Secular Dynamics around small bodies with Solar Radiation Pressure

J. Feng, X.-Y. Hou*

School of Astronomy and Space Sciences, Nanjing University, China

Abstract

This study focuses on the secular dynamics of orbital motion around asteroids with joint effects of C_{20} term and solar radiation pressure (SRP). With their combined influences, two specific orbits are firstly studied, namely the solar terminator orbit (STO) and the heliotropic orbit. Then, the global orbital motion other than the two specific ones is investigated. Finally, the influence of the asteroid's orbital eccentricity around the Sun on the secular dynamics is discussed.

Keywords: numerical, celestial mechanics, minor planets, asteroids

1. Introduction

With combined effects of the gravity and the SRP, the secular dynamics of orbital motion in the vicinity of celestial bodies has been a research focus for many years, as a result of but not limited to the following reasons: (1) particles/dusts close to or on the surface of the bodies are significantly affected by the SRP. They might escape from, collide with or stay for long time near the body [3, 10, 11, 19, 22, 26]; (2) small bodies in the solar system have become popular targets for deep

*Corresponding author

Email address: `silence@nju.edu.cn` (X.-Y. Hou)

1. INTRODUCTION

space explorations world widely. Weak gravity of these bodies and large area-to-mass ratio of the spacecraft requires the consideration of the joint effects for mission orbit design and control, etc. [5, 7, 27, 29, 34]; (3) the space debris have become an emergent research topic because of its exponential rate of increase and potential hazards to spacecraft. For the high area-to-mass ratio and high-altitude debris, their orbital motion is strongly affected by the SRP [4, 6, 31, 32, 35].

Using the point-mass gravity of the body, only considering the first-order effect of the SRP, and ignoring the body's orbital eccentricity around the Sun, Mignard [20] found that the averaged dynamics around the body is integrable and gave the integral in Keplerian elements [21]. Considering the body's orbital eccentricity and using the true anomaly instead of the time as the independent variables, Scheeres [27] proved that the equations of motion (EOM) can be simplified to the same form as those for case of the circular orbit. Using the averaged system, Richter and Keller [23] gave a simplified description of the solution based on a constant vector and two vectors spinning around it. Using the eccentricity vector and the angular momentum vector, Scheeres [29], Rosengren and Scheeres [24] derived the time explicit form of the solution, which is useful for practical applications.

With the joint effects of the SRP and the body's point-mass gravity, there are two specific kinds of orbits. The first type is the STO, whose inclination is around 90° and the orbital plane points toward or away from the Sun [30, 13]. STO is stable and allows stable quasi-periodic motion in its vicinity [2]. Actually, the STO is the halo family bifurcating from the planar Lyapunov family around the L2 point of the circular restricted three body problem (CRTBP) or Hill problem

1. INTRODUCTION

[1, 2, 9, 18, 14]. The other type is the Heliotropic orbit, whose orbital plane completely lies in the small body's orbital plane around the Sun and whose perigee always points toward or away from the Sun. It is not always stable and has been extensively studied for the problem of particle's motion near planets [20, 10, 11, 26, 12]. Recently, its potential applications in asteroid explorations [16, 25] and in Earth missions [6] have attracted researchers' attentions.

This study focuses on the orbital motion around the small solar system bodies. Because of their highly irregular shapes, the effect of non-spherical gravity field (more specifically, the zonal terms) on the long-term behaviors of the orbital motion around them cannot be ignored for motions not far away from the asteroids. Although there are studies on the long-term joint effects between non-spherical terms and the third body perturbations [35, 4], there are much less work on the long-term dynamics between non-spherical terms and the SRP. Scheeres [28] briefly discussed the influence of the second order and degree gravitational field on the stability of the STO. Lantukh et al. [16] shows that most non-planar frozen families (including the STO) are unstable when the C_{20} term and the SRP are simultaneously considered. In this paper, our investigation on the STO shows that even very small value of the C_{20} term introduces instability to the orbital motion. The divergence speed of the STO increases with the C_{20} values, therefore, the STO seems to be 'stable' within short-time scales if C_{20} is small. Interestingly, if C_{20} increases larger than a critical value, the orbit becomes stable instead (see Section 3.2 for details). This is a phenomenon never pointed out before. For prograde and retrograde Heliotropic orbit, some interesting phenomena are discovered by applying a different methodology from previous studies [16, 25] (see

2. DYNAMICAL MODELLING

Section 4 for details). And a better criterion for their stability is proposed. In addition to these two specific kinds of orbits, global dynamics is discussed for two subcases, i.e. the SRP dominant case and the C_{20} dominant case, respectively. Finally, the effect of the small body's orbit eccentricity around the Sun is briefly discussed.

The area-to-mass ratio used in this study ranges from 0.01 to 0.03 $\text{kg} \cdot \text{m}^{-2}$, corresponding to typical values of the spacecraft around a small body [34]. The radius of the small body is taken between 500 m and 1 km . Finally, to simplify the model, the shadow effect of the SRP is not considered in this study.

2. Dynamical Modelling

2.1. Solar Radiation Pressure

Denoting the area-to-mass ratio of the spacecraft as S/m , the solar radiation flux at a reference distance r_a from the Sun as $\rho_\odot$, the light speed as c , and the albedo of the asteroid as κ , the EOM of the spacecraft in the asteroid-centered inertial frame, shown in the left plot of Fig.1, is given as

$$\mathbf{R}'' = \frac{d^2 \mathbf{R}}{dt^2} = -G m_a \frac{\mathbf{r}}{r^3} - (1 + \kappa) \frac{\rho_\odot r_a^2}{c} \frac{S}{m} \frac{\Delta}{\Delta^3} \quad (1)$$

in which m_a is the mass of the asteroid and G is the gravitational constant. Taking the unit of length, mass and time as

$$[L] = a_e, [M] = m_a, [T] = \sqrt{\frac{[L]^3}{G [M]}} \quad (2)$$

2. DYNAMICAL MODELLING

Figure 1: Left: The illustration of SRP; Right: Sketch of the asteroid-centered frame rotating with the Sun and the angles g and h in Eq.18.

where a_e a symbol indicating the reference radius of the asteroid used as the length unit. In our work, it is defined by following Eq.21. **Introducing the following non-dimensional quantities**

$$\mathbf{r} = \frac{\mathbf{R}}{[L]}, \mathbf{r}' = \frac{\mathbf{R}'}{[L]}, \delta = \frac{\Delta}{[L]}, \tau = \frac{t}{[T]}$$

$$\ddot{\mathbf{r}} = \frac{d^2 \mathbf{r}}{d\tau^2} = \frac{[T]^2 d^2 \mathbf{r}}{dt^2} = \frac{[T]^2}{[L]} \frac{d^2 \mathbf{R}}{dt^2} = \frac{[T]^2}{[L]} \mathbf{R}''$$

and substituting them into Eq.1, a non-dimensional form of Eq.1 can be written as

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3} - \beta_{\odot} \frac{\delta}{\delta^3} = \frac{\partial V_0}{\partial \mathbf{r}} + \frac{\partial V_1}{\partial \mathbf{r}} \quad (3)$$

in which

$$V_0 = \frac{1}{r}, V_1 = -\frac{\beta_{\odot}}{\delta}, \beta_{\odot} = (1 + \kappa) \frac{\rho_{\odot} r_a^2 S}{c} \frac{a_e^2}{m G m_a} \quad (4)$$

2. DYNAMICAL MODELLING

and the SRP can be expanded in the following form

$$V_1 = -\frac{\beta_\odot}{\Delta} = -\frac{\beta_\odot}{r'} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_n(\cos \psi) \quad (5)$$

The term with $n = 0$ is a constant, and the terms with $n \geq 2$ are very small. Therefore, they are both ignored in this study and only the term with $n = 1$ is taken into account. The simplified SRP is written as

$$V_1 \approx -\beta_\odot \frac{r}{r'^2} \cos \psi = -\beta_\odot \frac{\mathbf{r} \cdot \mathbf{r}'}{r'^3} \quad (6)$$

2.2. The C_{20} and C_{22} terms

In the inertial frame with origin at the asteroid's barycenter and reference plane as the equatorial plane, the potential function of the C_{20} and C_{22} terms (where $S_{22} = 0$ by selecting the proper coordinate) **are usually expressed in Keplerian orbital elements of the motion around the asteroid, i.e. the inclination i , the longitude of the ascending node Ω , the argument of the pericenter ω and the true anomaly f .**

$$\begin{aligned} V_{20} &= -\frac{C_{20}}{2r^3} \left[\left(1 - \frac{3}{2}\sin^2 i\right) + \frac{3}{2}\sin^2 i \cos 2u \right] \\ V_{22} &= \frac{3C_{22}}{4r^3} \left[(1 + \cos i)^2 \cos(2u + 2\Omega_e) \right. \\ &\quad \left. + (1 - \cos i)^2 \cos(2u - 2\Omega_e) + 2\sin^2 i \cos 2\Omega_e \right] \end{aligned} \quad (7)$$

in which $u = f + \omega$ and Ω_e is defined as

$$\Omega_e = (\Omega - S_0) - n_e(t - t_0) \quad (8)$$

2. DYNAMICAL MODELLING

where n_e is the rotation rate of the asteroid, and S_0 is the initial phase angle of the x -axis of the asteroid's body-fixed frame with respect to the inertial frame at initial time epoch t_0 .

