Hedge inflation risk of specific purpose guarantee funds[1]

Ze Chen[a], Bingzheng Chen[b], Yi Hu[c2] and Hai Zhang [d]

[a] *School of Finance, Renmin University, Beijing, China;*
[b] *School of Economics and Management, Tsinghua University, Beijing, China;*
[c] *School of Finance, Nanjing Audit University, Nanjing, P.R. China;*
[d] *Department of Accounting & Finance, Strathclyde Business School, UK*

Abstract

Specific purpose guarantee funds (SPGFs) such as pension guarantee funds are popular among investors with both specific investment purpose and guaranteed return requirement, but receive little academic attention. We propose a practical purpose-oriented constant proportion portfolio insurance (PO-CPPI) strategy that optimally allocates its assets into a risk-free fund (floor) and a purpose-related portfolio (cushion) to maximize prospect theory investors' utility with consideration of their purpose-related inflation risk. Our closed-form solution of optimal PO-CPPI allocation derived in the continuous time case and Monte-Carlo simulations in the discrete-time and dynamic cases prove the superiority of PO-CPPI over general portfolio insurance strategies.

Keywords: CPPI; Portfolio insurance; Prospect theory; Specific purpose guarantee funds.

JEL Classification G11 G22

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[2]**Correspondance** Yi Hu, School of Finance, Nanjing Audit University, No. 86 West Yushan Road, Jiangpu Street, Pukou District, Nanjing, Jiangsu Province, 211800, P.R. China. Email: huyinau@nau.edu.cn.

1. Introduction

Specific purpose guarantee funds (SPGFs) are popular in practice among investors who have clear–cut usage of the investment outcomes, such as covering post-retirement living costs or hedging children's future education costs. For instance, pension guarantee funds like life-long withdraw variable annuity are popular among investors aiming to prepare for post-retirement living costs.[3] Another example is education-related guarantee funds that enable parents to contribute a monthly or lump-sum investment today to cover, say, their children's future education costs.[4] However, the investment strategies, hedging techniques, and performance of SPGFs are relatively ignored by academic researches.

The specific investment purpose, associated with which the inflation risk affects investors' perception of investment outcome to a large extent, distinguishes SPGFs from ordinary guarantee funds. This is because SPGF investors expect not only a minimum guaranteed return but also a stable purchasing power when these contracts expire. Hence, the rate of inflation related to the investors' specific purposes of using these investments, termed purpose-related inflation hereafter, plays an essential role when assessing the fund performance. An unfavourable case for education-related guarantee fund investors is that the fund has a moderate performance, while education-related costs surge. By contrast, the scenario in which the fund performs poorly but education-related inflation is moderate is, however, much more acceptable. Therefore, the purpose-related inflation risk is a major risk for SPGFs investors, and thus a challenge for SPGFs providers.

Neither the specific investment purpose nor the purpose-related inflation risk has been carefully addressed by most popular portfolio insurance strategies. Thus, it is necessary for SPGFs to design and employ proper portfolio insurance strategies that hedge the purpose-related inflation, an essential factor to be considered by SPGF investors when evaluating the fund performance. For this reason, we attempt to modify the popular constant proportion portfolio insurance (CPPI) strategy, that was first proposed by Black and Jones (1987) and Black and Perold (1992). We propose an adjusted CPPI strategy, termed the purpose-oriented CPPI (PO-CPPI) strategy, which takes into account investors' specific purpose. Differing from the standard CPPI's allocation into a risk-free asset (known as "floor") and a diversified market portfolio (known as "cushion"), our PO-CPPI portfolio invests its cushion into a purpose-related risky portfolio that better hedges investors' inflation risk.

Hedge specific inflation risk by a purpose-related risky portfolio is justified because of the high correlation between purpose-related inflation risk and the performance of certain industry stocks found in literature. Several empirical studies examine the relationship between price inflation and stock returns, finding a robust relation between certain types of

[3] Detailed introduction and example illustration of pension guarantee fund are to appear later in Section 2.

[4] There are many popular education-related guarantee funds in the market. For more detailed introduction, we refer to products like the Kiss Kids Education Plan of AIA (https://www.aia.com.hk/en/our-products/savings/kiss-kids-education-plan.html) and Children's Investment Plan of ICICI Prudential Life Insurance (https://www.iciciprulife.com/guaranteed-child-saving-plans/future-perfect.html).

inflation and their related industry-level investment, including financial sector stocks (Boyd et al., 2001), oil and gas industry stocks (Sadorsky, 2001; Apergis and Miller, 2009; Kang et al., 2015) and real estate stocks, inflation-linked liabilities and nominal bonds (Martellini et al., 2014), and real estate investment trusts (Rubens et al., 1989; Hoesli and Oikarinen, 2012; Bahram et al., 2004). Thus, it is straightforward to hedge inflation risk and maintain stable purchasing power for SPGF investors by investing in purpose-related assets.

Table 1: Floor and cushion of CPPI and PO-CPPI portfolios
This table reports the composition and objective of CPPI and PO-CPPI portfolios. The floor composition of both strategies is a risk-free fund. The cushion of CPPI is a diversified fund while that of PO-CPPI is a purpose-oriented one.

Composition	**Objective**	**CPPI**	**PO-CPPI**
Floor (risk-free)	guaranteed return	a risk-free fund	a risk-free fund
Cushion (risky)	excess return	a diversified fund	a purpose-oriented diversified fund

The main difference between the standard CPPI and our proposed PO-CPPI strategy lies in the underlying cushion portfolio of risky assets (see Table 1). Both CPPI and PO-CPPI allocate the portfolio into a risk-free floor to realize a guaranteed return and into a risky cushion to realize potential upside gains. The cushion of CPPI is typically a diversified portfolio of all stocks in the market, equivalent to the stock market index. By contrast, the cushion of PO-CPPI is a purpose-oriented and diversified fund. This purpose-oriented fund better hedges investors' purpose-related inflation risk by assigning more weights in stocks belonging to the purpose-related industry.[5]

Our study echoes with researches on improving the performance of CPPI and other portfolio insurance strategies from different aspects. Boulier and Kanniganti (2005) propose and evaluate some modifications of standard CPPI. Lee et al. (2008) adjust CPPI parameter based on the "momentum" of market performance and find such their variable proportion portfolio insurance strategy outperforms standard CPPI. Chen et al. (2008) propose a dynamic proportion portfolio insurance strategy by identifying the risk variables related to market conditions and use this to build an equation tree for the risk multiplier using genetic programming. Similarly, other works like Ameur and Prigent (2007), Balder et al. (2009), Hamidi et al. (2014), and Jiang et al. (2009) have modelled the multipliers of traditional portfolio insurance strategies as time-varying ones[6]. Happersberger et al. (2020) provide a comprehensive review and also propose a novel combination approach to estimate tail risk exposure of dynamic portfolio insurance strategies.

Moreover, we adopt the prospect theory utility framework in this study as extensive evidence shows that guarantee fund investors tend to be prospect theory investors. Dichtl

[5]We further explain how to construct the purpose-oriented fund later in Section 3 and Section 4.
[6]See Section 6 for more detalied review of related literature.

and Drobetz (2011) find that the popularity of portfolio insurance strategies can only be explained in a behavioural finance context, in which the investors are described by the prospect theory. Zakamouline (2014) also shows that the loss aversion of investors plays a crucial role in measuring portfolio performance. On the basis of these previous findings, we consider the PO-CPPI portfolio optimization under the prospect theory instead of the classical utility framework. To find the optimal allocation rule of the PO-CPPI strategy, we first derive the explicit final payoff distribution and then solve optimal formula in the continuous time case.

The main contribution of this work is proposing an innovative PO-CPPI for prospect theory SPGF investors and proving its superiority over general portfolio insurance strategies via numerical simulations. Having derived the optimal PO-CPPI allocation, we further conduct extensive Monte Carlo simulations which demonstrate the superiority of the PO-CPPI strategy over other benchmark strategies with considerations of the gap risk under the discrete time cases. We find that PO-CPPI strategy outperforms CPPI and other portfolio insurance strategies in various aspects, e.g. protecting against downside risk and achieving higher investment return, and that PO-CPPI achieves higher prospect theory utility for investors than other strategies. In addition, we extend the PO-CPPI strategy into a dynamic setting with consideration of dynamic risk modelling and propose a new dynamic strategy, namely, purpose-oriented dynamic proportion portfolio insurance (PO-DPPI). We theoretically present how to calculate the dynamic multiplier and prove the superiority of PO-DPPI over other dynamic strategies.

The remainder of this paper is structured as follows. Section 2 introduces a particular SPGF example, a pension guarantee fund in China, and explains the intuition behind the proposed PO-CPPI strategy. Section 3 sets up the model, while the innovative PO-CPPI strategy with closed-form solutions for its optimal allocation rules and leverage is presented in Section 4, followed by Monte Carlo simulations comparing the PO-CPPI strategy with several benchmark strategies in Section 5. We extend our PO-CPPI with a dynamic multiplier and summarize the key results in Section 6. Finally, Section 7 concludes. All proofs are gathered in the Online Appendix.

2. SPGF example illustration: Pension guarantee fund in China

In this section, we introduce a Chinese pension guarantee fund as a particular example of SPGFs. The aim of this section is to provide an intuitive explanation of PO-CPPI's motivation to invest in a purpose-related risky cushion. It is noteworthy that this example alone does not prove the superiority of PO-CPPI, but only presents a phenomenon that inspires this study.

Pension guarantee funds such as variable annuity products are a representative example of SPGF participated by investors to cover post-retirement living costs. Variable annuities are the most popular individuals pension savings products in many developed markets like the US and UK (Crawford et al., 2008; Steinorth and Mitchell, 2015).[7] Indeed, a variable annuity is a fund-linked insurance contract, which provides return guarantees on investor's policy account (Smith, 1982; Walden, 1985). The variable annuity provider, very often an insurance company, works as fund manager to implement dynamic investment strategies to the investor's policy account. According to variable annuity contract, the investor owns the value in the policy account and receives it as post-retirement incomes, which is protected at a guaranteed level. In return, the provider charges a fixed management fee year by year and thus the policy account value must be reported to the investor.

For instance, guaranteed minimum income benefit (GMIB) annuity ensures that investors receive a minimum payment after retirement regardless of the market conditions. The guaranteed amount of GMIB annuity is predetermined at the purchase of annuity product. Some GMIB annuities provide a lump-sum payment when the investors turn to retirement age, while some others provide lifelong pension payments to investors after their retirement (Bacinello et al., 2011; Milevsky and Salisbury, 2006). Overall, GMIB investors not only expect a minimum guaranteed return but also want the investment outcome to have high purchasing power with respect to their retirement purpose at maturity.

Therefore, the providers of pension guarantee funds are well motivated to employ proper strategies for maintaining good investor relations and attracting new investors. In the following, we compare the investment performance of the standard CPPI strategy and proposed PO-CPPI strategy for pension guarantee fund in China. It is noteworthy that our strategy joins the recent discussion on retirement investing strategies, see e.g. to properly secure minimum levels of replacement income in retirement, to offer both security and flexibility of retirement income in decumulation (see e.g. Martellini et al. (2019), Martellini et al. (2020), Mulvey et al. (2019), and many others).

2.1. Retirement-related inflation and retirement industry sector stocks

Using historical data, we first look at the retirement-related inflation and performance of retirement industry sector stocks in China. First of all, we notice that the price inflation for retirees is higher than the normal consumer price index (CPI) in China. The IAMAC-SinoLife Senior Living Cost Index (ISLCI), a price index issued by the Insurance Asset Management Association of China, measures the living costs of retirees in mainland China. Figure 1 compares the ISLCI with the CPI from January 2001 to May 2017. As illustrated, the average annual rate for the ISLCI is 3.86%, much higher than that of CPI, 2.46%. This difference comes from that the retirees' basket of goods is different from that of whole population, e.g. the medical- and health-related expenses are a major living cost for retirees.

