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Feedback delay control of highly nonlinear stochastic functional differential equations with discrete-time state observations

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Abstract

The purpose of this paper is to give the delay control based on discrete-time state observations to stabilize highly nonlinear hybrid stochastic functional differential equations (SFDEs). It is considered that time lag generated by the controller in each discrete observation should be different. The new controlled hybrid SFDEs are affected the variable delay caused by the controller, the distributed delay and the superlinear coefficients of the systems itself, which makes the problem handling more complicated. Then, a series of criteria for the exponential stability of the controlled SFDEs are obtained, and an upper bound for the discrete observation interval and variable delay is given. Finally, numerical example illustrate the proposed theoretical results.

KEYWORDS:
Discrete observation; Functional equation; Highly nonlinear; Delay control; Exponential stabilization

1 | INTRODUCTION

Functional differential equations are always used to describe systems whose states depend not only on the present but also on the past.[1,2,3] Considering the influence of random factors and the sudden changes of system structure and parameters, SFDEs with Markovian switching (also known as hybrid SFDEs), including stochastic delay differential equations (SDDEs) with Markovian switching, have been widely used to deal with practical problems. Stability and stabilization are the fundamental and important contents in SFDEs.[4,5,6,7,8,9,10]

However, most of the existing stability results require that the coefficients of the functional system must satisfy the linear growth condition.[11,12] In fact, in the real world, especially in ecosystems and financial systems, many SFDEs are highly nonlinear (that is, the coefficients of these systems do not satisfy the linear growth condition).[13,14,15] Hu et al.[16] discussed some asymptotic properties of the hybrid SDDEs whose coefficients are highly nonlinear. Feng et al.[17] further extended the above results to more general hybrid SFDEs and improved the stability conditions for a special class of nonautonomous functional systems. Fei et al.[18] studied the delay dependent stability theories of hybrid SDDEs with highly nonlinear to reduce the conservatism. Along this line, the theory and application of the stability of highly nonlinear hybrid SFDEs have also received a lot of attentions; see e.g., related works[19,20,21,22] and references therein.

On the other hand, for unstable hybrid stochastic differential equations (SDEs), Mao[23] designed a class of feedback controllers $u(t, x([t/\delta]\delta), r(t))$ only based on discrete state observations $x([t/\delta]\delta)$, which makes the controlled systems mean square exponentially stable, where the state $x(t) \in R^n$ and δ is the time interval between two observations, $[t/\delta]$ is is the integer part of t/δ, and the mode $r(t) \in \mathbf{\Theta} = \{1, 2, ..., N\}$ is a finite Markov chain. Obviously, such controllers $u(t, x([t/\delta]\delta), r(t))$ can not only save cost, but also be implemented more easily. Inspired by this, some scholars have extended this controller based on discrete observations to more general systems, and some have applied it to stochastic stabilization by intermittent control and have achieved many results.[24,25] Recently, based on discrete observation data $x([t/\delta]\delta)$, Fei and his collaborators[26,27] designed feedback controllers for highly nonlinear hybrid systems, and studied the asymptotic and exponential stability of the controlled systems.

Furthermore, considering that there may be a time lag δ_0 in the signal transmission of feedback control, Qiu et al.[28] designed a more realistic controller $u(t, x([t/\delta]\delta - \delta_0), r(t))$ to stabilize the unstable hybrid SDEs. In fact, delay control have been widely used in stochastic systems.[29,30,31,32,33]However, to the authors' best knowledge, there is little known on how to stabilize hybrid SFDEs with highly nonlinear by a delay feedback control based on discrete-time state observations. The problem becomes even harder when the time lag is a variable of time instead of a positive constant δ_0 as in the papers mentioned above. Comparing with the existing papers, we highlight the main works of this article are as follows.

- We consider that the control function u based on the discrete state values $x(k\delta)$ may produce different time lags δ_k at times $k\delta$, where $k = [t/\delta]$ with $k = 0, 1, 2, ...$ In this case, the controller $u(t, x(k\delta), r(t))$ works on interval $[k\delta+\delta_k, (k+1)\delta+\delta_{k+1})$. That is, affected by the variable delay δ_k, the working time of the controller in each discrete observation is variable rather than a constant δ. The work pattern of the controller is shown in Figure 1.1.

Figure 1.1: Sketch of the work pattern of $u(t, x(k\delta), r(t))$.

- Most of the existing papers use the comparison method to obtain the results of delay control based on discrete-time state observations. Specifically, when the continuous controller $u(t, x(t), r(t))$ can stabilize the unstable SDEs, compare $u(t, x([t/\delta]\delta - \delta_0), r(t))$ with $u(t, x(t), r(t))$ and obtain an upper bound of $\delta + \delta_0$ by using the property of flow. However, this comparison method not only requires that the equation is globally Lipschitz continuous, but also requires that the time lag must be a constant δ_0. Inspired by the work of Li et al.,[31] we will use Lyapunov functional method to find a better upper bound of $\delta + \delta_k$ for highly nonlinear hybrid SFDEs.

- The controlled highly nonlinear SFDEs are affected by both the distributed delay of the system itself and the variable delay δ_k caused by discrete observation signals. We used some new techniques to deal with the effects of different types and properties of time lags.

2 | NOTATIONS AND PROBLEM STATEMENT

Notations. Let $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a natural filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions. If G is a subset of Ω, denote by $\mathbb{I}_G$ its indicator function; that is, $\mathbb{I}_G(\omega) = 1$ if $\omega \in G$ and 0 otherwise. Let $R_+ = [0, \infty)$. If A is a vector or matrix, its transpose is denoted by A^T. For $x \in R^n$, $|x|$ denotes its Euclidean norm. For $A \in R^{n\times d}$, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its Frobenius norm. $A \leq 0$ $(A < 0)$ means that the matrix A is non-positive definite (negative definite). If A is a symmetric real-valued matrix $(A = A^T)$, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions ψ from $[-h, 0] \to R^n$ with the norm $\|\psi\| = \sup_{-h\leq s\leq 0} |\psi(s)|$. If $x(t)$ is an R^n-valued stochastic process, we let $x_t = \{x(t+s) : -h \leq s \leq 0\}$ for $t \geq 0$ whence x_t is a $C([-h, 0]; R^n)$-valued stochastic process.

Let $B(t) = (B_1(t), ..., B_d(t))^T$ be a d-dimensional Brownian motion defined on the probability space. For $t \geq 0$, let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbf{\Theta} = \{1, 2, ..., N\}$ with generator

$\Pi = (\pi_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \pi_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\pi_{ii} = -\sum_{j\neq i} \pi_{ij}$. We always assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Consider a nonlinear hybrid SFDE

$$dx(t) = f(t, x_t, r(t))dt + g(t, x_t, r(t))dB(t), t \geq 0 \tag{1}$$

with the initial data

$$\xi = \{\xi(s) : -h \leq s \leq 0\} \in C([-h, 0]; R^n) \text{ and } i_0 \in \mathbf{\Theta}, \tag{2}$$

where $h > 0$ is a system delay. Moreover,

$$f : R_+ \times C([-h, 0]; R^n) \times \mathbf{\Theta} \to R^n \text{ and } g : R_+ \times C([-h, 0]; R^n) \times \mathbf{\Theta} \to R^{n\times d}.$$

be both Borel measurable functions.

Let's give some hypotheses about the coefficients f and g.

Assumption 1. For any integer $b \geq 1$, there exists a real number $L_b > 0$ such that for all $(t, i) \in R_+ \times \mathbf{\Theta}$ and all $\psi, \phi \in C([-h, 0]; R^n)$ with $\|\psi\| \vee \|\phi\| \leq b$, it follows that

$$|f(t, \psi, i) - f(t, \phi, i)| \vee |g(t, \psi, i) - g(t, \phi, i)| \leq L_b\|\psi - \phi\|. \tag{3}$$

Moreover, for each $i \in \mathbf{\Theta}$, there are two probability measures μ_1 and μ_2 on $[-h, 0]$ as well as some numbers $L > 0$, $m_1 \geq 1$ and $m_2 \geq 1$ such that

$$|f(t, \psi, i)| \leq L(|\psi(0)| + |\psi(0)|^{m_1} + \int_{-h}^{0} |\psi(\theta)|d\mu_1(\theta) + \int_{-h}^{0} |\psi(\theta)|^{m_1} d\mu_1(\theta))$$

$$\text{and} \quad |g(t, \psi, i)| \leq L(|\psi(0)| + |\psi(0)|^{m_2} + \int_{-h}^{0} |\psi(\theta)|d\mu_2(\theta) + \int_{-h}^{0} |\psi(\theta)|^{m_2} d\mu_2(\theta)) \tag{4}$$

for all $(t, \psi) \in R_+ \times C([-h, 0]; R^n)$.

