bound of Theorem 1.5 for the process Y_t, now with $f(x) = x^{\frac{1}{4} - \frac{\theta}{2}}$ and $\boldsymbol{\beta} = \frac{\theta}{2}$. □

Proof of Theorem 1.6 (a). The proof of the upper bound is similar to Theorem 1.3 but instead we use (3.20) and note that this bound is uniform over the initial profiles u_0. We can then conclude

$$P\left(\mathcal{H}_T^{(\theta)}(u) \le \epsilon \,\Big|\, u(T - c_0\epsilon^{\frac{2}{\theta}}, \cdot)\right) \le \exp\left(-\frac{C}{|\log \epsilon|^{\frac{3}{2}}\epsilon^{\frac{1}{\theta}}}\right).$$

Now integrate over the profile at time $T - c_0\epsilon^{\frac{2}{\theta}}$. □

Proof of Theorem 1.6 (b). The proof is very similar to that of the proof of the upper bound of Theorem 1.5 (b). The only difference is that now $\mathcal{N}(t_i, t_{i+1}, X)$ is no longer Gaussian.

For $t_i \le s \le t_{i+1}$, we note that

$$\mathcal{N}(t_i, s, X) = \int_{t_i}^{s} \int_{\mathbf{T}} G_{t_{i+1}-r}(X, y) \cdot \sigma\left(r, y, u(r, y)\right) W(dydr)$$

is a martingale. Similar arguments to that of the proof of (3.24) and an application of the Markov property complete the proof.

□

8. Some extensions

In this section, we provide *support theorems* in the Hölder semi-norm, which are similar to the support theorem in the sup norm in [AJM]. We provide probabilities that the solution u stays close to a function in Hölder spaces $C^{\gamma,\beta}$ (see Theorems 8.1 and 8.2 for the precise statements). These theorems are of a different flavour from the support theorem proved in [BMSS95], where a description of the support set of the solution is given.

We first consider small ball probabilities of (1.1) with *nice* drifts. By means of a change of measure argument, we can show that all of our results are still valid when we add a bounded drift term to the equation. In other words, if instead of (1.1), we consider the following

$$\partial_t u(t,x) = \frac{1}{2}\,\partial_x^2 u(t,x) + g(t,x,u) + \sigma\big(t,x,u(t,x)\big)\cdot \dot W(t,x), \quad t\in\mathbf{R}_+,\, x\in\mathbf{T}, \tag{8.1}$$

where $g(t,x,u) : \mathbf{R}^+ \times \mathbf{T} \times \mathbf{R} \to \mathbf{R}$ is globally Lipschitz in the third variable and is bounded in absolute value by a constant $\mathbb{G}$, then Theorems 1.2, 1.3 and 1.4 still hold. This follows exactly the argument given in Section 2.2 of [AJM]. We also have the following result which is analogous to that of Theorem 1.2 of [AJM]. The proofs are similar to those presented in [AJM, Section 2.1] and are therefore omitted.

Proposition 8.1. *Consider the solution to* (1.1). *Let* $h : \mathbf{R}^+ \times \mathbf{T} \to \mathbf{R}$ *be a smooth function such that* $h, \partial_t h$ *and* $\partial_x^2 h$ *are uniformly bounded by a constant* H. *Let* $0 < \theta < \frac{1}{2}$ *and* $0 < \epsilon < 1$ *and suppose that* $\mathcal{H}_0^{(\theta)}(u-h) \le \frac{\epsilon}{2}\left(1 \wedge \frac{1}{2\Lambda}\right)$ *where* Λ *is given in* (1.4).

(a) *Suppose that the function* $\sigma(t,x,u)$ *is independent of* u *but satisfies Assumption 1.1. Then there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *depending on* $\mathscr{C}_1, \mathscr{C}_2, \theta$ *and* H *such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \le P\left(\sup_{0\le t\le T} \mathcal{H}_t^{(\theta)}(u-h) \le \epsilon\right) \le C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}}\right),$$

and

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \le P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u-h) \le \epsilon\right) \le C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}}\right).$$