2.3. The averaged dynamics

From the orbital geometry, it is known that

$$\cos \psi = A \cos f + B \sin f \quad (9)$$

$$\begin{aligned} A &= \frac{1}{4} (1 - \cos i) [(1 + \cos i') \cos (\omega - \theta + u') \\ &\quad + (1 - \cos i') \cos (\omega - \theta - u')] \\ &\quad + (1 + \cos i) [(1 + \cos i') \cos (\omega + \theta - u') \\ &\quad + (1 - \cos i') \cos (\omega + \theta + u')] \\ &\quad + 2 \sin i \sin i' [\cos (\omega - u') - \cos (\omega + u')] \\ B &= -\frac{1}{4} (1 - \cos i) [(1 + \cos i') \sin (\omega - \theta + u') \\ &\quad + (1 - \cos i') \sin (\omega - \theta - u')] \\ &\quad + (1 + \cos i) [(1 + \cos i') \sin (\omega + \theta - u') \\ &\quad + (1 - \cos i') \sin (\omega + \theta + u')] \\ &\quad + 2 \sin i \sin i' [\sin (\omega - u') - \sin (\omega + u')] \end{aligned} \quad (10)$$

in which

$$\theta = \Omega - \Omega', u' = f' + \omega' \quad (11)$$

and i', f', Ω, ω' are respectively the orbit inclination, true anomaly, longitude of the ascending node, argument of perigee of the Sun's orbit w.r.t. the asteroid.

2. DYNAMICAL MODELLING

Taking the asteroid's heliocentric orbital plane as the reference plane, we have $i', \Omega' = 0$. Further, ignoring the asteroid's orbital eccentricity e' and argument of perigee ω' , Eq.10 can be simplified as follows

$$\begin{aligned} A &= \frac{1}{4} [2(1 - \cos i) \cos(\omega - \Omega + M') \\ &\quad + 2(1 + \cos i) \cos(\omega + \Omega - M')] \\ B &= -\frac{1}{4} [2(1 - \cos i) \sin(\omega - \Omega + M') \\ &\quad + 2(1 + \cos i) \sin(\omega + \Omega - M')] \end{aligned} \quad (12)$$

Substituting Eq.9 into Eq.6 and then averaging, it can be obtained that

$$\langle V_1 \rangle = \frac{1}{T} \int_0^T V_1 dt = \frac{3}{2} \tilde{\beta}_\odot e a A \quad (13)$$

in which $T = a^{-\frac{3}{2}}$ is the non-dimensional period of the Keplerian orbit. a and e are the semi-major axis and orbit eccentricity of the Keplerian orbit, and

$$\tilde{\beta}_\odot = \frac{\beta_\odot}{a'^2} = (1 + \kappa) \frac{\rho_\odot}{c} \left(\frac{r_a}{a'} \right)^2 \frac{S}{m} \frac{a_e^2}{G m_a} \quad (14)$$

For a fast-rotating asteroid, the Ω_e in Eq.8 can be treated as a fast variable. Therefore $\langle V_{22} \rangle = 0$, and the average of V_{20} is given as

$$\langle V_{20} \rangle = \frac{C_{20}}{4a^3} (1 - e^2)^{-\frac{3}{2}} (1 - 3\cos^2 i) \quad (15)$$

Using the Delaunay variables

$$\begin{aligned} \tilde{l} &= M, \tilde{g} = \omega, \tilde{h} = \Omega \\ \tilde{L} &= \sqrt{a}, \tilde{G} = \tilde{L}\sqrt{1 - e^2}, \tilde{H} = \tilde{G} \cos i \end{aligned} \quad (16)$$

2. DYNAMICAL MODELLING

the Hamiltonian of the system is expressed as

$$\begin{aligned}
 H &= -\frac{1}{2\tilde{L}^2} - \langle V_1 \rangle - \langle V_{20} \rangle \\
 &= -\frac{1}{2\tilde{L}^2} - \frac{C_{20}}{4\tilde{L}^3\tilde{G}^3} \left(1 - \frac{3\tilde{H}^2}{\tilde{G}^2} \right) - \frac{3}{4}\tilde{\beta}_\odot\tilde{L}\sqrt{\tilde{L}^2 - \tilde{G}^2} \\
 &\quad \left[\left(1 - \frac{\tilde{H}}{\tilde{G}} \right) \cos(\tilde{g} - \tilde{h} + M') + \left(1 + \frac{\tilde{H}}{\tilde{G}} \right) \cos(\tilde{g} + \tilde{h} - M') \right]
 \end{aligned} \tag{17}$$

With the following canonical transform

$$\begin{aligned}
 l &= \tilde{l}, g = \tilde{g}, h = \tilde{h} - M' = h - n't \\
 L &= \tilde{L}, G = \tilde{G}, H = \tilde{H}
 \end{aligned} \tag{18}$$

the x -axis of the asteroid's body-fixed frame is transformed to the direction pointing towards the Sun, as indicated in the right frame of Fig.1. The Hamilton function is then transformed to

$$\begin{aligned}
 K &= -\frac{1}{2L^2} - \frac{C_{20}}{4L^3G^3} \left(1 - \frac{3H^2}{G^2} \right) - \frac{3}{4}\tilde{\beta}_\odot L\sqrt{L^2 - G^2} \\
 &\quad \left[\left(1 - \frac{H}{G} \right) \cos(g - h) + \left(1 + \frac{H}{G} \right) \cos(g + h) \right] - n'H
 \end{aligned} \tag{19}$$

As the reference plane is defined as the asteroid's equatorial plane and Eq.13 is derived by assuming the $x - y$ plane as the asteroid's orbital plane, Eq.19 is only valid when the equatorial plane and orbital plane of the asteroid coincide with each other, namely the asteroid's obliquity is 0° or 180° . According to Lantukh et al. [16], about 25 percent of small asteroids (**diameter < 30 km**) possess this property, which is related to the YORP effect Vokrouhlicky et al. [33]. The EOM

2. DYNAMICAL MODELLING

corresponding to the dynamical system expressed by Eq.19 is given as

$$\begin{aligned}
 \dot{g} &= \frac{3C_{20}}{4L^3G^4} \left(1 - \frac{5H^2}{G^2}\right) \\
 &+ \frac{3}{4}\tilde{\beta}_\odot \frac{LG}{\sqrt{L^2 - G^2}} \left[\left(1 - \frac{H}{G}\right) \cos(g - h) + \left(1 + \frac{H}{G}\right) \cos(g + h) \right] \\
 &- \frac{3}{4}\tilde{\beta}_\odot \frac{LH\sqrt{L^2 - G^2}}{G^2} [\cos(g - h) - \cos(g + h)] \\
 \dot{h} &= \frac{3C_{20}}{2L^3G^4} \frac{H}{G} + \frac{3}{4}\tilde{\beta}_\odot \frac{L\sqrt{L^2 - G^2}}{G} [\cos(g - h) - \cos(g + h)] - n' \\
 \dot{G} &= -\frac{3}{4}\tilde{\beta}_\odot L\sqrt{L^2 - G^2} \left[\left(1 - \frac{H}{G}\right) \sin(g - h) + \left(1 + \frac{H}{G}\right) \sin(g + h) \right] \\
 \dot{H} &= \frac{3}{4}\tilde{\beta}_\odot L\sqrt{L^2 - G^2} \left[\left(1 - \frac{H}{G}\right) \sin(g - h) - \left(1 + \frac{H}{G}\right) \sin(g + h) \right]
 \end{aligned} \tag{20}$$

2.4. Statement

Before proceeding to further details, we make a short statement here. The tri-axial shape model with semi axes $a \geq b \geq c$ for the asteroid is used. The density is ρ in this study, which takes the value of $2000 \text{ kg} \cdot \text{m}^{-3}$. The reference radius in this study is taken as

$$a_e = (abc)^{1/3} = a(\beta\gamma)^{1/3}, \beta = b/a, \gamma = c/a \tag{21}$$

where β and γ are shape parameters normalized by a . The C_{20} and C_{22} terms can be obtained as

$$\begin{aligned}
 C_{20} &= \frac{2c^2 - a^2 - b^2}{10a_e^2} = \frac{a^2(2\gamma^2 - \beta^2 - 1)}{a_e^2 10} \\
 C_{22} &= \frac{a^2 - b^2}{20a_e^2} = \frac{a^2(1 - \beta^2)}{a_e^2 20}
 \end{aligned} \tag{22}$$

3. STO

from which, it is easily to get that

$$a = \frac{a_e}{(\beta\gamma)^{1/3}}, C_{20} = \frac{(2\gamma^2 - \beta^2 - 1)}{10(\beta\gamma)^{2/3}}, C_{22} = \frac{1 - \beta^2}{20(\beta\gamma)^{2/3}} \quad (23)$$

According to Eq.14, for fixed values of S/m and the relative perturbative strength of the SRP with respect to the asteroid gravity is determined by the value of a_e . Fixing the value of a_e , the values of C_{20} and C_{22} terms can be varied by simultaneously adjusting the shape parameters β, γ .

