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UNCERTAINTY PROPAGATION FOR ORBITAL MOTION AROUND AN ASTEROID USING GENERALISED INTRUSIVE POLYNOMIAL ALGEBRA: APPLICATION TO THE DIDYMOS SYSTEM.

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ABSTRACT

The Hera mission plans to release a CubeSat into orbit around the binary asteroid system Didymos to investigate these bodies in more detail. Uncertainties play a key role in designing the trajectory for the CubeSat and thus a method is needed that is able to accurately and efficiently capture all the effects of uncertainties on the dynamics of this system. This paper introduces the Generalised Intrusive Polynomial Algebra (GIPA) as a novel method of producing a surrogate model representing this dynamical system under uncertainties. This method uses an algebra constructed using a basis of orthogonal polynomials that expand the dynamics over a certain set. The algebra is then used to propagate this set over time using conventional numerical integration methods. Both a Taylor polynomial basis and a Chebyshev basis are considered for this study. The results of the GIPA propagation are compared to the results from a Monte Carlo simulation to test its accuracy. GIPA is shown to be able to accurately and efficiently determine how the uncertainties evolve over time for a range of different initial conditions. Which is a useful tool that can be used in the robust design of spacecraft trajectories.

1 INTRODUCTION

Uncertainties are an important topic in orbital dynamics as they can have a significant effect on the long term evolution of the state of a spacecraft. The sources of these uncertainties can come from unknowns in the environment in which the spacecraft operates, e.g. unknown gravity field, and from technical uncertainties, e.g. sensor noise or actuator errors. To deal with these uncertainties in space trajectory design, mission planners use uncertainty quantification (UQ) and uncertainty propagation (UP) techniques to determine their size and distribution, and how these values evolve over time. Linearization of the dynamics allows the use of simple linear methods to perform the UP, however these methods fail to achieve the desired accuracy away from the linearization point for highly nonlinear problems. In this case, a Monte Carlo (MC) approach has often been the desired method for the analysis of the effect of uncertainties on the nonlinear dynamics. The MC method allows for accurate results, at the cost of long computation times. Furthermore, the MC method only provides a set of propagated samples at a certain time, which does not allow for a more wide set of dynamical systems tools to analyze the uncertain dynamics. These problems have caused other (semi-)analytical techniques for UQ and UP to be developed [1].

These methods can be classified into two categories: non-intrusive and intrusive techniques. Non-intrusive techniques use the dynamics as a black box and take the input and output of a set of samples for the analysis. Intrusive methods use the dynamical equations of the system to derive the evolution of the uncertainties over time. This requires more adaptation of the existing orbital dynamics programs, but gives the user more information and tools for the analysis of the dynamics. One of the most popular intrusive method used in the space sector is the Differential Algebra (DA) technique, which expands the dynamics in terms of Taylor polynomials and uses an algebra over these Taylor polynomials to propagate the uncertainties [2] [3]. This allows for an efficient approximation of the uncertain dynamics, as the degree n of the polynomials can be chosen according to the desired accuracy and the amount of nonlinearity present in the dynamics.

As a Taylor approximation does not distribute the errors evenly over the whole range of the expansion, and requires the dynamics to be $n + 1$ times differentiable, there has been an interest in finding other possible polynomial bases that can be used. In [4], a comparison was made between different polynomial bases for the algebra. It was found that a Chebyshev basis can achieve a higher accuracy, at the cost of a slight increase in computation time. The Chebyshev algebra was then used in an astrodynamics context in [5] and further explored for the case of the entry of an object from space into the Earth's atmosphere in [6]. Recently, a generalised setup was developed by [7], called Generalised Intrusive Polynomial Algebra (GIPA), in which a method was developed for which any set of orthogonal polynomials can be used to construct an algebra and perform the UP using that algebra.

This research will use the GIPA method to generate a surrogate model of the dynamics to perform UP for the state of a spacecraft orbiting a binary asteroid system. This case was specifically chosen as uncertainties are one of the main problems in small body mission design [8] and the dynamics for these system are complex and highly nonlinear. Being able to produce a model for the uncertain dynamics of this system allows mission planners and trajectory designers to not only propagate the uncertainties, but also determine how the dynamical structure is influenced by the uncertainties [9] and develop algorithms to cope with them. This research focuses only on the uncertain state and does not consider uncertainties in the environment. However, environmental parameters are handled in a similar matter and as the focus of this paper is on the functioning of GIPA for this system and not an in depth analysis of the resulting dynamics, these are not considered here.

This paper is structured as follows: first the GIPA method is explained in depth in section 2. Then the dynamical models and simulation setup are discussed more in section 3. The results of applying GIPA to these dynamics are then shown in section 4 and some final remarks regarding the results from this research are finally given in section 5.

2 GENERALISED INTRUSIVE POLYNOMIAL ALGEBRA

The uncertainty in the state of the spacecraft at a certain initial time t_0 can be viewed as a set $\Omega_{\mathbf{x}_0} \subseteq \mathbb{R}^d$, which consists of all possible states $\mathbf{x}_0 = \mathbf{x}(t_0)$ under the considered uncertainties. Given a non-autonomous and nonlinear dynamical system as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{1}$$

the state at a given time t can be found using:

$$\mathbf{x}(t) = \mathbf{x}_t = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\mathbf{x}, \tau) d\tau. \tag{2}$$

However, in the case of a set of initial conditions instead of a single point, the whole set of possible final states is given by:

$$\Omega_{\mathbf{x}_t} = \{\mathbf{x}(t) \mid \forall\, \mathbf{x}_0 \in \Omega_{\mathbf{x}_0}\}. \tag{3}$$