(b) *Suppose that* $\sigma(t,x,u)$ *is now dependent on* u *and satisfies both Assumptions 1.1 and 1.2. Then for any* $\eta > 0$, *there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *depending* *on $\mathscr{C}_1, \mathscr{C}_2, \theta$ and H such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}+\eta}}\right) \le P\left(\sup_{0\le t\le T} \mathcal{H}_t^{(\theta)}(u-h) \le \epsilon\right) \le C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}} |\log \epsilon|^{\frac{3}{2}}}\right),$$

and

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}+\eta}}\right) \le P\left(\sup_{x\in \mathbf{T}} \mathscr{H}_x^{(\theta)}(u-h) \le \epsilon\right) \le C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}} |\log \epsilon|^{\frac{3}{2}}}\right).$$

(c) *Suppose that $\sigma(t,x,u)$ is again dependent on u and satisfies both Assumptions 1.1 and 1.2. Then there is a $\mathcal{D}_0 > 0$ such that for all $\mathcal{D} < \mathcal{D}_0$, there exists positive constants $C_1, C_2, C_3, C_4 > 0$ depending on $\mathscr{C}_1, \mathscr{C}_2, \theta$ and H such that*

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \le P\left(\sup_{0\le t\le T} \mathcal{H}_t^{(\theta)}(u-h) \le \epsilon\right) \le C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}} |\log \epsilon|^{\frac{3}{2}}}\right),$$

and

$$C_1 \exp\left(-\frac{C_2 T}{\epsilon^{\frac{3}{\theta}}}\right) \le P\left(\sup_{x\in \mathbf{T}} \mathscr{H}_x^{(\theta)}(u-h) \le \epsilon\right) \le C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}} |\log \epsilon|^{\frac{3}{2}}}\right).$$

Definition 8.1. *We say that $f : [0,T] \times \mathbf{T} \to \mathbf{R}$ is in $C^{\gamma,\beta}$ if we have*

$$\|f\|_{C^{\gamma,\beta}} := |f(0,0)| + \sup_{\substack{0\le s,t\le T \\ x,y\in\mathbf{T} \\ (t,x)\ne(s,y)}} \frac{|f(t,x)-f(s,y)|}{|t-s|^\gamma + |x-y|^\beta} < \infty.$$

In other words, $C^{\gamma,\beta}$ is the set of functions that are uniformly bounded, Hölder continuous with the exponent γ in time and the exponent β in space.

Let $\psi : \mathbf{R} \to \mathbf{R}$ be a non-negative, symmetric and smooth function such that the support of ψ is in $[-1,1]$ and $\int_{\mathbf{R}} \psi(x)\,dx = 1$. For any positive integer n and $f \in C^{\gamma,\beta}$, we set $\psi_n(x) := n\psi(nx)$ and define

$$f_n(t,x) = \iint_{\mathbf{R}^2} \tilde{f}(s,y)\psi_n(x-y)\psi_n(t-s)dyds, \tag{8.2}$$

where $\tilde{f}$ is the periodization of f in the spatial variable x and we also define $\tilde{f}(s,x) = f(0,x)$ for $s<0$ and $x \in \mathbf{R}$ and $\tilde{f}(s,x) = f(T,x)$ for $s > T$ and $x \in \mathbf{R}$. We have the following bounds on the derivatives of the above function. The proof is straightforward and is therefore omitted.

Lemma 8.1. *Suppose $f \in C^{\gamma,\beta}$. Then there exists a constant $C > 0$ such that for all $x \in \mathbf{T}$ and $t \in [0,T]$*

$$\left|\frac{\partial f_n(t,x)}{\partial x}\right| \le Cn, \qquad \left|\frac{\partial f_n(t,x)}{\partial t}\right| \le Cn, \qquad \left|\frac{\partial^2 f_n(t,x)}{\partial x^2}\right| \le Cn^2.$$

The following lemma shows we can approximate $f \in C^{\gamma,\beta}$ by smooth mollifications of f.