When $m_1 = m_2 = 1$, condition (4) degenerates to linear growth condition, so the results in this paper are more general than those of the previous ones.[28,34] Meanwhile, Assumption 1 can not guarantee the existence of global solution for equation (1), we need to introduce a new condition, which can be traced back to Khasminskii's work.[35]

Assumption 2. Let m_1, m_2, μ_1, μ_2 be the same as in Assumption 1. Assume that there are some positive numbers $m, p, a_j, b_j, (j = 0, 1, 2)$ such that

$$p \geq (m_1 + 1) \vee (2m_2 - m_1 + 1), \quad m > (m_1 + 1) \vee (2m_2) \quad \text{and} \quad a_0 > a_1 + a_2 \tag{5}$$

while

$$\psi(0)^T f(t, \psi, i) + \frac{p-1}{2}|g(t, \psi, i)|^2 \leq -a_0|\psi(0)|^m + \sum_{j=1}^{2} a_j \int_{-h}^{0} |\psi(\theta)|^m d\mu_j(\theta)$$
$$+ b_0|\psi(0)|^2 + \sum_{j=1}^{2} b_j \int_{-h}^{0} |\psi(\theta)|^2 d\mu_j(\theta) \tag{6}$$

for all $(t, \psi, i) \in R_+ \times C([-h, 0]; R^n) \times \mathbf{\Theta}$.

Using Theorem 3.1 in the work of Feng et al.,[17] it can be seen from Assumptions 1 and 2 that functional equation (1) has a global continuous solution such that $\sup_{0\leq t<\infty} \mathbb{E}|x(t)|^p < \infty$. However, the equations that satisfy the above assumptions are not necessarily stable (see Example (45) in Section 5). Therefore, we need to design a more realistic controller u in the drift term to stabilize the unstable stochastic system (1). As mentioned before, the controller only observes at discrete time $k\delta$, and each observation may have different time lags δ_k. Here $u : R_+ \times R^n \times \Theta \to R^n$ is a Borel measurable function. Let $\bar{\delta} > \underline{\delta} \geq 0$, while we shall assume $\delta_k \in [\underline{\delta}, \bar{\delta}]$, $\delta + \bar{\delta} \leq h$ and $\delta + \underline{\delta} > \bar{\delta}$. Then we will discuss the controlled hybrid SFDE

$$dx(t) = [f(t, x_t, r(t)) + u(t, x(\varrho_t), r(t))]dt + g(t, x_t, r(t))dB(t), \quad t \geq 0, \tag{7}$$

stability, where

$$x(\varrho_t) = \begin{cases} 0, & \text{if } t \in [0, \delta_0), \\ x(k\delta), & \text{if } t \in [k\delta + \delta_k, (k+1)\delta + \delta_{k+1}),\ k = 0, 1, 2, ... \end{cases}$$

Let's give a hypothesis about our controller u .

Assumption 3. Assume that there is a real number $\zeta > 0$ such that

$$|u(t, x, i) - u(t, y, i)| \leq \zeta |x - y| \tag{8}$$

for all $(t, i) \in R_+ \times \Theta$ and $x, y \in R^n$. Moreover, assume that $u(t, 0, i) \equiv 0$ for all $(t, i) \in R_+ \times \Theta$.

Remark 1. (i) Obviously, the assumption of $\delta + \underline{\delta} > \bar{\delta}$ is to ensure that interval $[k\delta + \delta_k, (k+1)\delta + \delta_{k+1})$ is nonempty. Since the time lag caused by the controller and the time interval of discrete observation are easy to adjust in practical application, we have assumed $\delta + \bar{\delta} \leq h$. Actually, when the delay of the system h is very small, it is entirely possible that $\delta + \bar{\delta} > h$. Especially $h = 0$, the system (1) becomes a SDE, similar results have been given by Fei et al.[26]

(ii) From Assumption 3, we can see that the solutions of hybrid SFDEs (7) and (1) are equal on $[0, \delta_0)$, that is, the controller u has no effect $[0, \delta_0)$, which is more reasonable. Since we will discuss the asymptotic behavior of the SFDE (7), we only need to discuss this controlled system on $[\delta_0, \infty)$. Therefore, let's define a bounded function $\vartheta : R_+ \to [\underline{\delta}, \delta + \bar{\delta})$ by

$$\vartheta(t) = t - k\delta \quad \text{for } t \in [k\delta + \delta_k, (k+1)\delta + \delta_{k+1}),\ k = 0, 1, 2, ... \tag{9}$$

Thus the SFDE (7) can be rewritten as

$$dx(t) = [f(t, x_t, r(t)) + u(t, x(t - \vartheta(t)), r(t))]dt + g(t, x_t, r(t))dB(t), \tag{10}$$

on $t \geq \delta_0$. Obviously, due to the existence and uniqueness of the solution of SFDE (1), we may choose the corresponding initial data as

$$\hat{\xi} = \{\hat{\xi}(s) : -h + \delta_0 \leq s \leq \delta_0\} \in C([-h, 0]; R^n) \text{ and } i_{\delta_0} \in \Theta, \tag{11}$$

where $\hat{\xi}$ is the solution of SFDE (1) on $[-h + \delta_0, \delta_0]$.

(iii) For any $(t, x, i) \in R_+ \times R^n \times \Theta$, by Assumption 3, it is easy to show that

$$|u(t, x, i)| \leq \zeta |x|, \tag{12}$$

where ζ is defined in (8).

3 | BOUNDEDNESS

In the following section, we will discuss the existence and uniqueness of the solution and the moment boundedness of the new controlled system (10).

Theorem 1. Under Assumptions 1, 2 and 3, for any given initial data (11),

(i) the SFDE (10) has a unique global solution $x(t)$,

(ii) for p in condition (5), the solution $x(t)$ satisfies that

$$\sup_{\delta_0 \leq t < \infty} \mathbb{E}|x(t)|^p < \infty. \tag{13}$$

That is, the controlled system (10) is asymptotically bounded in pth moment.

Proof. Let $V(x) = |x|^p$. An operator $LV : [\delta_0, \infty) \times C([-h, 0]; R^n) \times \Theta \to R$ is defined by

$$\begin{aligned} LV(t, \psi, i) =& p|\psi(0)|^{p-2}\psi(0)^T[f(t, \psi, i) + u(t, \psi(-\vartheta(0)), i)] + \frac{p}{2}|\psi(0)|^{p-2}|g(t, \psi, i)|^2 \\ &+ \frac{p(p-2)}{2}|\psi(0)|^{p-4}|\psi(0)^T g(t, \psi, i)|^2. \end{aligned}$$

By Assumptions 2 and 3, we further get

$$\begin{aligned}LV(t,\psi,i) \leq & p|\psi(0)|^{p-2}\Big[\psi(0)^T f(t,\psi,i)+\frac{p-1}{2}|g(t,\psi,i)|^2+\psi(0)^T u(t,\psi(-\vartheta(0)),i)\Big]\\ \leq & -pa_0|\psi(0)|^{p+m-2}+pb_0|\psi(0)|^p+p\zeta|\psi(0)|^{p-1}|\psi(-\vartheta(0))|\\ & +p|\psi(0)|^{p-2}\sum_{j=1}^{2}\Big(a_j\int_{-h}^{0}|\psi(\theta)|^m d\mu_j(\theta)+b_j\int_{-h}^{0}|\psi(\theta)|^2 d\mu_j(\theta)\Big).\end{aligned}$$

From the Young inequality, it is easy to calculate that

$$\begin{aligned}pa_j|\psi(0)|^{p-2}\int_{-h}^{0}|\psi(\theta)|^m d\mu_j(\theta)= & \int_{-h}^{0}pa_j|\psi(0)|^{p-2}|\psi(\theta)|^m d\mu_j(\theta)\\ \leq & \frac{p(p-2)a_j}{p+m-2}|\psi(0)|^{p+m-2}+\frac{pma_j}{p+m-2}\int_{-h}^{0}|\psi(\theta)|^{p+m-2}d\mu_j(\theta),\\ pb_j|\psi(0)|^{p-2}\int_{-h}^{0}|\psi(\theta)|^2 d\mu_j(\theta)= & \int_{-h}^{0}pb_j|\psi(0)|^{p-2}|\psi(\theta)|^2 d\mu_j(\theta)\\ \leq & (p-2)b_j|\psi(0)|^p+2b_j\int_{-h}^{0}|\psi(\theta)|^p d\mu_j(\theta),\\ p\zeta|\psi(0)|^{p-1}|\psi(-\vartheta(0))|= & \Big(\frac{(p\zeta)^{p/(p-1)}|\psi(0)|^p}{(0.5\varepsilon p)^{1/(p-1)}}\Big)^{\frac{p-1}{p}}\Big(0.5\varepsilon p|\psi(-\vartheta(0))|^p\Big)^{\frac{1}{p}}\\ \leq & \frac{(p-1)(\zeta)^{p/(p-1)}}{(0.5\varepsilon)^{1/(p-1)}}|\psi(0)|^p+0.5\varepsilon|\psi(-\vartheta(0))|^p.\end{aligned}$$