To obtain this set through single point propagation, an infinite number of samples need to be propagated. The set can be approximated by a finite number of samples, but to achieve an acceptable accuracy, often a large number of samples are needed as the dynamical structure for a spacecraft orbiting an asteroid can be quite complex [10]. Another approach is to approximate the set by a polynomial $P_{n,d}(\mathbf{x}_0)$ with a degree of n in d variables, given then as follows:

$$\tilde{\Omega}_{\mathbf{x}_t} = P_{n,d}(\mathbf{x}_0) = \sum_{i=1}^{\mathcal{N}} c_i(t)\alpha_i(\mathbf{x}_0), \tag{4}$$

where $\alpha_i(\mathbf{x}_0)$ are a set of multivariate polynomial basis functions, $c_i(t_f)$ the corresponding coefficients, and $\mathcal{N} = \binom{n+d}{d}$ the number of terms of the polynomial. It is important to note that the sets $\Omega_{\mathbf{x}_0}$ and $\Omega_{\mathbf{x}_t}$ (and thus also $\tilde{\Omega}_{\mathbf{x}_t}$) do not require or contain any statistical information, they are purely a set representing all the possible states of the system at a certain time.
There are many methods that can generate the polynomial approximation given in Eq. (4). In this work, the Generalised Intrusive Polynomial Algebra (GIPA) method is considered, where $\tilde{\Omega}_{\mathbf{x}_t}$ is obtained by propagating a polynomial approximation of $\Omega_{\mathbf{x}_0}$ from t_0 to a final time t using a polynomial algebra given in a specifically chosen set of basis functions. This method is explained in further detail in the section 2.1.

2.1 Polynomial Algebra

A polynomial expansion for a continuous, piece-wise differentiable, function $f(\mathbf{x}) : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is given by:

$$f(\mathbf{x}) = \sum_{\mathbf{i},|\mathbf{i}|\leq n} c_{\mathbf{i}}\alpha_{\mathbf{i}}(\mathbf{x}) + r(\epsilon) \tag{5}$$

where $x \in \Omega$, $\Omega = [-1,1]^d$, $\mathbf{i} \in [0,n]^d$ is a vector of indices with $|\mathbf{i}|$ the sum of all the entries, and $r(\epsilon)$ is the remainder of the approximation after truncation. The ordering and manipulation of the indices is discussed in more detail in [11]. The domain of $[-1,1]$ can be extended to an arbitrary domain using a simple linear transformation, which doesn't affect the operations discussed here. Thus, the domain $[-1,1]$ is considered here for simplicity, while not losing any generality.
The polynomial $P_{n,d}(\mathbf{x})$ is part of a function space of polynomials $\mathcal{P}_{n,d}(\alpha)$ with basis α. The choice of basis for this space determines the specific properties that the polynomial approximation will have. The two bases that will be discussed in this work are the Taylor polynomial basis and the Chebyshev polynomial basis. Taylor polynomials offer a good approximation near the expansion point and provide analytic expressions for elementary functions (e.g. $\sin$, $\cos$), which as will be shown later in this section is an advantage. However, the accuracy of a Taylor polynomial quickly drops off further away from the expansion point (as can be seen in figure 1), and it requires the function that is approximated to be $n+1$ times differentiable. The Chebyshev polynomial approximation is uniformly convergent across the expansion interval, and is able to handle discontinuities in the dynamics better compared to a Taylor expansion [4]. The differences in error distribution between the two bases is shown in more detail in figure 1. The downside of a Chebyshev basis is the fact that no general expression for elementary functions exist, which can decrease the accuracy and increase the computation time, as will be shown in section 4. There are other bases like the Newton basis and the Hermite basis that might be more applicable to certain other types of problems, and can be implemented and tested in a similar manner to the bases described here.

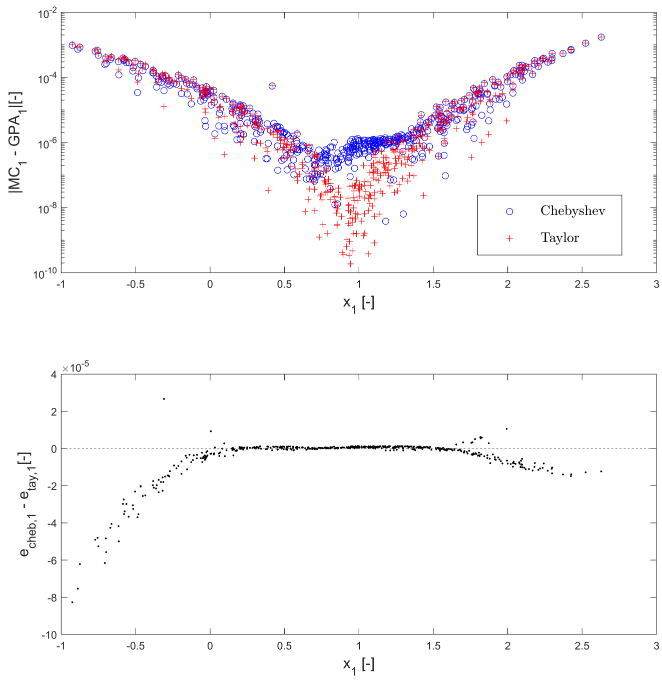

Figure 1: The error distribution of the two different GIPA bases as a function of the location in the set. The top figure shows the distribution, where the bottom figure shows the differences. Below the zero line signifies better Chebyshev performance, whereas above the line shows better Taylor performance.

