A DIVERGENCE-FREE STABILISED FINITE ELEMENT METHOD FOR THE EVOLUTIONARY NAVIER-STOKES EQUATIONS*

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Abstract. This work is devoted to the finite element discretisation of the incompressible Navier-Stokes equations. The starting point is a low-order stabilised finite element method using piecewise linear continuous discrete velocities and piecewise constant pressures. This pair of spaces needs to be stabilised, and, as such, the continuity equation is modified by adding a stabilising bilinear form based on the jumps of the pressure. This modified continuity equation can be rewritten in a standard way involving a modified different velocity field, which is as a consequence divergence-free. This modified velocity field is then fed back to the momentum equation making the convective term skew-symmetric. Thus, the discrete problem can be proven stable without the need to rewrite the convective field in its skew-symmetric way. Error estimates with constant independent of the viscosity are proven. Numerous numerical experiments confirm the theoretical results.

Key words. Evolutionary Navier-Stokes equations; stabilised finite element methods; divergence-free finite element method

AMS subject classifications. 35Q30, 65M12, 65M15, 65M60.

1. Introduction. The finite element approximation of the incompressible Navier-Stokes equations has been a very active area of research. Early attempts were gathered, among others, in the classic textbook [46], and the early series of papers by Heywood and Rannacher finishing with [32]. While the just mentioned references deal with inf-sup stable finite element methods, from a very early stage there was also a rapid development of stabilised finite element methods for this equation, with SUPG (see [11]) being one of the most popular ones.

The discretisation of the time-dependent Navier-Stokes equations is usually carried out by separating the discretisations in time and space. An often used approach (that we follow in this work) is to discretise the time derivative by time-advancing finite differences, and discretise the space variables by finite element methods. While for the time-dependent part of this problem it is usually considered enough to use an $\mathcal{A}$ -stable method (as the Euler scheme used in this paper), for the space discretisation the stability and convergence of the resulting finite element method relies on two facts that are true for the continuous problem. The first is the inf-sup condition, linking the divergence of the velocity space with the pressure space. During the last few decades there has been an immense amount of work on the validity of this condition at the discrete level (see, e.g., the relatively recent textbooks [9, 36] for extensive reviews). The second fact is the following: thanks to the incompressibility of the fluid, the convective term is skew-symmetric. Interestingly, in the discrete setting even if the pair of spaces is proven to be inf-sup stable, there is no guarantee that the discrete velocity is divergence-free, unless particular choices for finite element spaces are used. As a consequence the convective term is no longer skew-symmetric, with a possible

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impact on the stability of the discrete problem. One early solution to this consists in rewriting the convective term in a skew-symmetric way, as done as, for example, as early as [32]. While the two writings of the convective term are equivalent for the continuous problem, they lead to different discretisations, which might lead to technical difficulties at the discrete level, see, e.g., the discussions in [47] and [14]. A second approach to treat the convective term is to use pairs of finite element spaces that give an exactly divergence-free discrete velocity. For this latter choice, the convective term does not need to be rewritten before the discretisation is proposed in order to prove stability.

The topic of divergence-free spaces has been treated extensively. An early idea is to consider as velocity space the curl of a C^1 -conforming finite element space. One issue associated to this process is that some pre-processing needs to be applied to eliminate linearly dependent functions. Another alternative is to present pairs of spaces for which the divergence of the velocity space is a subspace of the pressure space (or even equal to the pressure space). For example, the early Scott-Vogelius element [44] considers $H^1(\Omega)$ -conforming velocities of degree k , while discontinuous pressures of degree $k - 1$ are used. The stability of this pair requires either special meshes, or a high-enough degree k (for example, $k \geq 4$ is needed in [29]). Concerning low-order conforming approximations, some recent progress has been made in producing divergence-free piecewise linear finite element methods. For example, in [27, 28] either enriched methods, or taking $k = 1$ on barycentrically refined meshes (also known as the Alfeld split [39], and Hsieh-Clough-Tocher triangulation, see the references quoted in [49, P. 461]) have been proven to be inf-sup stable, thus leading to divergence-free velocities. In addition, in two space dimensions, the pair using $k = 1$ is inf-sup stable in Powell-Sabin meshes [49] (see also the interesting discussion in [41] linking the stability of the lowest order quadrilateral element to the stability of the triangular element for $k = 1$ in criss-cross meshes). For three-dimensional meshes, as reported in [16], for the Anfled split the lowest order inf-sup stable case is $k = 4$ (cf. [48]), while for the Powell-Sabin split the the lowest inf-sup stable case is $k = 2$ [50]. However, the case considered in this paper, this is, taking $k = 1$ on general shape-regular meshes, stabilisation is a must. Another possibility is to relax the continuity requirements and present a discontinuous Galerkin method, as it has been done, for example, in [18], where an idea related to the one presented in this work is presented in the steady-state case. Alternatively, one can only relax the tangential continuity, and thus imposing $H(\text{div}, \Omega)$ -conforming methods as in, e.g., [43, 40, 30], where BDM elements are used for high-Reynolds number problems, the steady state Boussinesq problem, and Eulers equation, respectively. More recent developments in this direction include IGA [22] and VEM methods [7].