These, together with (5), yield

$$\begin{aligned}LV(t,\psi,i)\leq & -p\Big(a_0-\frac{(p-2)(a_1+a_2)}{p+m-2}\Big)|\psi(0)|^{p+m-2}+\sum_{j=1}^{2}K_j\int_{-h}^{0}|\psi(\theta)|^{p+m-2}d\mu_j(\theta)\\ & +0.5\varepsilon|\psi(-\vartheta(0))|^p+K_0|\psi(0)|^p+\sum_{j=1}^{2}2b_j\int_{-h}^{0}|\psi(\theta)|^p d\mu_j(\theta).\end{aligned}\tag{14}$$

where $K_0=pb_0+(p-2)(b_1+b_2)+\frac{(p-1)(\zeta)^{p/(p-1)}}{(0.5\varepsilon)^{1/(p-1)}}$, $K_j=\frac{pma_j}{p+m-2}, j=1,2.$

(i) Under condition (3), using the standard truncation method,[36] there exists a unique maximal local solution of equation (10) on $t\in[\delta_0,\rho_e)$, where ρ_e is the explosion time. Let k_0 be a sufficiently large positive constant for $\|\hat{\xi}\|<k_0$. To show that the local solution $x(t)$ is global, for each integer $k\geq k_0$, define the stopping time

$$\rho_k=\inf\{t\in[\delta_0,\rho_e):|x(t)|\geq k\},$$

where throughout this paper we set $\inf\emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). Obviously, ρ_k increases as $k\to\infty$ and $\rho_k\to\rho_\infty\leq\rho_e$ a.s. If we can deduce that $\rho_\infty=\infty$ a.s., then $\rho_e=\infty$ a.s., which implies the desired result (i). This is also equivalent to prove that there is $\lim_{k\to\infty}P(\rho_k\leq t)\to 0$. By the Itô formula, we obtain

$$\mathbb{E}V(x(t\wedge\rho_k))=V(x(\delta_0))+\mathbb{E}\int_{\delta_0}^{t\wedge\rho_k}LV(s,x_s,r(s))ds.\tag{15}$$

Recalling (5), we can rewrite (14) as

$$LV(t,\psi,i)\leq-p(a_0-a_1-a_2)|\psi(0)|^{p+m-2}+\sum_{j=1}^{2}K_j(\int_{-h}^{0}|\psi(\theta)|^{p+m-2}d\mu_j(\theta)-|\psi(0)|^{p+m-2})$$

$$+0.5\varepsilon|\psi(-\vartheta(0))|^p+(K_0+2b_1+2b_2)|\psi(0)|^p+\sum_{j=1}^{2}2b_j(\int_{-h}^{0}|\psi(\theta)|^p d\mu_j(\theta)-|\psi(0)|^p)$$

$$\leq\sum_{j=1}^{2}K_j(\int_{-h}^{0}|\psi(\theta)|^{p+m-2}d\mu_j(\theta)-|\psi(0)|^{p+m-2})+0.5\varepsilon|\psi(-\vartheta(0))|^p$$

$$+\sum_{j=1}^{2}2b_j(\int_{-h}^{0}|\psi(\theta)|^p d\mu_j(\theta)-|\psi(0)|^p)+C_1,$$

where $C_1:=\max_{s\geq 0}\Big[-p(a_0-a_1-a_2)s^{p+m-2}+(K_0+2b_1+2b_2)s^p\Big]$. Hence, we deduce that

$$\mathbb{E}V(x(t\wedge\rho_k))\leq|x(0)|^p+C_1t+\sum_{j=1}^{2}K_j\mathbb{E}\int_{\delta_0}^{t\wedge\rho_k}\Big(\int_{-h}^{0}|x(s+\theta)|^{p+m-2}d\mu_j(\theta)-|x(s)|^{p+m-2}\Big)ds$$

$$+0.5\varepsilon\mathbb{E}\int_{\delta_0}^{t\wedge\rho_k}|x(s-\vartheta(s))|^p ds+\sum_{j=1}^{2}2b_j\mathbb{E}\int_{\delta_0}^{t\wedge\rho_k}\Big(\int_{-h}^{0}|x(s+\theta)|^p d\mu_j(\theta)-|x(s)|^p\Big)ds. \tag{16}$$

Using the Fubini theorem, we may give the following estimate

$$\begin{aligned}\int_{\delta_0}^{t\wedge\rho_k}\Big(\int_{-h}^{0}|x(s+\theta)|^{p+m-2}d\mu_j(\theta)-|x(s)|^{p+m-2}\Big)ds&=\int_{-h}^{0}d\mu_j(\theta)\int_{\delta_0+\theta}^{t\wedge\rho_k+\theta}|x(s)|^{p+m-2}ds-\int_{\delta_0}^{t\wedge\rho_k}|x(s)|^{p+m-2}ds\\&\leq\int_{-h}^{0}d\mu_j(\theta)\int_{\delta_0-h}^{t\wedge\rho_k}|x(s)|^{p+m-2}ds-\int_{\delta_0}^{t\wedge\rho_k}|x(s)|^{p+m-2}ds\\&=\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^{p+m-2}ds.\end{aligned}$$

Similarly,

$$\int_{\delta_0}^{t\wedge\rho_k}\Big(\int_{-h}^{0}|x(s+\theta)|^p d\mu_j(\theta)-|x(s)|^p\Big)ds\leq\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^p ds.$$

Substituting these into (16) gives

$$\begin{aligned}\mathbb{E}|x(t\wedge\rho_k)|^p\leq&0.5\varepsilon\mathbb{E}\int_{\delta_0}^{t\wedge\rho_k}|x(s-\vartheta(s))|^p ds+C(t)\leq 0.5\varepsilon\mathbb{E}\int_{\delta_0}^{t}|x(s-\vartheta(s))|^p\mathbb{1}_{[\delta_0,\rho_k]}(s)ds+C(t)\\=&0.5\varepsilon\int_{\delta_0}^{t}\mathbb{E}\Big(|x(s-\vartheta(s))|^p\mathbb{1}_{[\delta_0,\rho_k]}(s)\Big)ds+C(t),\end{aligned}$$

where

$$C(t)=|x(0)|^p+C_1t+(K_1+K_2)\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^{p+m-2}ds+2(b_1+b_2)\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^p ds.$$

Obviously, for all $s\geq\delta_0$, we deduce that $0\leq s-\vartheta(s)\leq s$, which implies

$$\mathbb{E}\Big[|x(s-\vartheta(s))|^p\mathbb{1}_{[\delta_0,\rho_k]}(s)\Big]\leq\sup_{0\leq w\leq s}\mathbb{E}|x(w\wedge\rho_k)|^p.$$

Therefore, we have

$$\mathbb{E}|x(t\wedge\rho_k)|^p \le 0.5\varepsilon\int_{\delta_0}^{t}\sup_{0\le w\le s}\mathbb{E}|x(w\wedge\rho_k)|^p ds + C(t),$$

which implies that

$$\sup_{\delta_0\le w\le t}\mathbb{E}|x(w\wedge\rho_k)|^p \le 0.5\varepsilon\int_{\delta_0}^{t}\sup_{0\le w\le s}\mathbb{E}|x(w\wedge\rho_k)|^p ds + C(t).$$

Further, we get

$$\begin{aligned}\sup_{0\le w\le t}\mathbb{E}|x(w\wedge\rho_k)|^p \le& \sup_{0\le w\le \delta_0}\mathbb{E}|x(w)|^p + \sup_{\delta_0\le w\le t}\mathbb{E}|x(w\wedge\rho_k)|^p\\ \le&\|\hat{\xi}\|^p + 0.5\varepsilon\int_{\delta_0}^{t}\sup_{0\le w\le s}\mathbb{E}|x(w\wedge\rho_k)|^p ds + C(t).\end{aligned}$$

It follows from the Gronwall inequality immediately that

$$\sup_{0\le w\le t}\mathbb{E}|x(w\wedge\rho_k)|^p \le (C(t)+\|\hat{\xi}\|^p)e^{0.5\varepsilon(t-\delta_0)}.$$

Hence

$$\mathbb{P}(\rho_k\le t)k^p \le \sup_{0\le w\le t}\mathbb{E}|x(w\wedge\rho_k)|^p \le (C(t)+\|\hat{\xi}\|^p)e^{0.5\varepsilon(t-\delta_0)},$$

which implies that

$$\limsup_{k\to\infty}\mathbb{P}(\rho_k\le t)\le \lim_{k\to\infty}\frac{(C(t)+\|\hat{\xi}\|^p)e^{0.5\varepsilon(t-\delta_0)}}{k^p}=0,$$

as required.