After the basis is chosen, the next step of GIPA is to add a set of arithmetic operations $\otimes$, which correspond to the commonly known arithmetic operations $\oplus \in \{+, -, \cdot, /\}$. Then, for any operation between two functions f_a and f_b, the corresponding operations for their polynomial approximations, F_A and F_B, is given as follows:

$$f_a \oplus f_b \sim F_A \otimes F_B. \tag{6}$$

The polynomial space $\mathcal{P}_{n,d}(\alpha)$ and the operations on that space together form an algebra ($\mathcal{P}_{n,d}(\alpha)$, $\otimes$) of size $\mathcal{N}$, for which any polynomial part of this algebra is completely defined by its coefficients $c = \{c_{\mathbf{i}} : |\mathbf{i}| < n\}$.
Ref. [12] argued that for any polynomial basis, it is beneficial to transform the basis to the monomial basis, given by ϕ, after the expansion of the initial set as this will reduce the computational cost and only requires one set of operations to be implemented. The downside is that the coefficients can get much larger compared to other bases and the change of basis operation can be ill-conditioned. However, it was shown that despite this, the monomial basis can still give accurate results [6]. The final order of operations is as follows: first, the initial set is expanded in the desired polynomial basis. Second, the basis is transformed to the monomial basis (if possible) by:

$$\nu : \mathcal{P}_{n,d}(\alpha) \rightarrow \mathcal{P}_{n,d}(\phi). \tag{7}$$

Finally, the monomial algebra is constructed to propagate the initial set $\Omega_{\mathbf{x}_0}$ to the desired final set $\Omega_{\mathbf{x}_t}$.
The "general" part of GIPA is due the fact that irregardless of the choice of polynomial basis, the operations remain the same due to the fact that a change of basis to monomials is used after the initial

expansion. These operations are further discussed here.
The operations of subtraction and addition for the monomial algebra are as follows: given two polynomials $A(\mathbf{x})$ and $B(\mathbf{x})$, with coefficients a and b respectively, the result of addition and subtraction are:

$$c = a \pm b. \tag{8}$$

Where the resulting polynomial $C(\mathbf{x})$ can then be constructed from the coefficients c. The multiplication operation is given by:

$$A(\mathbf{x}) \cdot B(\mathbf{x}) = (\sum_{\mathbf{i},|\mathbf{i}|\leq n} a_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}) \cdot (\sum_{\mathbf{i},|\mathbf{i}|\leq n} b_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}), \tag{9}$$

where all the resulting monomials with orders higher than n are truncated by setting them to zero to keep the resulting polynomial order equal to the order of $A(\mathbf{x})$ and $B(\mathbf{x})$.
The composition operator can be used to create a set of elementary functions (including e.g. a division and exponent operation), and is defined as follows: given a multivariate function $g(\mathbf{x})$ and a d-dimensional multivariate function $\mathbf{y}(\mathbf{x})$, with their respective polynomial approximations $G(\mathbf{x})$ and $\mathbf{Y}(\mathbf{x})$, the composition operator is given by:

$$g(\mathbf{y}(\mathbf{x})) \sim G(\mathbf{y}) \circ \mathbf{Y}(\mathbf{x}). \tag{10}$$

Thus, a general definition is given by:

$$\circ : \mathcal{P}_{\nu,\delta}(\phi) \times [\mathcal{P}_{n,d}(\phi)]^{\delta} \to \mathcal{P}_{n,d}(\phi). \tag{11}$$

To be able to use the elementary functions given by $h(y)$, e.g. $\{1/y, \sin(y), \cos(y), \exp(y), \log(y), \text{etc.}\}$ in the polynomial algebra, the composition operator is used as follows:

$$h(\mathbf{f}(\mathbf{x})) \sim H(y) \circ F(\mathbf{x}). \tag{12}$$

where $H(y)$ is the univariate polynomial approximation of $h(y)$ and $f(\mathbf{x})$ a multivariate function with $F(\mathbf{x})$ its polynomial approximation. In this case:

$$\circ : \mathcal{P}_{n,1}(\phi) \times \mathcal{P}_{n,d}(\phi) \to \mathcal{P}_{n,d}(\phi). \tag{13}$$

The way in which $H(y) \sim h(y)$ is approximated has a large impact on the accuracy and functionality of the polynomial algebra. For the Taylor algebra, the functions are expanded using the well known order-n MacLaurin expansion, which expands around a central point and doesn't require information about the interval over which it is approximated. For the Chebyshev algebra, an expansion on the interval $I = [\underline{y}, \bar{y}]$ is performed using an order 100 Chebyshev interpolation. The Chebyshev series is then obtained by truncating the interpolation to a certain order, and converting it to the monomial basis. A problem with this is that a good estimation of I is needed to perform the approximation of $h(y)$ over this interval, which can give an overestimation of the range for certain algorithms. This problem is discussed in more detail in [12]. If a certain dynamics expression has a lot of elementary functions, this range overestimation can become a large problem.

2.2 Propagation

Consider now a numerical propagation scheme that can propagate the state through the dynamical system given in Eq. (1), given by:

$$\psi(\mathbf{x}_k) \rightarrow \mathbf{x}_{k+1}. \tag{14}$$

The basic idea is to represent all the elements of the state vector $\mathbf{x}$ as elements of the polynomial algebra $(\mathcal{P}_{n,d}(\phi), \otimes)$, then all operations and elementary functions used in ψ and Eq. (1) are also represented using the algebra. Finally, ψ is used to propagate the system in time using the same methods for a single state vector $\mathbf{x}$, but now propagating the full set represented by $\mathbf{X}$.
An example can be given using the Euler scheme, given by:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{f}(\mathbf{x}_k)\Delta t. \tag{15}$$

All the elements of the initial state vector $\mathbf{x}_0$ are expanded in the algebra to give $\mathbf{X}_0$. $\mathbf{f}(\mathbf{x}_k)$ then has all its arithmetic operations and elementary functions changed to the corresponding operations in the polynomial algebra to give at each timestep:

$$\mathbf{X}_{k+1} = \mathbf{X}_k + \mathbf{F}_k\Delta t. \tag{16}$$

where $\mathbf{F}_k$ is the dynamics function represented in the polynomial algebra, evaluated using $\mathbf{X}_k$. This is done at each timestep until the final desired time is reached, at which point $\mathbf{X}_{k_f}$ represents the approximation of the set of final states resulting from the set of initial states, given by Eq.(4).
An often discussed disadvantage of intrusive methods is that they require large adaptions to existing software and methods. However, for GIPA it is possible to use the same orbit propagators and dynamical models found in most astrodynamics applications, and only change the underlying elementary functions and arithmetic operations. This can often be done using templates and overloading, which is possible in a range of modern coding languages. The implementation of the algebra for this work uses the open-source SMART-UQ software package [13].