One further difficulty of the Navier-Stokes equations is that in many applications the Reynolds number can be extremely large. For this reason, the development of error estimates that do not blow up when the viscosity ν tends to zero has been an overarching topic in this area. In [31], probably the first paper that proves robust bounds, the SUPG method is analysed for space-time finite elements, where velocity and pressure spaces consist of continuous piecewise linear functions in space and discontinuous piecewise linear functions in time. A rate of convergence of order $3/2$ is proved for the $L^2(\Omega)$ error of the velocity. In [12] a continuous interior penalty method is considered based on equal order approximations of the velocity and the pressure. Error bounds of order $r + 1/2$ are obtained for the $L^2(\Omega)$ error of the velocity, r being the degree of the local polynomials used. In [42] robust error bounds of order r (instead of $r + 1/2$) for the plain Galerkin method with the Scott-Vogelius pair are

proved for the velocity error in the $L^2(\Omega)$ norm. The same rate of convergence is obtained for the Galerkin method with grad-div stabilisation in [19]. Robust bounds for the $L^2(\Omega)$ velocity error of the same order, r , and also of order $r + 1/2$ (depending on the added stabilisation terms) can be found in [20] for non inf-sup stable elements with local projection stabilisation. Finally, in [15, 24] robust error bounds of order r are proved for equal order elements with only symmetric pressure stabilisation.

Based on the discussion carried out in the previous paragraphs, in this work our aim is to present a $H^1(\Omega)$ -conforming method that avoids the rewriting of the convective field, and to prove robust error estimates for it. Our starting point is the pair $\mathbb{P}_1^2 \times \mathbb{P}_0$, i.e., piecewise linear continuous velocities and piecewise constant pressures, supplied with pressure jump stabilisation to compensate for the inf-sup deficiency, an idea already present in early works in stabilisation [33]. Jump stabilisation is sometimes considered to affect the local mass conservation of the velocity field, and that is one of the motivations for the work [38] where the jumps of the pressure are added locally in macro-patches. In addition, the conforming velocity can not be exactly divergence-free since it is well-known that the pair $\mathbb{P}_1^2 \times \mathbb{P}_0$ locks. So, in this work we exploit the following idea already hinted in [6] (and exploited in [2] for a steady-state problem): since the stabilised finite element method modifies the continuity equation by adding jump terms to it, the jumps of the pressure can be lifted using the lowest order Raviart-Thomas space to rewrite the continuity equation in a standard way, but for an enriched velocity field involving the degrees of freedom of both velocity and pressure¹. This new (hidden) velocity field is proven divergence-free, and then it is fed back to the convective term in the momentum equation, thus making it skew-symmetric. So, after this process we have, on the one hand avoided a rewriting of the convective field, and on the other, recovered (at zero cost) a solenoidal velocity field that enjoys the same convergence properties as the conforming original velocity field. Finally, replacing the standard velocity field by the re-constructed one has a stabilising effect, as our numerical experiments show that it is not necessary to add extra stabilisation controlling the dominating convection, hence helping to reduce the complexity of the final method.

The plan of the paper is as follows. Notation and several preliminary results are given in Section 2. The finite element method used in this work is presented in Section 3, where two variants of it are discussed, and its main properties are summarised. Then, in Section 4 the error analysis is performed, showing error estimates that do not depend on the viscosity of the fluid. As was already mentioned, qualitatively similar estimates can be proven using the skew-symmetric writing of the convective term, and that is sketched in Section 5. The method's performance is assessed in a series of numerical results in Section 6, and some conclusions are drawn in Section 7. A brief discussion on the fact that avoiding rewriting the convective term has a positive impact in the conservation properties is laid down in the appendix.

2. Preliminaries and Notation. We will use standard notation for Sobolev spaces. For $D \subset \mathbb{R}^d$, $d \in \{2, 3\}$, an open and bounded domain, and $1 \leq p \leq \infty$, we will denote $W^{m,p}(D)$ ($W_0^{m,p}(D)$) as the space of all (generalised) functions in $L^p(D)$

¹In a way, this process reverses the one of deriving a stabilised finite element method starting from an enrichment of the velocity space proposed in [5]. In fact, this approach builds the enriched finite element velocity starting from the stabilised finite element method. In addition, although this idea can be extended to higher order spaces by modifying the lifting of the pressure jumps only slightly, we believe that its main interest lies in the low order case since for high enough degree the Scott-Vogelius pair is inf-sup stable and provides a divergence-free velocity.

whose derivatives up to order m belong to $L^p(D)$ (and with zero trace on ∂D). The norm (seminorm) in $W^{m,p}(\Omega)$ is denoted by $\|\cdot\|_{m,p,D}$ ($|\cdot|_{m,p,D}$), and when $p = 2$ we denote $W^{m,2}(D) = H^m(D)$, with norm $\|\cdot\|_{m,D}$. Also, we adopt the convention $H^0(D) = L^2(D)$ and denote by $L_0^2(D)$ the set of functions in $L^2(D)$ with zero mean-value in D . The inner product in $L^2(D)$ is denoted by $(\cdot, \cdot)_D$ and the subscript is dropped whenever $D = \Omega$. In addition, $H(\text{div}, D)$ stands for the space of $L^2(D)^d$ functions with divergence belonging to $L^2(D)$.

The following Sobolev's embedding [1] will be used in the analysis: Let $s \in (1, \infty)$. For $1 \leq p < d/s$ let q be such that $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. Then, there exists a positive constant C , independent of s , such that

$$(1) \quad \|v\|_{0,q',\Omega} \leq C\|v\|_{s,p,\Omega}, \quad \frac{1}{q'} \geq \frac{1}{q}, \quad v \in W^{s,p}(\Omega).$$

If $p > d/s$ the above relation is valid for $q' = \infty$. We will also make use of the following particular inequalities that are valid for $d \in \{2, 3\}$ (see [1]):

$$(2) \quad \|v\|_{0,2d,\Omega} \leq C\|v\|_{s,\Omega} \quad s \geq 1, \quad \|v\|_{0,\frac{2d}{d-1},\Omega} \leq C\|v\|_{s,\Omega} \quad s \geq 1/2.$$

Similar embedding inequalities hold for vector-valued functions.