Second, we also observe that retirement industry sector stocks outperform the stock market during the time horizon when retirees' inflation is higher than CPI. In this example, the stocks of retirement industry sector are the purpose-related stocks because the most investment outcome is to be spent for retirement. Using historical data from the Shanghai

[7] We refer to Ledlie et al. (2008) for a description of the main characteristics of variable annuity products and the development of their market as well.

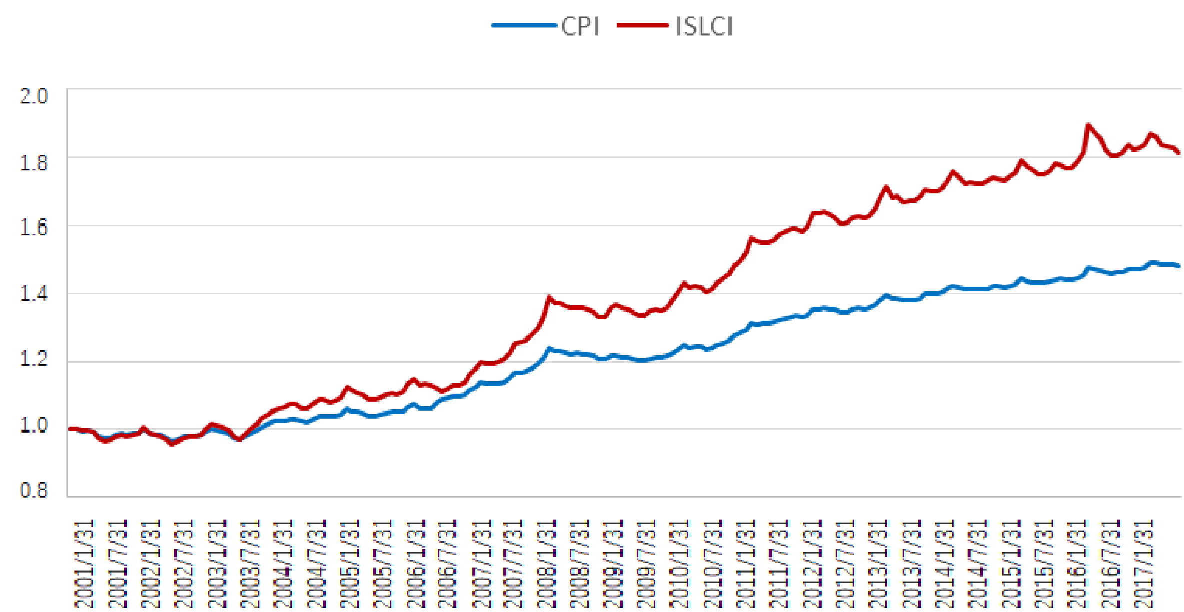

Figure 1: Historical ISLCI and CPI in China
This figure presents the historical ISLCI and CPI in China from January 2001 to May 2017. Source: Insurance Asset Management Association of China.

Stock Exchange, Figure 2 compares the performance of the stock market index with that of the retirement industry sector index from January 2006 to May 2017.[8] Figure 2 shows that the retirement industry sector index achieves a higher return than the stock market index.

2.2. Purpose-related cushion via adding investment weight in retirement industry stocks

As introduced earlier, extensive literature have found that there is a high correlation between purpose-related inflation and the performance of certain industry stocks. The observation in Figure 1 and Figure 2 coincides with this finding. In the particular case of China, a pension guarantee fund would achieve a higher purchasing power if it invests a larger proportion of its funds in the retirement industry sector index. It is natural that the purpose-related stocks provide a good hedge instrument, therefore, the standard CPPI can be improved by investing more in retirement industry sector stocks in its cushion.

Insurance companies that underwrite GMIB annuity liabilities in China commonly adopt CPPI techniques to achieve the guaranteed return. According to the Variable Annuity Fund Management Interim Regulation issued by the former China Insurance Regulation Commission in 2011,[9] insurance companies should adopt either internal option-based hedging

[8]Both indexes are calculated using the daily traded stocks issued by the Shanghai Stock Exchange. We report this period as the Shanghai Stock Exchange started issuing the retirement industry sector index since January 2006.

[9]On 8 April 2018, this commission and the banking regulator, the China Banking Regulatory Commission

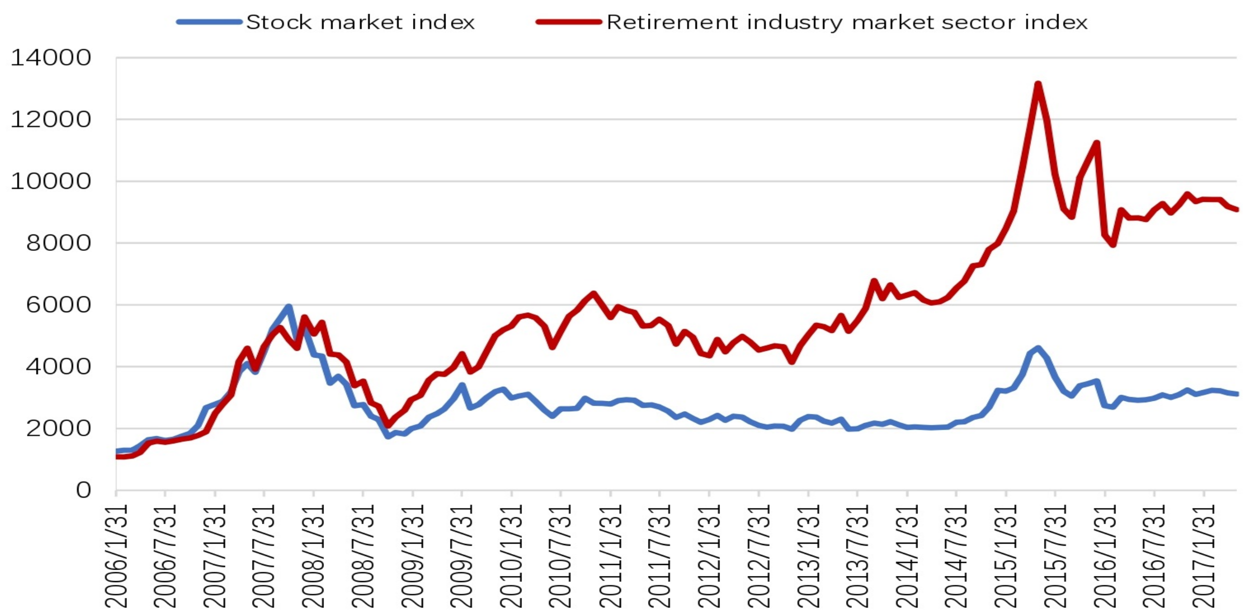

Figure 2: Historical performance of the stock market index and retirement industry sector index in China
This figure presents the historical performance of the stock market index and retirement industry sector index listed on the Shanghai Stock Exchange from January 2006 to May 2017. Source: WIND data.

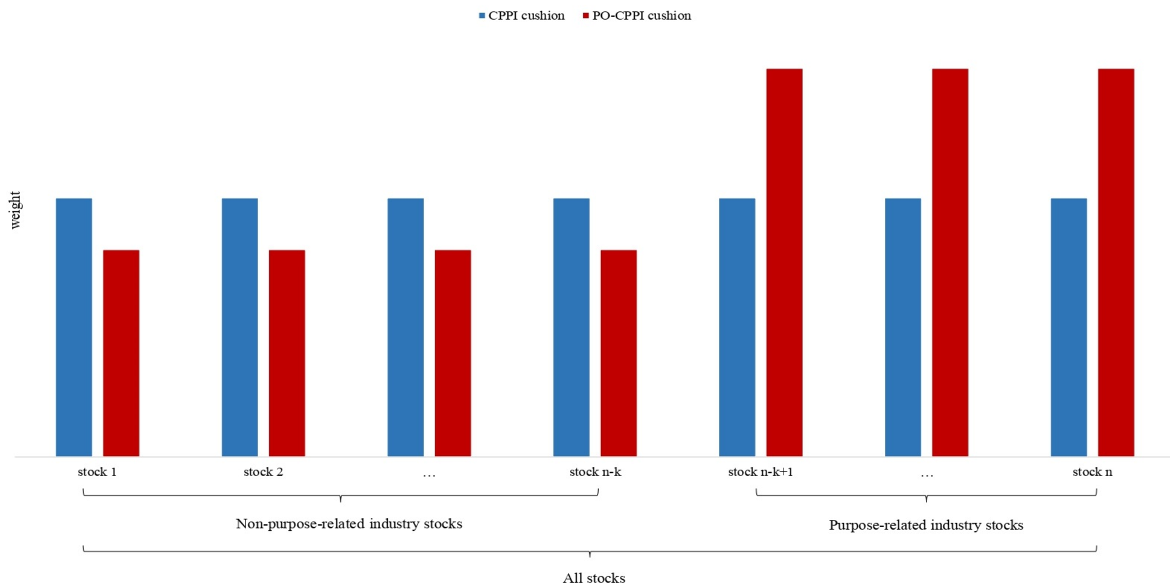

Figure 3: Cushion portfolio of the CPPI and PO-CPPI strategies
This figure presents the cushion portfolio of the CPPI and PO-CPPI strategies.

or CPPI technique to manage their variable annuity funds. In practice, CPPI is more commonly adopted as it does not require a sophisticated derivative market.[10]

Against this background, we propose PO-CPPI strategy that hedges investors' specific purpose by investing in a purpose-related cushion. The PO-CPPI's cushion increases the proportion of purpose-related stocks, and thus crowds out that of the remaining non-purpose-related ones (see Figure 3). Simply put, the PO-CPPI cushion can be seen as a portfolio of both the stock market index (a diversified fund equally investing in all stocks) and the purpose-related sector market index (a diversified fund equally investing in the purpose-related sector stocks).[11]

Figure 4 compares the performance of the CPPI and PO-CPPI from January 2012 to December 2014.[12] In this particular example of pension guarantee fund in China, it is not

officially merged into the China Banking and Insurance Regulatory Commission to resolve problems such as unclear responsibilities and cross-regulation.

[10]In addition to variable annuities, the hedging techniques have been applied to many insurance liabilities, see e.g. Barigou et al. (2019); Barigou and Dhaene (2019); Chen et al. (2020, 2021); Dhaene et al. (2017).

[11]Our PO-CPPI strategy can be linked to the discussions on fund-separations as the SPGF investors differ from the classical mean-variance ones. Since Tobin (1958) and Markowitz (1959) propose the two-fund separation under a mean-variance framework, many works have questioned mutual fund separation and proposed three-fund or even K-fund separation for many different utility forms (Cairns et al., 2006; Dahlquist et al., 2016; Deguest et al., 2018; Dybvig and Liu, 2018; Hakansson, 1969; Merton, 1973; Pye, 1967; Ross, 1978; Samuelson, 1967).