3 DYNAMICS

This section introduces the dynamical models that govern the motion of the spacecraft within the Didymos binary system. All the physical values and parameters are taken from [14] and shown in table 1. The total equations of motion consist of the primary body's gravity, the secondary body's gravity, the solar radiation pressure (SRP), and the Sun's gravity:

$$\mathbf{F}^I_{tot} = \mathbf{F}^I_{g,prim} + \mathbf{F}^I_{g,sec} + \mathbf{F}^I_{SRP} + \mathbf{F}^I_{g,Sun}, \tag{17}$$

where $\mathbf{F}^I$ is the force expressed in an inertial reference frame. The inertial reference frame taken here has its origin at the center of mass of the primary and its three axes correspond to the principle moments of inertia of the main body. The motion of the system around the Sun is not considered as the simulation times are relatively short. Furthermore, the rotational model of the primary is assumed to be a uniform rotation about its z-axis, which is in close agreement with measurements from recent observations [15]. A body fixed reference frame is then also defined which coincides with the inertial frame at $t = 0$, and a transformation between the two frames can be given with a rotation matrix about the z-axis using the rotational rate of the primary.

The gravitational model used here is the spherical harmonics model up to degree and order 4, as it allows for more fidelity compared to just the $J2$-perturbation model and is less computationally complex than the polyhedron model while still producing accurate predictions. As only the ellipsoidal shape of the two bodies is given (see table 1), an algorithm to convert this shape to a spherical harmonics field is used, which is given in [16]. As was mentioned in section 2.1, the use of a large

Table 1: Physical parameters of the Didymos system, taken from [14].

Total system mass	$5.278{\cdot}10^{11}$ kg $\pm$ $0.54{\cdot}10^{11}$ kg
Mass ratio	0.0093 $\pm$ 0.0013
Heliocentric eccentricity	0.38 $\pm$ $7.7{\cdot}10^{-9}$
Heliocentric semimajor axis	1.64 AU $\pm$ 9.8e-9 AU
Heliocentric inclination	3.41 degrees +/- 2.4e-6 degrees
Primary	
Diameter	780 m $\pm$ 3 m
Rotational Period	2.26 h $\pm$ 0.0001 h
Ellipsoid semi-axes (a, b, c)	399, 392, 380 m
Secondary	
Diameter	164 m $\pm$ 18 m
Ellipsoid semi-axes (a, b, c)	103, 79, 66 km
Binary eccentricity	0.03 (upper limit)
Binary semimajor axis	1.19 km
Binary inclination	0.0 degrees

amount of elementary functions like $\sin$ and $\cos$ can lead to an overestimation problem. As the most commonly used expression for the spherical harmonics gravity acceleration requires a change of variables to spherical coordinates, there are a significant amount of elementary functions used. Thus, in this case a different formulation for the spherical harmonics acceleration is used, given by [17], which relies less on elementary functions.
The gravitational force of the Sun acting on the spacecraft is taken to be a central gravitational field, with the heliocentric position of Didymos given in table 1. Furthermore, the SRP model used in this case is the cannonball radiation pressure model, which assumes a constant reference area facing the Sun, and the magnitude being proportional to the inverse square of the distance towards the Sun [18].

3.1 Simulation Setup

To determine the effectiveness of GIPA with a Taylor and Chebyshev basis for a spacecraft in a binary asteroid system like Didymos, several simulation settings have to be considered. First, for the nominal trajectory a terminator orbit will be chosen, as this has been known to be a stable choice for situations where the SRP has a significant impact on the dynamics [19]. The requirements for the initial conditions for a circular terminator orbit are as follows [20]:

$$\Omega_s = 90^\circ/270^\circ + \nu_h, \tag{18}$$
$$i_s = 90^\circ - i_h, \tag{19}$$
$$a_s < \frac{1}{4}\sqrt{\frac{\mu\beta}{G_1}}, \tag{20}$$

where Ω_s is the right ascension of the ascending node of the spacecraft, ν_h is the true anomaly of the heliocentric orbit, i_s the inclination of the spacecraft, i_h the inclination of the heliocentric orbit, a_s the spacecraft's semi-major axis, μ the asteroids gravitational parameter, β the mass-to-area ratio of the spacecraft, and G_1 a constant equal to $1 \cdot 10^8 \ \frac{kg \cdot km^3}{s^2 \cdot m^2}$. The resulting maximum semi-major axis is equal to 8.13 km, thus initially orbits with a semi-major axis of 7 km will be considered to ensure no escape (considering no state uncertainties). Most of the simulations will be run for 10 days as this is significantly longer than the time in between planned ΔV maneuvers (3-4 days [21]) and thus can give results on how the orbit will evolve due to its natural dynamics in those time spans, and what the risk is if those maneuvers cannot be executed. A Runge-Kutta 4 fixed step-size integrator is used to integrate the equations of motion, with 1 hour steps. The initial uncertainties first considered are 1% on the initial semi-major axis and orbital velocity. The effect of larger uncertainties will be shown in section 4.2. All the values are adimensionalized and scaled by dividing by the position and velocity of Didymoon.