We next define the following trilinear form, linked to the convective term in the Navier-Stokes equation below

$$(3) \quad b(\mathbf{u}, \mathbf{v}, \mathbf{w}) := (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}),$$

and using the previous Sobolev embeddings the following results for b can be proven.

LEMMA 1. *There exists $C > 0$ such that, for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ regular enough such that the norms in the right-hand side are finite, the following holds*

$$(4) \quad |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} \|\mathbf{w}\|_{1,\Omega},$$

$$(5) \quad |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{2,\Omega} \|\mathbf{w}\|_{0,\Omega},$$

$$(6) \quad |b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq C \|\mathbf{u}\|_{0,\Omega} \|\mathbf{v}\|_{2,\Omega} \|\mathbf{w}\|_{1,\Omega}.$$

Proof. The continuity (4) is standard and its proof can be found in [26]. To prove (5) we use Hölder's inequality and (2) and arrive at

$$|b(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq \|\mathbf{u}\|_{0,2d,\Omega} \|\nabla \mathbf{v}\|_{0,\frac{2d}{d-1},\Omega} \|\mathbf{w}\|_{0,\Omega} \leq C \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{2,\Omega} \|\mathbf{w}\|_{0,\Omega},$$

which is (5). The proof of (6) follows using the same arguments, changing the roles of $\mathbf{u}$ and $\mathbf{w}$. $\square$

2.1. The problem of interest. Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a bounded domain with polyhedral and Lipschitz boundary $\partial\Omega$. In this work we will approximate the following evolutionary incompressible Navier-Stokes equations

$$(7) \quad \begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} & \text{in } (0, T] \times \Omega, \\ \nabla \cdot \mathbf{u} &= 0 & \text{in } (0, T] \times \Omega, \\ \mathbf{u}(0, \cdot) &= \mathbf{u}_0(\cdot) & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

Here, $\mathbf{u}$ denotes the velocity field, p the kinematic pressure, $\nu > 0$ the kinematic viscosity coefficient, $\mathbf{u}_0$ a given initial velocity (that will be supposed solenoidal), and $\mathbf{f}$ represents the external body accelerations acting on the fluid.

170 The weak form of (7) is: Find $(\mathbf{u}, p) \in L^2((0, T], H_0^1(\Omega)^d) \times L^2((0, T], L_0^2(\Omega))$ such
 171 that, $\mathbf{u}(0, \cdot) = \mathbf{u}_0(\cdot)$ in Ω , and for almost all $t \in (0, T)$

$$172 \quad (8) \quad \begin{aligned} (\partial_t \mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^d, \\ (q, \nabla \cdot \mathbf{u}) &= 0 \quad \forall q \in L_0^2(\Omega). \end{aligned}$$

173 2.2. Triangulations, finite element spaces, and interpolation operators.

174 We now introduce notation and basic results related to the finite element method
 175 presented in this manuscript. We first consider $\{\mathcal{T}_h\}_{h>0}$, a family of simplicial, con-
 176 forming, shape-regular triangulations such that $h \rightarrow 0$. Just to simplify the presenta-
 177 tion we will assume that the meshes are quasi-uniform. For an element $K \in \mathcal{T}_h$, $|K|$
 178 denotes its area (volume), h_K denotes its diameter, and $h = \max\{h_K : K \in \mathcal{T}_h\}$. We
 179 also denote the unit normal outward to K by $\mathbf{n}_K$. Over this triangulation, we define
 180 the following finite element spaces

$$181 \quad (9) \quad V_h := \{v_h \in C^0(\Omega) : v_h|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\} \quad , \quad V_{h,0} := V_h \cap H_0^1(\Omega),$$

$$182 \quad (10) \quad \mathbf{V}_h := V_h^d \quad , \quad \mathbf{V}_{h,0} := \mathbf{V}_h \cap H_0^1(\Omega)^d,$$

$$183 \quad (11) \quad Q_h := \{q_h \in L_0^2(\Omega) : q_h|_K \in \mathbb{P}_0(K), \forall K \in \mathcal{T}_h\}.$$

185 The set of facets of the triangulation $\mathcal{T}_h$ will be denoted by $\mathcal{F}$, and we decompose $\mathcal{F}$
 186 as $\mathcal{F} = \mathcal{F}_I \cup \mathcal{F}_\partial$, where $\mathcal{F}_I$ stands for the set of internal facets, and $\mathcal{F}_\partial$ is the set of
 187 boundary facets. For any collection of elements D , we denote $\mathcal{F}_D := \{\gamma \in \mathcal{F} : \gamma \cap D =$
 188 $\gamma\}$. For any facet $\gamma \in \mathcal{F}$ we denote by $|\gamma|$ its measure and $h_\gamma := \text{diam}(\gamma)$. On an
 189 internal facet $\gamma = K \cap K'$ we define the jump of a piecewise smooth function as

$$190 \quad (12) \quad \llbracket q \rrbracket_\gamma := (q|_K \mathbf{n}_K)|_\gamma + (q|_{K'} \mathbf{n}_{K'})|_\gamma.$$

191 For an element $K \in \mathcal{T}_h$ and each $\gamma \in \mathcal{F}$ we define the neighborhoods

$$192 \quad (13) \quad \omega_K := \{K' \in \mathcal{T}_h : K' \cap K \neq \emptyset\}, \quad \omega_\gamma := \{K' \in \mathcal{T}_h : K' \cap \gamma = \gamma\}.$$

193 In what follows $i_h : H^1(\Omega) \rightarrow V_h$ denotes the variant of the Clment interpolation
 194 operator (that can also be defined from $H_0^1(\Omega)$ with values in $V_{h,0}$) described in [10,
 195 4.8]. This operator satisfies the following interpolation bound (see [26, Theorem A.4],
 196 or [10, Eq. (4.8.10)]): there exists $C > 0$ such that

$$197 \quad (14) \quad \|v - i_h(v)\|_{m,p,K} \leq Ch_K^{n-m} |v|_{n,p,\omega_K} \quad \forall p \in [1, \infty], \quad 0 \leq n \leq m \leq 2.$$