(ii) By (14), using the Itô formula to function $e^{\varepsilon t}|x|^p$ gives

$$\begin{aligned}d(e^{\varepsilon t}V(x(t))) =& e^{\varepsilon t}(LV(t,x_t,r(t))+\varepsilon V(x(t)))dt + pe^{\varepsilon t}|x(t)|^{p-2}x(t)^T g(t,x_t,r(t))dB(t)\\ \le& e^{\varepsilon t}\Big[-p\Big(a_0-\frac{(p-2)(a_1+a_2)}{p+m-2}\Big)|x(t)|^{p+m-2}+\sum_{j=1}^{2}K_j\int_{-h}^{0}|x(t+\theta)|^{p+m-2}d\mu_j(\theta)\\ &+0.5\varepsilon|x(t-\vartheta(t))|^p+(\varepsilon+K_0)|x(t)|^p+\sum_{j=1}^{2}2b_j\int_{-h}^{0}|x(t+\theta)|^p d\mu_j(\theta)\Big]dt\\ &+pe^{\varepsilon t}|x(t)|^{p-2}x(t)^T g(t,x_t,r(t))dB(t). \end{aligned}\tag{17}$$

Define

$$V_1(t)=\sum_{j=1}^{2}K_j\int_{-h}^{0}\int_{t+\theta}^{t}e^{\varepsilon(s-\theta)}|x(s)|^{p+m-2}ds\,d\mu_j(\theta),\quad\text{and}\quad V_2(t)=\sum_{j=1}^{2}2b_j\int_{-h}^{0}\int_{t+\theta}^{t}e^{\varepsilon(s-\theta)}|x(s)|^{p}ds\,d\mu_j(\theta).$$

By the differential calculation, we obtain

$$\begin{aligned}dV_1(t)=&\sum_{j=1}^{2}K_j\Big(\int_{-h}^{0}e^{\varepsilon(t-\theta)}|x(t)|^{p+m-2}d\mu_j(\theta)-\int_{-h}^{0}e^{\varepsilon t}|x(t+\theta)|^{p+m-2}d\mu_j(\theta)\Big)dt\\ \le&\sum_{j=1}^{2}K_j\Big(e^{\varepsilon(t+h)}|x(t)|^{p+m-2}-e^{\varepsilon t}\int_{-h}^{0}|x(t+\theta)|^{p+m-2}d\mu_j(\theta)\Big)dt.\end{aligned}$$

Similarly

$$dV_2(t)\le\sum_{j=1}^{2}2b_j\Big(e^{\varepsilon(t+h)}|x(t)|^p-e^{\varepsilon t}\int_{-h}^{0}|x(t+\theta)|^p d\mu_j(\theta)\Big)dt.$$

These, together with (17), give

$$d(e^{\varepsilon t}V(x(t)) + V_1(t) + V_2(t)) \leq e^{\varepsilon t}\Big[H(x(t)) + 0.5\varepsilon|x(t-\vartheta(t))|^p\Big]dt + pe^{\varepsilon t}|x(t)|^{p-2}x^T(t)g(t,x_t,r(t))dB(t), \tag{18}$$

where

$$H(s) = -\left(pa_0 - \frac{p(a_1+a_2)(p-2)}{p+m-2} - e^{\varepsilon h}(K_1+K_2)\right)|s|^{p+m-2} + \left(\varepsilon + K_0 + (2b_1+2b_2)e^{\varepsilon h}\right)|s|^p.$$

Recalling (5) and the definition of K_j, we may choose $\varepsilon > 0$ so small such that

$$a_0 - \frac{(a_1+a_2)(p-2)}{p+m-2} - e^{\varepsilon h}(K_1+K_2) > 0.$$

Then, let $C_2 = \sup_{s\geq 0} H(s)$, we can rewrite (18) as

$$d(e^{\varepsilon t}V(x(t)) + V_1(t) + V_2(t)) \leq e^{\varepsilon t}(C_2 + 0.5\varepsilon|x(t-\vartheta(t))|^p)dt + pe^{\varepsilon t}|x(t)|^{p-2}x^T(t)g(t,x_t,r(t))dB(t). \tag{19}$$

Integrating the above inequality from δ_0 to t, and taking expectation lead to

$$\begin{aligned}\mathbb{E}e^{\varepsilon t}|x(t)|^p \leq& \mathbb{E}(e^{\varepsilon t}V(x(t)) + V_1(t) + V_2(t))\\ \leq& V(x(\delta_0)) + V_1(\delta_0) + V_2(\delta_0) + \mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}\Big[C_2 + 0.5\varepsilon|x(s-\vartheta(s))|^p\Big]ds\\ \leq& C_3 + C_2\frac{e^{\varepsilon t}}{\varepsilon} + 0.5\sup_{0\leq s\leq t}\mathbb{E}|x(s)|^p e^{\varepsilon t},\end{aligned}$$

where $C_3 := V(x(\delta_0)) + V_1(\delta_0) + V_2(\delta_0)$. Subsequently,

$$\mathbb{E}|x(t)|^p \leq C_3 + \frac{C_2}{\varepsilon} + 0.5\sup_{0\leq s\leq t}\mathbb{E}|x(s)|^p.$$

This implies

$$\sup_{0\leq s\leq t}\mathbb{E}|x(s)|^p \leq \sup_{\delta_0\leq s\leq t}\mathbb{E}|x(s)|^p + \|\hat{\xi}\|^p \leq C_3 + \frac{C_2}{\varepsilon} + 0.5\sup_{0\leq s\leq t}\mathbb{E}|x(s)|^p + \|\hat{\xi}\|^p.$$

Then, we have

$$\sup_{0\leq s\leq t}\mathbb{E}|x(s)|^p \leq 2(C_3 + \frac{C_2}{\varepsilon} + \|\hat{\xi}\|^p) := C_4. \tag{20}$$

Letting $t \to \infty$, we therefore obtain the desired result (13). □

Remark 2. It is obvious from Theorem 1 that the solution of equation (7) with initial data (2) is unique and asymptotically bounded on $[0,\infty)$. Similarly, the stabilization results in the following section hold for SFDE (7). Therefore, in all the assumptions in the next section, we let $(t,\psi,i) \in R_+ \times C([-h,0];R^n) \times \mathbf{\Theta}$.

4 | EXPONENTIAL STABILIZATION

In this section, we will give some criteria related to the control term u to obtain the exponential stability of the controlled SFDE (10), and these criteria will be constructed by M-matrix. For the definition and basic properties of M-matrix, the reader may refer to section 2.6 in the work of Mao and Yuan.[4] Next, we give a condition related to M-matrix.

Assumption 4. For each $i \in \mathbf{\Theta}$, assume that there exist $b_{i0}, \hat{b}_{i0} \in R$ and some positive constants $a_{i0}, \hat{a}_{i0}, a_{ij}, \hat{a}_{ij}, b_{ij}, \hat{b}_{ij}$ $(j = 1,2)$ for both

$$\psi(0)^T[f(t,\psi,i) + u(t,\psi(0),i)] + \frac{1}{2}|g(t,\psi,i)|^2$$

$$\leq -a_{i0}|\psi(0)|^m + \sum_{j=1}^{2} a_{ij} \int_{-h}^{0} |\psi(\theta)|^m d\mu_j(\theta) + b_{i0}|\psi(0)|^2 + \sum_{j=1}^{2} b_{ij} \int_{-h}^{0} |\psi(\theta)|^2 d\mu_j(\theta) \tag{21}$$

and

$$\begin{aligned}&\psi(0)^T[f(t,\psi,i) + u(t,\psi(0),i)] + \frac{m_1}{2}|g(t,\psi,i)|^2 \\ &\leq -\hat{a}_{i0}|\psi(0)|^m + \sum_{j=1}^{2} \hat{a}_{ij} \int_{-h}^{0} |\psi(\theta)|^m d\mu_j(\theta) + \hat{b}_{i0}|\psi(0)|^2 + \sum_{j=1}^{2} \hat{b}_{ij} \int_{-h}^{0} |\psi(\theta)|^2 d\mu_j(\theta)\end{aligned} \tag{22}$$

to hold for all $(t,\psi) \in R_+ \times C([-h,0];R^n)$. Moreover,

$$\Lambda_1 := -2\text{diag}(b_{10},...,b_{N0}) - \Pi \;\text{ and }\; \Lambda_2 := -(m_1+1)\text{diag}(\hat{b}_{10},...,\hat{b}_{N0}) - \Pi \tag{23}$$

are nonsingular M-matrices.

Remark 3. From the assumptions in Section 2, we can see that the above conditions are very easy to achieve in practice. For example, let's take $u(t,x,i) = Ax$, where A is a symmetric $n \times n$ real-valued matrix such that $\lambda_{\max}(A) \leq -2b_0$ (u obviously satisfies Assumption 3). Then

$$x^T u(t,x,i) \leq -2b_0|x|^2, \quad \forall (t,x,i) \in R_+ \times R^n \times \mathbf{\Theta}.$$

Combining this and Assumption 6 clearly shows that both conditions (21) and (22) can be satisfied, and

$$\Lambda_1 = 2\text{diag}(b_0,...,b_0) - \Pi \quad \text{and} \quad \Lambda_2 = (m_1+1)\text{diag}(b_0,...,b_0) - \Pi$$

are both nonsingular M matrices, which implies that all the conditions of Assumption 4 are satisfied.

Using the properties of M-matrices, there exist positive constants η_i and $\hat{\eta}_i$ such that

$$(\eta_1,...,\eta_N)^T := \Lambda_1^{-1}(1,...,1)^T, \quad (\hat{\eta}_1,...,\hat{\eta}_N)^T := \Lambda_2^{-1}(1,...,1)^T, \tag{24}$$

where Λ_1 and Λ_2 have specified in Assumption 4.