4 RESULTS

Two metrics will be used to determine the accuracy of the GIPA: the root mean square error (RMSE) and the maximum error. A Monte Carlo simulation with $N_s = 1000$ samples will be used to compare against, for which the two performance metrics will then be calculated as follows:

$$RMSE = \sqrt{\frac{1}{N_s}\sum_{i=1}^{N_s}(\hat{x}_i - x_i)^2}, \tag{21}$$

$$E_m = \max_{1 \leq i \leq N_s} |\hat{x}_i - x_i|, \tag{22}$$

where $\hat{x_i}$ is the GIPA calculated state and x is the MC state at the same point in time.
First, the effects of changing various parameters and approaches within the GIPA will be analyzed and discussed. The impact both of the polynomial expansion degree and the range estimation method are determined. Secondly, the impact of choosing different initial conditions for the orbit and GIPA are shown. The main focus will be on the initial uncertain set size, semi-major axis, and eccentricity.

4.1 GIPA Parameters

The degree of the polynomial expansion is an important factor that firstly needs to be investigated for this specific problem. Figure 2 shows both the RMSE and maximum error as a function of the polynomial degree. Furthermore, it shows the runtime for the application on a computer with an Intel© 8th generation core i7 processor and 16Gb of RAM. For reference, the runtime for the Monte Carlo simulation is also shown in the figure. It can be seen that at lower degrees the Taylor basis performs better, and at a degree of 5 for both bases the RMSE flattens out and no significant gains are obtained if the polynomial expansion degree is increased. The maximum error does seem to improve slightly if the degree is higher. This does come at the cost of a significant increase in runtime between a degree 4 and degree 5 polynomial compared to the increase in accuracy. However, if then figure 3 is observed, it can be seen that earlier in the propagation, the RMSE of the degree 5 polynomial is much lower and only becomes comparable with the degree 4 polynomial towards the end of the simulation. As was mentioned in section 2.1, a good estimation of the range I of the expansion is needed to estimate the elementary functions. The fastest method for this is to take the sum and subtraction of all the coefficients to obtain the maximum and minimum, respectively, for each state variable. In [12] it was shown that this method can lead to large overestimation errors. Thus another method is tested here, where the range is obtained by randomly selecting n samples and evaluating them using the expansion. From this, the maximum and minimum values are then saved and used as the range.

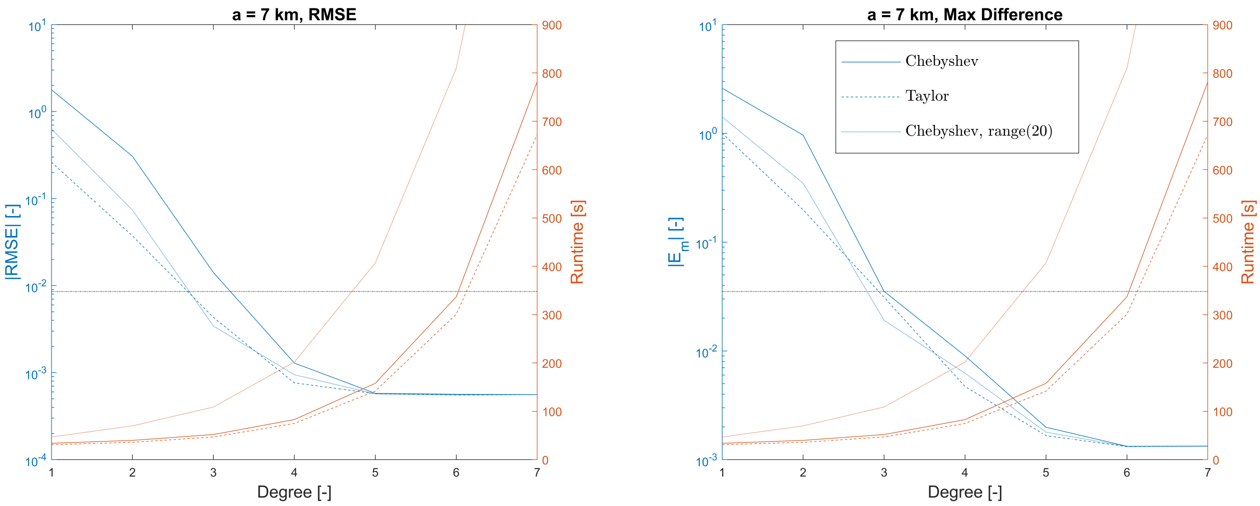

Figure 2: The accuracy after 10 days of propagation for different polynomial degree expansions and coefficient range estimation algorithms. The horizontal line represent the MC runtime.

Figure 2 and 3 both show how these methods give different results when using 20 samples. It can be seen that the sample based method requires a significantly longer runtime, but at lower degrees produces a much better result. However, at a degree of 5 and higher, the sample based method does not seem to result in much improvement compared to the coefficient based method. If the evolution of the accuracy is observed in figure 3, then for a polynomial degree of 4 the increase in accuracy at earlier times is clearly visible. However, for a polynomial degree of 5, it can be clearly seen that a maximum accuracy has already been reached and that changing the range estimation function doesn't influence the accuracy much. For the coming results, a polynomial of degree 5 will be considered with normal coefficient based range estimation, unless stated otherwise.