Lemma 8.2. *Let $f:[0,T]\times\mathbf{T}\to\mathbf{R}$ be in $C^{\gamma,\beta}$ for some $\gamma,\beta\in(0,1]$. Consider the sequence of smooth functions $\{f_n\}_{n=1}^{\infty}$ defined by (8.2). Let $\beta_1\in(0,\beta)$ and $\gamma_1\in(0,\gamma)$. Then, for any fixed $\epsilon>0$, there exist constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that we have*

$$\sup_{0\le t\le T}\mathcal{H}_t^{\left(\frac{1}{2}-\beta_1\right)}(f_n-f)\le\epsilon\quad as\quad n\ge C_1(\epsilon), \tag{8.3}$$

and

$$\sup_{n\in\mathbf{T}}\mathscr{H}_x^{\left(\frac{1}{2}-2\gamma_1\right)}(f_n-f)\le\epsilon\quad as\quad n\ge C_2(\epsilon). \tag{8.4}$$

Proof. We start by making the following observation. Since for each $t\ge0$, $f(t,\cdot)$ is Hölder(β) continuous, for $\beta_1<\beta$ we have

$$\frac{|f(t,x)-f(t,y)|}{|x-y|^{\beta_1}}\le C_1|x-y|^{\beta-\beta_1},$$

where C_1 is a positive constant that is independent of t. We also have

$$\begin{aligned}\frac{|f_n(t,x)-f_n(t,y)|}{|x-y|^{\beta_1}}&=\frac{\left|\iint[f(s,x-z)-f(s,y-z)]\psi_n(z)\psi_n(t-s)dzds\right|}{|x-y|^{\beta_1}}\\&\le C_1|x-y|^{\beta-\beta_1}.\end{aligned}$$

We therefore obtain

$$\sup_{0\le t\le T}\left|\frac{(f_n(t,x)-f(t,x))-(f_n(t,y)-f(t,y))}{|x-y|^{\beta_1}}\right|\le\epsilon$$

whenever we choose $|x-y|\le\left(\frac{\epsilon}{2C_1}\right)^{\frac{1}{\beta-\beta_1}}$. We now consider $f(t,x)-f_n(t,x)$:

$$\begin{aligned}f(t,x)-f_n(t,x)&=\iint[f(t,x)-f(s,y)]\psi_n(t-s)\psi_n(x-y)\,dyds\\&=\iint\left[f(t,x)-f(t-s,x-y)\right]\psi_n(s)\psi_n(y)\,dyds\\&=\iint\left[\frac{f(t,x)-f(t-s,x)}{|s|^{\gamma}}\right]|s|^{\gamma}\psi_n(s)\psi_n(y)\,dyds\\&\quad+\iint\left[\frac{f(t-s,x)-f(t-s,x-y)}{|y|^{\beta}}\right]|y|^{\beta}\psi_n(s)\psi_n(y)\,dyds.\end{aligned}$$

Since $f\in C^{\gamma,\beta}$, $\psi_n(x)=0$ if $|x|>1/n$ and $\int\psi_n(x)\,dx=1$, there exists some constant $C_2>0$ such that

$$\sup_{0\le t\le T}\sup_{x\in\mathbf{T}}|f(t,x)-f_n(t,x)|\le C_2\left(n^{-\gamma}+n^{-\beta}\right). \tag{8.5}$$

Hence, for all $t \in [0,T]$ and $x, y \in \mathbf{T}$ satisfying $|x-y| \geq \left(\frac{\epsilon}{2C_1}\right)^{\frac{1}{\beta-\beta_1}}$, there exists some constant $C_3 > 0$ which only depends on β, β_1 such that

$$\begin{aligned}\left|\frac{(f_n(t,x)-f(t,x))-(f_n(t,y)-f(t,y))}{|x-y|^{\beta_1}}\right| &\leq 2\left(\frac{2C_1}{\epsilon}\right)^{\beta_1/(\beta-\beta_1)} \sup_{x\in\mathbf{T}} |f(t,x)-f_n(t,x)| \\ &\leq C_3\left(n^{-\gamma}+n^{-\beta}\right)\epsilon^{\beta_1/(\beta_1-\beta)}.\end{aligned}$$

We have therefore proved (8.3) for all large enough $n \geq C_1(\epsilon)$ where

$$C_1(\epsilon) := \max\left\{\left(2C_3\epsilon^{\beta/(\beta_1-\beta)}\right)^{1/\gamma}, \left(2C_3\epsilon^{\beta/(\beta_1-\beta)}\right)^{1/\beta}\right\}. \tag{8.6}$$

For (8.4), we follow the same proof above but switch β by γ to get (8.4). Here, we need $n \geq C_2(\epsilon)$ where

$$C_2(\epsilon) := \max\left\{\left(2C_3\epsilon^{\gamma/(\gamma_1-\gamma)}\right)^{1/\gamma}, \left(2C_3\epsilon^{\gamma/(\gamma_1-\gamma)}\right)^{1/\beta}\right\}. \tag{8.7}$$