In this paper, the form of Lyapunov functional with M-matrix is as follows

$$\hat{U}(t,\hat{x}_t,\hat{r}_t) = U(x(t),r(t)) + \varpi \int_{-\delta^*}^{0} \int_{t+s}^{t} \Phi(w,x_w,r(w))dwds \tag{25}$$

for $t \geq \delta_0$, where $\hat{x}_t := \{x(t+\theta) : -2h \leq \theta \leq 0\}$, $\hat{r}_t := \{r(t+\theta) : -2h \leq \theta \leq 0\}$, $\delta^* = \delta + \bar{\delta}$, ϖ is a positive number to be determined later, U and $\Phi(t,x_t,r(t))$ have been defined by

$$U(x,i) = \eta_i|x|^2 + \hat{\eta}_i|x|^{m_1+1} \tag{26}$$

and

$$\Phi(t,x_t,r(t)) = \delta^*|f(t,x_t,r(t)) + u(t,x(t-\vartheta(t)),r(t))|^2 + |g(t,x_t,r(t))|^2,$$

respectively. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $\delta_0 \leq t < \delta_0 + 2h$, we set $x(\theta) = x(-h)$ for $\theta \in [\delta_0 - 2h, -h)$ and $r(\theta) = i_0$ for $\theta \in [\delta_0 - 2h, 0)$. Similarly, we set

$$f(t,\psi,i) = f(0,\psi,i),\; g(t,\psi,i) = g(0,\psi,i),\; u(t,\psi(-\vartheta(0)),i) = u(0,\psi(-\vartheta(0)),i)$$

for $(t,\psi,i) \in [\delta_0 - 2h, 0) \times C([-h,0];R^n) \times \mathbf{\Theta}$. The following Lemma directly follows from the generalized Itô formula[4] and the basic differential operation.

Lemma 1. For $t \geq \delta_0$, $\hat{U}(t,\hat{x}_t,\hat{r}_t)$ is an Itô stochastic process with its Itô differential

$$d\hat{U}(t,\hat{x}_t,\hat{r}_t) = \Big[\mathcal{L}U(t,x_t,r(t)) + \varpi\delta^*\Phi(t,x_t,r(t)) - \varpi \int_{t-\delta^*}^{t} \Phi(s,x_s,r(s))ds\Big]dt + d\hat{M}(t), \tag{27}$$

where $\mathcal{L}U : [\delta_0,\infty) \times C([-h,0];R^n) \times \mathbf{\Theta} \to R$ is defined as

$$\mathcal{L}U(t,x_t,r(t)) = 2\eta_{r(t)}\Big[x(t)^T[f(t,x_t,r(t)) + u(t,x(t-\vartheta(t)),r(t))] + \frac{1}{2}|g(t,x_t,r(t))|^2\Big]$$

$$+ (m_1+1)\hat{\eta}_{r(t)}|x(t)|^{m_1-1}\Big[x(t)^T[f(t,x_t,r(t))+u(t,x(t-\vartheta(t)),r(t))]+\frac{1}{2}|g(t,x_t,r(t))|^2\Big] + \frac{(m_1^2-1)}{2}\hat{\eta}_{r(t)}|x(t)|^{m_1-3}|x(t)^Tg(t,x_t,r(t))|^2+\sum_{j=1}^{N}\pi_{r(t)j}(\eta_j|x(t)|^2+\hat{\eta}_j|x(t)|^{m_1+1}),$$

as well as $\hat{M}(t)$ is a local continuous martingale with $\hat{M}(\delta_0)=0$.

Obviously, we deduce that

$$\begin{aligned}\mathcal{L}U(t,x_t,r(t)) &\le 2\eta_{r(t)}\Big[x(t)^T[f(t,x_t,r(t))+u(t,x(t-\vartheta(t)),r(t))]+\frac{1}{2}|g(t,x_t,r(t))|^2\Big]+(m_1+1)\hat{\eta}_{r(t)}|x(t)|^{m_1-1}\\ &\times\Big[x(t)^T[f(t,x_t,r(t))+u(t,x(t-\vartheta(t)),r(t))]+\frac{m_1}{2}|g(t,x_t,r(t))|^2\Big]+\sum_{j=1}^{N}\pi_{r(t)j}(\eta_j|x(t)|^2+\hat{\eta}_j|x(t)|^{m_1+1})\\ &\le LU(t,x_t,r(t))+(2\eta_{r(t)}+(m_1+1)\hat{\eta}_{r(t)}|x(t)|^{m_1-1})x(t)^T(u(t,x(t-\vartheta(t)),r(t))-u(t,x(t),r(t)))\end{aligned}$$

where $LU: R_+\times C([-h,0];R^n)\times\mathbf{\Theta}\to R$ is defined by

$$\begin{aligned}LU(t,\psi,i)&=2\eta_i\Big[\psi(0)^T[f(t,\psi,i)+u(t,\psi(0),i)]+\frac{1}{2}|g(t,\psi,i)|^2\Big]+(m_1+1)\hat{\eta}_i|\psi(0)|^{m_1-1}\\ &\times\Big[\psi(0)^T[f(t,\psi,i)+u(t,\psi(0),i)]+\frac{m_1}{2}|g(t,\psi,i)|^2\Big]+\sum_{j=1}^{N}\pi_{ij}(\eta_j|\psi(0)|^2+\hat{\eta}_j|\psi(0)|^{m_1+1}).\end{aligned} \tag{28}$$

Let's give the first stability result of this paper.

Theorem 2. Let Assumptions 1, 2, 3, 4 hold. Assume that there exist positive numbers $\kappa,\alpha_1,\alpha_2,\alpha_3$ and $\beta_j, j=1,...,7$, as well as a function $W(x)\in C(R^n;R_+)$, such that

$$\beta_1+\beta_2<1,\quad \beta_3+\beta_4<1,\quad \beta_5|\psi(0)|^{m+m_1-1}\le W(\psi(0))\le\beta_6+\beta_7|\psi(0)|^{m+m_1-1} \tag{29}$$

and

$$\begin{aligned}LU(t,\psi,i)&+\alpha_1|f(t,\psi,i)|^2+\alpha_2|g(t,\psi,i)|^2+\alpha_3\big(2\eta_i|\psi(0)|+(m_1+1)\hat{\eta}_i|\psi(0)|^{m_1}\big)^2\\ &\le-\kappa\Big[|\psi(0)|^2-\beta_1\int_{-h}^{0}|\psi(\theta)|^2d\mu_1(\theta)-\beta_2\int_{-h}^{0}|\psi(\theta)|^2d\mu_2(\theta)\Big]\\ &\quad -W(\psi(0))+\beta_3\int_{-h}^{0}W(\psi(\theta))d\mu_1(\theta)+\beta_4\int_{-h}^{0}W(\psi(\theta))d\mu_2(\theta)\end{aligned} \tag{30}$$

for all $(t,\psi,i)\in R_+\times C([-h,0];R^n)\times\mathbf{\Theta}$. Assume also δ^* is sufficiently small for

$$\delta^*<\frac{\sqrt{\kappa\alpha_3(1-\beta_1-\beta_2)}}{2\zeta^2}\ \text{ and }\ \delta^*\le\frac{\sqrt{\alpha_1\alpha_3}}{\sqrt{2}\zeta}\wedge\frac{\alpha_2\alpha_3}{\zeta^2}\wedge\frac{1}{4\sqrt{2}\zeta}. \tag{31}$$

Then, for any $\hat{p}\in[2,p)$ and initial data (11), the solution of the SFDE (10) obeys

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^{\hat{p}})<0. \tag{32}$$

Proof. We divide the proof into four steps.

Step 1. By Assumption 3, then recalling (28), we deduce that

$$\mathcal{L}U(t,x_t,r(t))\le LU(t,x_t,r(t))+\alpha_3\big[2\eta_{r(t)}|x(t)|+(m_1+1)\hat{\eta}_{r(t)}|x(t)|^{m_1}\big]^2+\frac{\zeta^2}{4\alpha_3}|x(t)-x(t-\vartheta(t))|^2.$$

Then

$$d\hat{U}(t,\hat{x}_t,\hat{r}_t)\le\mathbb{L}\hat{U}(t,\hat{x}_t,\hat{r}_t)dt+d\hat{M}(t), \tag{33}$$

in which

$$\mathbb{L}\hat{U}(t,\hat{x}_t,\hat{r}_t)=LU(t,x_t,r(t))+\alpha_3\big[2\eta_{r(t)}|x(t)|+(m_1+1)\hat{\eta}_{r(t)}|x(t)|^{m_1}\big]^2$$

$$+\frac{\zeta^2}{4\alpha_3}|x(t)-x(t-\vartheta(t))|^2+\varpi\delta^*\Phi(t,x_t,r(t))-\varpi\int_{t-\delta^*}^{t}\Phi(s,x_s,r(s))ds. \tag{34}$$

Moreover, using Assumptions 1, 3 and Theorem 1, we have

$$\sup_{\delta_0\le t<\infty}\mathbb{E}|\mathbb{L}\hat{U}(t,\hat{x}_t,\hat{r}_t)|<\infty. \tag{35}$$