3. STO

According to the research of Mignard and Henon [21], Richter and Keller [23], and Scheeres [27], the system of Eq.19 is completely integrable if only considering the SRP (i.e. $C_{20} = 0$). Scheeres et al. [30] found two types of equilibrium points (EPs), which correspond respectively to the STO and the Heliotropic orbit that are described in the Introduction. Before preceding to further details, we make clear that we are dealing with the averaged Hamiltonian Eq.19, i.e. the mean (or the averaged) but not the instantaneous orbital elements of the orbits. As a result, an EP of Eq.20 only means that the averaged orbit remains fixed w.r.t. the Sun-asteroid line (see Eq.18). In the physical space, the orbit is still moving around the asteroid because the mean anomaly M (which does appear in Eq.20 due to the averaging process) constantly changes with time. This section will discuss the effect of the C_{20} term on the STO, and leave the Heliotropic orbit to the next section.

3. STO

 Table 1: Different combinations of g and h satisfying Eq.25

Situation	1	2	3	4	5	6	7(STO)	8(STO)
g	0	π	π	0	$\pi/2$	$-\pi/2$	$\pi/2$	$-\pi/2$
h	0	π	0	π	$-\pi/2$	$\pi/2$	$\pi/2$	$-\pi/2$

As an EP of Eq.20, the STO satisfies

$$\dot{g} = \dot{h} = \dot{G} = \dot{H} = 0 \quad (24)$$

According to Eq.20, the last two equalities of Eq.24 ($\dot{G} = \dot{H} = 0$) require

$$\left(1 - \frac{H}{G}\right) \sin(g - h) = 0, \quad \left(1 + \frac{H}{G}\right) \sin(g + h) = 0 \quad (25)$$

Since the STO is non-planar, it is obvious that $1 \mp H/G \neq 0$, which means $\sin(g \mp h) = 0$. Assuming $g \in (-\pi, \pi]$, $h \in (-\pi, \pi]$, several possible cases of g, h values are given in Table1. However, for some specific combinations of C_{20} and $\tilde{\beta}_{\odot}$, not all the situations in Table1 can make the first two equalities of Eq.24.

For cases 1,2 (or 3,4) in Table1, the following equations can be obtained

$$\begin{aligned} \dot{g} = f_1(G, H) &= \frac{3C_{20}}{4L^3G^4} \left(1 - \frac{5H^2}{G^2}\right) \pm \frac{3}{2}\tilde{\beta}_{\odot} \frac{LG}{\sqrt{L^2 - G^2}} = 0 \\ \dot{h} = f_2(G, H) &= \frac{3C_{20}}{2L^3G^4} \frac{H}{G} - n' = 0 \end{aligned} \quad (26)$$

in which the upper and lower signs correspond to cases 1, 2 and 3, 4 respectively. For cases 5, 6 (or 7, 8) in Table1, the following equations can be obtained

$$\begin{aligned} \dot{g} = f_1(G, H) &= \frac{C_{20}}{2L^6G^2} \left(1 - \frac{5H^2}{G^2}\right) \pm \tilde{\beta}_{\odot} \frac{H}{\sqrt{L^2 - G^2}} = 0 \\ \dot{h} = f_2(G, H) &= \frac{3C_{20}}{2L^3G^4} \frac{H}{G} \mp \frac{3}{2}\tilde{\beta}_{\odot} \frac{L\sqrt{L^2 - G^2}}{G} - n' = 0 \end{aligned} \quad (27)$$

3. STO

in which the upper and lower sign correspond to situations 5-6 and 7-8 respectively. Since $C_{20} < 0$, cases 3, 4 are always impossible, and cases 1, 2, 5, 6 are possible depending on the values of C_{20} and $\tilde{\beta}_\odot$. The stability of the EP for cases 1, 2, 5, 6 also changes with the values of C_{20} and $\tilde{\beta}_\odot$. Readers can refer to the work of Lantukh et al. [16] for more details on these EPs. Cases 7, 8 are always possible, and they are the STOs which are extensively studied by previous researchers and also the focus of this section.

3.1. Eccentricity and inclination

When $C_{20} = 0$, it can be obtained from Eq.27 that $H = 0$, namely the orbital inclination is 90° . From the second equation of Eq.27, it can be derived

$$G = \left(L^2 + \left(\frac{2n'}{3\tilde{\beta}_\odot} \right)^2 \right)^{-1/2} L^2 \quad (28)$$

Similar with the definition in Mignard and Henon [21], Scheeres [29], the solar radiation angle Λ is introduced as

$$\tan \Lambda = \frac{3\tilde{\beta}_\odot L}{2n'} \quad (29)$$

Then Eq.29 is actually equal to

$$e = \cos \Lambda \quad (30)$$

which indicates that the orbital eccentricity and inclination are constant for the given orbital radius r_a of the asteroid and S/m of the spacecraft. However, the situation is different after taking into account the C_{20} term. Taking $a_e = 500 m$ and $r_a = 1 AU$ as an example, Fig.2 gives the variations of $e = \sqrt{1 - (G/L)^2}$ as

3. STO

Figure 2: Curves of e and i of STO as a function of the C_{20} term for $S/m = 0.001, 0.01, 0.1 \text{ kg}^{-1} \cdot \text{m}^2$, respectively. In each plot, the black, red dashed, blue dot and pinkish red dashed dot lines represent $a = 2, 3, 5, 10$, respectively.

a function of the C_{20} term for $S/m = 0.001, 0.01, 0.1 \text{ kg}^{-1} \cdot \text{m}^2$, respectively. It can be observed that:

1. With the increase of the C_{20} term, the eccentricity of the STO decreases in general. However, the decreasing tendency of e is less obvious when the orbital altitude is high, as the influence of the C_{20} term weakens with the increasing orbital altitude;
2. The introduction of the C_{20} term makes the inclination of STO larger than 90° . For high-altitude orbit, the inclination increases with the increase of the C_{20} term, but this phenomenon is not obvious with the increase of orbital altitude. For low-altitude orbits, the inclination does not change monotonously as a function of the C_{20} term.

3. STO

3.2. Orbital Stability

Denoting $\mathbf{X} = (g, h, G, H)$, Eq.20 can be written in the following form

$$\dot{\mathbf{X}} = F(\mathbf{X}) \quad (31)$$

which can be expanded at the EP denoted as X_e (i.e. the STO with $g = h = \pi/2$ and G, H given by Eq.27. Only considering the linearized term, the linearized equation can be obtained as

$$\Delta \dot{\mathbf{X}} = A \Delta \mathbf{X} = \left. \frac{\partial F(\mathbf{X})}{\partial \mathbf{X}} \right|_{\mathbf{X}=\mathbf{X}_e} \Delta \mathbf{X} \quad (32)$$

in which matrix A is a 4×4 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \quad (33)$$

The expressions of $a_{ij}, i, j = 1 \dots 4$ are given in Appendix. For $g = h = \pi/2$, we have $a_{11} = a_{12} = a_{21} = a_{22} = 0$ and $a_{33} = a_{34} = a_{43} = a_{44} = 0$. The eigenvalues of A satisfy

$$s(\lambda) = \lambda^4 + \bar{b}\lambda^2 + \bar{c} = 0 \quad (34)$$

in which

$$\begin{aligned} \bar{b} &= -(a_{23}a_{32} + a_{24}a_{42} + a_{13}a_{31} + a_{14}a_{41}) \\ \bar{c} &= a_{13}a_{24}(a_{31}a_{42} - a_{41}a_{32}) - a_{14}a_{23}(a_{31}a_{42} - a_{41}a_{32}) \end{aligned} \quad (35)$$

3. STO

It can be seen that the sign of $\bar{c} - \bar{b}^2/4$ determines the eigenvalues of the linearized system. For $C_{20} = 0$, it is easily obtained that $\bar{c} - \bar{b}^2/4 = 0$. The system has two pairs of roots with the same eigenvalues

$$\lambda_{1,2} = \lambda_{3,4} = \pm i\omega = \pm i \left(\frac{n'}{\cos \Lambda} \right) \quad (36)$$

In general, the solution of Eq.34 in this case should include Poisson terms in the form of $t \cos \lambda_i t$ or $t \sin \lambda_i t$, but the situation here is particular. Eq.20 in this situation is explicitly written as

$$\begin{aligned} \Delta \dot{g} &= -\frac{3}{2} \tilde{\beta}_\odot \frac{L^3}{G^2} \sqrt{L^2 - G^2} \cdot \Delta H, \Delta \dot{h} = -\frac{3}{2} \tilde{\beta}_\odot \frac{L^3}{G^2} \sqrt{L^2 - G^2} \cdot \Delta G \\ \Delta \dot{G} &= \frac{3}{2} \tilde{\beta}_\odot L \sqrt{L^2 - G^2} \cdot \Delta h, \Delta \dot{H} = \frac{3}{2} \tilde{\beta}_\odot L \sqrt{L^2 - G^2} \cdot \Delta g \end{aligned} \quad (37)$$

It is observed that for this specific situation this two-dimensional linearized system is completely separated and its solution can be written in the following form

$$\begin{aligned} \Delta g &= \alpha \cos(\omega t + \phi_1), \Delta H = \kappa \alpha \sin(\omega t + \phi_1) \\ \Delta h &= \beta \cos(\omega t + \phi_2), \Delta G = \kappa \beta \sin(\omega t + \phi_2) \end{aligned} \quad (38)$$

in which

$$\kappa = L \sin \Lambda \cos \Lambda \quad (39)$$

The frequency ω is given by Eq.36. Eq.38 indicates that the STO is stable, which is coincident with the conclusions of previous research [21, 29]. However, for $C_{20} \neq 0$, the stability of the STO changes. Fixing $a_e = 500 \text{ m}$ and $S/m =$

3. STO

Figure 3: The curves of $s(\lambda)$ for different S/m , a and C_{20} . Left: $S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$, $a = 5$; Middle: $S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$, $a = 2$; Right: $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$, $a = 2$.

