Weak Transient Signal Detection
Via a Polynomial Eigenvalue Decomposition

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Problem & Model

- A number of broadband stationary sources \( s_\ell[n], \ell = 1, \ldots, L \), illuminate an \( M \)-element sensor array;
- each transfer path is modelled by a vector of impulse responses \( \mathbf{a}_\ell[n] \in \mathbb{C}^M \);
- stationary additive, spatially and temporally uncorrelated noise \( \mathbf{v}[n] \in \mathbb{C}^M \);

\[
\mathbf{x}[n] = \sum_{\ell=1}^{L} \mathbf{a}_\ell[n] \ast s_\ell[n] + \mathbf{v}[n]
\]
Problem & Model

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- each transfer path is modelled by a vector of impulse responses \( a_\ell[n] \in \mathbb{C}^M \);
- stationary additive, spatially and temporally uncorrelated noise \( v[n] \in \mathbb{C}^M \);
- a broadband transient signal \( s_{L+1}[n] \) enters the scene at some point in time;
- aim: we want to detect the onset of this transient signal, which may be weak in power [10];
- assumption: \( M > L \).

\[
x[n] = \sum_{\ell=1}^{L+1} a_\ell[n] * s_\ell[n] + v[n]
\]
Model

Each source, \( s_\ell[n] \), contributes to the data vector \( x[n] = [x_1[n], \ldots, x_M[n]]^T \) via a steering vector \( a_\ell[n] = [A_{\ell,1}[n], \ldots A_{\ell,M}[n]]^T \);

compact with \( A[n] = [a_1[n] \ldots a_L[n]] \) and \( s[n] = [s_1[n], \ldots, s_L[n]]^T \):

\[
\]

space-time covariance: \( R[\tau] = \mathcal{E}\{x[n]x^H[n-\tau]\} \):

\[
R[\tau] = A[\tau] \ast \mathcal{E}\{s[n]s^H[n-\tau]\} \ast A^H[-\tau] + \mathcal{E}\{v[n]v^H[n-\tau]\} \\
= A[\tau] \ast \Gamma[\tau] \ast A^H[-\tau] + \sigma_v^2 I_M \delta[\tau] .
\]
Cross-Spectral Density Matrix

- Transfer function matrix $A(z) = \sum_n A[n]z^{-n}$ (short $A(z) \circ A[n]$) is a polynomial in $z \in \mathbb{C}$;
- Cross-spectral density $R(z) \circ R[\tau]$:
  \[ R(z) = A(z)\Gamma(z)A^P(z) + \sigma_v^2I_M; \]
- Parahermitian property:
  \[ R^P(z) = R^H(1/z^*) = R(z); \]
- When evaluated for a specific normalised angular frequency $\Omega_0$: $R_0 = R(z)|_{z=e^{j\Omega_0}}$;
- $R_0$ is a constant matrix that describes a narrowband problem;
- $R_0$ is Hermitian $\rightarrow$ eigenvalue decomposition (EVD) $R_0 = Q_0\Lambda_0Q_0^H$. 
Narrowband EVD and Subspace Decomposition

- We assume an ordered EVD \( R_0 = Q_0 \Lambda_0 Q_0^H \), where \( \Lambda_0 = \text{diag}\{\lambda_1, \ldots, \lambda_M\} \) with \( \lambda_\ell \geq \lambda_{\ell+1}, \ell = 1, \ldots, (M - 1) \);

- partitioning enables a subspace decomposition:

\[
R_0 = \begin{bmatrix} Q_s & Q_n \end{bmatrix} \begin{bmatrix} \Lambda_s + \sigma_v^2 I_L & \sigma_v^2 I_{M-L} \\ \sigma_v^2 I_{M-L} & \sigma_v^2 I_{M-L} \end{bmatrix} \begin{bmatrix} Q_s^H \\ Q_n^H \end{bmatrix}
\]

- source enumeration: eigenvalues above noise floor = number of uncorrelated sources;

- \( y[n] = Q_n^H x[n] \in \mathbb{C}^{M-L} \) only contains noise;

- power in \( y[n] \): \( \mathcal{E}\{\|y[n]\|_2^2\} = (M - L) \sigma_v^2 \) because of orthonormality of \( Q \).
Broadband EVD

- Space-time covariance $R[\tau]$ or equivalently the CSD matrix $R(z)$ are only diagonalised by the EVD for a specific value $\tau$ or $z$;
- for an analytic $R(z)$ that is not derived from multiplexed data, there exists a parahermitian matrix EVD [12, 11]

$$R(z) = Q(z)\Lambda(z)Q^P(z) ; \tag{3}$$

- $\Lambda(z)$ is diagonal, parahermitian, analytic, and unique;
- eigenvectors in $Q(z)$ are paraunitary, analytic, and unique up to an arbitrary allpass function;
- paraunitarity $Q(z)Q^P(z) = Q^P(z)Q(z) = I$ implies losslessness;
- a number of algorithms can approximate (3) [6, 7, 8, 15, 13, 14].
Broadband Subspace Decomposition