[12]Our historical illustration in the selected sample period does not justify the superiority of the PO-CPPI

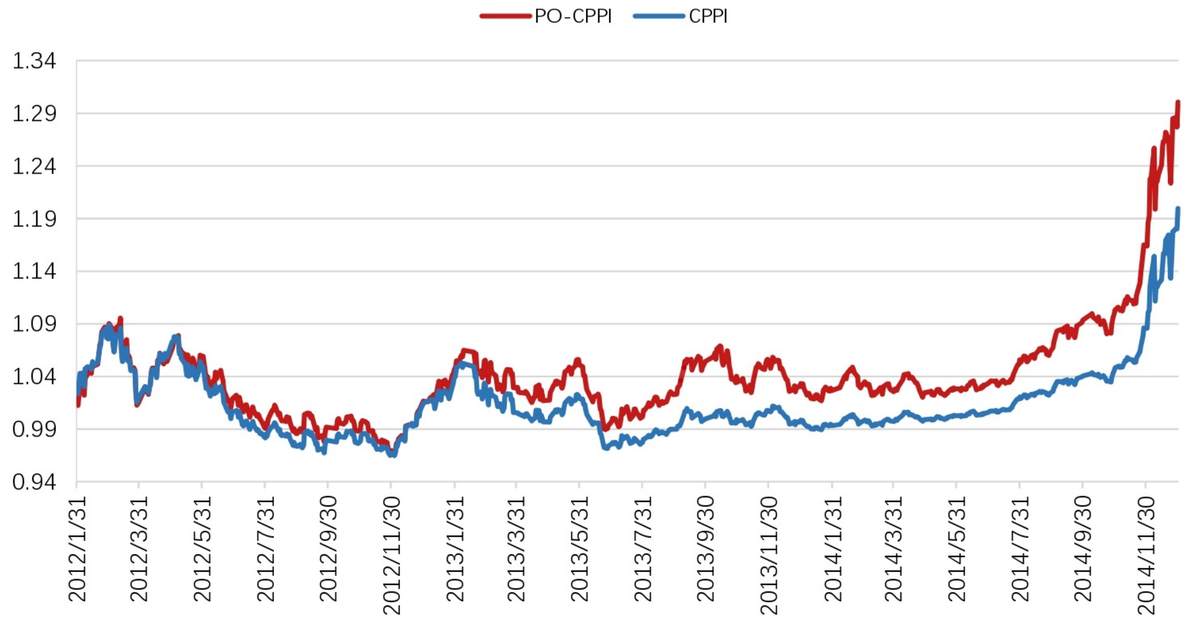

Figure 4: Performance of CPPI and PO-CPPI strategies in Chinese market
This figure presents the performance of the CPPI and PO-CPPI strategies in Chinese market from January 2012 to December 2014.

surprising that the PO-CPPI strategy outperforms the standard CPPI strategy, as the PO-CPPI's cushion portfolio invests a larger proportion in the retirement industry sector.

However, the aforementioned case of China alone does not justify the superiority of the proposed PO-CPPI strategy. Otherwise, the opposite case, namely that the stock market index outperforms the retirement industry sector index, would lead to the opposite result that the standard CPPI outperforms PO-CPPI. To fully justify the superiority of the PO-CPPI portfolio, we consider the following from both the theoretical and realistic cases:

- In Sections 3 and 4, we theoretically introduce the PO-CPPI strategy with its cushion being modeled as a combination of the stock market index and the purpose-related industry sector index. The optimal combination ratio derived shows that the optimal cushion of PO-CPPI invests more in the purpose-related industry stocks.

- In Section 5, we further show that the superiority remains in the practical case via a comprehensive numerical analysis. Having considered the gap risk resulted from portfolio rebalancing under discrete-time case, PO-CPPI is proven to dominate other portfolio insurance strategies for prospect theory investors.

strategy or take into account its tail risk performance. We later provide rigid proof of the superiority of our proposed PO-CPPI in Section 4 and Section 5. The historical backtest simulations via Bootstrap approach in Section 6 show the robustness of the superiority of PO-CPPI strategy in Chinese market without depending on the choice of time window.

- In Section 6, from the perspective of tail risk management, PO-DPPI with a dynamic multiplier based on the tail risk modelling is proposed and we show PO-DPPI outperforms other dynamic strategies.

3. Theoretical framework

In this section, we introduce the theoretical settings of the financial market, the purpose-related inflation risk, and the prospect theory utility.

3.1. Financial market

First, we introduce the financial market, which consists of a risk-free asset and n stocks. We denote the price of the risk-free asset by S_t^f and capture its stochastic differential process as

$$\frac{dS_t^f}{S_t^f} = rdt,$$

where r is the risk-free interest rate. All individual stocks are risky assets. The dynamic processes of the individual stock price $S_t^{(j)}$, numbered with $j = 1, 2, ..., n$, is given by the classical Black–Scholes (BS) processes:

$$\frac{dS_t^{(j)}}{S_t^{(j)}} = \mu_{(j)} dt + \sigma_{(j)}\ dZ_{(j)},\ \text{for } j = 1, 2, ..., n,$$

where $\mu_{(j)}$ and $\sigma_{(j)}$ are the growth and volatility parameters of the BS process, respectively. Here, $Z_{(j)}$ is the Brownian motion (BM) that drives the stock price $S_t^{(j)}$.

For a given SPGF, we assume that there are k of n stocks, numbered from $n - k + 1$ to n, belonging to an industry that is closely related to investors' specific purpose. The number of purpose-related stocks k is known and satisfies $k < n$. The economic meaning is that these k stocks belong to the industry sector that is purpose-related and better hedges investors' inflation risk than others stocks. In the example of pension guarantee funds, the purpose-related market sector includes the retirement-related industry stocks. Specifically, the price processes of these k stocks are more related to purpose-related inflation than the other $n - k$ stocks.

As the standard CPPI's cushion invests in a diversified fund, here we introduce a stock market index, denoted by S_t^I. This stock market index invests in all n stocks in the financial market in a diversified manner. Then, the dynamic price process of S_t^I can be represented as

$$\begin{aligned}\frac{dS_t^I}{S_t^I} &= \frac{1}{n}\sum_{j=1}^{n}\frac{dS_t^{(j)}}{S_t^{(j)}} \\ &= \mu_I dt + \sigma_I^i dZ_i,\end{aligned}$$

where μ_I and $\sigma_I^i dZ_i$ are determined by synthesizing the dynamic processes of n individual stocks. That is, Z_i synthesizes the risk sources of all individual stocks. Similarly, we define the purpose-related market sector index, denoted by S_t^P, which represents a diversified fund of all k purpose-related stocks at time t. That is,

$$\frac{dS_t^P}{S_t^P} = \frac{1}{k}\sum_{j=n-k+1}^{n} \frac{dS_t^{(j)}}{S_t^{(j)}} \tag{1}$$

$$= \mu_P dt + \sigma_P dZ_P, \tag{2}$$

where Z_P is synthesized from the k individual stocks' BMs. By decomposing Z_P into two orthogonal BMs, we can rewrite the process in equation (2) as

$$\frac{dS_t^P}{S_t^P} = \mu_P dt + \sigma_P^i dZ_i + \sigma_P^p dZ_p,$$

where Z_i and Z_p are orthogonal, and $\sigma_P dZ_P = \sigma_P^i dZ_i + \sigma_P^p dZ_p$.

For simplicity, we hereafter refer to the market index fund S_t^I and the specific purpose-related index S_t^P as I-fund and P-fund, respectively. Note that Z_P is correlated with Z_i as we have $dZ_P dZ_i = \frac{\sigma_P^i}{\sqrt{(\sigma_P^i)^2+(\sigma_P^p)^2}} dt$. This correlation results from the common risk sources of the I-fund and the P-fund. From the mathematical perspective, Z_p can be viewed as the risk factor that only affects the price process of S_t^P.

3.2. Purpose-related inflation index

SPGFs investors have a planed specific usage of the investment outcome, such as post-retirement costs. For given SPGFs, we capture the inflation of purpose-related expenses by an index Y_t and refer to it as the *purpose-related inflation index*. As Y_t represents the price level, we normalize $Y_0 = 1$ for simplicity. Generally, it is expected that $Y_T > 1$ at maturity due to inflation.

It is expected that Y_t is jointly driven by stock market risk and other idiosyncratic risk that has not been traded in the stock market. We further assume that the price process of Y_t is given by

$$\frac{dY_t}{Y_t} = \mu_Y dt + \sigma_Y dZ_Y, \tag{3}$$

where Z_Y can be decomposed into the following three orthogonal BMs:

$$\sigma_Y dZ_Y = \sigma_Y^s dZ_i + \sigma_Y^p dZ_p + \sigma_Y^e dZ_e. \tag{4}$$

The decomposition in equation (4) shows that Y_t cannot be perfectly hedged by either the I-fund or P-fund. On the basis of the process in equation (3), we yield the formula of Y_t via stochastic integration:

$$\begin{aligned} Y_t &= Y_0 \exp\{(\mu_Y - \frac{1}{2}(\sigma_Y^s)^2 - \frac{1}{2}(\sigma_Y^p)^2 - \frac{1}{2}(\sigma_Y^e)^2)t + \sigma_Y^s W_s(t) + \sigma_Y^p W_p(t) + \sigma_Y^e W_e(t)\} \\ &= Y_0 \exp\{\tilde{\mu}_Y t + \sigma_Y W_Y(t)\}, \end{aligned}$$

where $Y_0 = 1$, $\tilde{\mu}_Y = \mu_Y - \frac{1}{2}(\sigma_Y^i)^2 - \frac{1}{2}(\sigma_Y^p)^2 - \frac{1}{2}(\sigma_Y^e)^2$ and $\sigma_Y W_Y(t) = \sigma_Y^s W_s(t) + \sigma_Y^p W_p(t) + \sigma_Y^e W_e(t)$. Here, $W_s(t)$, $W_p(t)$ and $W_e(t)$ are the integration of the three BMs, dZ_i, dZ_p and dZ_e, from time 0 to t.

So far, we have introduced the dynamic process of S_t^I, S_t^P, and Y_t which are driven by the three BMs of Z_i, Z_p, and Z_e. Similar to Martellini et al. (2012), we adopt a multi-dimension vector form to represent the prices to make our mathematical settings easier to follow. Specifically, we use the superscripts to denote three orthogonal BMs and the subscripts to represent I-fund, P-fund and inflation index. To write down their differential processes in a unified multi-dimension vector form, we define

$$dN_t = \mu_N dt + \sigma_N^s dZ_i + \sigma_N^p dZ_p + \sigma_N^e dZ_e,$$

where $N_t = S_t^I,\ S_t^P,\ Y_t$, and $N = I,\ P,\ Y$ correspondingly. In addition, we further define the volatility loading vector of the process of N_t as $\overrightarrow{\sigma}_N = (\sigma_N^s,\ \sigma_N^p,\ \sigma_N^e)$. Thus, from the processes in equations (1), (2) and (4), we have $\overrightarrow{\sigma}_i = (\sigma_I^i,\ 0,\ 0)$, $\overrightarrow{\sigma}_p = (\sigma_P^i,\ \sigma_P^p,\ 0)$ and $\overrightarrow{\sigma}_y = (\sigma_Y^s,\ \sigma_Y^p,\ \sigma_y^e)$.

3.3. Prospect theory utility

In this study, we adopt the prospect theory utility. This is mainly due to the solid findings of Dichtl and Drobetz (2011) that guarantee funds investors are more appropriately captured by the prospect theory. They justified the popularity of portfolio insurance strategies in the prospect theory utility context, while expected utility theory struggles to provide an explanation. In contrast to expected utility, prospect theory investors behave differently in evaluating potential gains and losses: (i) they evaluate the investment outcome by its deviation from a reference point; (ii) they value potential gains and losses asymmetrically (i.e. the marginal utility of the potential is higher than that of the gain); and (iii) instead of using the statistical probabilities, they overweight events with low probability of occurrence, but underweight 'average' events.