4.2 Initial Conditions

In this section, the influence of choosing different uncertainty values (represented in GIPA by the size of the to be propagated set) and orbital parameters is discussed. First, a range of possible initial set sizes is investigated for the state variables. The relative set size is considered, where the sizes are determined as a percentage of the initial semi-major axes and orbital velocity for the position and velocity state variable respectively. A grid of relative set sizes with values $\sigma_p = [1\%, 2\%, 3\%, 4\%, 5\%]$, $\sigma_v = [1\%, 3\%, 5\%]$ is taken, where σ represents the radius of the uncertainty ellipsoid.
The results of the accuracy at the final time for the whole grid for both the Taylor and Chebyshev bases are shown in figure 4. It can be seen that the difference in accuracy between the $(1\%, 1\%)$ size and the $(5\%, 5\%)$ (corresponding to $(70m, 7mm/s)$ and $(350m, 3.5cm/s)$ respectively) differs two to three orders of magnitude in both RMSE and maximum error. There is, furthermore, no clear distinction in sensitivity to either the position or velocity state variables with respect to an increase in RMSE. For lower sizes, the difference in accuracy between the bases is small. However, when increasing the set size, the relative difference between the accuracy of both bases shows that the Chebyshev basis performs better. This is due to the fact that the Taylor polynomial loses accuracy further from the central expansion point, whereas the Chebyshev uniformly converges over the whole set. For the results discussed afterwards, a set size of $(1\%, 1\%)$ is chosen. The conclusions taken there still hold for larger set sizes, as only the absolute accuracy shown might differ.
Two other dynamical parameters that have a large influence on the dynamics are discussed here: the

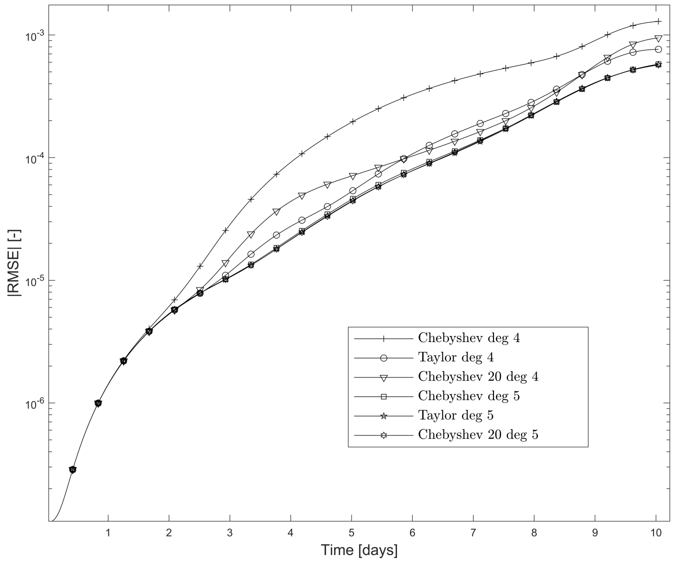

Figure 3: The change in accuracy over time for various GIPA settings.

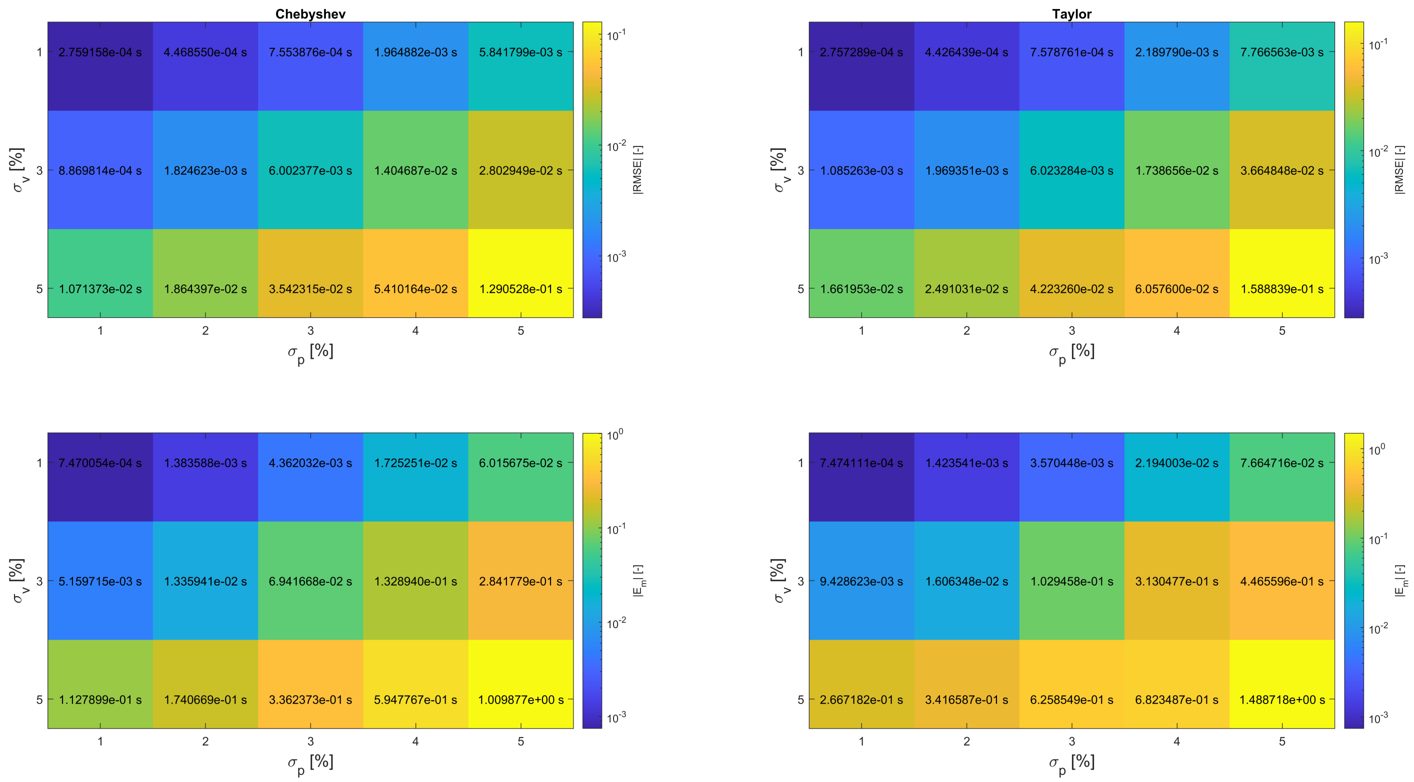

Figure 4: Grid of several initial set sizes and the corresponding RMSE and maximum error (norm over whole state vector) at the final time.