This completes the proof of the lemma. □

Remark 8.1. *It is easy to see from* (8.3), (8.4) *and* (8.5) *that every $f \in C^{\gamma,\beta}$ can be approximated by its smooth mollification f_n in $\|\cdot\|_{C^{\gamma_1,\beta_1}}$ for all $\gamma_1 < \gamma$ and $\beta_1 < \beta$. That is, for any $\epsilon > 0$ and for every $f \in C^{\gamma,\beta}$, there exists a constant $C(\epsilon) > 0$ such that*

$$\|f_n - f\|_{C^{\gamma_1,\beta_1}} \leq \epsilon \qquad \text{for } n \geq C(\epsilon). \tag{8.8}$$

We can now drop the assumption that h is smooth in Proposition 8.1. We obtain bounds on the small ball probabilities when $h \in C^{\gamma,\beta}$. The upper bounds remain the same as before, but the lower bounds now depend on β and γ. We now have to treat the spatial and temporal regularities of $u-h$ differently. We first consider the spatial difference of $u-h$, i.e., $\mathcal{H}_t^{(\theta)}(u-h)$.

Theorem 8.1. *Consider the solution to* (1.1). *Let $0 < \theta < \frac{1}{2}$ and $0 < \epsilon < 1$. Suppose $h : [0,T] \times \mathbf{T} \to \mathbf{R}$ is in $C^{\gamma,\beta}$ with $\frac{1}{2} - \theta < \beta \leq 1$ and $\gamma \in (0,1]$. We also assume $\mathcal{H}_0^{(\theta)}(u-h) \leq \frac{\epsilon}{4}$. Then we have the following:*

(a) *Suppose that the function $\sigma(t,x,u)$ is independent of u but satisfies Assumption 1.1. Then there exist positive constants $C_1, C_2, C_3, C_4 > 0$ depending on $\mathscr{C}_1, \mathscr{C}_2, \theta, \beta$ and γ such that*

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}}} + \frac{1}{\epsilon^{\frac{4(\beta\vee\gamma)}{\gamma(\beta+\theta-1/2)}}}\right]\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}}\right).$$

(b) *Suppose that $\sigma(t,x,u)$ is now dependent on u and satisfies both Assumptions 1.1 and 1.2. Then for any $\eta > 0$, there exist positive constants $C_1, C_2, C_3, C_4 > 0$ depending on $\mathscr{C}_1, \mathscr{C}_2, \theta, \beta$ and γ such that*

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}+\eta}} + \frac{1}{\epsilon^{\frac{4(\beta\vee\gamma)}{\gamma(\beta+\theta-1/2)}}}\right]\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right).$$

(c) *Suppose that $\sigma(t,x,u)$ is again dependent on u and satisfies both Assumptions 1.1 and 1.2. Then there is a $\mathcal{D}_0 > 0$ such that for all $\mathcal{D} < \mathcal{D}_0$, there exists positive constants $C_1, C_2, C_3, C_4 > 0$ depending on $\mathscr{C}_1, \mathscr{C}_2, \theta, \beta$ and γ such that*

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}}} + \frac{1}{\epsilon^{\frac{4(\beta\vee\gamma)}{\gamma(\beta+\theta-1/2)}}}\right]\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h) \leq \epsilon\right) \leq C_3 \exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right).$$

Proof. The proof is similar to the proof of Theorem 1.2 of [AJM], but here we use the approximation procedure presented above. That is, we set $\beta_1 := \frac{1}{2} - \theta$ in Lemma 8.2 and $n = C_1(\epsilon)$ where $C_1(\epsilon)$ is given in (8.6), and define a smooth function h_n by (8.2). Then, we have

$$\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(h_n - h) \leq \frac{\epsilon}{2}. \tag{8.9}$$

Since $\mathcal{H}_t^{(\theta)}(u-h) \leq \mathcal{H}_t^{(\theta)}(u-h_n) + \mathcal{H}_t^{(\theta)}(h-h_n)$, (8.9) implies that

$$P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h_n) \leq \frac{\epsilon}{2}\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h) \leq \epsilon\right).$$