The form of prospect theory utility has been widely discussed in the literature. It is commonly adopted that prospect theory investors are loss averse and have an S-shaped utility function, which is concave for gains and convex for losses. The investment outcome can deviate either positively or negatively from a reference point. Following Dichtl and Drobetz (2011) and Tversky and Kahneman (1992) among others, the loss averse utility function is thus defined as follows:

$$\nu(\Delta V) = \begin{cases} (\Delta V)^{\gamma} & \text{for } \Delta V \geqslant 0 \\ -\lambda(-\Delta V)^{\gamma} & \text{for } \Delta V < 0 \end{cases}, \tag{5}$$

where ΔV is the deviation from the reference point, $1 > \gamma > 0$ and $\lambda \geqslant 1$. The parameter λ captures loss aversion, indicating that investors consider losses more than twice as important as gains. Tversky and Kahneman (1992) and some following literature suggest that $\gamma \approx 0.88$ and $\lambda \approx 2.25$.[13] Moreover, instead of weighting the values in function (5) with their statistical

[13] We also adopt these common parameters for the simulation analysis in Section 5.

probabilities, Lattimore et al. (1992) suggest that the probability weighting function is given by :

$$w_{\delta,\beta} := \frac{\delta \cdot p^{\beta}}{\delta \cdot p^{\beta} + (1-p)^{\beta}}$$
$$:= \begin{cases} w^{+}(p) = \frac{\delta^{+} \cdot p^{\beta^{+}}}{\delta^{+} \cdot p^{\beta^{+}} + (1-p)^{\beta^{+}}} & \Delta x \geqq 0 \\ w^{-}(p) = \frac{\delta^{-} \cdot p^{\beta^{-}}}{\delta^{-} \cdot p^{\beta^{-}} + (1-p)^{\beta^{-}}} & \Delta x < 0 \end{cases}. \tag{6}$$

The probability weighting function in equation (6) distinguishes between two essential features based on the following parameters: (1) the parameter β mainly controls curvature, and (2) the parameter δ mainly controls elevation. These two parameters incorporate the experimental observation that prospect theory investors tend to overweight small probability events. The empirical results in Abdellaoui (2000) shows $\delta^{+} = 0.65$, $\delta^{-} = 0.84$, $\beta^{+} = 0.6$, and $\beta^{-} = 0.65$. Therefore, we adopt this classical form of prospect theory utility that is found to be in line with theory and experimental evidence (Gurevich et al., 2009; Prelec, 2000).

Throughout this paper, we consider that a popular type of guaranteed investment outcome of SPGFs that equals to the principal investment made by the investors. We denote the reference point at maturity T, or the principal investment, denoted by P_T. In other words, the investment outcome of SPGFs is at least above the predetermined guaranteed level and may have some upside potential gains as well. Now, we formulate the SPGF investor's prospect utility. The SPGF investor's prospect theory utility at maturity T is defined as follows:

$$U(V_T, Y_T) = \begin{cases} (\frac{V_T - P_T}{Y_T})^{\gamma} & \text{for } V_T \geqslant P_T \\ -\lambda \cdot (-\frac{V_T - P_T}{Y_T})^{\gamma} & \text{for } V_T < P_T \end{cases}, \tag{7}$$

where γ is the risk-averse parameter of SPGF investors; and the probability weighting function of the utility form in equation (7) is in line with the equation (6).

We remark on the adopted utility function. On the one hand, we introduce the guaranteed amount as the reference point in the utility function as suggested by Dichtl and Drobetz (2011). They found that the reference point of prospect theory investors is the guaranteed investment. On the other hand, we take the investor's purpose-related inflation risk into account. Due to the specific investment purpose, SPGFs investors' utility is not only determined by the investment outcome, but also deflated by purpose-related expense inflation at maturity, Y_T. The price index Y_T at maturity would dramatically affect investors' real wealth or perception of SPGFs outcomes. The impact of purpose-related inflation on consumption is captured as the denominator in the utility, which is a commonly adopted approach in economic studies.

4. PO-CPPI strategy

In this section, we briefly review the standard CPPI strategy under our theoretical framework in Section 4.1, followed by a detailed construction of the innovative PO-CPPI strategy in Section 4.2. Finally, Section 4.3 discusses the utility maximization problem of prospect theory investors and provides explicit results.

4.1. Standard CPPI strategy

CPPI strategy is widely used in many guarantee funds. Its portfolio maintains exposure to the upside potential while providing a capital guarantee against downside risk. At any time t, the CPPI portfolio value V_t consists of investment in a risk-free fund (floor) and a diversified fund of risky stocks (cushion). Denote P_t and C_t as the floor and cushion invested at time t, respectively. Then, we have

$$V_t = P_t + C_t, \ t \in [0, T].$$

The major duty of CPPI portfolio is to guarantee a fixed payoff P_T at maturity. A typical CPPI floor strategy at time t is a fixed-rate floor, which is given by

$$P_t = e^{-d(T-t)} P_T, \ t \in [0, T],$$

where d is the fixed rate of the floor strategy with $d \leqslant r$ and P_T is the guaranteed amount. Further, $d \leqslant r$ represents the conservative cases that allocate more than minimum ratios to the risk-free asset. The most common floor strategy is to allocate the minimum amount of ratio to the risk-free asset, i.e. $d = r$. In this case, the floor amount at time t is determined $P_t = e^{-r(T-t)} P_T, \ t \in [0, T]$.

The cushion, the difference between the portfolio value and floor, is invested in a diversified fund of all stocks, i.e. the I-fund. CPPI portfolio usually levels its cushion C_t to chase higher returns. Its leverage ratio is denoted by m, which stays constant and is called as "constant proportion". Then, CPPI portfolio's exposure in stock market E_t equals

$$E_t = mC_t = m(V_t - P_t), \ t \in [0, T],$$

where $m \geqslant 1$. The constant proportion m is determined at time 0 and stays constant during the investment horizon. The cushion value C_t fluctuates with the market value of I-fund. Once it approaches zero, the entire CPPI portfolio will be invested only in the risk-free asset until maturity, to maintain the guarantee return.

Therefore, the standard CPPI strategy is a "two-fund separation" investment. At any time t,

- if $V_t > P_t$, the portfolio allocates the amount P_t to the risk-free fund and the amount C_t to the I-fund with leverage m;
- if $V_t \leqslant P_t$, the entire portfolio is invested in the risk-free fund.

It is notable that the CPPI portfolio value V_t never falls below the guaranteed floor if using time-continuous rebalancing.

4.2. Proposed PO-CPPI

To better hedge the purpose-related inflation risk of SPGFs, we propose a PO-CPPI strategy. PO-CPPI is a modified CPPI strategy that dynamically rebalances its portfolio amount between the risk-free asset and a purpose-oriented risky fund. Following the notations of CPPI strategy, we denote P_t and C_t as the floor and cushion, respectively.

Unlike CPPI, the cushion of PO-CPPI portfolio is more purpose-oriented. As introduced earlier, the cushion of PO-CPPI is not a simply diversified fund of all stocks in the market, but a combination of two diversified risky funds (the I-fund and P-fund). Denote α as the proportion of cushion to be invested in the I-fund. Then, the remaining $1-\alpha$ part is assigned to the P-fund. The evolution of SPGF portfolio value at time t, denoted by V_t, is given by

$$\begin{aligned} dV_t &= E_t[\alpha \frac{dS_t^I}{S_t^I} + (1-\alpha)\frac{dS_t^P}{S_t^P}] + P_t \frac{dS_f}{S_f} - (m-1)C_t \frac{dS_f}{S_f} \\ &= E_t[\alpha \frac{dS_t^I}{S_t^I} + (1-\alpha)\frac{dS_t^P}{S_t^P}] + V_t r dt - E_t r dt, \end{aligned}$$

where $E_t = mC_t$ is exposure to the risky asset and $(m-1)C_t r dt$ is the leverage cost of the period t.

We now summarize the distribution of V_t in the following proposition.

Proposition 1. *Under the continuous time setting, for $t \in [0, T]$ the PO-CPPI portfolio value at time t follows the distribution:*

$$V_t = P_t + C_0 \exp(B_t - \frac{1}{2}At) + (r-d)\int_0^t \exp\{B_t - B_\xi - \frac{1}{2}A(t-\xi)\}P_\xi d\xi, \tag{8}$$

and the expected portfolio value of the PO-CPPI portfolio at time t is

$$E(V_t) = P_t + C_0 e^{\mu_B t} + (r-d)p_0 e^{\mu_B t}\frac{1-e^{(d-\mu_B)t}}{\mu_B - d}, \tag{9}$$

where $A = m^2\alpha^2\sigma_i^2 + m^2(1-\alpha)^2\sigma_P^2 + 2m^2\alpha(1-\alpha)\sigma_P^i\sigma_i$;
$B_t = \{r + m[\alpha\sigma_i\theta_i + (1-\alpha)\sigma_P\theta_P]\}t + m[\alpha\sigma_i W_s(t) + (1-\alpha)\sigma_P W pt]$; *and* $P_0 + C_0 = V_0$.

For simplicity and without loss of generality, hereafter we consider only the most common floor strategy, with $d = r$, for the proposed PO-CPPI strategy. Then, the PO-CPPI portfolio value at time t in equation (8) becomes:

$$V_t = P_t + C_0 \exp\{\{r + m[\alpha\sigma_i\theta_i + (1-\alpha)\sigma_P\theta_P] - \frac{1}{2}A\}t + m\alpha\sigma_i W_s(t) + (1-\alpha)\sigma_P W_{Pt}\}. \tag{10}$$

Moreover, since the standard CPPI can be viewed as a special case of PO-CPPI with $\alpha = 1$, the distribution of CPPI portfolio value at time t is

$$V_t = P_t + C_0 \exp(B_t - \frac{1}{2}m^2\sigma_i^2 t) + (r-d)\int_0^t \exp\{B_t - B_\xi - \frac{1}{2}m^2\sigma_i^2(t-\xi)\}P_\xi d\xi, \tag{11}$$

where $B_t = \{(r + m\sigma_i\theta_i)t + m\sigma_i QW_s(t)\}$, and $t \in [0, T]$.

Now, the challenge for us is to demonstrate that PO-CPPI strategy improves the performance of CPPI for SPGFs. To do so, we solve the optimal proportion α^* and then further convincingly prove the proportion of P fund, $1 - \alpha$, is greater than zero.

4.3. Optimal PO-CPPI allocation rules

Consider an SPGF manager aims to maximize the prospect theory investor's utility $U(V_T, Y_T)$ in equation (7) by choosing the leverage m and I-fund proportion α at the start of the fund. The optimization problem is given by

$$\underset{m,\alpha}{Max}\ E[U(V_T, Y_T)|\mathcal{F}_0]. \tag{12}$$

To determine the optimal allocation (α^*, m^*), we first introduce the purpose-related risk aversion-adjusted (PRA) return for preparation. The PRA return is a modified indicator that reflects the effect of risk aversion and purpose-related inflation risk on evaluating the I-fund and P-fund.

Definition 1. *For the stock market index I-fund and purpose-related market sector P-fund, the PRA return is*

$$\mu_N^{(\gamma)} = \mu_N - \gamma \overrightarrow{\sigma}_N \cdot \overrightarrow{\sigma}_y, \ \ N = I \text{ or } P,$$

where $\overrightarrow{\sigma}_N = (\sigma_N^s, \sigma_N^p, \sigma_N^e)$ and $\overrightarrow{\sigma}_y = (\sigma_Y^s, \sigma_Y^p, \sigma_Y^e)$.

By definition, the possible range of γ for prospect theory investors is $0 < \gamma \leqslant 1$ and $\overrightarrow{\sigma}_N \cdot \overrightarrow{\sigma}_y$ is always positive. Thus, we always have $\mu_N^{(\gamma)} \leqslant \mu_N$ and the PRA return can be viewed to be "punished". For the I-fund or P-fund, the "punishment" of the PRA return increases with investors' risk aversion and the targeted fund's volatility correlation with purpose-related inflation Y_t.