$0.03 \text{ kg}^{-1} \cdot \text{m}^2$, Fig.3a and 3b give the curves of $s(\lambda)$ for different values of the C_{20} term at $a = 5$ and $a = 2$, respectively. The bold black line corresponds to the curve for $C_{20} = 0$ in each plot. In Fig.3a, there are two pairs of complex eigenvalues for $\bar{c} > \bar{b}^2/4$, which is the case for $C_{20} \neq 0$. Therefore, the STO is unstable under the perturbation of the C_{20} term. In Fig.3b, it is found that the STO is still unstable because of $\bar{c} - \bar{b}^2/4 > 0$ for small value of the C_{20} term. However, when the absolute value of the C_{20} term increases to a critical value (between -0.04 and -0.05), the sign of $\bar{c} - \bar{b}^2/4$ changes from positive to negative, which indicates the STO becomes stable again. Further, if the value of S/m is reduced to $0.01 \text{ kg}^{-1} \cdot \text{m}^2$, the critical value of the C_{20} term also decreases (between -0.01 and -0.02), as illustrated in Fig.3c. Numerical simulations are carried out to verify these analysis. For the sake of brevity, details are excluded here.

Concerning the stability of the STO under the joint effects of the C_{20} term and the SRP, following conclusions can be made from Fig.3:

1. When the C_{20} term is small, the STO is unstable under its perturbation. The inclination gradually diffuses away from its initial equilibrium value which

4. HELIOTROPIC ORBIT

is close to 90° ;

2. For low-altitude motion, the STO can become stable again if the C_{20} term is larger than a critical value. The critical value decreases with the decreasing of the value of S/m . This also indicates that for a fixed C_{20} value, the STO tends to be stable for smaller SRP perturbation.

In summary, the introduction of even a very small C_{20} term usually brings instability to the STO. On the other hand, when C_{20} is large enough, the STO becomes stable again.

4. Heliotropic Orbit

For the planar Heliotropic orbit, Eq.25 can be satisfied by one of the following conditions

$$\begin{cases} H = G, & g + h = 0 \text{ or } \pi \\ H = -G, & g - h = 0 \text{ or } \pi \end{cases} \quad (40)$$

The first and second cases correspond to a prograde and retrograde orbit, respectively. For both cases, the orbits lie in the asteroid's orbital plane. They are both called heliotropic orbit. The perigee points towards Sun for $g \pm h = 0$ and away from the Sun for $g \pm h = \pi$. For the joint effects of zonal harmonic terms and the SRP on the heliotropic orbit, investigations have been made by Lantukh et al. [16]. However, they treat these orbits as EPs of Eq.20. From following discussions, it can be seen that this treatment is inappropriate. In this section, the stability of these orbits is re-investigated and a better criterion is given.

4. HELIOTROPIC ORBIT

4.1. Prograde heliotropic orbit

For the prograde Heliotropic orbit, we have three conditions: (1) $g + h = 0$ or π , (2) $H = G$, and (3) $\dot{G} = 0$. According to Eq.20, when the first two conditions are satisfied, the third condition is automatically satisfied. The condition $g + h = 0$ or π requires

$$-\frac{3C_{20}}{2L^3G^4} \pm \frac{3}{2}\tilde{\beta}_{\odot}L\frac{G}{\sqrt{L^2 - G^2}} - n' = 0 \quad (41)$$

in which the upper and lower signs corresponds to $g + h = 0$ and $g + h = \pi$, respectively. The exact value of G can be solved from Eq.41. However, Eq.41 only indicates $\dot{g} + \dot{h} = 0$, but usually $\dot{g} = -\dot{h} \neq 0$. Therefore, heliotropic orbits are not EPs of the system Eq.20, and the method of evaluating the stability of the STO in the above section cannot be applied to the heliotropic orbit.

4.1.1. Orbital eccentricity

For $C_{20} = 0$, Eq.41 only allows $g + h = 0$. This case only allows $G = L \cos \Lambda$, and the eccentricity is obtained as

$$e = \sin \Lambda \quad (42)$$

which is a constant for a fixed value of the SRP. However, when $C_{20} \neq 0$, e changes its value for different values of C_{20} . Taking $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$ and $a_e = 500 \text{ m}$ as an example, Fig.4 gives the curves of the following functions

4. HELIOTROPIC ORBIT

Figure 4: Curves of functions $g_1(G)$, $g_2(G)$, $g_3(G)$ with $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$ and $a_e = 500 \text{ m}$ at different values of the C_{20} term. The unit of the horizontal axis is $\sqrt{L}$

$g_1(G)$, $g_2(G)$, $g_3(G)$ for different values of the C_{20} term

$$\begin{aligned} g_1(G) &= \frac{3}{2} \tilde{\beta}_\odot L \frac{G}{\sqrt{L^2 - G^2}} - n', \\ g_2(G) &= -\frac{3}{2} \tilde{\beta}_\odot L \frac{G}{\sqrt{L^2 - G^2}} - n', \\ g_3(G) &= \frac{3C_{20}}{2L^3 G^4}. \end{aligned} \quad (43)$$

The condition $g+h=0$ requires $g_1 = g_3$, and the condition $g+h=\pi$ requires $g_2 = g_3$. Obviously, only for very small values of C_{20} , the condition $g+h=0$ can be satisfied. Such small C_{20} values are usually impossible for natural celestial bodies. Therefore in practice, the introduction of the C_{20} term only allows the case $g+h=\pi$ in which the perigee points away from the Sun. As a result in the following part, we only consider the case of $g+h=\pi$.

4. HELIOTROPIC ORBIT

Figure 5: Upper row: the eccentricity of the prograde Heliotropic orbit as a function of the C_{20} term at $S/m = 0.001, 0.01, 0.1 \text{ kg}^{-1} \cdot \text{m}^2$, respectively. The black, red dashed, blue dotted and pinkish red dot dashed lines denote $a = 2, 3, 5, 10$, respectively. Lower row: variation of the periastron radius r_p as a function of a at different values of S/m and the C_{20} term. The black, red dashed, blue dotted and pinkish red dot dashed lines denote $C_{20} = -0.001, -0.01, -0.1$, respectively.

The upper row of Fig.5 gives the eccentricity of the prograde Heliotropic orbit as a function of the C_{20} term, at $S/m = 0.001, 0.01, 0.1 \text{ kg}^{-1} \cdot \text{m}^2$, respectively. From these three frames, it is observed that the orbit eccentricity increases with decreasing C_{20} and increasing S/m . The lower row of Fig.5 illustrates the variation of the periastron radius r_p as a function of a at different values of S/m and the C_{20} term. It can be concluded that: (1) For small C_{20} values, the height of the perigee decreases with increasing a . (2) For large C_{20} values, the height of the perigee first increases and then decreases.

4. HELIOTROPIC ORBIT

4.1.2. Orbital stability

As mentioned previously, the prograde Heliotropic orbit is not the EP of Eq.20, as in general $\dot{g} \neq 0, \dot{h} \neq 0$. It can be obtained from the constraint $h + g = \pi$ that

$$h = \pi - g, \dot{h} = -\dot{g} \quad (44)$$

which is substituted into the first two equations of Eq.20. Together with the condition of $H = G$, the following relation is acquired

$$\dot{g} = -\dot{h} = n' - \frac{3C_{20}}{2L^3G^4} - \frac{3}{2}\tilde{\beta}_{\odot} \frac{L\sqrt{L^2 - G^2}}{G} \sin^2 g = M - N \sin^2 g \quad (45)$$

in which

$$M = n' - \frac{3C_{20}}{2L^3G^4} > 0, N = \frac{3}{2}\tilde{\beta}_{\odot} \frac{L\sqrt{L^2 - G^2}}{G} > 0 \quad (46)$$

It is discussed in two different cases:

Case 1: If $M > N$, the following integral is obtained

$$\tan g = \sqrt{\frac{M}{M - N}} \tan \left[\sqrt{M(M - N)} (t - t_0) \right] \quad (47)$$

where t_0 is a constant. Obviously, it is a periodic function with a period of

$$T_p = \frac{2\pi}{\sqrt{M(M - N)}} \quad (48)$$

Theoretically, the initial value of g can be arbitrarily chosen. Taking $S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$, $C_{20} = -0.05$ and $a = 4$ as an example, Fig.8a gives the integrals within one period at different initial conditions. Since the curves have the same

4. HELIOTROPIC ORBIT

shapes but only different initial phase angles, they actually corresponds to only one solution which is given by Eq.47. Fig.8b shows the time history curve of $g + h$ for a nearby orbit around this heliotropic orbit with initial conditions as

$$g_0 = \pi/2, h_0 = \pi - g_0, G_0 = G^*, H_0 = G^* \times 0.999 \quad (49)$$

in which G^* is obtained from Eq.41. The value of $g + h$ is librating around 180° , which means that the Heliotropic orbit is stable.