By Lemma 8.1, there exists a constant C so that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{T}} \left|\left(\partial_t - \partial_x^2\right) h_n(t,x)\right| \leq Cn^2. \tag{8.10}$$

Here, $\left(\partial_t - \frac{1}{2}\partial_x^2\right) h_n(t,x)$ is the drift term when we consider the differential form of $u(t,x) - h_n(t,x)$. That is, if we let $\tilde{u}_n := u - h_n$, then $\tilde{u}$ satisfies

$$\begin{aligned}
\partial_t \tilde{u}_n &= \partial_t u - \partial_t h_n \\
&= \frac{1}{2}\partial_x^2 u + \sigma(t,x,u)\dot{W} - \partial_t h_n \\
&= \frac{1}{2}\partial_x^2 \tilde{u}_n - \left(\partial_t h_n - \frac{1}{2}\partial_x^2 h_n\right) + \tilde{\sigma}_n(t,x,\tilde{u}_n)\dot{W},
\end{aligned}$$

where $\tilde{\sigma}_n(t,x,z) := \sigma(t,x,z+h_n(t,x))$. A close inspection of the proof of Theorem 1.2 of [AJM] shows that in the case of (a)

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}}} + n^4\right]\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h) \leq \epsilon\right).$$

Here, n^4 comes from (8.10). Recalling the choice of n finishes the proof of the lower bound in part (a). The arguments for the lower bounds in (b) and (c) are similar.

Let us now consider the upper bounds. First, we prove the upper bound in part (a). Here we also use the approximation procedure. That is, we choose and fix n large enough to get (8.9). Then, by triangle inequality, we have

$$P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h) \leq \frac{\epsilon}{2}\right) \leq P\left(\sup_{0\leq t\leq T} \mathcal{H}_t^{(\theta)}(u-h_n) \leq \epsilon\right).$$

Let $v_n(t,x) := u(t,x) - h_n(t,x)$. Then, v satisfies

$$\partial_t v_n(t,x) = \frac{1}{2}\partial_x^2 v_n(t,x) - g_n(t,x) + \sigma\big(t,x\big)\cdot \dot{W}(t,x), \quad t\in\mathbf{R}_+,\ x\in\mathbf{T}, \tag{8.11}$$

where $g_n(t,x) = \left(\partial_t - \partial_x^2\right) h_n(t,x)$. To get the upper bound in part (a), we just follow the proof of the upper bound of Theorem 1.2 (a). Note that the upper bound of Theorem 1.2 (a) is obtained once (3.10) is proved. The only difference from (8.11) to (1.1) is that we have the additional drift term $g_n(t,x)$ in (8.11). However, the drift term does not have any effect in obtaining the upper bound. More precisely, similar to (3.10), we need to show that there exists a constant $C > 0$ such that

$$\operatorname{Var}\left(\bar{\Delta}_j \,\middle|\, \mathcal{G}_{j-1}\right) \geq C\epsilon^{1/\theta}, \tag{8.12}$$

where

$$\begin{aligned}\bar{\Delta}_j := \tilde{\Delta}_j &+ \Big[\left(G_{t_1} * u_0\right)(x+\delta) - \left(G_{t_1} * u_0\right)(x)\Big] \\ &+ \left[\int_0^{t_1} \int_{\mathbf{T}} \left(G_{t_1-s}(y - x_j - \delta) - G_{t_1-s}(y - x_j - \delta)\right) g_n(s,y)\, dy\, ds\right],\end{aligned}$$

and $\tilde{\Delta}_j$ is given in (3.6). Here, since g_n is deterministic, we have

$$\operatorname{Var}\left(\bar{\Delta}_j \,\middle|\, \mathcal{G}_{j-1}\right) = \operatorname{Var}\left(\tilde{\Delta}_j \,\middle|\, \mathcal{G}_{j-1}\right).$$

Thus, (3.10) implies (8.12), which leads to the upper bound in part (a), that is the same upper bound in Theorem 1.2.