Step 2. Let $\varpi=\zeta^2/\alpha_3$. We derive from (33) and (35) that

$$e^{\varepsilon t}\mathbb{E}\hat{U}(t,\hat{x}_t,\hat{r}_t)\le\hat{U}(\delta_0,\hat{x}_{\delta_0},\hat{r}_{\delta_0})+\mathbb{E}\int_{\delta_0}^{t}e^{\varepsilon s}(\varepsilon\hat{U}(s,\hat{x}_s,\hat{r}_s)+\mathbb{L}\hat{U}(s,\hat{x}_s,\hat{r}_s))ds \tag{36}$$

for any $t\ge\delta_0$. Using condition (31), it is easy to show that $\frac{2(\delta^*)^2\zeta^2}{\alpha_3}\le\alpha_1$ and $\frac{\delta^*\zeta^2}{\alpha_3}\le\alpha_2$, then by elementary inequality and (12), we have

$$\begin{aligned}\varpi\delta^*\Phi(s,x_s,r(s))&\le\frac{2(\delta^*)^2\zeta^2}{\alpha_3}|f(s,x_s,r(s))|^2+\frac{\delta^*\zeta^2}{\alpha_3}|g(s,x_s,r(s))|^2+\frac{2(\delta^*)^2\zeta^2}{\alpha_3}|u(s,x(s-\vartheta(s)),r(s))|^2\\&\le\alpha_1|f(s,x_s,r(s))|^2+\alpha_2|g(s,x_s,r(s))|^2+\frac{2(\delta^*)^2\zeta^4}{\alpha_3}|x(s-\vartheta(s))|^2.\end{aligned}$$

Substituting this into (34) and using condition (30) give

$$\begin{aligned}\mathbb{L}\hat{U}(s,\hat{x}_s,\hat{r}_s)\le&LU(s,x_s,r(s))+\alpha_1|f(s,x_s,r(s))|^2+\alpha_2|g(s,x_s,r(s))|^2\\&+\alpha_3\big[2\eta_{r(s)}|x(s)|+(m_1+1)\hat{\eta}_{r(s)}|x(s)|^{m_1}\big]^2+\frac{2(\delta^*)^2\zeta^4}{\alpha_3}|x(s-\vartheta(s))|^2\\&+\frac{\zeta^2}{4\alpha_3}|x(s)-x(s-\vartheta(s))|^2-\frac{\zeta^2}{\alpha_3}\int_{s-\delta^*}^{s}\Phi(w,x_w,r(w))dw\\\le&-\kappa\Big[|x(s)|^2-\beta_1\int_{-h}^{0}|x(s+\theta)|^2d\mu_1(\theta)-\beta_2\int_{-h}^{0}|x(s+\theta)|^2d\mu_2(\theta)\Big]-W(x(s))\\&+\beta_3\int_{-h}^{0}W(x(s+\theta))d\mu_1(\theta)+\beta_4\int_{-h}^{0}W(x(s+\theta))d\mu_2(\theta)+\frac{2(\delta^*)^2\zeta^4}{\alpha_3}|x(s-\vartheta(s))|^2\\&+\frac{\zeta^2}{4\alpha_3}|x(s)-x(s-\vartheta(s))|^2-\frac{\zeta^2}{\alpha_3}\int_{s-\delta^*}^{s}\Phi(w,x_w,r(w))dw.\end{aligned}$$

By (31), noting that $\zeta\delta^*\le\frac{1}{4\sqrt{2}}$, we obtain that

$$\frac{2(\delta^*)^2\zeta^4}{\alpha_3}|x(s-\vartheta(s))|^2\le\frac{4(\delta^*)^2\zeta^4}{\alpha_3}|x(s)|^2+\frac{\zeta^2}{8\alpha_3}|x(s)-x(s-\vartheta(s))|^2.$$

It is easy to see that

$$\begin{aligned}\mathbb{L}\hat{U}(s,\hat{x}_s,\hat{r}_s)\le&-\Big(\kappa-\frac{4(\delta^*)^2\zeta^4}{\alpha_3}\Big)|x(s)|^2+\kappa\beta_1\int_{-h}^{0}|x(s+\theta)|^2d\mu_1(\theta)+\kappa\beta_2\int_{-h}^{0}|x(s+\theta)|^2d\mu_2(\theta)\\&-W(x(s))+\beta_3\int_{-h}^{0}W(x(s+\theta))d\mu_1(\theta)+\beta_4\int_{-h}^{0}W(x(s+\theta))d\mu_2(\theta)\\&+\frac{3\zeta^2}{8\alpha_3}|x(s)-x(s-\vartheta(s))|^2-\frac{\zeta^2}{\alpha_3}\int_{s-\delta^*}^{s}\Phi(w,x_w,r(w))dw.\end{aligned}$$

Substituting this into (36) gives

$$e^{\varepsilon t}\mathbb{E}\hat{U}(t,\hat{x}_t,\hat{r}_t) \leq \hat{U}(\delta_0,\hat{x}_{\delta_0},\hat{r}_{\delta_0}) + \mathbb{E}\int_{\delta_0}^{t} \varepsilon e^{\varepsilon s}\hat{U}(s,\hat{x}_s,\hat{r}_s)ds + \Upsilon_1 + \Upsilon_2 + \Upsilon_3 - \Upsilon_4, \tag{37}$$

where

$$\Upsilon_1 = \mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}\Big[-\Big(\kappa - \frac{4(\delta^*)^2\zeta^4}{\alpha_3}\Big)|x(s)|^2 + \kappa\beta_1\int_{-h}^{0}|x(s+\theta)|^2 d\mu_1(\theta) + \kappa\beta_2\int_{-h}^{0}|x(s+\theta)|^2 d\mu_2(\theta)\Big]ds,$$

$$\Upsilon_2 = \mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}\Big[-W(x(s)) + \beta_3\int_{-h}^{0} W(x(s+\theta))d\mu_1(\theta) + \beta_4\int_{-h}^{0} W(x(s+\theta))d\mu_2(\theta)\Big]ds,$$

$$\Upsilon_3 = \frac{3\zeta^2}{8\alpha_3}\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}|x(s)-x(s-\vartheta(s))|^2 ds,$$

$$\Upsilon_4 = \frac{\zeta^2}{\alpha_3}\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}\Big(\int_{s-\delta^*}^{s}\Phi(w,x_w,r(w))dw\Big)ds.$$

Step 3. Applying the substitution technique, we have

$$\int_{\delta_0}^{t}\int_{-h}^{0} e^{\varepsilon s}|x(s+\theta)|^2 d\mu_k(\theta)ds = \int_{-h}^{0} e^{-\varepsilon\theta}d\mu_k(\theta)\int_{\delta_0}^{t} e^{\varepsilon(s+\theta)}|x(s+\theta)|^2 ds \leq e^{\varepsilon h}\int_{-h}^{0} d\mu_k(\theta)\int_{\delta_0-h}^{t} e^{\varepsilon s}|x(s)|^2 ds$$

$$\leq e^{\varepsilon h}\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^2 ds + e^{\varepsilon h}\int_{\delta_0}^{t} e^{\varepsilon s}|x(s)|^2 ds.$$

Thus

$$\Upsilon_1 \leq \kappa(\beta_1+\beta_2)e^{\varepsilon h}\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^2 ds - \Big[\kappa - \frac{4(\delta^*)^2\zeta^4}{\alpha_3} - \kappa(\beta_1+\beta_2)e^{\varepsilon h}\Big]\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}|x(s)|^2 ds. \tag{38}$$

Similarly,

$$\Upsilon_2 \leq e^{\varepsilon h}(\beta_3+\beta_4)\int_{\delta_0-h}^{\delta_0} W(\hat{\xi}(s))ds - [1 - e^{\varepsilon h}(\beta_3+\beta_4)]\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}W(x(s))ds. \tag{39}$$

By the Fubini theorem,

$$\Upsilon_3 = \frac{3\zeta^2}{8\alpha_3}\int_{\delta_0}^{t} e^{\varepsilon s}\mathbb{E}|x(s)-x(s-\vartheta(s))|^2 ds.$$

Appling the Itô isometry and the Hölder inequality, we get

$$\mathbb{E}|x(s)-x(s-\vartheta(s))|^2 \leq 2\mathbb{E}\int_{s-\vartheta(s)}^{s}\Big(\delta^*|f(w,x_w,r(w)) + u(w,x(w-\vartheta(w)),r(w))|^2 + |g(w,x_w,r(w))|^2\Big)dw$$

$$\leq 2\mathbb{E}\int_{s-\delta^*}^{s}\Big(\delta^*|f(w,x_w,r(w)) + u(w,x(w-\vartheta(w)),r(w))|^2 + |g(w,x_w,r(w))|^2\Big)dw,$$

which implies

$$\Upsilon_3 \leq 3/4\Upsilon_4. \tag{40}$$

Plugging (38) , (39) and (40) into (37), we have

$$e^{\varepsilon t}\mathbb{E}\hat{U}(t,\hat{x}_t,\hat{r}_t) \le C_5 + \mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}\varepsilon\hat{U}(s,\hat{x}_s,\hat{r}_s)ds - \Big[\kappa - \frac{4(\delta^*)^2\zeta^4}{\alpha_3} - \kappa(\beta_1+\beta_2)e^{\varepsilon h}\Big]\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}|x(s)|^2 ds$$
$$- [1 - e^{\varepsilon h}(\beta_3+\beta_4)]\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}W(x(s))ds - 1/4\Upsilon_4, \tag{41}$$

where $C_5 = \hat{U}(\delta_0,\hat{x}_{\delta_0},\hat{r}_{\delta_0}) + \kappa(\beta_1+\beta_2)e^{\varepsilon h}\int_{\delta_0-h}^{\delta_0}|\hat{\xi}(s)|^2 ds + e^{\varepsilon h}(\beta_3+\beta_4)\int_{\delta_0-h}^{\delta_0}W(\hat{\xi}(s))ds$.