Case 2: If $M < N$, the following integral is obtained from Eq.45

$$\tan g = \sqrt{\frac{M}{N-M}} \tanh \left[\sqrt{\frac{N-M}{M}} (t - t_0) \right] \quad (50)$$

where t_0 is the integral constant and its upper limit is given as

$$\lim_{t \rightarrow +\infty} \tan g = \sqrt{\frac{M}{N-M}} \quad (51)$$

For different values of g , Eq.50 describes completely different trajectories. Taking $S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$, $C_{20} = -0.05$ and $a = 5$ as an example, Fig.8c gives the evolutions of two trajectories with very close initial values of g .

$$g_0 = g_c \pm \Delta g, g_c = \pi - \sin^{-1} \sqrt{\frac{M}{N}}, \Delta g = 0.01\pi \quad (52)$$

Fig.8c means that even for infinitely small initial values of Δg , the final g can approach very different values, which clearly indicates instability. Fig.8d shows an orbit nearby this Heliotropic orbit with initial conditions also given by Eq.49. The value of $g + h$ no longer librates around a value, indicating the instability of the Heliotropic orbit.

4. HELIOTROPIC ORBIT

Figure 6: For a) and c): Solution of Eq.30 for different initial conditions of g ; For b) and d): The evolution of $g + h$ of an orbit near the Heliotropic orbit. For a) – d), $S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$, $C_{20} = -0.05$. For a) and b), $a = 4$; for c) and d), $a = 5$.

4. HELIOTROPIC ORBIT

 Table 2: Stability of Heliotropic orbit. Definitions of M , N , M' are given by Eq.46 and 57.

Prograde	$M/N > 1$ (i.e. $a < a_c$)	stable
	$M/N < 1$ (i.e. $a > a_c$)	unstable
Retrograde	$M'/N > 1$ (i.e. $a < a_c^1$)	stable
	$0 < M'/N < 1$ (i.e. $a_c^1 < a < a_c^2$)	unstable
	$M'/N < 0$ (i.e. $a > a_c^2$)	stable

To summarize, the value of M/N can be used as a criterion for determining the stability of the heliotropic orbit, which is stable for $M/N > 1$ and unstable for $M/N < 1$. Given fixed values of S/m and the C_{20} term, there exists a critical value a_c , and $a < a_c$ corresponds to $M/N > 1$ while $a > a_c$ corresponds to $M/N < 1$ (referring to Table2).

4.2. Retrograde heliotropic orbit

For the retrograde Heliotropic orbit, the following three conditions should be satisfied: (1) $g - h = 0$ or π , (2) $H = -G$, and (3) $\dot{G} = 0$. Similar with the prograde case, the third condition is automatically satisfied if the first two conditions are satisfied. The condition $g - h = 0$ or π requires

$$-\frac{3C_{20}}{2L^3G^4} \pm \frac{3}{2}\tilde{\beta}_\odot \frac{LG}{\sqrt{L^2 - G^2}} + n' = 0 \quad (53)$$

with the upper and lower signs for $g - h = 0$ and $g - h = \pi$, respectively. The value of G can be solved from Eq.53. However, Eq.53 only means $\dot{g} - \dot{h} = 0$, but in general $\dot{g} = \dot{h} \neq 0$. Therefore, retrograde heliotropic orbits are also not EPs of the system Eq.20, and the method of evaluating the stability of the STO cannot be

4. HELIOTROPIC ORBIT

applied to them.

4.2.1. Orbital eccentricity

For $C_{20} = 0$, Eq.53 only allows $g - h = \pi$, from which $G = L \cos(\Lambda)$. As a result, $e = \sin(\Lambda)$ can be obtained. For $C_{20} \neq 0$, taking $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$ and $a_e = 500 \text{ m}$ as an example, divide Eq.53 into the three following functions $g_4(G), g_5(G), g_6(G)$

$$\begin{aligned} g_4(G) &= \frac{3}{2} \tilde{\beta}_\odot L \frac{G}{\sqrt{L^2 - G^2}} + n', \quad g_5(G) = -\frac{3}{2} \tilde{\beta}_\odot L \frac{G}{\sqrt{L^2 - G^2}} + n', \\ g_3(G) &= \frac{3C_{20}}{2L^3G^4} \end{aligned} \quad (54)$$

Therefore, $g_4 = g_3$ and $g_5 = g_3$ correspond to the cases of $g - h = 0$ and $g - h = \pi$, respectively. The curves of $g_4(G), g_5(G)$ have same shapes as those of $g_1(G), g_2(G)$ in Fig.4, only with a constant difference of $2n'$. Due to the fact of $g_4 > 0, g_3 < 0$, and $g_3 \neq g_4, g - h = 0$ can never happen for the retrograde heliotropic orbit. Moreover, as $g_5 - g_2 > 0$, the value of G for $g_5 = g_3$ is larger than the value for $g_2 = g_3$, for the same values of S/m and the C_{20} term. As a result, the retrograde heliocentric orbit always has a smaller eccentricity than the prograde orbit. Fig.7 gives the eccentricity (the upper row) and perigee (the lower row) of the retrograde Heliotropic orbit as a function of the C_{20} term, at $S/m = 0.001, 0.01, 0.1 \text{ kg}^{-1} \cdot \text{m}^2$ and for different values of a , respectively. It can be seen that the eccentricity has similar variations with the prograde case. For the middle and right frames with larger S/m values, they have similar shapes as the middle and right frames of Fig.5. However, for the left frames with a smaller

4. HELIOTROPIC ORBIT

Figure 7: Corresponding to Fig.5, but for the retrograde heliotropic orbit.

S/m value, the orbit eccentricity for the retrograde case is obviously smaller than the prograde orbit, and the perigee height increases with increasing a .

4.2.2. Orbital stability

Similarly, the retrograde heliotropic orbit is not the EP of Eq.20. It can be obtained from the constraint $g - h = \pi$ that

$$h = g - \pi, \dot{h} = \dot{g} \quad (55)$$

which is substituted into the first two equations of Eq.20. Together with the condition of $H = -G$, the following relation is acquired

$$\dot{g} = -\frac{3C_{20}}{2L^3G^4} - n' - \frac{3}{2}\tilde{\beta}_{\odot} \frac{L\sqrt{L^2 - G^2}}{G} \sin^2 g = M' - N \sin^2 g \quad (56)$$

4. HELIOTROPIC ORBIT

in which N has the same definition in Eq.45 and M' is defined as

$$M' = -\frac{3C_{20}}{2L^3G^4} - n' \quad (57)$$

whose sign changes with parameters. There are three cases listed as follows:

Case 1: If $M' < 0$, the integral of Eq.56 is obtained

$$\tan g = \sqrt{\frac{M'}{M' - N}} \tan \left[-\sqrt{M'(M' - N)} (t - t_0) \right] \quad (58)$$

where t_0 is an integral constant. Obviously, it is a periodic function with a period of

$$T'_p = \frac{2\pi}{\sqrt{M'(M' - N)}} \quad (59)$$

Case 2: If $M' > 0$ and $M' < N$, the integral is similar to Case 2 of the prograde case and given as

$$\tan g = \sqrt{\frac{M'}{N - M'}} \tanh \left[\sqrt{\frac{N - M'}{M'}} (t - t_0) \right] \quad (60)$$

where t_0 is the integral constant and its upper limit is given as

$$\lim_{t \rightarrow +\infty} \tan g = \sqrt{\frac{M'}{N - M'}} \quad (61)$$

Case 3: If $M' > N > 0$, the integral is similar with Case 1 of the prograde case and is given as

$$\tan g = \sqrt{\frac{M'}{M' - N}} \tan \left[\sqrt{M'(M' - N)} (t - t_0) \right] \quad (62)$$

4. HELIOTROPIC ORBIT

Figure 8: Left: the variation of M'/N as the function of a for $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$ and $C_{20} = -0.005$; Right (three plots): the evolution of $g - h$ at $a = 2, 3, 6$.

in which t_0 is also an integral constant. Obviously, it is a periodic function with the same expression as Case 1 for the period.

With $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$ and $C_{20} = -0.005$, Fig.8 (upper left) gives the variation of M'/N as the function of a . From the discussions for cases 1-3, it is known that the orbit is unstable for $0 < M'/N < 1$. Denote the value of a at $M'/N = 1$ and $M'/N = 0$ as the critical values a_c^1 and a_c^2 , respectively. Taking three values of $a = 2, 3, 6$ locating in the range $(0, a_c^1]$, $(a_c^1, a_c^2]$, $(a_c^2, +\infty]$ respectively, and the following initial conditions

$$g_0 = \pi/3, h_0 = g_0 - \pi, G_0 = G^*, H_0 = -G^* \times 0.999 \quad (63)$$

in which G^* is obtained from Eq.53, the right three plots of Fig.8 show the evolutions of $g - h$ of these orbits, whose behavior agree with analysis.