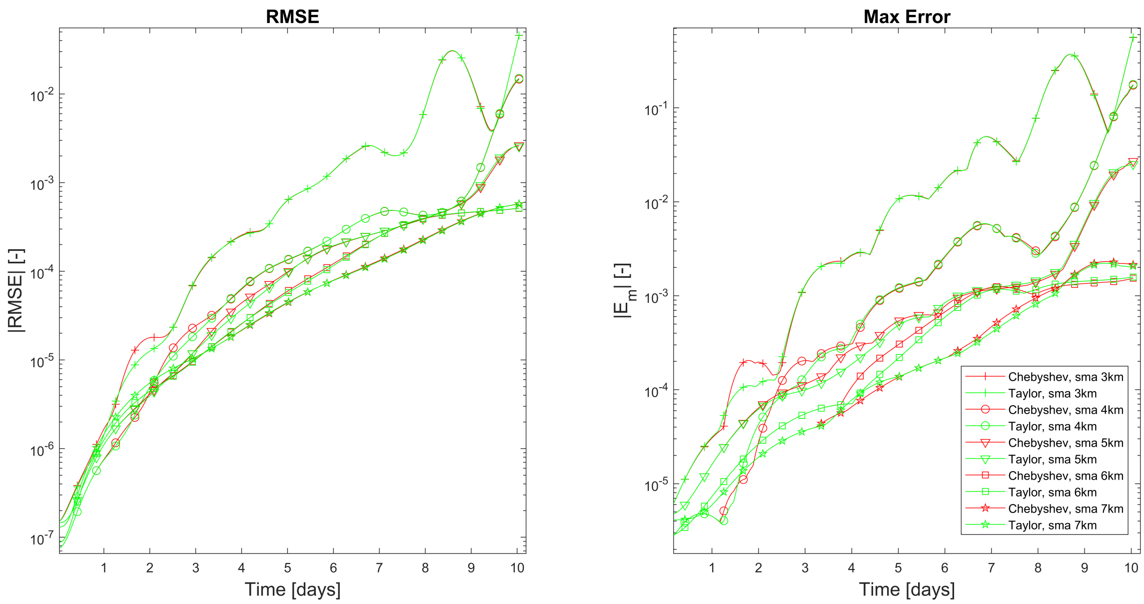

Figure 5: The influence of the starting semi-major axis on the accuracy development over time. The green lines are for the Taylor basis and the red lines for the Chebyshev basis.

initial semi-major axis a and eccentricity e. Figure 5 shows the influence of changing the initial a. There, it can be seen that there is clear trend between the accuracy and the altitude. As the altitude decreases, the dynamics become more complex and the accuracy of the GIPA decreases. Except for slight variations at earlier times, the performance of both bases is comparable at all values of a.
In figure 6, a is fixed at $7km$ and the eccentricity is increased in 5 steps. For higher eccentricity values it can be seen that the accuracy significantly decrease as the spacecraft moves closer to periapsis. For smaller values of the eccentricity, it can be seen that after crossing periapsis the accuracy increases again. This is shown in more detail in 3 dimensions in figure 7 for $e = 0.3$. The RMSE initially increases before approaching and reaching the peak when at the periapsis. Here it can be seen in figure 7 that the accuracy, especially at the ends of the set, is relatively low compared to previous steps. However, instead of remaining at the same RMSE or increasing RMSE afterwards, the RMSE decreases again when approaching apoapsis. This shows clearly that the accuracy is highly dependent on the complexity of the dynamics, which is higher near the bodies. Compared with the influence of a, the two bases show much more difference in their performance. Before approaching the periapsis, the performance is similar, however the Taylor polynomial retains much more accuracy after periapsis passage compared to the Chebyshev basis. This could be due to the fact that the Chebyshev basis builds up overestimation error over time, which it cannot lose again after periapsis. For the Chebyshev basis at $e = 0.5$ and $e = 0.7$, a jump in RMSE and maximum error happens after a period of time. The specific source of these jumps is unknown and thus caution needs to be taken when using GIPA for these types of highly eccentric orbits.

5 FINAL REMARKS

In this paper the GIPA method has been applied to uncertainty propagation for a spacecraft orbiting the binary asteroid system Didymos. This intrusive method requires adapting the dynamics, but produces a full model of the dynamics which allows for a more in depth analysis of the effects of uncertainties