Thanks to the concept of the PRA return, we now solve the global optimal allocation parameters m^* and α^* of the PO-CPPI portfolio in the continuous time case as follows.

Proposition 2. *The optimal allocation parameters m^* and α^* of PO-CPPI portfolio satisfy*

$$F(m^*) = 0,$$

$$\alpha^* = \alpha^*(m^*),$$

where

$$\begin{aligned} F(m) = {} & \alpha^*(m)\mu_I^{(\gamma)} + (1 - \alpha^*(m))\mu_p^{(\gamma)} - r + (\mu_I^{(\gamma)} - \mu_P^{(\gamma)})\alpha^*(m) \\ & + (\gamma - 1)\{[\frac{(\sigma_P^i)^2 + (\sigma_P^p)^2 - \sigma_p^s\sigma_I^i}{(\sigma_I^i - \sigma_P^i)^2 + (\sigma_P^p)^2} + (\sigma_P^p)^2 + (\sigma_P^i)^2]m - \sigma_Y^i\sigma_P^i - \sigma_Y^p\sigma_P^p\}, \end{aligned}$$

$$\alpha^*(m) = \frac{(\sigma_P^i)^2 + (\sigma_P^p)^2 - \sigma_p^s\sigma_I^i}{(\sigma_I^i - \sigma_P^i)^2 + (\sigma_P^p)^2} + \frac{\mu_I^{(\gamma)} - \mu_P^{(\gamma)}}{(\sigma_i - \sigma_p^s)^2 + (\sigma_P^p)^2}\frac{1}{(1 - \gamma)m}.$$

We first have a look at the optimal PO-CPPI allocation rule of one special example, where SPGFs are participated by risk-neutral investors. This particular case is a special example of Proposition 2 with $\gamma = 1$. In this risk-neutral case, the optimal PO-CPPI allocations α^* and m^* in Proposition 2 are simplified as follows:

$$\alpha^* = \begin{cases} 1 & if\ \mu_I^{(1)} - \mu_P^{(1)} > 0 \\ \alpha,\ \alpha \in [0,1] & if\ \mu_I^{(1)} - \mu_P^{(1)} = 0 \\ 0 & if\ \mu_I^{(1)} - \mu_P^{(1)} < 0 \end{cases}, \tag{13}$$

and

$$m^* = \begin{cases} M & if\ \alpha^* \mu_I^{(1)} + (1-\alpha^*)\mu_P^{(1)} - r > 0 \\ m,\ m \in [1, M] & if\ \alpha^* \mu_I^{(1)} + (1-\alpha^*)\mu_P^{(1)} - r = 0 \\ 1 & if\ \alpha^* \mu_I^{(1)} + (1-\alpha^*)\mu_P^{(1)} - r < 0 \end{cases}, \tag{14}$$

where $\alpha\mu_I^{(1)} + (1-\alpha)\mu_p^{(1)} - r$ is the excess PRA return of the cushion fund. The optimal allocation in equation (13) shows the principle of PO-CPPI's optimal allocation rule in the special case: 1. the optimal proportion α^* largely depends on the comparison of the PRA return $\mu_I^{(1)}$ with $\mu_P^{(1)}$, 2. and the optimal leverage m^* is determined by the portfolio's average excess PRA return, $\alpha\mu_I^{(1)} + (1-\alpha)\mu_p^{(1)} - r$. It is noteworthy that in this particular example the optimal m^* and α^* are binary in most cases and exhibit no monotonicity relationship. However, the pattern of this particular example is not the general case.

Proposition 2 illustrates the optimal allocation parameters m^* and α^* without considering the limited possible range of the parameters in practice. However, in the real-world scenario, there exists an upper bound of the leverage m because of the regulation and fund's limited borrowing capability. Further, the range of α is often limited to $[0, 1]$ because of short-sale constraints. Thus, in practice the parameter ranges are $m \in [1, M]$ and $\alpha \in [0, 1]$, where M is the maximum possible value of m under the regulation.[14]

In the following, now we explore the general relation between the optimal allocation parameters m^* and α^* with parameter ranges. We first determine the optimal proportion α^* invested in the I-fund for the given leverage m, and then study how α^* changes with m. The following corollary reveals that the monotonicity of $\alpha^*(m)$ depends on the relativity of the PRA return of I-fund and P-fund, $\mu_I^{(\gamma)} - \mu_P^{(\gamma)}$.

Corollary 1. *The optimal ratio $\alpha^*(m)$ of PO-CPPI portfolio satisfies:*
(1) if $\mu_I^{(\gamma)} > \mu_P^{(\gamma)}$, then $\alpha^*(m)$ is a decreasing function of m;
(2) if $\mu_I^{(\gamma)} < \mu_P^{(\gamma)}$, then $\alpha^*(m)$ is an increasing function of m;
(3) if $\mu_I^{(\gamma)} = \mu_P^{(\gamma)}$, then α^* is not correlated with m.

[14] In practice, the leverage m is limited and even regulated. According to Balder et al. (2009) and Dichtl and Drobetz (2011), leverage m has a significant impact on CPPI portfolio's outcome, and it is normally below 10. In this study, we follow that $m \leqslant 10$.

Proof. First, equation (2) implies that the $\alpha^*(m)$ is a function of m in the form of $\frac{1}{m}$ when $\mu_I^{(\gamma)} > \mu_P^{(\gamma)}$. Obviously, it is a decreasing function of m. Then, the remaining proofs are similar and trivial. □

Corollary 1 shows the "diversification" effect of optimal allocations: $\alpha^*(m)$ gradually shifts to the fund with less PRA as the leverage ratio m increases. Intuitively, a larger ratio α invested in the fund with lower PRA return could lower the investment performance but also decrease the volatility of cushion portfolio in the meanwhile. This interesting feature of the optimal PO-CPPI allocation indicates that it employs diversification effect to offset the risk caused by high leverage.

On the basis of the monotonicity of $\alpha^*(m)$, we further show the relationship between the optimal proportion α^* and given leverage m when they are bounded. Assume M is the upper bound of leverage and there are short-sale constraints, then the parameter ranges are $m \in [1, M]$ and $\alpha \in [0, 1]$. With these bounds, we have Corollary 2, whose proof is trivial.

Corollary 2. *Consider the constraints that leverage $m \in [1, M]$ and ratio $\alpha \in [0, 1]$, the optimal ratio $\alpha_c^*(m)$ of PO-CPPI portfolio becomes*

$$\alpha_c^*(m) = \begin{cases} 1 & \text{if } \alpha^*(m) > 1 \\ \alpha^*(m) & \text{if } \alpha^*(m) \in [0, 1] \\ 0 & \text{if } \alpha^*(m) < 0 \end{cases} .$$

where $\alpha^*(m)$ is given by Proposition 2.

So far, we have only considered the optimal allocation in the continuous time case, under which the portfolio is continuously rebalanced. As the portfolio value never falls below the floor in this case, SPGF investor's loss aversion characteristic does not have an opportunity to play a role in the optimal portfolio allocation.[15]

5. Simulation analysis

So far we have now shown PO-CPPI dominates CPPI in the continuous time case. However, whether PO-CPPI outperforms CPPI or other portfolio strategies in the discrete case is unclear. Therefore, in this section we conduct an extensive numerical analysis to test the superiority of the proposed PO-CPPI strategy. We first present the Monte Carlo simulation design in Section 5.1, followed by detailed definitions of the benchmark strategies and performance measures in Section 5.2. Then, the main results on the performance of PO-CPPI compared with the other benchmark strategies such as CPPI are presented in Section 5.3. We further some robustness checks by adopting alternative ways of modelling in Section 5.4.

[15] Due to the limited length, here we cannot put our analysis of the optimal allocation ratio $\alpha^*(m)$ in the discrete time case. When the portfolio cannot be rebalanced continuously, our simulation results show that the monotonic relationship between $\alpha^*(m)$ and m remains robust in the discrete time case, with considerations of gap risk and loss aversion. Thus, the result indicates that the optimal allocation ratio $\alpha^*(m)$ in Proposition 2 is an effective strategy to implement in the real world scenario.

5.1. Simulation design

5.1.1. Simulation procedure

The Monte Carlo simulations are designed on a step-by-step basis as follows:

1. We consider a wide range of market possibilities with 10 economic scenarios with different fund returns, volatilities and inflation settings.
2. We run 100,000 simulations for each scenario and report the performance of PO-CPPI and the other benchmark strategies from the Monte Carlo simulation.
3. We use different performance measures to evaluate the 100,000 outcomes of all the strategies in each scenario. The measures include the protection ratio, return distribution measures, and investors' prospect theory utility[16].
4. Paired t-tests are applied to compare prospect theory investors' utilities (for both $\lambda = 1$ and $\lambda = 2.25$) under the different portfolio insurance strategies.

According to the model in Section 3, the I-fund, P-fund and purpose-related inflation index follow multivariate correlated BS processes. Before running the Monte Carlo simulations, some key parameters must thus be assigned: the return and volatility of the I-fund (μ_I and $\overrightarrow{\sigma}_i$); the return and volatility of the P-fund (μ_P and $\overrightarrow{\sigma}_p$); and the growth rate and volatility of purpose-related expense risk (μ_Y and $\overrightarrow{\sigma}_y$).

5.1.2. Parameter calibration

We assign the parameter values based on the existing literature. In our simulation, we consider a stochastic setting of interest rate. Specifically, we adopt the classical Vasicek model to describe the evolution of interest rates, whose stochastic differential equation is as follows:

$$dr_t = a(b - r_t)dt + cdZ_t^r,$$

where Z_t^r is a BM process. Vasicek model incorporates mean reversion and has been adopted widely in finance literature (Vasicek, 1977). Following the similar estimations in Hull (2003), we have $a = 0.136$, $b = 0.015$ and $c = 0.0119$. Such set of parameters imply that the interest rate is mean-reverting with a mean of 1.5%.[17]

According to Dimson et al. (2008), the mean annual equity excess return for developed stock markets was approximately 7%. In addition, Dimson et al. (2008) find that long-run stock return volatility is roughly 20% per year, and this proportion has been used in the simulations by Benninga (1990) and Figlewski et al. (1993). Thus, in our simulation, we estimate that a high state of stock market excess return is 6.5% (mean return 8%) and a low state is 4.5% (mean return 6%) and that stock market volatility ranges from 20% to 30%.

[16]The parameters of prospect theory utility follows Tversky and Kahneman (1992) and Dichtl and Drobetz (2011) and we assign $\gamma = 0.88$, $\lambda = 2,25$, $\delta^+ = 0.65$, $\delta^- = 0.84$, $\beta^+ = 0.6$, and $\beta^- = 0.65$.

[17]The calibration of interest rate is from the consideration of low interest rate enviroment faced by many economies. Our simulation results are robust to other levels of interest rate, e.g. a fixed rate of 4.5% in Arnott and Bernstein (2002) within the investment period.

We consider a five-year investment horizon and 250 trading days per year by extending the commonly-adopted one-year evaluation of the performance in many works (see e.g. Benartzi and Thaler (1995) and Dichtl and Drobetz (2011)). We normalize the initial SPGF value V_0 to 100. The guarantee level of the SPGF is set to be 100% of principal amount ($P_T = V_0$). All portfolio insurance strategies in the simulation adopt a base case leverage of $m = 5$, which is commonly used in practice (Herold et al., 2005). In particular, we implement the PO-CPPI strategy with the derived optimal allocation ratio $\alpha^*(5)$.