Similar to the prograde case, the value of M'/N can be used as a criterion for the stability of the retrograde heliotropic orbit. The orbit is stable for $M'/N > 1$ and $M'/N < 0$, and unstable for $0 < M'/N < 1$. Given a certain S/m and the C_{20} term, there exists two critical values $a_c^1 < a_c^2$ of a . The situations of $a < a_c^1$,

5. THE GLOBAL DYNAMICS

$a_c^1 < a < a_c^2$ and $a > a_c^2$ correspond to $M'/N > 1$, $0 < M'/N < 1$ and $M'/N < 0$, respectively (referring to Table2).

5. The global dynamics

In Sections 3 and 4, two special types of orbits are studied under the joint effects of the C_{20} term and the SRP, including their orbital geometry and stability. In this section, the global dynamics of Eq.19 is discussed qualitatively by applying the Poincaré section, as the system is an autonomous Hamiltonian with two degrees of freedom.

5.1. Poincaré section

Firstly, Eq.19 is rewritten as

$$\begin{aligned} & \frac{3C_{20}}{4L^3G^5}H^2 - \left[\frac{3}{4} \frac{\tilde{\beta}_\odot L \sqrt{L^2 - G^2}}{G} (\cos(g+h) - \cos(g-h)) + n' \right] H \\ & - \frac{C_{20}}{4L^3G^3} - \frac{1}{2L^2} - \frac{3}{4} \tilde{\beta}_\odot L \sqrt{L^2 - G^2} (\cos(g-h) + \cos(g+h)) \\ & - \bar{K} = 0 \end{aligned} \quad (64)$$

in which $\bar{K}$ is the value of the given Hamiltonian. For fixed values of $\bar{K}$ and G, g, h , there are two solutions of Eq.64, denoted as $H_1 \leq H_2$. Generally, H_1 corresponds to a larger orbital inclination and its corresponding orbit can be viewed as retrograde, while H_2 corresponds to a prograde orbit. Taking the STO as an example, the Hamiltonian is

$$\bar{K} = K_{ter} = -\frac{1}{2L^2} - \frac{C_{20}}{4L^3G^3} \left(1 - \frac{3H^2}{G^2} \right) + \frac{3}{2} \frac{\tilde{\beta}_\odot LH \sqrt{L^2 - G^2}}{G} - n'H \quad (65)$$

5. THE GLOBAL DYNAMICS

in which G, H is obtained from Eq.27. According to Eq.25 and Tab.1, the situation of $\sin(g \pm h) = 0$ is the focus and the following Poincaré sections can be chosen:

$$\begin{aligned}
 A) \ h = \pi/2, H = H_1; \quad B) \ h = 0, H = H_1; \\
 C) \ h = -\pi/2, H = H_1; \quad D) \ h = \pi, H = H_1; \\
 E) \ h = \pi/2, H = H_2; \quad F) \ h = 0, H = H_2 \\
 G) \ h = -\pi/2, H = H_2; \quad H) \ h = \pi, H = H_2
 \end{aligned} \tag{66}$$

For all of them, the values of g, G are recorded when the trajectory crosses these sections. Taking section $A)$ as an example, fixing $h = \pi/2$, the sample initial values of g, G are selected from $g \in [0, 2\pi)$ and $i \in [0, 2\pi)$, $e \in [0, 1)$. For a given value $\bar{K}$ of the Hamiltonian, H_1, H_2 can be determined from Eq.64 and $H = H_1$ is taken. The section can be obtained by doing the integration. Since Eq.20 has the following symmetry

$$g, h \Leftrightarrow -g, -h; \quad g, h \Leftrightarrow g + \pi, h + \pi, \tag{67}$$

only sections $A), B), E), F)$ need to be studied and the remaining four sections are obtained from this symmetry. Moreover, except some specific orbit, e.g. the STO with h around $\pm\pi/2$, most orbits have their h ranging between $[0, 2\pi)$. Therefore, only one of the four sections $A) - D)$ is required for the retrograde orbit. Similarly, only one section is needed from sections $E) - H)$ for the prograde orbit. Sections $A)$ and $E)$ are used for the following qualitative analysis of the global dynamics.

5. THE GLOBAL DYNAMICS

Figure 9: The sections $A)$ with different Hamiltonians at $C_{20} = -0.01, a = 5, S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$. The x - and y -axis represent g (in rad) and i (in deg), respectively.

5.2. The SRP-dominant case

Taking $C_{20} = -0.01, a = 5, S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$ as an example, the orbital motion is dominated by the SRP, as the C_{20} term is small and the orbital altitude is high.

5.2.1. Retrograde orbit

In Fig.9, the upper row gives the sections of $A)$ with different Hamiltonians $\bar{K} = K_{ter} + \Delta K$. The lower row illustrates the projection of inclination i on these sections, some parts of which overlap due to the fact that the variable i is not a canonical variable. This phenomenon can be avoided if the canonical variable H is used. However, the variable H depends on both e and i . It is not intuitive for observing i and is not applied here.

The upper left plot corresponds to the section of $\bar{K} = K_{ter}$. On a first look,

5. THE GLOBAL DYNAMICS

the region with large G values is regular but actually it is chaotic. Starting at the point denoted as 'start' in the figure, the following intersection points on this section gradually disperse to the whole range of the accessible values for (g, G) and the chaotic behavior appears. The divergence rate of e is proportional to the value of the C_{20} term. With the decrease of the Hamiltonian (i.e. the increase of inclination), two regular regions appear on the section. For $\Delta K = -0.005$, they are labelled as red and blue respectively on the section. The red one corresponds to a stable near-planar retrograde orbit, and the blue one has its $g - h$ oscillating around 0° . From the corresponding plot in the lower row, the inclination of the blue orbit ranges between 130° and 160° . Unfortunately, the orbits in these regular regions all have large e (for $a = 5$) and collide with the asteroid, meaning that they are of no practical applications.

5.2.2. Prograde orbit

Similar to Fig.9, Fig.10 gives the sections of E) with different Hamiltonians $\bar{K} = K_{ter} + \Delta K$ for the prograde orbit. Similar to the retrograde case, the orbit is generally chaotic for larger values of $\bar{K}$, i.e., larger orbit inclinations. When $\bar{K}$ decreases, i.e., the orbit inclination decreases, stable orbits appear. However, for this study case, these orbits also have large values of e and impact on the asteroid.

5.2.3. Conclusions

In the SRP-dominant case, following conclusions can be made from above simulations: (1) When i is near 90° , the orbital motion is chaotic. The orbit eccentricity vary between 0 and 1, and the orbit inclination can vary between 0° and 180° . (2) When the orbit inclination is close to 0° or 180° , stable prograde or

5. THE GLOBAL DYNAMICS

Figure 10: The sections E) with different Hamiltonians at $C_{20} = -0.01, a = 5, S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$. The x - and y -axis represent g (in rad) and i (in deg), respectively.

retrograde orbits exist, but they generally have large orbit eccentricities and may collide with the asteroid. As a result, no long-term stable orbits with small orbit eccentricities exist. Nevertheless, the eccentricities of orbits close to the STO with small initial e diverge slowly during the first several orbital periods of the asteroids which usually last for several years, so they can be used for short-duration asteroid missions.

5.3. The C_{20} -dominant case

Taking $C_{20} = -0.05, a = 2, S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$ as the study case, the orbital motion is dominated by the C_{20} term, as the orbital altitude is very low and close to the asteroid.

5. THE GLOBAL DYNAMICS

Figure 11: The sections A) with different Hamiltonians at $C_{20} = -0.01, a = 5, S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$. The x -axis represents g (in rad) and y -axis represents G , respectively.

5.3.1. Retrograde orbit

Fig.11 gives the sections of A) with different Hamiltonians $\bar{K} = K_{ter} + \Delta K$. The projections of corresponding inclination on these sections are given in Fig.12.

Compared with Fig.9, the chaotic region in Fig.11 and Fig.12 shrinks. The inclination can be divided into two regions at the critical inclination 117° :

1. For $i \in [90^\circ, 117^\circ)$, the orbits are generally chaotic, but two main stable regions exist around the resonances $\dot{h} = 0$ (h is around 90°) and $\dot{g} + \dot{h} = 0$ ($g + h$ is around 180°), respectively. However, the resonance $\dot{h} = 0$ is not clearly visible in Fig.11 and Fig.12, so they are magnified and shown in the first two plots of Fig.13. The STO and its nearby orbits (i.e. the orbits with both h and g librating around 90°) only occupy a small region of this stable region. They are denoted as 'Terminator' in Fig.13. The lower two plots of

5. THE GLOBAL DYNAMICS

Figure 12: Corresponding to Fig.11, but the y-axis represents the orbit inclination in deg.

Fig.13 also show the evolutions of Ω, e, i for an example orbit (denoted as 'Example' in the upper two plots of Fig.13) other than the STO type orbit in this region.

- For $i \in (117^\circ, 180^\circ)$, the motion is generally stable. Specifically, two sub-regions can be distinguished. The first is the resonance region $\dot{g} - \dot{h} = 0$, for which $g - h$ is around 0° and 180° and the inclination ranges from 125° to 180° . The orbits in this region either vibrate around $g - h = 0$ or 180° , or follow the non-resonant motion. The second is the region near the critical inclination (i.e. $i \in (117^\circ, 125^\circ)$), which also includes other resonances, e.g. $\dot{g} = 0, 3\dot{g} - \dot{h} = 0$. Some parts of this region are chaotic due to the overlap of these resonances.