198 The following stability bound appears as a corollary of (14): there exists $C > 0$ such
 199 that for all $v \in W^{m,p}(\Omega)$ (see [10, Corollary 4.8.15])

$$200 \quad (15) \quad \|i_h(v)\|_{m,p,K} \leq C \|v\|_{m,p,\omega_K} \quad \forall p \in [1, \infty] \quad \text{for } m = 0, 1.$$

201 Also, with the help of the following discrete trace inequality: for all $\gamma \in \mathcal{F} \cap \partial K$ with
 202 $K \in \mathcal{T}_h$, there exists $C > 0$ independent of h such that

$$203 \quad (16) \quad \|v\|_{0,\gamma} \leq C (h_\gamma^{-\frac{1}{2}} \|v\|_{0,K} + h_\gamma^{\frac{1}{2}} |v|_{1,K}) \quad \forall v \in H^1(\Omega),$$

204 the following bound can be proven for i_h

$$205 \quad (17) \quad \|v - i_h(v)\|_{0,\gamma} \leq Ch^{m-\frac{1}{2}} |v|_{m,\omega_K} \quad \forall v \in H^m(\Omega), \quad m \in \{1, 2\}.$$

In addition, we define the $L^2(\Omega)$ -orthogonal projection $\Pi_0 : L^2(\Omega) \rightarrow Q_h$ by

$$(18) \quad \Pi_0(q)|_K := |K|^{-1}(q, 1)_K.$$

Using the standard approximation properties of Π_0 (see, e.g. [21]), the following important estimate can be proven

$$(19) \quad \|q - \Pi_0(q)\|_{0,\Omega} + \left\{ \sum_{\gamma \in \mathcal{F}_I} h_\gamma \|\llbracket q - \Pi_0 q \rrbracket_\gamma\|_{0,\gamma}^2 \right\}^{\frac{1}{2}} \leq Ch |q|_{1,\Omega} \quad \forall q \in H^1(\Omega),$$

where the constant $C > 0$ does not depend on h .

We finally recall the following inverse inequality ([17, Theorem 3.2.6]): for every $k \geq 1$ there exists $C > 0$ such that, for each $v_h \in \mathbb{P}_k(K)$,

$$(20) \quad \|v_h\|_{m,p,K} \leq Ch_K^{n-m-d(\frac{1}{q}-\frac{1}{p})} \|v_h\|_{n,q,K} \quad 0 \leq n \leq m \leq 1, \quad 1 \leq q \leq p \leq \infty.$$

2.3. Preliminary results on stabilisation. In this work the space $\mathbf{V}_{h,0}$ defined in (10) will be used to approximate the velocity, while Q_h , defined in (11), will approximate the pressure. This pair of spaces is not inf-sup stable, but does satisfy the following weakened inf-sup condition (see [8, Lemma 2.2] for its proof): There exists $\beta_0 > 0$, independent of h , such that for all $q_h \in Q_h$ the following holds

$$(21) \quad \|q_h\|_{0,\Omega} \leq \beta_0 \left(\sup_{\mathbf{v}_h \in \mathbf{V}_{h,0} \setminus \{0\}} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{0,\Omega}} + \left\{ \sum_{\gamma \in \mathcal{F}_I} h_\gamma \|\llbracket q_h \rrbracket_\gamma\|_{0,\gamma}^2 \right\}^{\frac{1}{2}} \right).$$

In [8] this last result was essential to justify the authors' definition of stabilised finite element method for the Stokes problem using $\mathbf{V}_{h,0} \times Q_h$. The method proposed in this work uses (21) in a more straightforward way to define a stabilised finite element method (thus departing from [8]). In fact, we use the last lemma to justify the definition of stabilising term proposed as early as [33]. More precisely, we define the following stabilising bilinear form

$$(22) \quad s_{\text{pres}}(p_h, q_h) = \sum_{\gamma \in \mathcal{F}_I} \tau_\gamma (\llbracket p_h \rrbracket_\gamma, \llbracket q_h \rrbracket_\gamma)_\gamma,$$

where τ_γ is the stabilisation parameter given by $\tau_\gamma = \frac{h_\gamma}{12}$.

Remark 2. The value $\frac{1}{12}$ in the definition of τ_γ can be replaced by any positive constant. The choice $\frac{1}{12}$ is inspired from [5], where that value was obtained for $d = 2$ as a result of a multiscale enrichment of the velocity field.

The final ingredient in the definition of the method is a lifting of the pressure jumps that will be used to build a solenoidal velocity field. This lifting is defined using the lowest order Raviart-Thomas space with basis functions defined in the following way: for every facet $\gamma \in \mathcal{F}$ we choose one unit normal vector $\mathbf{n}_\gamma$, with the only condition that it points outwards Ω if $\gamma \in \mathcal{F}_\partial$. Associated with this choice of normal vector, the lowest-order Raviart-Thomas basis function φ_γ is defined by

$$(23) \quad \varphi_\gamma(\mathbf{x}) = \pm \frac{|\gamma|}{d|K|} (\mathbf{x} - \mathbf{x}_\gamma),$$

in every $K \in \omega_\gamma$, where $\mathbf{x}_\gamma \in K$ is the node of K opposed to the facet γ , and the sign depends on whether $\mathbf{n}_\gamma$ points in or out of K . Outside ω_γ , φ_γ is extended by zero. These basis functions have been very well studied since their introduction (see, e.g., [9] and the references therein). In this work we will use mainly the following properties of them: $\varphi_\gamma \in H(\text{div}, \Omega)$ (not to $H^1(\Omega)^d$), $\|\varphi_\gamma\|_{\infty, K} \leq C$ for all $K \in \omega_\gamma$, and, for all $\gamma' \in \mathcal{F}$ the following holds

$$(24) \quad \varphi_\gamma \cdot \mathbf{n}_{\gamma'} = \begin{cases} 1 & \text{if } \gamma' = \gamma, \\ 0 & \text{else.} \end{cases}$$