Step 4. Using the elementary inequality and (29), we give

$$|x|^{m_1+1} \le |x|^2 + |x|^{m+m_1-1} \le |x|^2 + \frac{W(x)}{\beta_5}.$$

Recalling the definition of $\hat{U}$ yields

$$\varpi_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le e^{\varepsilon t}\mathbb{E}\hat{U}(t,\hat{x}_t,\hat{r}_t) \le C_5 - \Big[1 - e^{\varepsilon h}(\beta_3+\beta_4) - \frac{\varepsilon\varpi_3}{\beta_5}\Big]\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}W(x(s))ds$$
$$- \Big[\kappa - \frac{4(\delta^*)^2\zeta^4}{\alpha_3} - \kappa(\beta_1+\beta_2)e^{\varepsilon h} - \varepsilon\varpi_2 - \varepsilon\varpi_3\Big]\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}|x(s)|^2 ds + \Upsilon_5 - 1/4\Upsilon_4, \tag{42}$$

where $\varpi_1 = \min_{i\in\Theta}\eta_i$, $\varpi_2 = \max_{i\in\Theta}\eta_i$, $\varpi_3 = \max_{i\in\Theta}\hat{\eta}_i$, and

$$\Upsilon_5 = \frac{\varepsilon\zeta^2}{\alpha_3}E\int_{\delta_0}^{t} e^{\varepsilon s}\Big(\int_{-\delta^*}^{0}\int_{s+v}^{s}\Phi(w,x_w,r(w))dwdv\Big)ds.$$

It is straightforward to show that

$$\Upsilon_5 \le \frac{\varepsilon\zeta^2}{\alpha_3}\mathbb{E}\int_{\delta_0}^{t} e^{\varepsilon s}\Big(\delta^*\int_{s-\delta^*}^{s}\Phi(w,x_w,r(w))dw\Big)ds = \varepsilon\delta^*\Upsilon_4.$$

We may choose $\varepsilon > 0$ to be so small such that

$$\varepsilon\delta^* \le \frac{1}{4},$$
$$\kappa(\beta_1+\beta_2)e^{\varepsilon h} + \varepsilon\varpi_2 + \varepsilon\varpi_3 \le \kappa - \frac{4(\delta^*)^2\zeta^4}{\alpha_3},$$
$$\frac{\varepsilon\varpi_3}{\beta_5} + e^{\varepsilon h}(\beta_3+\beta_4)) \le 1.$$

Plugging these into (42), we have

$$\mathbb{E}|x(t)|^2 \le \frac{C_5}{\varpi_1}e^{-\varepsilon t}, \quad \forall t \ge \delta_0.$$

Finally, for any $2 \le \hat{p} < p$, applying the Hölder inequality gives

$$\begin{aligned}\mathbb{E}|x(t)|^{\hat{p}} &= \mathbb{E}\Big[\Big(|x(t)|^2\Big)^{(p-\hat{p})/(p-2)}\Big(|x(t)|^p\Big)^{(\hat{p}-2)/(p-2)}\Big] \\ &\le \Big(C_5/\varpi_1\Big)^{(p-\hat{p})/(p-2)}C_4^{(\hat{p}-2)/(p-2)}e^{-\varepsilon t(p-\hat{p})/(p-2)},\end{aligned} \tag{43}$$

which completes the proof. □

By the similar method in the work of Fei et al,[26] from the rules of Theorem 2, we deduce that the SFDE (10) is also exponentially stable in almost surely sense.

Theorem 3. Under the same Assumptions of Theorem 2, for any initial value (11), the solution of the SFDE (10) obeys

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) < 0 \ \ a.s. \tag{44}$$

5 | EXAMPLE

To illustrate applications of our theory clearly, we give a scalar stochastic integro-differential equation (SIDE)

$$dx(t) = f(t, x_t, r(t))dt + g(t, x_t, r(t))dB(t), \tag{45}$$

in which coefficients are defined by

$$f(t, x_t, 1) = x(t)\Big(1 - 5x^2(t) + \int_{-h}^{0} x^2(t+\theta)d\mu_1(\theta)\Big),\ g(t, x_t, 1) = \int_{-h}^{0} x^2(t+\theta)d\mu_2(\theta),$$

$$f(t, x_t, 2) = x(t)\Big(0.5 - 4x^2(t) + \int_{-h}^{0} x^2(t+\theta)d\mu_1(\theta)\Big),\ g(t, x_t, 2) = \int_{-h}^{0} x^{5/3}(t+\theta)d\mu_2(\theta), \tag{46}$$

$h = 1$, $d\mu_1(\theta) = \frac{e^\theta}{1-e^{-1}}d\theta$ and $d\mu_2(\theta) = d\theta$ on $\theta \in [-1, 0]$ are probability measures, and $r(t) \in \boldsymbol{\Theta} = \{1, 2\}$ is a Markov chain with its generator

$$\Pi = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{47}$$

This equation is widely discussed in population models (see e.g., related works[37,38] and the reference therein). After some calculations, it is obvious that equation (45) satisfies Assumptions 1 and 2, which means that the SIDE (45) has a unique global solution. However, letting $x(t) = 3 + 3\sin(t)$ on $t \in [-1, 0]$ and $r(0) = 2$, from the numerical simulation of the computer, we can see that hybrid stochastic integro-differential equation (45) is not stable. This result can be clearly illustrated in Figure 5.1.

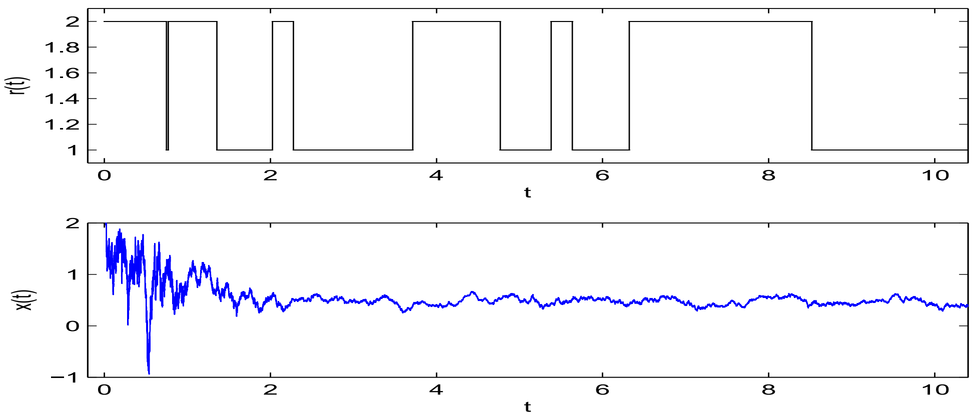

Figure 5.1: By the truncated Euler–Maruyama method with step size 10^{-4}, the computer simulation of the sample paths of the Markov chain and the equation (45) with $h = 1$.

Next, we will give the control function and verify our previous assumptions one by one. Firstly, the control function $u : R_+ \times R \times \boldsymbol{\Theta} \to R$ define as follows

$$u(t, x, 1) = -3x, \quad u(t, x, 2) = -2x, \tag{48}$$

which impies the condition (8) hold with $\zeta = 3$. By Theorem 1, the controlled SIDE

$$dx(t) = [f(t, x_t, r(t)) + u(t, x(\varrho_t), r(t))]dt + g(t, x_t, r(t))dB(t) \tag{49}$$

has a unique global solution on $[0, \infty)$, such that

$$\sup_{0 \le t < \infty} \mathbb{E}|x(t)|^p < C_4, \quad \forall p \ge 6.$$

For $(t, \psi, i) \in R_+ \times C([-1,0]; R^n) \times \mathbf{\Theta}$, we have

$$\psi(0)[f(t,\psi,i) + u(t,\psi(0),i)] + \frac{1}{2}|g(t,\psi,i)|^2$$
$$\leq \begin{cases} -4.5\psi^4(0) + 0.5\int_{-1}^0 \frac{\psi^4(\theta)e^\theta}{1-e^{-1}}d\theta + 0.5\int_{-1}^0 \psi^4(\theta)d\theta - 2\psi^2(0), & \text{if } i = 1, \\ -3.5\psi^4(0) + 0.5\int_{-1}^0 \frac{\psi^4(\theta)e^\theta}{1-e^{-1}}d\theta + 0.3333\int_{-1}^0 \psi^4(\theta)d\theta - 1.5\psi^2(0) + 0.1617\int_{-1}^0 \psi^2(\theta)d\theta, & \text{if } i = 2, \end{cases}$$

and

$$\psi(0)[f(t,\psi,i) + u(t,\psi(0),i)] + \frac{m_1}{2}|g(t,\psi,i)|^2$$
$$\leq \begin{cases} -4.5\psi^4(0) + 0.5\int_{-1}^0 \frac{\psi^4(\theta)e^\theta}{1-e^{-1}}d\theta + 1.5\int_{-1}^0 \psi^4(\theta)d\theta - 2\psi^2(0), & \text{if } i = 1, \\ -3.5\psi^4(0) + 0.5\int_{-1}^0 \frac{\psi^4(\theta)e^\theta}{1-e^{-1}}d\theta + \int_{-1}^0 \psi^4(\theta)d\theta - 1.5\psi^2(0) + 0.5\int_{-1}^0 \psi^2(\theta)d\theta, & \text{if } i = 2. \end{cases}$$