It is mentioned here that the resonance changes from $\dot{g} + \dot{h} = 0$ for $i < 117^\circ$ to $\dot{g} + \dot{h} = 0$ for $i > 117^\circ$, due to the fact that the sign of $\dot{g}$ changes at the critical

5. THE GLOBAL DYNAMICS

Figure 13: a) The zoom-in of the upper left plot at small e of Fig.11; b) The zoom-in of the near polar region of Fig.12; c) the evolution of Ω of the example orbit; d) the evolution of e, i of the example orbit. The unit of the x -axis is the orbital period of the asteroid around Sun.

5. THE GLOBAL DYNAMICS

inclination.

5.3.2. Prograde orbit

Similar to Fig.11 and Fig.12, Fig.14 and Fig.15 give the sections of E with different Hamiltonians $\bar{K} = K_{ter} + \Delta K$ and the projections of corresponding inclination i on these sections, respectively.

Similar to the retrograde case, the inclination can also be divided into two regions at the critical inclination 63° :

1. For $i \in (63^\circ, 90^\circ]$, the orbits are generally chaotic, but two main stable regions exist around the resonances $\dot{h} = 0$ (h is around 90°) and $\dot{g} - \dot{h} = 0$ ($g - h$ is around 180°), respectively. However, the resonance $\dot{h} = 0$ is also not clearly visible in Figs.14 and 15, but the magnified plots are no longer provided. Similarly, the resonance changes from $\dot{g} - \dot{h} = 0$ for $i < 63^\circ$ to $\dot{g} + \dot{h} = 0$ for $i > 63^\circ$, due to the fact that the sign of $\dot{g}$ changes at the critical inclination.
2. For $i \in (0, 63^\circ)$, the motion is generally stable. Specifically, two sub-regions can be distinguished. The first region is dominated by the resonance $\dot{g} + \dot{h} = 0$, for which $g + h$ is around 0° and 180° and the inclination ranges from 0° to 55° . The orbits in this region either librate around $g - h \approx 0$ or 180° , or follow the non-resonant motion. The second is the region near the critical inclination (i.e. $i \in (55^\circ, 63^\circ]$), which also includes other resonances, e.g. $\dot{g} = 0$, $3\dot{g} + \dot{h} = 0$, $2\dot{g} + \dot{h} = 0$. Some part of this region is chaotic due to the overlap of these resonances.

5. THE GLOBAL DYNAMICS

Figure 14: The sections $E)$ with different Hamiltonians at $C_{20} = -0.01, a = 5, S/m = 0.03 \text{ kg}^{-1} \cdot \text{m}^2$. The x -axis represents g in rad and y -axis represents G .

Figure 15: Corresponding to Fig.14, but the y -axis represents the orbit inclination in deg.

6. DISCUSSION

5.4. Conclusions

For the C_{20} -dominant case, following conclusions can be made from above numerical simulations:

1. When i is near 90° , the orbital motion can be stable, but not restricted by the STO type orbits, i.e. both g and h librate around specific values. Nevertheless, the orbital plane of the orbits in the stable region points towards or away from the Sun for $h \approx \pi/2$ and $h \approx 3\pi/2$, respectively, which satisfies $\dot{h} = 0$;
2. For $i \in (63^\circ, 117^\circ]$, the orbital motion is chaotic, except for the resonance regions $\dot{h} = 0$, $\dot{g} - \dot{h} = 0$ and $\dot{g} + \dot{h} = 0$;
3. For $i \in (0, 63^\circ)$ and $i \in (117^\circ, 180^\circ)$, the orbital motion is generally stable.

Depending on the strength of the SRP, there are different extents of variations in the orbit eccentricity. Generally, the variations in the orbit eccentricity for polar orbits are smaller than those for the near-planar orbits. So we can expect practically applicable orbits for polar orbits. Nevertheless, if the strength of the SRP is much smaller than that of the C_{20} term, the extent of variations in the orbit eccentricity caused by the SRP is also small. In this case, we can also expect practical stable near-planar orbits.

6. Discussion

The previous analysis is based on the assumption that the asteroid has circular orbit around the Sun. However, most near Earth asteroids usually have large heliocentric eccentricities, between 0.2 and 0.35. Simply ignoring it probably

6. DISCUSSION

affects the accurate analysis of the dynamics. Introducing the asteroid's orbit eccentricity, for the planar motion neglecting the deducing details, the new averaged Hamiltonian is obtained as

$$\begin{aligned}
 K = & -\frac{1}{2L^2} - \frac{C_{20}}{4L^3G^3} - \frac{3\tilde{\beta}_{\odot}a}{8(1-e'^2)}e \{ 2\alpha(1-\cos i)\cos(\omega-\Omega+f') \\
 & + 2\alpha(1+\cos i)\cos(\omega+\Omega-f') \\
 & + 2(1-\cos i)e'[\cos(\omega-\Omega)+\cos(\omega-\Omega+2f')] \\
 & + 2(1+\cos i)e'[\cos(\omega+\Omega-2f')+\cos(\omega+\Omega)] \\
 & + (1-\cos i)e'^2/2[\cos(\omega-\Omega-f')+\cos(\omega-\Omega+3f')] \\
 & + (1+\cos i)e'^2/2[\cos(\omega+\Omega-3f')+\cos(\omega+\Omega+f')] \}
 \end{aligned} \tag{68}$$

in which e' and f' are the eccentricity and true anomaly of the asteroid's orbit around the Sun, respectively, and $\alpha = 1 + e'^2/2$. If the C_{20} term is ignored, Eq.68 is integrable according to the work by Scheeres [27, 28], Rosengren and Scheeres [24]. However, considering the C_{20} term, the prograde orbit has the following Hamiltonian

$$\begin{aligned}
 K = & -\frac{1}{2L^2} - \frac{C_{20}}{4L^3G^3} - \frac{3\tilde{\beta}_{\odot}ae}{8(1-e'^2)} \{ 4\alpha\cos(\tilde{\omega}-f') \\
 & + 4e'[\cos(\tilde{\omega}-2f')+\cos\tilde{\omega}] + e'^2[\cos(\tilde{\omega}-3f')+\cos(\tilde{\omega}+f')] \}
 \end{aligned} \tag{69}$$

in which $\tilde{\omega} = \omega + \Omega$. Introducing the following canonical transform

$$\tilde{g} = \omega + \Omega, \tilde{G} = G - L \tag{70}$$

6. DISCUSSION

the EOM is written as

$$\begin{aligned}
 \dot{\tilde{g}} &= \frac{3C_{20}}{4L^3\tilde{G}^4} - \frac{3\tilde{\beta}_{\odot}a}{8(1-e'^2)} \frac{\partial e}{\partial \tilde{G}} \left\{ 4\alpha \cos(\tilde{g} - f') \right. \\
 &\quad \left. + 4e' [\cos(\tilde{g} - 2f') + \cos \tilde{g}] + e'^2 [\cos(\tilde{g} - 3f') + \cos(\tilde{g} + f')] \right\} \\
 \dot{\tilde{G}} &= -\frac{3\beta_{\odot}L\sqrt{L^2 - \tilde{G}^2}}{8(1-e'^2)} \left\{ 4\alpha \sin(\tilde{g} - f') \right. \\
 &\quad \left. + 4e' [\sin(\tilde{g} - 2f') + \sin \tilde{g}] + e'^2 [\sin(\tilde{g} - 3f') + \sin(\tilde{g} + f')] \right\}
 \end{aligned} \tag{71}$$

For any initial condition, Eq.71 can be integrated and the Poincaré section is generated by recording the canonical pair $\tilde{g}, \tilde{G}$ when $f' = 2k\pi, k \in \mathbb{Z}$ is satisfied. Fig.18 gives some examples with $S/m = 0.01 \text{ kg}^{-1} \cdot \text{m}^2$, $a_e = 1 \text{ km}$, $a = 3 \text{ km}$ at $e' = 0, 0.1, 0.5, 0.7$, respectively. From left to right, the C_{20} term takes values of $-0.001, -0.005, -0.01, -0.05$, respectively. The following conclusions can be made from these figures:

1. Comparison between different rows of figures indicates that the phase space is significantly changed when e' is large, in form of the resonance splitting (the resonance $\tilde{g} - M' = \pi$ into sub-resonances, arising from the high-order terms [17]), and the expansion of the chaotic region. In the chaotic region, some analytical solutions for the planar motion such as the one developed by Casanova et al. [4] is no longer valid; It also can be seen that e' needs to be considered if it has large values. The incorporation of e' in the modelling and analysis is carried out by our recent research [8].
2. For a certain SRP, Comparison between different rows of figures indicates that the regular region expands with the increase of the C_{20} term, when

7. CONCLUSIONS

the planar motion is dominated by the C_{20} term with the SRP as a small perturbation.