With the help of these basis functions we present the final ingredient that will be used later, namely, the linear mapping $\mathcal{L} : H^1(\Omega)^d \times Q_h \rightarrow H(\text{div}, \Omega)$ given by

$$(25) \quad \mathcal{L}(\mathbf{v}, q) := \mathbf{v} + \sum_{\gamma \in \mathcal{F}_I} \tau_\gamma \llbracket q \rrbracket_\gamma \cdot \mathbf{n}_\gamma \varphi_\gamma,$$

where τ_γ is the stabilisation parameter defined earlier. This operator is closely related to the stabilisation form $s_{\text{pres}}(\cdot, \cdot)$. In fact, the following result is the first example of such a link. Its proof is very close to the one from [2, Lemma 3], so we omit it.

LEMMA 3. *There exists $C > 0$, independent of h , such that for all $q_h \in Q_h$ the following holds*

$$(26) \quad \|\mathcal{L}(\mathbf{0}, q_h)\|_{0, \Omega} \leq C h s_{\text{pres}}(q_h, q_h)^{\frac{1}{2}}.$$

3. The stabilised finite element method. We start by introducing a partition of the interval $(0, T)$ into N equally spaced time instances $0 = t_0 < t_1 < \dots < t_N = T$ and introduce the time step $\Delta t = \frac{T}{N}$ in such a way that $t_j = t_{j-1} + \Delta t$, $j = 1, \dots, N$. For a function g depending on time we denote $g^j = g(t_j, \cdot)$.

Below we describe two variants of the method proposed in this work. The main feature of both is the replacement of the convective field by a divergence-free one built using the continuity equation and the mapping $\mathcal{L}$ defined in (25).

A. An implicit Euler scheme: We consider as initial condition $\mathbf{u}_h^0 = i_h(\mathbf{u}_0)$. Then, for $n = 0, \dots, N-1$, the implicit Euler method consists of finding $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_{h,0} \times Q_h$ such that

$$(27) \quad \begin{cases} \left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h \right) + \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) + b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\ - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) + (q_h, \nabla \cdot \mathbf{u}_h^{n+1}) + s_{\text{pres}}(p_h^{n+1}, q_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_{h,0} \times Q_h$, where the stabilisation bilinear form $s_{\text{pres}}(\cdot, \cdot)$ is given by (22).

As it was mentioned earlier, the main feature of this method is the replacement of $\mathbf{u}_h^{n+1}$ by $\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1})$ in (27). The justification of this idea is provided in the next result, whose proof follows ideas presented in [2, Theorem 1 and Lemma 6].

LEMMA 4. *Let $n \in \{1, \dots, N\}$ and let $(\mathbf{u}_h^n, p_h^n) \in \mathbf{V}_{h,0} \times Q_h$ be any solution of (27). Then*

$$(28) \quad \nabla \cdot \mathcal{L}(\mathbf{u}_h^n, p_h^n) = 0 \quad \text{in } \Omega.$$

In addition, if $p^n \in H^1(\Omega)$, the field $\mathcal{L}(\mathbf{u}_h^n, p_h^n)$ satisfies the following

$$(29) \quad \sum_{K \in \mathcal{T}_h} |\mathbf{u}^n - \mathcal{L}(\mathbf{u}_h^n, p_h^n)|_{m, K}^2 \leq C (|\mathbf{u}^n - \mathbf{u}_h^n|_{m, \Omega}^2 + h^{1-m} s_{\text{pres}}(p_h^n - p^n, p_h^n - p^n)),$$

for $m = 0, 1$. Finally, for each $\gamma \in \mathcal{F}$, let $\mathbf{t}_\gamma$ be a unit vector tangent to γ . Then, the conformity error for $\mathcal{L}(\mathbf{u}_h^n, p_h^n)$ can be bounded as follows

$$(30) \quad \Delta t \sum_{j=1}^n \sum_{\gamma \in \mathcal{F}} \|\mathcal{L}(\mathbf{u}_h^j, p_h^j) \cdot \mathbf{t}_\gamma\|_{0,\gamma}^2 \leq Ch \sum_{j=1}^n \Delta t s_{\text{pres}}(p_h^j, p_h^j).$$

Remark 5. Several remarks can be made concerning Method (27):

1. The last result strengthens the relationship between the mapping $\mathcal{L}$ and the stabilising term s_{pres} . In fact, an essential part of the proof of (28) consists in rewriting the second equation in (27) as the following standard continuity equation

$$(31) \quad (q_h, \nabla \cdot \mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1})) = 0 \quad \forall q_h \in Q_h,$$

and then use that $\nabla \cdot \mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in Q_h$. Thus, the definition of $\mathcal{L}$ arises naturally in the process of rewriting the second equation in (27) as (31).

2. The fact that the modified convective field $\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1})$ is solenoidal allows us to use the original writing of the convective form, without rewriting it in its skew-symmetric way. In the steady-state case this implies that the existence of solutions can be proven without assuming that the mesh is refined enough, and for the time-dependent case stability can be proven for any mesh and time step.

3. The last result states as well that, at zero cost, a divergence-free velocity field can be built starting from the solution of (27). In addition, thanks to (29) this field enjoys the same convergence properties as $(\mathbf{u}_h, p_h)$. It is important to stress that this new velocity field is not a postprocessing, but it is rather hardwired in the definition of the method.