5.1.3. Stock market scenarios

As discussed in Section 4, the optimal PO-CPPI allocation depends mainly on the relative superiority between the I-fund and P-fund. To analyze PO-CPPI's performance, we thus consider five possible market scenarios. In our setting, the price of each individual stock follows the BS process. Specifically, we consider there are 125 individual stocks in the market, among which 25 stocks belong to investors' purpose-related industry sector, e.g. the retirement industry sector for pension guarantee funds.[18] The I-fund portfolio equally invests in all 125 individual stocks, whereas the P-fund equally invests in all 25 industry sector stocks. It is noteworthy that the cushions of CPPI and PO-CPPI portfolios both invest in all 125 stocks though they have different weights for each stock.[19]

We consider a fixed P-fund (with a 7% mean return and 25% volatility) and five states of I-fund with different mean return and volatility (see Table 2). In the first four states, I-fund exhibits relative a higher (lower) expected return and a higher (lower) volatility than the fixed P-fund; and in the fifth state the expected return and volatility of the I-fund are the same as the fixed P-fund. We summarize the five scenarios as follows:

- Scenario 1 (Scenario 3): The I-fund has a lower return and a lower (higher) volatility than the fixed P-fund;
- Scenario 2 (Scenario 4): The I-fund has a higher return and a lower (higher) volatility than the fixed P-fund; and
- Scenario 5: The return and volatility of the I-fund equal to those of the fixed P-fund.

[18] The value of 125 and 25 are approximated by the ratio of retirement industry stocks in the whole stock market. Our simulation results are robust to some other alternative ratios.

[19] Such setting would rule out the possible alternative explanation that the simulation result that PO-CPPI is a successful modified strategy comes from the diversification effect. We also find the superiority of PO-CPPI strategy when we directly generate the synthesized BS processes of I-fund and P-fund, the results of which are reported in the Online Appendix.

Table 2: Four market scenarios
This table reports the four market scenarios of I-fund, including its mean return and volatility. In all scenarios, the performance of P-fund is fixed with a mean return and volatility of 7% and 25%, respectively.

I-fund	Expected return			
	Low		High	
Volatility				
Low	Scenario 1:	Expected return: 6% Volatility: 20%	Scenario 2:	Expected return: 8% Volatility: 20%
High	Scenario 3:	Expected return: 6% Volatility: 30%	Scenario 4:	Expected return: 8% Volatility: 30%

5.1.4. Inflation scenarios

Other than the stock market conditions, the inflation index also plays an important role in these economic scenarios. Unlike highly volatile stock markets, the purpose-related inflation follows a much steadier process as we know the purpose-related expense price is inflated at maturity almost for sure. We distinguish the low and high inflation states in each market condition scenario as follows:

- Low inflation state. In the low inflation state, we assume that the mean growth rate and volatility of the inflation index Y_t are 3% and 1%, respectively.
- High inflation state. In the high inflation state, we assume that the mean growth rate and volatility of the inflation index Y_t are 8% and 1%, respectively.

As introduced earlier, the inflation index Y_t has different correlations with the I-fund and P-fund. From the findings on the correlation between price inflation and related risky asset in literature, in the simulation we assume that the correlation between Y_t and the I-fund return is 16.7%, whereas that of the P-fund is 50%. Thus, in total we have 10 different economic scenarios (5 market scenarios $\times$ 2 inflation states). In the simulation, we run $100,000$ simulations for each scenario.

5.2. Benchmark strategies

We first select a variety of benchmark strategies including CPPI, time-invariant portfolio protection (TIPP) strategy and stop-loss, and risk-free cash investment that have guaranteed levels. To test the superiority of the proposed PO-CPPI, we compare PO-CPPI against other benchmark strategies on the basis of each simulated scenario.

CPPI strategies. We consider two benchmark CPPI strategies, CPPI-I and CPPI-P, with a difference in their cushion. Similar to standard CPPI, CPPI-I strategy invests its cushion in the I-fund, while CPPI-P invests in the P-fund. The testing of both CPPI-I and CPPI-P strategies is to rule out the possibility that the superiority of PO-CPPI over standard CPPI may come from investing in the P-fund instead of the strategy modification design.

TIPP strategy. TIPP strategy is proposed by Estep and Kritzman (1988) to protect not only the investor's initial wealth but also any interim capital gains during the investment. Instead of having a fixed-rate floor like CPPI, TIPP's floor rises with the value of the portfolio during the investment period. TIPP portfolio's exposure to stock market E_t is

$$E_t = mC_t = m(V_t - P_t),\ t \in [0, T],$$

and its floor is

$$P_t = \max(e^{-r(T-t)}P_T,\ f \cdot V_t),\ t \in [0, T],$$

where f is the predetermined protection ratio of the portfolio value V_t. $f \cdot V_t$ shows the 'ratcheting up' effect of TIPP, which transfers gains in the risky asset to the risk-free asset irreversibly once interim capital gains arise.

Stop loss (SL) strategy. SL strategy is one of the simplest portfolio insurance strategies. SL strategy initially invests all fund wealth V_0 in risky assets, and maintains the position as long as the market value of the portfolio exceeds the net present value of its floor $V_t \geqslant P_t$. Once the market value of the portfolio reaches or falls below the discounted floor $V_t < P_t$, all the risky portfolio positions are cleared off and reinvested in the risk-free asset until maturity.

Cash investment strategy. Cash investment strategy simply invests all the fund wealth V_0 in the risk-free fund (cash asset) throughout the investment horizon.

5.3. Simulation results

In this section, we compare the performances of the PO-CPPI with other strategies using Monte Carlo simulation results. Several performance measures are applied to the simulated investment outcomes of all the aforementioned strategies in each economic scenario, including the annual mean return, volatility, Sharpe ratio, and Value-at-Risk (VaR), Expected shortfall (ES), protection ratio[20], the prospect theory investors' utility with loss aversion coefficients $\lambda = 1$ and $\lambda = 2.25$[21]. Specifically, we adopt the $\text{VaR}_{1\%}$[22] and $\text{ES}_{1\%}$[23].

Table 3 presents the performance measures applied to the return distribution of each strategy. Note that the inflation states only influences the prospect theory investors' utility

[20] Protection ratio is the probability that a strategy successfully protects its guarantee (Huu Do, 2002). It measures the strategy's ability to sustain a pre-specified guaranteed return. Like other studies, we calculate the protection ratio on a yearly basis.

[21] Consistent with Tversky and Kahneman (1992) and Dichtl and Drobetz (2011), we assign $\gamma = 0.88$ in the in equation (5). Similar to the approach Dichtl and Drobetz (2011) adopted to analyze the role of loss aversion, we employ two utilities with loss aversion parameters $\lambda = 1$ and $\lambda = 2.25$. $\lambda = 1$ indicates the no loss aversion case that investors treat the loss and gain equally, while $\lambda = 2.25$ is the most common loss aversion found in the literature.

[22] To calculate $\text{VaR}_{1\%}$, we first sort the realized portfolio values in ascending order and $\text{VaR}_{1\%}$ is the 1% percentile (exactly 1% with poorer performance than it).

[23] $\text{ES}_{1\%}$ measures the average return of a strategy's bottom 1% performance scenarios. We first sort the realized portfolio values in ascending order and then calculate the average return of the poorest-performing 1%. ES focuses on the left tail of the distribution and measures the ability to control downside risk.

without affecting other performance measures, Table 4 illustrates the mean prospect theory utility values. Paired t-tests of prospect theory investors' utility, on the basis of the $100,000$ simulated portfolio outcomes, are conducted to compare PO-CPPI with the benchmark strategies.

5.3.1. Downside risk protection

Sustaining a guaranteed return and preventing loss are the main functions of portfolio insurance strategies. Table 4 shows that the active portfolio insurance strategies, including the PO-CPPI, CPPI and TIPP have much higher tail risk management skills than the passive ones (e.g. compare the $\text{VaR}_{1\%}$ under the Scenario 1). Overall, Table 3 shows that PO-CPPI almost has the best performance of managing downside risk. Specifically, it exhibits the highest protection ratio and lowest extreme loss (e.g. $\text{VaR}_{1\%}$ and $\text{ES}_{1\%}$) among all strategies and under all scenarios. That is, PO-CPPI is a competitive strategy at preventing downward return and sustaining guarantee. In addition, Table 4 reports that PO-CPPI dominates all the benchmark strategies in the non-loss-averse prospect utility case with $\lambda = 1$ and for the loss-averse prospect utility with $\lambda = 2.25$. This might be associated with PO-CPPI strategy's advantage in controlling downside risk since the strategies with higher protection ratio such as CPPI and TIPP experience much less fall of utility value than SL, when loss-aversion coefficient increases from 1 to 2.25.

Moreover, we employ Omega and Kappa, the ratio of portfolio performance's average of the gains above a threshold to the average of the losses below the same threshold, to measure the tail risk prevention. For any given threshold, the portfolio with a higher Omega or Kappa performs better than that with a lower one (Ameur and Prigent, 2014; Bertrand and Prigent, 2011). Denote portfolio's final outcome by X and threshold by L, Omega can be written as

$$\Omega_X(L) = \frac{E[(X-L)^+]}{E[(L-X)^+]},$$

and Kappa considered in Kaplan and Knowles (2004) is defined by

$$Kappa_X(L) = \frac{E[(X)-L]}{\left(E\left[[(L-X)^+]^l\right]\right)^{\frac{1}{l}}},$$

which becomes the Sharpe Omega measure for $l = 1$, and Sortino ratio for $l = 2$.[24]

To explore the relative performance of different strategies, we plot Omega and Kappa measures as functions of the threshold, L, under the Scenario 1. Similar to the literature such as Bertrand and Prigent (2011) and Jiang et al. (2009), we consider the range of threshold L to be $[100, 105]$. Figure 5 and Figure 6 illustrate the Omega and Kappa measures of all the simulated strategies, respectively. We observe that PO-CPPI strategy has higher values of Omega and Kappa measures for almost all L. That means, both functions suggest that PO-CPPI is superior to other strategies in protecting against downside risk.

[24] We illustrate the result of Kappa with 2, and the conclusion of $l = 1$ is cosistent and thus not reported.

Table 3: Simulation results
This table reports the Monte Carlo simulation results of the performance measures for the proposed PO-CPPI strategy and other benchmark strategies.