Subsequently

$$b_{10} = \hat{b}_{10} = -2, \quad b_{20} = \hat{b}_{20} = -1.5,$$

while

$$\Lambda_1 = \begin{pmatrix} 5 & -1 \\ -1 & 4 \end{pmatrix} \text{ and } \Lambda_2 = \begin{pmatrix} 9 & -1 \\ -1 & 7 \end{pmatrix},$$

which means that the Assumption 4 holds. By (24), we observe that

$$\eta_1 = 0.2632,\ \eta_2 = 0.3158,\ \hat{\eta}_1 = 0.1290,\ \hat{\eta}_2 = 0.1613.$$

The function U defined by (26) has the form

$$U(x,i) = \begin{cases} 0.2632x^2 + 0.1290x^4, & \text{if } i = 1, \\ 0.3158x^2 + 0.16131x^4, & \text{if } i = 2. \end{cases}$$

Using (28), we have

$$LU(t,\psi,i) \leq \begin{cases} -1.9785\psi^6(0) + 0.1720\int_{-1}^0 \frac{\psi^6(\theta)e^\theta}{1-e^{-1}}d\theta + 0.5161\int_{-1}^0 \psi^6(\theta)d\theta - 3.3684\psi^4(0) \\ \quad +0.2632\int_{-1}^0 \frac{\psi^4(\theta)e^\theta}{1-e^{-1}}d\theta + 0.2632\int_{-1}^0 \psi^4(\theta)d\theta - \psi^2(0), & \text{if } i = 1, \\ -1.9355\psi^6(0) + 0.2151\int_{-1}^0 \frac{\psi^6(\theta)e^\theta}{1-e^{-1}}d\theta + 0.4302\int_{-1}^0 \psi^6(\theta)d\theta - 3.0492\psi^4(0) \\ +0.3158\int_{-1}^0 \frac{\psi^4(\theta)e^\theta}{1-e^{-1}}d\theta + 0.3718\int_{-1}^0 \psi^4(\theta)d\theta - \psi^2(0) + 0.1053\int_{-1}^0 \psi^2(\theta)d\theta, & \text{if } i = 2. \end{cases}$$

Setting $\alpha_1 = 0.05$, $\alpha_2 = 0.6$ and $\alpha_3 = 1.5$, we have

$$LU(t,\psi,i) + \alpha_1|f(t,\psi,i)|^2 + \alpha_2|g(t,\psi,i)|^2 + \alpha_3\left(2\eta_i|\psi(0)| + (m_1+1)\bar{\eta}_i|\psi(0)|^{m_1}\right)^2$$
$$\leq -0.3892\psi^2(0) + 0.3053\int_{-1}^{0} \psi^2(\theta)d\theta - W(\psi(0)) + 0.1782\int_{-1}^{0} \frac{W(\psi(\theta))e^\theta}{1-e^{-1}}d\theta + 0.7995\int_{-1}^{0} W(\psi(\theta))d\theta, \tag{50}$$

where $W(x) = 2.00183x^4 + 0.64558x^6$. That is, conditions (29) and (30) are also hold, and condition (31) becomes $\delta^* < 0.0567$. Using Theorems 2 and 3, when $\delta^* = \delta + \bar{\delta} < 0.0567$, we can obtain that the controlled SFDE (49) is not only exponentially stable in $L^{\hat{p}}$ ($\hat{p} \geq 2$), but also almost surely exponentially stable.

Now, let's design δ and δ_k in two cases, and verify our results through simulation.

Case I. When the controller has good performance, the time lag generated by the controller is small, we can choose a larger discrete observation time interval. To perform a numerical simulation, we set $h = 1, \delta = 0.04$, δ_k is a uniform distribution on $[0, 0.01]$, and the same initial value as before. The sample paths of the Markov chain and the solution of the equation (49) are shown in Figure 5.2.

Case II. If the time lag caused by the controller is large, we need to make more frequent discrete observations. When condition $\underline{\delta} + \delta > \bar{\delta}$ is satisfied, $\bar{\delta} > \delta$ still makes the controlled system (49) exponentially stable. To perform a numerical simulation, we set $h = 1, \delta = 0.02$, δ_k is a uniform distribution on $[0.02, 0.03]$, and the same initial value as before. The sample paths of the Markov chain and the solution of the equation (49) are shown in Figure 5.3.

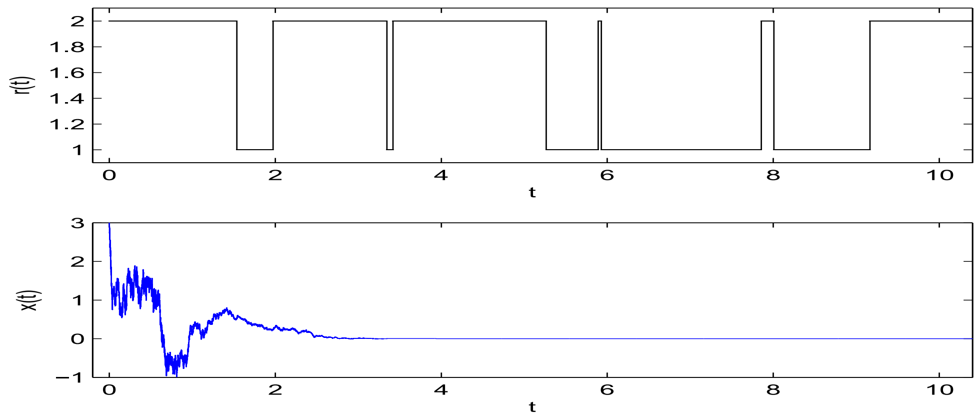

Figure 5.2: By the truncated Euler–Maruyama method with step size 10^{-4}, the computer simulation of the sample paths of the Markov chain and the equation (49) with $h = 1, \delta = 0.04$, δ_k is a uniform distribution on $[0, 0.01]$.

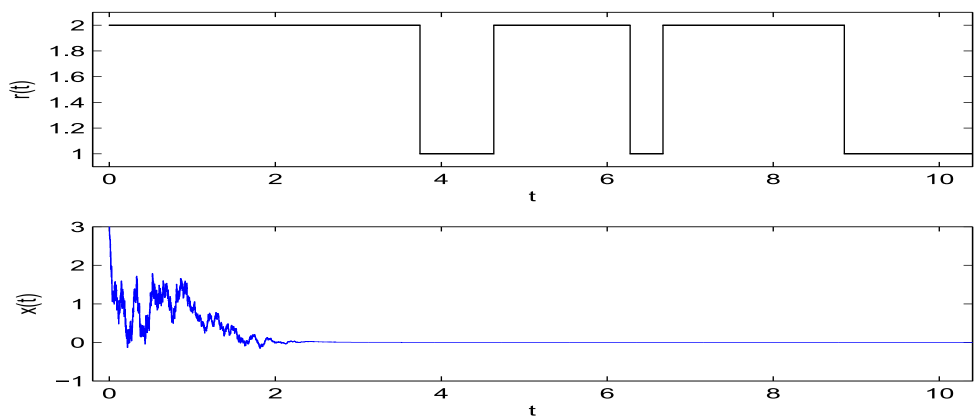

Figure 5.3: By the truncated Euler–Maruyama method with step size 10^{-4}, the computer simulation of the sample paths of the Markov chain and the equation (49) with $h = 1, \delta = 0.02$, δ_k is a uniform distribution on $[0.02, 0.03]$.

6 | CONCLUSION

In this article, we have discussed how to obtain the delay control based on discrete observations to make a given class of hybrid SFDEs stable, and we have also given an upper bound of observation interval and feedback delay. Different from the previous literature, this hybrid equation (1) is infinite dimensional system, and this coefficients are highly linear. The new controlled hybrid stochastic system (7) contains not only continuous states and discrete modes, but also new discrete states, so this equation is more complex to deal with. We have established the existence of the global solution and the pth boundedness of moment of the controlled system, and then we have obtained the $\hat{p}$th moment exponential stability and almost surely stability of the systems by using Lyapunov functional method. Finally, we have used a scalar hybrid integro-differential equation as an example to verify our results.

Combining the results of this paper with the work of Li et al.,[39] we can further discuss the feedback control problem of stochastic functional differential equations driven by G-Brownian motion.

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