B. A semi-implicit Euler scheme: We consider as initial condition $\mathbf{u}_h^0$ as the (unique) solution of

$$(32) \quad (\nabla \mathbf{u}_h^0, \nabla \mathbf{v}_h) - (p_h^0, \nabla \cdot \mathbf{v}_h) = (\nabla \mathbf{u}_0, \nabla \mathbf{v}_h) \quad \forall \mathbf{v} \in \mathbf{V}_{h,0},$$

$$(33) \quad (q_h, \nabla \cdot \mathbf{u}_h^0) + s_{\text{pres}}(p_h^0, q_h) = 0 \quad \forall q_h \in Q_h.$$

We start noticing that, since $(\mathbf{u}_h^0, p_h^0)$ is a finite element approximation of $(\mathbf{u}_0, 0)$, supposing $\mathbf{u}_0 \in H^2(\Omega)^d$ and using standard techniques (see, e.g., [5, 8]) the following error bound can be proven

$$(34) \quad \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{1,\Omega}^2 + s_{\text{pres}}(p_h^0, p_h^0) \leq Ch^2 \|\mathbf{u}_0\|_{2,\Omega}^2.$$

For $n = 0, \dots, N-1$, the semi-implicit Euler method consists of finding the pair $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_{h,0} \times Q_h$ such that

$$(35) \quad \begin{cases} \left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h \right) + \nu (\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) + b(\mathcal{L}(\mathbf{u}_h^n, p_h^n), \mathbf{u}_h^{n+1}, \mathbf{v}_h) \\ - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) + (q_h, \nabla \cdot \mathbf{u}_h^{n+1}) + s_{\text{pres}}(p_h^{n+1}, q_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_{h,0} \times Q_h$, where the stabilisation bilinear form is given by (22).

Remark 6. Arguing as in Lemma 4 and using the second equation in (35) we get

$$(36) \quad \nabla \cdot \mathcal{L}(\mathbf{u}_h^j, p_h^j) = 0, \quad j \geq 1.$$

In addition, thanks to the choice of initial condition (32)-(33) then $\nabla \cdot \mathcal{L}(\mathbf{u}_h^0, p_h^0) = 0$, and then (36) is also valid for $j = 0$. So, the linear system to be solved in each time step of (35) is well-posed. Moreover, (29) and (30) in Lemma 4 are also valid for this semi-implicit variant.

4. Convergence analysis. This section will be devoted to prove the convergence of the solutions of (27) and (35) to the exact solution $(\mathbf{u}, p)$ of (8). The main goal will be to prove error estimates with constants that don't depend on any negative powers of ν . We suppose that the solution $(\mathbf{u}, p)$ of (8) is regular enough such as all the norms used below are finite. In particular, from now on we will assume that $p \in L^\infty(0, T; H^1(\Omega))$ in such a way that

$$s_{\text{pres}}(p^n, q_h) = 0 \quad \forall q_h \in Q_h \text{ for } n = 1, \dots, N.$$

In addition, an argument that will be used throughout this section is the following discrete Gronwall inequality, proved originally in [32].

LEMMA 7. Let $k, B, a_j, b_j, c_j, \gamma_j, j = 0, \dots, n$, be nonnegative numbers such that

$$a_n + k \sum_{j=0}^n b_j \leq k \sum_{j=0}^n \gamma_j a_j + k \sum_{j=0}^n c_j + B, \quad \text{for } n \geq 0.$$

Suppose that $k\gamma_j < 1$, for all j , and set $\sigma_j = (1 - k\gamma_j)^{-1}$. Then

$$(37) \quad a_n + k \sum_{j=0}^n b_j \leq \exp \left(k \sum_{j=0}^n \sigma_j \gamma_j \right) \left\{ k \sum_{j=0}^n c_j + B \right\}, \quad \text{for } n \geq 0.$$

4.1. An error estimate for the velocity. We will only detail the convergence result for the implicit Euler method (27), as the analysis for the semi-implicit variant (35) is analogous. We start by decomposing the error as follows

$$(38) \quad \hat{\mathbf{e}}_h^n = \hat{\mathbf{u}}_h^n - \mathbf{u}^n, \quad \mathbf{e}_h^n = \hat{\mathbf{u}}_h^n - \mathbf{u}_h^n, \quad \hat{\lambda}_h^n = \hat{p}_h^n - p^n, \quad \lambda_h^n = \hat{p}_h^n - p_h^n,$$

where $\hat{\mathbf{u}}_h^n = i_h(\mathbf{u}^n)$ and $\hat{p}_h^n = \Pi_0(p^n)$ for every $n \in \{1, \dots, N\}$. Thanks to the choice of the initial condition we get that $\mathbf{e}_h^0 = \mathbf{0}$. Using (27) and the weak problem (8) we obtain the following error equation

$$(39) \quad \begin{aligned} & \left(\frac{\mathbf{e}_h^{n+1} - \mathbf{e}_h^n}{\Delta t}, \mathbf{v}_h \right) + \nu (\nabla \mathbf{e}_h^{n+1}, \nabla \mathbf{v}_h) + b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) - (\lambda_h^{n+1}, \nabla \cdot \mathbf{v}_h) \\ & - b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{v}_h) + (q_h, \nabla \cdot \mathbf{e}_h^{n+1}) + s_{\text{pres}}(\lambda_h^{n+1}, q_h) \\ & = - \left(\partial_t \mathbf{u}^{n+1} - \frac{\hat{\mathbf{u}}_h^{n+1} - \hat{\mathbf{u}}_h^n}{\Delta t}, \mathbf{v}_h \right) + \nu (\nabla \hat{\mathbf{e}}_h^{n+1}, \nabla \mathbf{v}_h) - (\hat{\lambda}_h^{n+1}, \nabla \cdot \mathbf{v}_h) \\ & + (q_h, \nabla \cdot \hat{\mathbf{e}}_h^{n+1}) + s_{\text{pres}}(\hat{p}_h^{n+1}, q_h) + b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{v}_h) - b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{v}_h). \end{aligned}$$