For the upper bound in part (b), we follow the proof of the upper bound in Theorem 1.3 (a). That is, we add $-h(t,x)$ to the mild form of $u(t,x)$ and also add $-h(t,x)$ to $V^{(\beta)}$ and $V^{(\beta,l)}$ in (3.17) and (3.18), then follow the proof of the upper bound in Theorem 1.3 (a). It is easy to see that Propositions 3.1 and 3.2, and Lemma 3.3 still hold. In addition, as in (3.24), we can regard $(u(t_1,x_{2j+1}) - u(t_1,x_{2j})) - (h(t_1,x_{2j+1}) - h(t_1,x_{2j}))$ as

$$M_0 - (h(t_1,x_{2j+1}) - h(t_1,x_{2j})) + B_{\langle M\rangle_{t_1}}.$$

Following the same proof of (3.24), we obtain

$$P\left(\left|(u(t_1,x_{2j+1}) - u(t_1,x_{2j})) - (h(t_1,x_{2j+1}) - h(t_1,x_{2j}))\right| \leq 5\epsilon^{1/2\theta}\right) \leq \gamma, \tag{8.13}$$

where γ is given in (3.24).

The proof of the upper bound in part (c) is exactly the same as the one for the upper bound in part (b). □

Similar to Theorem 8.1, we now provide small ball probabilities of the temporal Hölder semi-norms of $u-h$. We skip the proof since one can basically follow the proof of Theorem 8.1.

Theorem 8.2. *Consider the solution to* (1.1). *Let* $0 < \theta < \frac{1}{2}$ *and* $0 < \epsilon < 1$. *Suppose* $h : [0,T] \times \mathbf{T} \to \mathbf{R}$ *is in* $C^{\gamma,\beta}$ *with* $\frac{1}{4} - \frac{\theta}{2} < \gamma \leq 1$ *and* $\beta \in (0,1]$. *We also assume* $\mathscr{H}_0^{(\theta)}(u-h) \leq \frac{\epsilon}{4\Lambda}$. *Then we have the following:*

(a) Suppose that the function $\sigma(t,x,u)$ *is independent of* u *but satisfies Assumption 1.1. Then there exist positive constants* $C_1, C_2, C_3, C_4 > 0$ *depending on* $\mathscr{C}_1, \mathscr{C}_2, \theta, \beta$ *and*

γ such that

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}}}+\frac{1}{\epsilon^{\frac{4(\beta\vee\gamma)}{\beta\left(\gamma+\frac{\theta}{2}-\frac{1}{4}\right)}}}\right]\right) \le P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u-h)\le\epsilon\right) \le C_3\exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}}\right).$$

(b) *Suppose that $\sigma(t,x,u)$ is now dependent on u and satisfies both Assumptions 1.1 and 1.2. Then for any $\eta>0$, there exist positive constants $C_1,C_2,C_3,C_4>0$ depending on $\mathscr{C}_1,\mathscr{C}_2,\theta,\beta$ and γ such that*

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}+\eta}}+\frac{1}{\epsilon^{\frac{4(\beta\vee\gamma)}{\beta\left(\gamma+\frac{\theta}{2}-\frac{1}{4}\right)}}}\right]\right) \le P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u-h)\le\epsilon\right) \le C_3\exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right).$$

(c) *Suppose that $\sigma(t,x,u)$ is again dependent on u and satisfies both Assumptions 1.1 and 1.2. Then there is a $\mathcal{D}_0>0$ such that for all $\mathcal{D}<\mathcal{D}_0$, there exists positive constants $C_1,C_2,C_3,C_4>0$ depending on $\mathscr{C}_1,\mathscr{C}_2,\theta,\beta$ and γ such that*

$$C_1 \exp\left(-C_2 T\left[\frac{1}{\epsilon^{\frac{3}{\theta}}}+\frac{1}{\epsilon^{\frac{4(\beta\vee\gamma)}{\beta\left(\gamma+\frac{\theta}{2}-\frac{1}{4}\right)}}}\right]\right) \le P\left(\sup_{x\in\mathbf{T}} \mathscr{H}_x^{(\theta)}(u-h)\le\epsilon\right) \le C_3\exp\left(-\frac{C_4 T}{\epsilon^{\frac{3}{\theta}}|\log\epsilon|^{\frac{3}{2}}}\right).$$

We end with a remark.

Remark 8.2. *Support theorems involving the Hölder semi-norm used in Theorem 1.1 can be obtained by a combination of Theorems 8.1 and 8.2. We leave these to the reader.*

Acknowledgement: This work was initiated when M.F. and M.J. visited POSTECH in September 2019. They would like to thank the mathematics department for the hospitality during their visit. M.J. was partially supported by SERB MATRICS grant MTR/2020/000453 and a CPDA grant from I.S.I., and K.K. is partially supported by the NRF (National Research Foundation of Korea) grants 2019R1A5A1028324 and 2020R1A2C4002077.