- The parahermitian matrix EVD $R(\tilde{z}) = Q(z)\Lambda(\tilde{z})Q^P(z)$ enables a broadband subspace decomposition:

$$R(\tilde{z}) = \begin{bmatrix} Q_s(z) & Q_n(z) \end{bmatrix} \Lambda_s(\tilde{z}) \begin{bmatrix} Q_s(z) \\ Q_n(z) \end{bmatrix} + \sigma_v^2 I_L \sigma_v^2 I_{M-L}$$

- $Q[n] \circ Q(z)$ describes a lossless filter bank;
- data vector component in the noise-only subspace: $y[n] = Q_n^H[-n] * x[n]$;
- again, it can be shown that ideally $\mathbb{E}\{\|y[n]\|_2^2\} = (M - L)\sigma_v^2$. 
‘Syndrome’ Idea

- We estimate $\mathbf{R}(z)$ over a window of data, with $L < M$ stationary sources present;
- compute parahermitian matrix EVD, perform source enumeration, and determine the eigenvectors spanning the noise-only subspace, $\mathbf{Q}_n(z)$;
- if an additional source $s_{L+1}[n]$ enters the scene, it will likely protrude into the noise-only subspace;
- we therefore monitor the syndrome vector

$$y[n] = \mathbf{Q}_n^H[-n] \ast \mathbf{x}[n]$$

for a change in power, or for any structured / correlated components.
Intuitive Example I

- $M = 6$ sensors, $L = 3$ stationary sources; weak transient source at $n = 5000$;
- monitoring a sensor output $x_1[n]$:

![Graphs of $x_1[n]$ and $\|x[n]\|^2$ over discrete time $n$.](image)
Intuitive Example II

- $M = 6$ sensors, $L = 3$ stationary sources; weak transient source at $n = 5000$;
- monitoring a syndrome element $y_1[n]$:

![Graph (a)](image1)

- $z_1[n]$ vs. discrete time $n$

![Graph (b)](image2)

- $\|z[n]\|^2$ vs. discrete time $n$
Proposed Approach

- We use the statistics and evaluated parahermitian matrix EVD of a previous time window, and utilise the broadband noise-only subspace spanned by the columns of $Q_n(z)$;

- being analytic, $Q_n(z)$ can typically be approximated well by low-order polynomials, and is relatively inexpensive to implement;

- because of the processing, elements of the syndrome vector $y[n]$ are spatially and temporally correlated;

- decimation by $D$ can break temporal correlation and further reduces complexity;

- we can average over consecutive syndrome vectors to increase discrimination;

- $\xi^{(K)}_{n,D}$ is generalised $\chi^2$ distributed if temporal correlation is suppressed [9, 1].
Results I — Statistics

- $M = 6$ sensors, $L = 2$ stationary sources, transfer functions determined by radio propagation model for dense urban environment (polynomial order $\approx 40$);
- statistics of output for $I_0$: no transient versus $I_1$: transient present:

![Graph showing the comparison between $I_0$ and $I_1$.]
Results II — Sources and Propagation Environment

- Power of contributions for realistic channel scenario:

<table>
<thead>
<tr>
<th>signal</th>
<th>power</th>
</tr>
</thead>
<tbody>
<tr>
<td>source 1</td>
<td>0.0000 dB</td>
</tr>
<tr>
<td>source 2</td>
<td>-4.3494 dB</td>
</tr>
<tr>
<td>source 3</td>
<td>-13.2865 dB</td>
</tr>
<tr>
<td>noise</td>
<td>-16.2865 dB</td>
</tr>
</tbody>
</table>
Results III — Discrimination vs Decision Time

- Averaging increasingly separates the distributions for $I_0$ and $I_1$ — measured as discrimination $D$: derived from the ROC [5];

- Averaging also increases the time to compute $\xi_{n,D}(T)$ — decision time $T$ (for a 20MHz channel);

- $N$ here is the window over which the space-time covariance is estimated [2, 3, 4].
Summary

- We have proposed a broadband subspace approach to detect the presence of weak transient signals;
- this is based on second order statistics of sensor array data — the space-time covariance matrix — and a polynomial matrix EVD;
- this covariance matrix and its decomposition can be computed off-line;
- a subspace decomposition for the noise-only subspace determines a syndrome vector;
- in the absence of a transient signal, this syndrome only contains noise;
- a transient signal is likely to protrude into the noise-only subspace, and a change in energy can be detected even if the signal is weak;
- discrimination can be traded off against decision time;
- further work: (i) impact of time-varying channels, and (ii) forensic investigation of the transient source once detected.
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