Taking $(\mathbf{v}_h, q_h) = (\mathbf{e}_h^{n+1}, \lambda_h^{n+1})$ in (39), using the identity $(a-b)a = \frac{1}{2}(a^2 - b^2 + (a-b)^2)$ and the triangle inequality leads to the following bound

$$(40) \quad \begin{aligned} & \frac{\|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2}{2\Delta t} - \frac{\|\mathbf{e}_h^n\|_{0,\Omega}^2}{2\Delta t} + \nu \|\nabla \mathbf{e}_h^{n+1}\|_{0,\Omega}^2 + s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1}) \\ & \leq \left| \left(\partial_t \mathbf{u}^{n+1} - \frac{\hat{\mathbf{u}}_h^{n+1} - \hat{\mathbf{u}}_h^n}{\Delta t}, \mathbf{e}_h^{n+1} \right) \right| + |s_{\text{pres}}(\hat{p}_h^{n+1}, \lambda_h^{n+1})| + |(\hat{\lambda}_h^{n+1}, \nabla \cdot \mathbf{e}_h^{n+1})| \\ & + |(\lambda_h^{n+1}, \nabla \cdot \hat{\mathbf{u}}_h^{n+1})| + |\nu (\nabla \hat{\mathbf{e}}_h^{n+1}, \nabla \mathbf{e}_h^{n+1})| \\ & + |b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{e}_h^{n+1}) - b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{e}_h^{n+1})| = I + \dots + VI. \end{aligned}$$

We next bound term-by-term in the above inequality. We start by noticing that the assumption $p^{n+1} \in H^1(\Omega)$ and (19) give

$$(41) \quad s_{\text{pres}}(\hat{p}_h^{n+1}, \hat{p}_h^{n+1}) = s_{\text{pres}}(\hat{\lambda}_h^{n+1}, \hat{\lambda}_h^{n+1}) \leq Ch^2 \|p^{n+1}\|_{0,\Omega}^2.$$

So, using the Cauchy-Schwarz and Young inequalities, integration by parts, $\nabla \cdot \mathbf{u} = 0$, and the approximation properties (14), (17) and (19) we obtain the following bounds

$$\begin{aligned} II &\leq s_{\text{pres}}(\hat{p}_h^{n+1}, \hat{p}_h^{n+1}) + \frac{1}{4} s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1}) \leq Ch^2 \|p^{n+1}\|_{1,\Omega}^2 + \frac{s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1})}{4}, \\ IV &= \sum_{\gamma \in \mathcal{F}_I} ((\hat{\mathbf{u}}_h^{n+1} - \mathbf{u}^{n+1}) \cdot \mathbf{n}_\gamma, \llbracket \lambda_h^{n+1} \rrbracket_\gamma)_\gamma \leq Ch^2 \|\mathbf{u}^{n+1}\|_{2,\Omega}^2 + \frac{s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1})}{4}, \\ V &\leq Ch^2 \nu |\mathbf{u}^{n+1}|_{2,\Omega}^2 + \frac{\nu}{2} \|\nabla \mathbf{e}_h^{n+1}\|_{0,\Omega}^2. \end{aligned}$$

In addition, due to the definition of Π_0 and the fact that $\nabla \cdot \mathbf{e}_h^{n+1} \in Q_h$, then $III = 0$.

The term I is linked to the time discretisation error. Using a Taylor series with integral remainder, Cauchy-Schwarz's inequality, and (14) we get

$$\begin{aligned} I &= \left| \left(\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} - \partial_t \mathbf{u}^{n+1} + \frac{\hat{\mathbf{e}}_h^{n+1} - \hat{\mathbf{e}}_h^n}{\Delta t}, \mathbf{e}_h^{n+1} \right) \right| \\ &= \left| \left(\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \partial_{tt} \mathbf{u}(s)(t_n - s) ds + \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \partial_t \hat{\mathbf{e}}_h(s) ds, \mathbf{e}_h^{n+1} \right) \right| \\ &\leq \frac{2}{\Delta t} \left\{ \int_{\Omega} \left(\int_{t_n}^{t_{n+1}} \partial_{tt} \mathbf{u}(s)(t_n - s) ds \right)^2 dx + \int_{\Omega} \left(\int_{t_n}^{t_{n+1}} \partial_t \hat{\mathbf{e}}_h(s) ds \right)^2 dx \right\}^{\frac{1}{2}} \|\mathbf{e}_h^{n+1}\|_{0,\Omega} \\ &\leq \frac{2}{\Delta t} \left\{ \frac{\Delta t^3}{3} \|\partial_{tt} \mathbf{u}\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^d)}^2 + \Delta t h^2 \|\partial_t \mathbf{u}\|_{L^2(t_n, t_{n+1}; H^1(\Omega)^d)}^2 \right\}^{\frac{1}{2}} \|\mathbf{e}_h^{n+1}\|_{0,\Omega} \\ &\leq C \left\{ \Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^d)}^2 + \frac{h^2}{\Delta t} \|\partial_t \mathbf{u}\|_{L^2(t_n, t_{n+1}; H^1(\Omega)^d)}^2 \right\}^{\frac{1}{2}} \|\mathbf{e}_h^{n+1}\|_{0,\Omega}. \end{aligned}$$