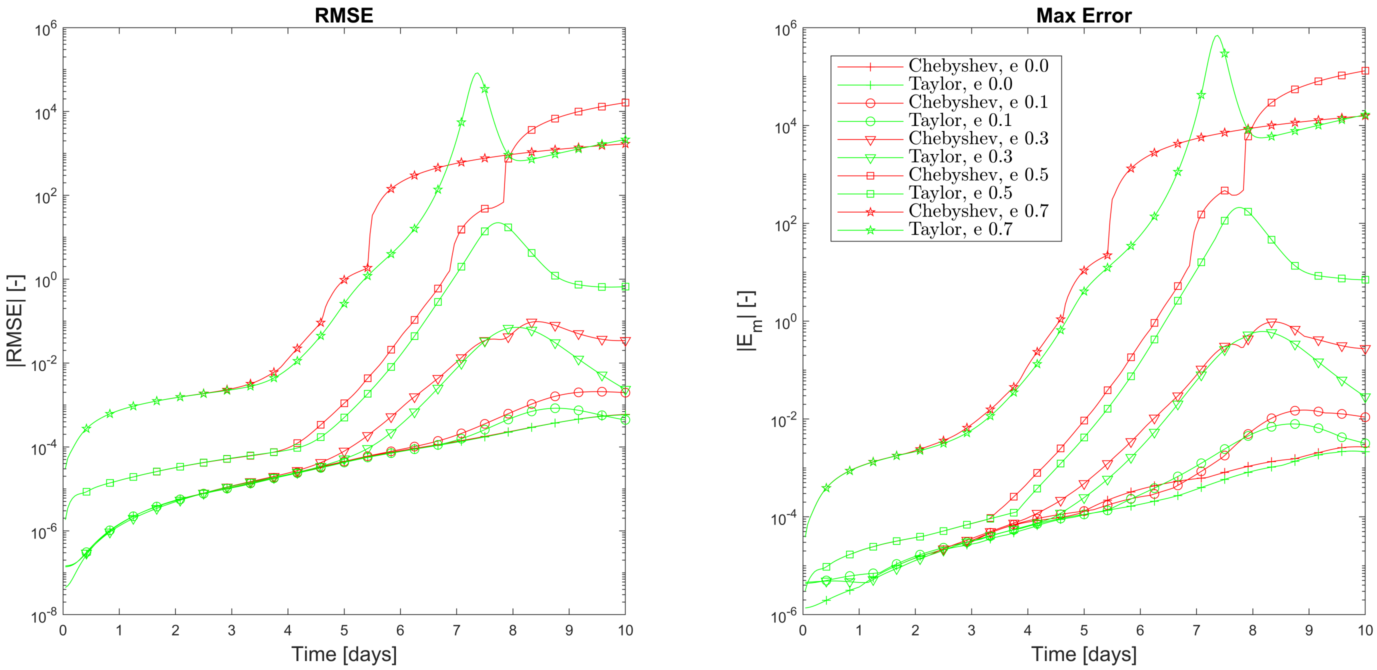

Figure 6: The influence of the starting eccentricity on the accuracy development over time at a semi-major axis of 7 km.

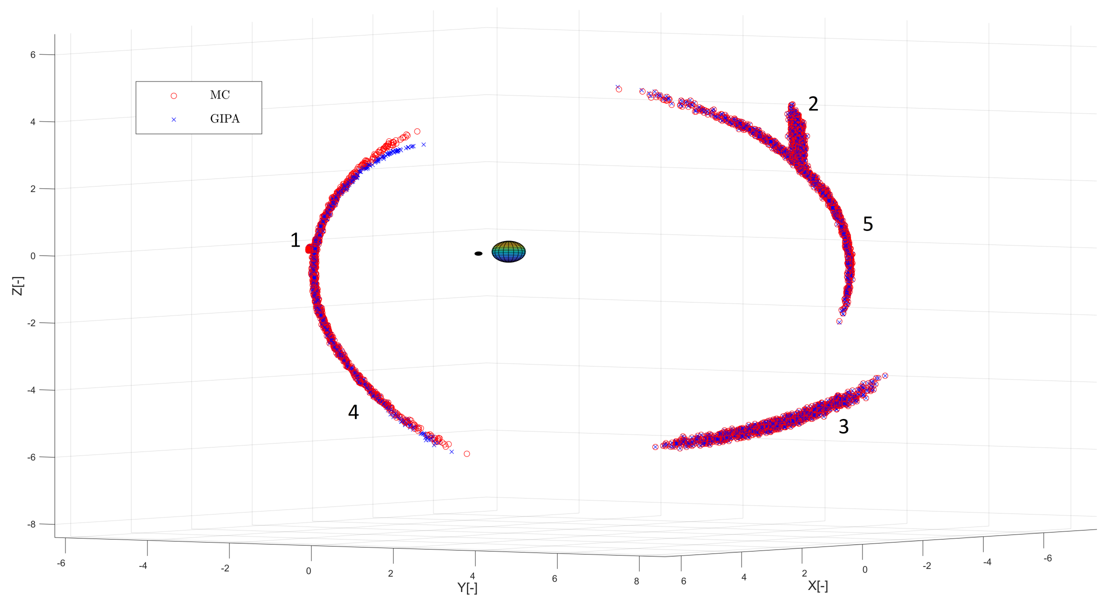

Figure 7: Monte Carlo versus Taylor basis GIPA of the $e = 0.3$ orbit for 5 points in time. The order in time for the different steps is represented by the number next to it.

on the state of the spacecraft. The focus in this paper was on how well GIPA is able to analyze uncertainty in the initial state of the spacecraft and how this uncertainty evolves over time. Both a Chebyshev basis and a Taylor basis for the GIPA were used and their differences were discussed in more detail.

Initially, the influence of the different GIPA settings on the accuracy compared to a Monte Carlo result were discussed. It was determined that for lower degrees, the Taylor basis showed better results, but after increasing the degree to 5, both bases reached similar performance and flattened out in terms of RMSE. For lower maximum errors the degree could be increased but this would increase the runtime significantly, thus a degree of 5 was chosen to be optimal in this case.

Additionally, it was found that increasing the size of the uncertainties had a significant effect on the accuracy of GIPA. This effect was worse for the Taylor basis, as there it was found that the accuracy at higher sizes was worse compared to the Chebyshev basis.

In terms of the effect of the initial conditions on the performance of the GIPA, it was determined that close orbits and/or highly eccentric orbits show overall lower accuracy than higher, more circular orbits. This is due to larger changes in the dynamics over time and the higher magnitude accelerations experienced for these accelerations. A sudden jump was observed for the propagation of highly eccentric orbits with a Chebyshev basis, for which the cause needs to be investigated more in depth.

Concluding, this study showed the potential of the GIPA tool in analysing the nonlinear dynamics under uncertainties. For highly eccentric orbits the accuracy is relatively low, thus further studies can focus on improving the accuracy of GIPA for these types of orbits. Furthermore, the use of the surrogate function produced by GIPA to analyze the dynamical structure of specific systems, and possibilities of using the, on-ground produced, GIPA model for on-board applications can be explored in more detail.

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