References

[AJM] Siva Athreya, Mathew Joseph, and Carl Mueller, *Small ball probabilities and a support theorem for the stochastic heat equation* `https://arxiv.org/pdf/2006.07978.pdf`, To appear in Annals of Probability.

[BMSS95] Vlad Bally, Annie Millet, and Marta Sanz-Solé, *Approximation and support theorem in Hölder norm for parabolic stochastic partial differential equations*, Ann. Probab. **23** (1995), no. 1, 178–222. MR 1330767

[CJK13] Daniel Conus, Mathew Joseph, and Davar Khoshnevisan, *On the chaotic character of the stochastic heat equation, before the onset of intermittency*, The Annals of Probability **41** (2013), no. 3B, 2225–2260.

[DKM+09] Robert Dalang, Davar Khoshnevisan, Carl Mueller, David Nualart, and Yimin Xiao, *A minicourse on stochastic partial differential equations*, Lecture Notes in Mathematics, vol. 1962, Springer-Verlag, Berlin, 2009, Held at the University of Utah, Salt Lake City, UT, May 8–19, 2006, Edited by Khoshnevisan and Firas Rassoul-Agha. MR 1500166 (2009k:60009)

[HSWX20] Randall Herrell, Renming Song, Dongsheng Wu, and Yimin Xiao, *Sharp space-time regularity of the solution to stochastic heat equation driven by fractional-colored noise*, Stoch. Anal. Appl. **38** (2020), no. 4, 747–768. MR 4112745

[KL93] J. Kuelbs and W. V. Li, *Small ball estimates for brownian motion and the brownian sheet*, J. Theoret. Probab. **6** (1993), 547–597.

[KLS95] J. Kuelbs, W. V. Li, and Qi Man Shao, *Small ball probabilities for Gaussian processes with stationary increments under Hölder norms*, J. Theoret. Probab. **8** (1995), no. 2, 361–386. MR 1325856

[LaM17] RafałLatał a and Dariusz Matlak, *Royen's proof of the Gaussian correlation inequality*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 2169, Springer, Cham, 2017, pp. 265–275. MR 3645127

[Lot17] Sergey Lototsky, *Small ball probabilities for the infinite-dimensional ornstein-uhlenbeck process in sobolev spaces.*, Stoch. Partial Differ. Equ. Anal. Comput **5** (2017), no. 2.

[LS01] W. V. Li and Q.-M. Shao, *Gaussian processes: inequalities, small ball probabilities and applications*, Stochastic processes: theory and methods, vol. 19, North-Holland, Amsterdam, 2001, pp. 533–597.

[Mar04] A. Martin, *Small ball assymptotics for the stochastic wave equation*, J. Theoret. Probab. **17** (2004), no. 3, 693–703.

[RB00] A. Ramachandra Rao and P. Bhimasankaram, *Linear algebra*, second ed., Texts and Readings in Mathematics, vol. 19, Hindustan Book Agency, New Delhi, 2000. MR 1781860

[Roy14] Thomas Royen, *A simple proof of the Gaussian correlation conjecture extended to some multivariate gamma distributions*, Far East J. Theor. Stat. **48** (2014), no. 2, 139–145. MR 3289621

[TX17] Ciprian Tudor and Yimin Xiao, *Sample paths of the solution to the fractional-colored stochastic heat equation*, Stochastic and Dynamics **17** (2017), no. 1.

[Wal86] John B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV—1984, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986, pp. 265–439. MR 876085 (88a:60114)

Mohammud Foondun, Department of Mathematics and Statistics, 26 Richmond St, Glasgow G1 1XH, United Kingdom

Email address: mohammud.foondun@strath.ac.uk

Mathew Joseph, Statmath Unit, Indian Statistical Institute, 8th Mile Mysore Road, Bangalore 560059, India

Email address: m.joseph@isibang.ac.in

Kunwoo Kim, Pohang University of Science and Technology (POSTECH), Pohang, Gyeongbuk, South Korea 37673

Email address: kunwoo@postech.ac.kr