	PO-CPPI	CPPI-I	CPPI-P	TIPP	SL	Cash
Scenario 1: I-fund expected return = 6%, volatility = 20%						
Mean Return (%)	3.12	2.78	3.14	2.65	2.76	1.50
Volatility (%)	5.94	8.30	8.40	4.40	10.79	0.07
Sharpe Ratio	0.07	0.04	0.06	0.05	0.04	0.00
Protection Ratio (%)	90.08	83.62	84.41	84.5	60.77	97.78
$VaR_{1\%}$ (%)	-0.62	-0.71	-0.7	-0.69	-4.51	-6.54
$ES_{1\%}$ (%)	-0.78	-0.87	-0.86	-0.85	-5.13	-8.54
Scenario 2: I-fund expected return = 8%, volatility = 20%						
Mean Return (%)	3.72	3.69	3.10	3.40	3.56	1.50
Volatility (%)	8.00	11.22	7.94	5.74	13.56	0.07
Sharpe Ratio	0.08	0.07	0.06	0.08	0.06	0.00
Protection Ratio (%)	90.28	86.02	84.43	86.8	63.07	97.75
$VaR_{1\%}$ (%)	-0.63	-0.69	-0.71	-0.67	-4.42	-6.62
$ES_{1\%}$ (%)	-0.79	-0.85	-0.87	-0.82	-5.04	-8.61
Scenario 3: I-fund expected return = 6%, volatility = 30%						
Mean Return (%)	3.18	2.56	3.23	2.46	2.60	1.50
Volatility (%)	7.15	11.89	8.85	5.87	15.37	0.07
Sharpe Ratio	0.06	0.03	0.06	0.04	0.03	0.00
Protection Ratio (%)	87.92	74.32	84.52	76.02	57.38	97.78
$VaR_{1\%}$ (%)	-0.66	-0.78	-0.7	-0.76	-5.29	-6.66
$ES_{1\%}$ (%)	-0.82	-0.95	-0.87	-0.92	-6.01	-8.7
Scenario 4: I-fund expected return = 8%, volatility = 30%						
Mean Return (%)	3.47	3.30	3.20	3.09	3.24	1.50
Volatility (%)	8.08	15.56	8.68	7.54	19.41	0.07
Sharpe Ratio	0.07	0.05	0.06	0.06	0.04	0.00
Protection Ratio (%)	88.11	76.98	84.44	78.45	59.15	97.69
$VaR_{1\%}$ (%)	-0.65	-0.76	-0.7	-0.74	-5.17	-6.61
$ES_{1\%}$ (%)	-0.81	-0.92	-0.86	-0.9	-5.92	-8.64
Scenario 5: I-fund expected return = 7%, volatility = 25%						
Mean Return (%)	3.29	3.01	3.17	2.83	3.01	1.50
Volatility (%)	6.79	11.33	8.27	5.77	14.56	0.07
Sharpe Ratio	0.07	0.04	0.06	0.06	0.04	0.00
Protection Ratio (%)	88.94	80.02	84.39	81.21	59.82	97.75
$VaR_{1\%}$ (%)	-0.64	-0.75	-0.71	-0.72	-4.85	-6.53
$ES_{1\%}$ (%)	-0.81	-0.91	-0.87	-0.89	-5.56	-8.66

Table 4: Utility of simulation results and paired t-tests
This table reports the Monte Carlo simulation results of the scaled utility value for prospect theory investors under the two inflation levels. The results of paired t-tests are also reported. The null hypothesis in the paired t-tests is that the prospect theory utility of a benchmark strategy is equal to that of the PO-CPPI strategy.

	PO-CPPI	CPPI-I	CPPI-P	TIPP	SL	Cash
Scenario 1: I-fund expected return = 6%, volatility = 20%						
Low Inflation State						
Utility ($\lambda = 1$)	54.75	46.85***	52.69***	46.91***	43.78***	37.23***
Utility ($\lambda = 2.25$)	54.31	46.20***	52.06***	46.31***	37.31***	35.68***
High Inflation State						
Utility ($\lambda = 1$)	44.31	38.12***	42.66***	38.26***	35.50***	30.22***
Utility ($\lambda = 2.25$)	43.96	37.60***	42.15***	37.77***	30.23***	28.97***
Scenario 2: I-fund expected return = 8%, volatility = 20%						
Low Inflation State						
Utility ($\lambda = 1$)	62.02	58.97***	51.17***	58.00***	54.82***	36.93***
Utility ($\lambda = 2.25$)	61.60	58.40***	50.54***	57.47***	48.53***	35.36***
High Inflation State						
Utility ($\lambda = 1$)	52.28	50.27***	42.15***	48.97***	46.05***	30.22***
Utility ($\lambda = 2.25$)	51.93	49.80***	41.64***	48.54***	40.95***	28.95***
Scenario 3: I-fund expected return = 6%, volatility = 30%						
Low Inflation State						
Utility ($\lambda = 1$)	52.66	41.48***	51.38*	42.07***	39.09***	37.08***
Utility ($\lambda = 2.25$)	52.15	40.61***	50.75**	41.24***	31.49***	35.52***
High Inflation State						
Utility ($\lambda = 1$)	44.14	33.29***	43.72	34.04***	31.97***	30.12***
Utility ($\lambda = 2.25$)	43.72	32.58***	43.22	33.37***	25.80***	28.85***
Scenario 4: I-fund expected return = 8%, volatility = 30%						
Low Inflation State						
Utility ($\lambda = 1$)	59.28	53.03***	52.83***	52.93***	49.91***	37.16***
Utility ($\lambda = 2.25$)	58.78	52.22***	52.20***	52.16***	42.49***	35.62***
High Inflation State						
Utility ($\lambda = 1$)	48.06	42.86***	43.36***	42.75***	39.99***	30.07***
Utility ($\lambda = 2.25$)	47.66	42.20***	42.86***	42.13***	33.96***	28.79***
Scenario 5: I-fund expected return = 7%, volatility = 25%						
Low Inflation State						
Utility ($\lambda = 1$)	56.06	49.81***	51.25***	49.81***	46.75***	37.13***
Utility ($\lambda = 2.25$)	55.59	49.07***	50.62***	49.11***	39.77***	35.59***
High Inflation State						
Utility ($\lambda = 1$)	46.15	40.03***	42.99***	39.92***	37.83***	30.14***
Utility ($\lambda = 2.25$)	45.76	39.42***	42.48***	39.35***	32.17***	28.87***

* $p \leq 0.05$, ** ≤ 0.01, *** ≤ 0.001.

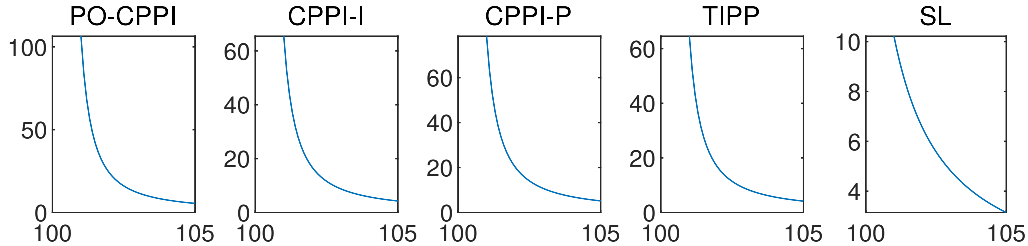

Figure 5: The Omega function in Scenario 1
This figure presents the Omega function for the threshold in the range $[100, 105]$ of different strategies in Scenario 1.

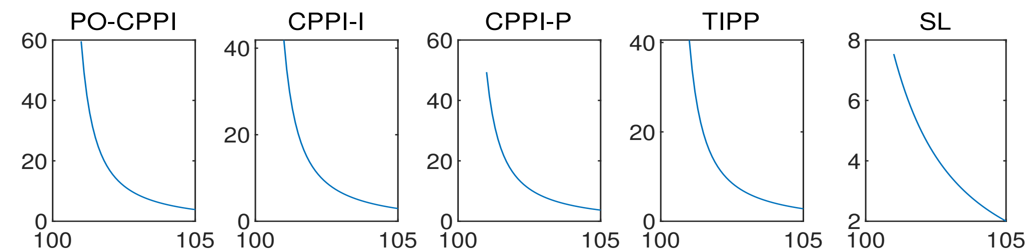

Figure 6: The Kappa function in Scenario 1
This figure presents the Kappa function for the threshold in the range $[100, 105]$ of different strategies in Scenario 1.

5.3.2. PO-CPPI versus CPPI

In this study, we are particularly interested in comparing PO-CPPI and CPPI. As introduced above, Table 3 has shown that PO-CPPI strategy outperforms CPPI strategy in most cases in terms of managing downside risk and achieving upside potentials. We supplement the evidence that PO-CPPI has a higher average annual return than CPPI in almost all scenarios. Such superiority of PO-CPPI can also be reflected from other measures such as $\text{VaR}_{1\%}$ and $\text{ES}_{1\%}$. Thus, in general PO-CPPI is proven to be a more competent strategy than CPPI, with both better downside risk protection and higher returns.

In addition, PO-CPPI is the most preferred strategy for prospect theory investors, and thus dominates CPPI. Table 4 illustrates that prospect theory utilities ($\lambda = 1$ and $\lambda = 2.25$) of PO-CPPI significantly dominate those of both CPPI-I and CPPI-P in almost all scenarios (except CPPI-P strategy in Scenario 3). This result indicates that the proposed PO-CPPI strategy is preferred to standard CPPI for prospect theory investors. Hence, the superiority of PO-CPPI over CPPI verifies the success to modify standard CPPI by hedging purpose-related inflation risk.

5.4. Robustness checks

So far, our model and simulation have considered the dynamics of risky assets to be continuously driven by BS processes. In this section, we conduct two robustness checks by adopting alternative approaches to simulate the risky asset price. Specifically, we adopt the Bootstrap simulation using the historical data and the generalized autoregressive conditional heteroskedasticity (GARCH) process of stock prices.

It is noteworthy that the portfolio insurance strategies under BS and the alternative frameworks face gap risks, because the portfolio is only rebalanced at discrete time-points. That is, the portfolio value may fall below the guarantee level during two rebalancement, no matter the prices of risky assets evolve continuously or discretely during the interval.

5.4.1. Historical test via Bootstrap simulation

We supplement the simulation results by applying the Bootstrap method to historical data in Chinese stock market from January 2017 to December 2020[25]. Consistent with our previous approach, the simulation randomly selects 125 stocks (I-fund) from the whole market with 25 P-fund stocks from the retirement-industry sector. In addition, we adopt the fluctuating deposit rate during the selected horizon as the risk-free return.

We employ the classical Bootstrap method using the above historical data to generate 100,000 simulation paths.[26] The comparison of utility under the Bootstrap simulation is reported in Table 5.[27] We observe that under the historical tests the PO-CPPI still dominates several other strategies in the aspects of prospect theory utilities. This result implies the finding that PO-CPPI dominates other strategies, including CPPI-I and CPPI-P, remains robust under historical test using Chinese market data.

5.4.2. Modelling risky assets by GARCH model

A common setting in literature on portfolio insurance to model the discrete-time frame of risky asset dynamics is via the family of GARCH models (see e.g. Ameur and Prigent (2014) and Hamidi et al. (2014)). We supplement a robustness check to compare PO-CPPI with other strategies under the discrete-time GARCH model setting. To be specific, we consider that the returns of both I-fund and P-fund follows the most common AR(1)-GARCH(1,1) process[28]:

$$r_t = c + \phi_1 r_{t-1} + \xi_t, \tag{15}$$

$$\sigma_t^2 = \kappa + \gamma_1 \sigma_{t-1}^2 + \eta_1 \xi_{t-1}, \tag{16}$$

where $\xi_t = \sigma_t \epsilon_t$ and ϵ_t are standard Gaussian white noise process. We use the historical daily stock returns to estimate the parameters of GARCH models. We randomly select 125 stocks which includes 25 P-fund stocks and estimate each stock's parameters for the AR(1)-GARCH(1,1) model. Then, the estimated GARCH parameters are used to simulate $100,000$ asset price paths.

[25] As we conduct this simulation, the daily inflation index of ISLCI is only available for this period.

[26] Dichtl et al. (2017) presents a systematic comparison of portfolio insurance strategies using Bootstrap-based approach.

[27] The results of other performance measures are similar and thus not reported.

[28] We also have examined some other GARCH family processes and the conclusions are found to be similar.

Table 5: Bootstrap and GARCH model simulation results
This table reports the prospect theory utility values of the Bootstrap and GARCH model simulation results. The results of paired t-tests are also reported. The null hypothesis in the paired t-test is that the prospect theory utility of a benchmark strategy is equal to that of the PO-CPPI strategy.

	PO-CPPI	CPPI-I	CPPI-P	TIPP	SL	Cash
Panel A: Bootstrap simulation						
Utility ($\lambda = 1$)	100.96	67.58***	77.62***	73.56***	71.89***	34.85***
Utility ($\lambda = 2.25$)	100.95	67.57***	77.61***	73.54***	68.23***	34.85***
Panel B: GARCH model simulation						
Utility ($\lambda = 1$)	102.28	72.52***	85.23***	68.33**	89.57***	39.60***
Utility ($\lambda = 2.25$)	101.07	70.66***	82.90***	66.80***	87.30***	37.96***

* p $\leq$ 0.05, ** $\leq$ 0.01, *** $\leq$ 0.001.