For the retrograde orbit, its Hamiltonian is written as

$$K = -\frac{1}{2L^2} - \frac{C_{20}}{4L^3G^3} - \frac{3\tilde{\beta}_{\odot}ae}{8(1-e'^2)} \left\{ 4\alpha \cos(\tilde{\omega}' + f') \right. \\ \left. + 4e' [\cos(\tilde{\omega}' + 2f') + \cos\tilde{\omega}] + e'^2 [\cos(\tilde{\omega}' + 3f') + \cos(\tilde{\omega}' - f')] \right\} \quad (72)$$

in which $\tilde{\omega}' = \omega - \Omega$. Similar conclusions as Fig.16 can be made, as it is symmetric w.r.t. Eq.69 in f' .

7. Conclusions

This study investigates the long-term dynamical behavior of orbital motion around asteroid, under the joint effects of the C_{20} term and the SRP. Two types of special orbits are studied, namely the STO and the Heliotropic orbit. For the former one, the following observations are made:

1. For a certain SRP, the introduction of the C_{20} term brings instability to the STO and makes its inclination larger than 90° , referring to Fig.2 for details;
2. For a certain SRP, although the C_{20} term makes the STO unstable, the orbit becomes stable again if the C_{20} term is large enough, referring to Fig.3 for details.

For the Heliotropic orbit, our study shows that:

7. CONCLUSIONS

Figure 16: The Poincaré section of Eq.71 for $e' = 0, 0.1, 0.5, 0.7$ and $C_{20} = -0.001, -0.005, -0.01, -0.05$.

7. CONCLUSIONS

1. For the prograde orbit, the introduction of the C_{20} term changes its $g + h$ from 0 to π (referring to Fig.4) and varies its eccentricity (referring to Fig.5 and 7);
2. For the prograde orbit, there is a critical value a_c , and the orbit is stable for $a < a_c$ and unstable for $a > a_c$ (referring to Fig.6); for the retrograde orbit, there exist two critical values $a_c^1 < a_c^2$, and the orbit is stable for $a < a_c^1$ or $a > a_c^2$ and unstable for $a_c^1 < a < a_c^2$.

By applying the Poincaré section, the global dynamics is investigated for two situations: the SRP dominates over the perturbative C_{20} term and the C_{20} term dominates the perturbative SRP. Following conclusions are made:

1. For the SRP-dominating case, even small value of the C_{20} term can bring global chaos to the phase space. Nevertheless, when i is close to 0° or 180° , stable orbits with large orbit eccentricity exist (referring to Figs.9 and 10). They collide with the asteroid due to the large orbit eccentricity. In contrast, the near polar orbits are unstable but with a slow divergence rate of the eccentricity. These orbits have potential applications for asteroid missions before the orbit eccentricity grows too large. In summary, small-sized particles or probes cannot stay around the asteroid for long time with the dominant SRP effect;
2. For the C_{20} -dominating case, the motion is generally stable (referring to Figs.12 and 15) for $i \in (0, 63^\circ)$ and $i \in (117^\circ, 180^\circ)$. For $i \in (63^\circ, 90^\circ)$ and $(90^\circ, 117^\circ)$, the stable regions exist around resonances $\dot{g} - \dot{h} = 0$ and $\dot{g} + \dot{h} = 0$, respectively. Near polar region includes stable resonance region

7. CONCLUSIONS

$\dot{h} = 0$ (namely pointing towards or away from the Sun), among which STO exists (referring to Fig.13). These near-polar stable orbits are practically applicable due to their small eccentricity.

Finally, the effect of the asteroid's orbital eccentricity e' around the Sun on the long-term orbital motion around the asteroid is discussed for the planar motion. Comparing with the system without considering the asteroid's orbit eccentricity, the large value e' breaks the integrability and also changes the phase space structure.

The method developed here can also be applied to study the orbital motion around large celestial bodies, for which situation the solar gravitation needs to be considered. Our research simplifies the dynamics in terms of ignoring high-order zonal harmonics, the obliquity of the asteroid and the tesseral harmonics, which have influence on the dynamics quantitatively or qualitatively. Although the tesseral harmonics only have short-term effect but will affect the long-term property of motion close to the resonance region. In our previous work [15], the combined effects of the C_{20} and C_{22} terms on the long term orbital motion around the asteroid was investigated. After taking the SRP into account, new characteristics of the dynamics need to be evaluated for asteroids with different rotation rates.

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7. CONCLUSIONS

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7. CONCLUSIONS

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7. CONCLUSIONS

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7. CONCLUSIONS

Appendix

$$\begin{aligned}
 a_{11} &= -\frac{3}{4}\tilde{\beta}_{\odot}\frac{LG}{\sqrt{L^2-G^2}}\left[\left(1-\frac{H}{G}\right)\sin(g-h)+\left(1+\frac{H}{G}\right)\sin(g+h)\right] \\
 &\quad +\frac{3}{4}\tilde{\beta}_{\odot}\frac{LH\sqrt{L^2-G^2}}{G^2}[\sin(g-h)-\sin(g+h)] \\
 a_{12} &= \frac{3}{4}\tilde{\beta}_{\odot}\frac{LG}{\sqrt{L^2-G^2}}\left[\left(1-\frac{H}{G}\right)\sin(g-h)-\left(1+\frac{H}{G}\right)\sin(g+h)\right] \\
 &\quad -\frac{3}{4}\tilde{\beta}_{\odot}\frac{LH\sqrt{L^2-G^2}}{G^2}[\sin(g-h)-\sin(g+h)] \\
 a_{13} &= \frac{3C_{20}}{2L^3G^5}\left(\frac{15H^2}{G^2}-2\right)+\frac{3}{2}\tilde{\beta}_{\odot}\frac{L^3H}{G^3\sqrt{L^2-G^2}}[\cos(g-h)-\cos(g+h)] \\
 &\quad +\frac{3}{4}\tilde{\beta}_{\odot}\frac{L^3}{(L^2-G^2)^{3/2}}\left[\left(1-\frac{H}{G}\right)\cos(g-h)+\left(1+\frac{H}{G}\right)\cos(g+h)\right] \\
 a_{14} &= -\frac{15C_{20}H}{2L^3G^6}-\frac{3}{4}\tilde{\beta}_{\odot}\frac{L^3}{G^2\sqrt{L^2-G^2}}[\cos(g-h)-\cos(g+h)] \\
 a_{21} &= -\frac{3}{4}\tilde{\beta}_{\odot}\frac{L\sqrt{L^2-G^2}}{G}[\sin(g-h)-\sin(g+h)] \\
 a_{22} &= \frac{3}{4}\tilde{\beta}_{\odot}\frac{L\sqrt{L^2-G^2}}{G}[\sin(g-h)+\sin(g+h)] \\
 a_{23} &= -\frac{15C_{20}H}{2L^3G^6}-\frac{3}{4}\tilde{\beta}_{\odot}\frac{L^3}{G^2\sqrt{L^2-G^2}}[\cos(g-h)-\cos(g+h)] \\
 a_{24} &= \frac{3C_{20}}{2L^3G^5} \\
 a_{31} &= -\frac{3}{4}\tilde{\beta}_{\odot}L\sqrt{L^2-G^2}\left[\left(1-\frac{H}{G}\right)\cos(g-h)+\left(1+\frac{H}{G}\right)\cos(g+h)\right] \\
 a_{32} &= \frac{3}{4}\tilde{\beta}_{\odot}L\sqrt{L^2-G^2}\left[\left(1-\frac{H}{G}\right)\cos(g-h)-\left(1+\frac{H}{G}\right)\cos(g+h)\right] \\
 a_{33} &= \frac{3}{4}\tilde{\beta}_{\odot}\frac{LG}{\sqrt{L^2-G^2}}[\sin(g-h)+\sin(g+h)] \\
 &\quad -\frac{3}{4}\tilde{\beta}_{\odot}\frac{L^3H}{\tilde{G}^2\sqrt{L^2-G^2}}[\sin(g-h)-\sin(g+h)] \\
 a_{34} &= \frac{3}{4}\tilde{\beta}_{\odot}\frac{L\sqrt{L^2-G^2}}{G}[\sin(g-h)-\sin(g+h)]
 \end{aligned}
 \tag{73}$$

7. CONCLUSIONS

$$\begin{aligned}
 a_{41} &= \frac{3}{4} \tilde{\beta}_{\odot} L \sqrt{L^2 - G^2} \left[\left(1 - \frac{H}{G} \right) \cos(g - h) - \left(1 + \frac{H}{G} \right) \cos(g + h) \right] \\
 a_{42} &= -\frac{3}{4} \tilde{\beta}_{\odot} L \sqrt{L^2 - G^2} \left[\left(1 - \frac{H}{G} \right) \cos(g - h) + \left(1 + \frac{H}{G} \right) \cos(g + h) \right] \\
 a_{43} &= -\frac{3}{4} \tilde{\beta}_{\odot} \frac{LG}{\sqrt{L^2 - G^2}} [\sin(g - h) - \sin(g + h)] \\
 &\quad + \frac{3}{4} \tilde{\beta}_{\odot} \frac{L^3 H}{G^2 \sqrt{L^2 - G^2}} [\sin(g - h) + \sin(g + h)] \\
 a_{44} &= -\frac{3}{4} \tilde{\beta}_{\odot} \frac{L \sqrt{L^2 - G^2}}{G} [\sin(g - h) + \sin(g + h)]
 \end{aligned} \tag{74}$$