We rerun the simulation analysis. The results reported in Table 5 show that our main findings remain robust. PO-CPPI strategy still outperforms other strategies by providing higher investors' utilities under the GARCH model. Generally, the findings are consistent with our main conclusions, which are expected as the rebalancements of portfolios under GARCH and BS simulations are both discretely in time.

6. Discussion: PO-DPPI strategy with a dynamic multiplier

Dynamic proportion portfolio insurance (DPPI) strategies are designed by using dynamic multipliers to provide dynamic tail risk protection. Specifically, the multiplier m_t of DPPI strategy that remains constant for the time interval $[t, t+1]$ in the discrete rebalancing setting, dynamically varies with the market condition. The value of the dynamic multiplier m_t is based on a risk measure ρ_t, which dynamically gauges the return of risky portfolio, e.g. VaR and ES. It is intuitive that the advantage of dynamic strategies is to timely adjust their exposures to risky assets according to the market conditions.

Happersberger et al. (2020) provide a comprehensive analysis of literature on comparing different ways to dynamically determine the risky investment exposure of strategies. Commonly, DPPI sets the maximal value of dynamic multiplier by the following form,

$$m_t \leq \frac{1}{\rho_t(r_{t+1})} \tag{17}$$

to provide effective downside protection. Here, r_{t+1} is the return of cushion's underlying asset during the interval $[t, t+1]$, a random variable at time t. Previous literature has applied several different risk measures such as VaR and ES to determine the conditional dynamic multiplier, the criteria of which could ensures that the gap risk is well maintained (Ameur and Prigent, 2007; Balder et al., 2009; Hamidi et al., 2008, 2009; Jiang et al., 2009). For instance, Hamidi et al. (2009) propose to define the multiple as a function of an extended

Dynamic AutoRegressive Quantile model of the Value-at-Risk (DARQ-VaR); Balder et al. (2009) study the CPPI strategy in a discrete–time setting with consideration of gap risk and discuss the criteria of under the ES measure. Thus, the common adopted (the upper bound of) dynamic multiples determined under the VaR and ES measures are $m_t = \frac{1}{VaR_t(r_{t+1})}$ and $m_t = \frac{1}{ES_t(r_{t+1})}$. In the following, we extend the PO-CPPI strategy by considering a dynamic conditional multiple which controls the portfolio's tail risk. In line with literature, we call this strategy as PO-DPPI strategy.

6.1. PO-DPPI with dynamic multiplier based on tail risk modelling

The value of underlying portfolio of PO-DPPI's cushion is denoted by S_t, which consists of α proportion of S_t^I and $1-\alpha$ proportion of S_t^P. The differential process of the $S_\alpha(t)$ is

$$\begin{aligned}\frac{dS_t(\alpha)}{S_t(\alpha)} &= \alpha\frac{dS_t^I}{S_t^I} + (1-\alpha)\frac{dS_t^P}{S_t^P} \\ &= \mu_\alpha dt + \sigma_\alpha dZ_\alpha, \end{aligned} \tag{18}$$

where $\mu_\alpha = \alpha\mu_I + (1-\alpha)\mu_P$, $\sigma_\alpha = \sqrt{\alpha^2\sigma_I^2 + (1-\alpha)^2\sigma_P^2 + 2\alpha(1-\alpha)\rho_{I,P}\sigma_I\sigma_P}$, and dZ_α is a synthetic BM process. Suppose the time interval between two rebalancement is Δ_t. Then, the time-$(t+1)$ value of cushion C_{t+1} is given as:

$$C_{t+1} = C_t + \Delta C_t,$$

where

$$\Delta C_t = m_t C_t \frac{\Delta S_t(\alpha)}{S_t(\alpha)} - (m_t - 1)C_t r_t \Delta_t.$$

Using a similar logic of determining DPPI's multiplier in the literature (see e.g. Ameur and Prigent (2007), Ameur and Prigent (2014) and Hamidi et al. (2014)), we have the following Proposition.

Proposition 3. *At any time t, to guarantee that the probability of depletion of cushion C_{t+1} is lower than p with $Pr[C_{t+1} \geq 0] \geq 1-p$, PO-DPPI portfolio's dynamic multiplier m_t should satisfy:*

$$m_t \leq m_p \triangleq \frac{1 + r_t\Delta_t}{1 + r_t\Delta_t - \exp((\mu_\alpha - \frac{1}{2}\sigma_\alpha^2)\Delta_t + \sigma_\alpha\sqrt{\Delta_t}\Phi^{-1}(p))}, \tag{19}$$

where r_t is the risk-free interest rate and $\Phi^{-1}(p)$ is the p-quantile of a standard normal distribution.

It is notable that the right hand side of equation (19) relies on the determination of α. It is a natural setting to consider that PO-DPPI strategy adopts the optimal $\alpha^*(m)$ given in the equation (13) for the time interval $[t, t+1]$. Then, it is obvious that both sides of equation (19) vary with the multiplier m and thus PO-DPPI's multiplier are obtained by solving the equations (19). This is different from the approach of determining the (upper

Table 6: Simulation results of dynamic strategies
This table reports the Monte Carlo simulation results of the prospect theory utility values of PO-DPPI and other dynamic strategies. The results of paired t-tests are also reported. The null hypothesis in the paired t-test is that the prospect theory utility of a benchmark strategy is equal to that of the PO-DPPI strategy.

Panel A	p-PO-DPPI	VaR_p-DPPI-I	VaR_p-DPPI-P	VaR_p-TVPP
Utility ($\lambda = 1$)	134.39	94.66***	131.11	88.77***
Utility ($\lambda = 2.25$)	132.18	92.12***	127.66*	76.12***
Panel B	VaR_p-PO-DPPI	VaR_p-DPPI-I	VaR_p-DPPI-P	VaR_p-TVPP
Utility ($\lambda = 1$)	134.38	94.66***	131.11	88.77***
Utility ($\lambda = 2.25$)	132.17	92.12***	127.66*	76.12***
Panel C	ES_p-PO-DPPI	ES_p-DPPI-I	ES_p-DPPI-P	ES_p-TVPP
Utility ($\lambda = 1$)	132.66	94.65***	131.10	116.51***
Utility ($\lambda = 2.25$)	130.33	92.11***	127.65*	107.72***

* p $\leq$ 0.05, ** $\leq$ 0.01, *** $\leq$ 0.001.

bound of) DPPI's dynamic multiplier in literature in which $m_p = \rho[r_{S_t}]^{-1}$ is directly set by the asset and risk measure ρ.

Furthermore, we also consider the common VaR approach for tail risk protection and get the following equation:

$$m_t = VaR_p[r_{S_t}(\alpha)]^{-1}, \tag{20}$$

which is consistent with the common way of determining (the upper bound of) the multiplier m_t in the literature of dynamic tail risk protection strategies like DPPI. In addition, when we apply the p-quantile expected shortfall (ES) measure in literature, m_p becomes

$$m_t = ES_p[r_{S_t}(\alpha)]^{-1}. \tag{21}$$

Similarly, the corresponding PO-DPPI strategy adopts $\alpha^*(m)$ in the equation (13).

Thus, the dynamic multipliers m_t implied by the equations (19), (20), and (21) lead to PO-DPPI strategies that adopt different principles for tail risk prevention. Specifically, we consider three PO-DPPI strategies using these three solved dynamic multipliers:

1. p-PO-DPPI with its dynamic multiplier m_t solved in equation (19);
2. VaR_p-PO-DPPI with its dynamic multiplier m_t in equation (20);
3. ES_p-PO-DPPI with its dynamic multiplier m_t in equation (21).

In the following, we further conduct numerical simulations to compare the performance of PO-DPPI strategies with that of other dynamic strategies. Specifically, we consider the level of risk management with $p = 1\%$.

6.2. Performance of PO-DPPI strategy

We examine the performance of dynamic PO-DPPI strategy under the AR(1)-GARCH(1,1) setting. To ensure comparability, we consider that the multiplier of CPPI and TIPP strategies should also become dynamic. Specifically, the multipliers of DPPI-I, DPPI-P and time-variant portfolio protection (TVPP) strategies are implied by the VaR or ES risk protection rules, i.e. $m_t = VaR_p[r_{S_t}]^{-1}$ and $m_t = ES_p[r_{S_t}]^{-1}$. We use the prefix notation "VaR_p-" and "ES_p-" to denote the dynamic strategies under these two types of multipliers.

We conduct Monte Carlo simulations to compare the performance of PO-DPPI and other benchmark dynamic strategies. Following the GARCH simulation approach in Section 5.4.2, we still consider that the I-fund and P-fund consist of randomly-chosen 125 and 25 stocks from the Chinese stock market, respectively. Table 6 reports the simulation results. Panel A reports the comparison between the performance of p-PO-DPPI and other VaR_p-based dynamic strategies. Similarly, in Panel B and C the benchmark strategies become VaR_p-PO-DPPI and ES_p-PO-DPPI strategies. Overall, the significantly higher utility level indicates that PO-DPPI outperforms the DPPI and TVPP dynamic strategies. To sum, our findings suggest that the superiority of PO-CPPI and PO-DPPI should come from their hedging of purpose-related inflation risk, with or without considerations of dynamic risk modelling.

7. Conclusion

Although various SPGFs with minimum guaranteed returns provide investors investment outcomes for specific usages like retirement savings, their currently adopted investment strategies are suboptimal. Finding appropriate portfolio insurance strategies has become a challenge for SPGF providers. In this study, we construct an innovative PO-CPPI strategy to improve the performance of SPGFs for prospect theory investors. Different from the standard CPPI strategy, PO-CPPI's cushion is a purpose-related fund which optimally combines the stock market index (I-fund) and purpose-related fund index (P-fund). Overall, we find that the proposed PO-CPPI outperforms other examined strategies in terms of hedging against downside risk and improving investors' prospect utility, under both the continuous and discrete time cases. The theoretical and numerical examinations of PO-DPPI strategy with multiplier based on dynamic risk modelling are also investigated.

In the continuous time case, we theoretically prove that the proposed PO-CPPI dominates CPPI by deriving the explicit optimal cushion allocation of PO-CPPI. The theoretical analysis discusses the relationship between the PO-CPPI's optimal proportion α in I-fund and the leverage ratio m. We show that the optimal PO-CPPI's investment in the purpose-related P-fund contributes to superior performance and higher investors' utility. In the discrete time case with gap risk, we adopt Monte Carlo simulation to compare the performances of PO-CPPI with other benchmark strategies. Our numerical analysis illustrates that PO-CPPI achieves a relatively higher mean return, better portfolio protection rate, and larger prospect theory utility. Moreover, the advantage of PO-CPPI increases with loss aversion, indicating that it is more preferred by prospect theory investors. Our main conclusions remain robust when using stochastic interest rate process, modelling stock prices with both the historical simulation approach with bootstrap and GARCH processes. Further, we extend PO-CPPI into a dynamic setting with a corresponding PO-DPPI strategy which still dominates other dynamic benchmark strategies.

As we have only considered the improvement of standard CPPI strategy by investing additionally in the SPGF's purpose-related assets, we leave the extension of this framework into other portfolio insurance strategies for future research. Moreover, it could be interesting to investigate the form of utilities under which the improved strategies in literature are the optimal. The investigation of such utility functions, though possibly exist but may not be in explicit forms, may provide some additional view on portfolio insurance strategies.

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