It only remains to bound the nonlinear term. First, using that the velocity field $\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1})$ is solenoidal (and then $b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \mathbf{e}_h^{n+1}, \mathbf{e}_h^{n+1}) = 0$), we get

$$\begin{aligned} VI &= b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \mathbf{u}_h^{n+1}, \mathbf{e}_h^{n+1}) - b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) + b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) \\ &\quad - b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{e}_h^{n+1}) \\ &= b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) - b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) + b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) \\ &\quad - b(\mathbf{u}^{n+1}, \mathbf{u}^{n+1}, \mathbf{e}_h^{n+1}) \\ &= b(\mathcal{L}(\mathbf{0}, p_h^{n+1}), \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) - b(\mathbf{e}_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}, \mathbf{e}_h^{n+1}) + b(\hat{\mathbf{u}}_h^{n+1}, \hat{\mathbf{e}}_h^{n+1}, \mathbf{e}_h^{n+1}) \\ &\quad + b(\hat{\mathbf{e}}_h^{n+1}, \mathbf{u}^{n+1}, \mathbf{e}_h^{n+1}) \\ (43) \quad &= \mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \mathfrak{D}. \end{aligned}$$

Using Hölder's inequality, Lemma 3, Young's inequality, the inverse inequality (20), (41), and (15) we get

$$\begin{aligned} \mathfrak{A} &\leq \|\mathcal{L}(\mathbf{0}, p_h^{n+1})\|_{0,\Omega} \|\nabla \hat{\mathbf{u}}_h^{n+1}\|_{\infty,\Omega} \|\mathbf{e}_h^{n+1}\|_{0,\Omega} \\ &\leq Ch s_{\text{pres}}(p_h^{n+1}, p_h^{n+1})^{\frac{1}{2}} \|\nabla \hat{\mathbf{u}}_h^{n+1}\|_{\infty,\Omega} \|\mathbf{e}_h^{n+1}\|_{0,\Omega} \\ &\leq \frac{1}{4} s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1}) + Ch^2 \|p^{n+1}\|_{1,\Omega}^2 + C \|\mathbf{u}^{n+1}\|_{\infty,\Omega}^2 \|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2. \end{aligned}$$

The remaining terms are bounded using similar arguments. More precisely, the use of Hölder's inequality, (15), (5) in Lemma 1, and (14) yield to

$$\begin{aligned} \mathfrak{B} &\leq \|\nabla \hat{\mathbf{u}}_h^{n+1}\|_{\infty,\Omega} \|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2 \leq C \|\nabla \mathbf{u}^{n+1}\|_{\infty,\Omega} \|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2, \\ \mathfrak{C} &\leq \|\hat{\mathbf{u}}_h^{n+1}\|_{\infty,\Omega} \|\nabla \hat{\mathbf{e}}_h^{n+1}\|_{0,\Omega} \|\mathbf{e}_h^{n+1}\|_{0,\Omega} \leq Ch^2 |\mathbf{u}^{n+1}|_{2,\Omega}^2 + \|\mathbf{u}^{n+1}\|_{\infty,\Omega}^2 \|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2, \\ \mathfrak{D} &\leq \|\hat{\mathbf{e}}_h^{n+1}\|_{1,\Omega} \|\mathbf{u}^{n+1}\|_{2,\Omega} \|\mathbf{e}_h^{n+1}\|_{0,\Omega} \leq Ch^2 \|\mathbf{u}^{n+1}\|_{2,\Omega}^4 + \|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2. \end{aligned}$$

375 Gathering the above bounds for $\mathfrak{A}, \dots, \mathfrak{D}$ the following bound for VI appears

$$376 \quad VI \leq \frac{1}{4} s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1}) + C \{1 + \|\nabla \mathbf{u}^{n+1}\|_{\infty, \Omega} + \|\mathbf{u}^{n+1}\|_{\infty, \Omega}^2\} \|e_h^{n+1}\|_{0, \Omega}^2$$

$$377 \quad (44) \quad + Ch^2 (\|\mathbf{u}^{n+1}\|_{2, \Omega}^4 + \|\mathbf{u}^{n+1}\|_{2, \Omega}^2 + \|p^{n+1}\|_{1, \Omega}^2).$$

379 Hence, replacing the bounds for $I, \dots, VI$ into (40) and multiplying both sides by
 380 $2\Delta t$ we arrive at the following bound for every $j = 0, \dots, N-1$,

$$(45) \quad \begin{aligned} & \|e_h^{j+1}\|_{0, \Omega}^2 - \|e_h^j\|_{0, \Omega}^2 + \Delta t \left(\nu \|\nabla e_h^{j+1}\|_{0, \Omega}^2 + \frac{1}{2} s_{\text{pres}}(\lambda_h^{j+1}, \lambda_h^{j+1}) \right) \\ & \leq \tilde{C} \Delta t \left(1 + \|\nabla \mathbf{u}^{j+1}\|_{\infty, \Omega} + \|\mathbf{u}^{j+1}\|_{\infty, \Omega}^2 \right) \|e_h^{j+1}\|_{0, \Omega}^2 + C \Delta t^2 \|\partial_{tt} \mathbf{u}\|_{L^2(t_j, t_{j+1}; L^2(\Omega)^d)}^2 \\ & + C \Delta t h^2 \left(\|\mathbf{u}^{j+1}\|_{2, \Omega}^4 + (1 + \nu) \|\mathbf{u}^{j+1}\|_{2, \Omega}^2 + \|p^{j+1}\|_{1, \Omega}^2 \right) + Ch^2 \|\partial_t \mathbf{u}\|_{L^2(t_j, t_{j+1}; H^1(\Omega)^d)}^2 \end{aligned}$$

382 where $\tilde{C}$ is independent of $h, \Delta t$, and ν . Finally, for every $n \in \{0, \dots, N\}$, adding the
 383 above bound from $j = 0$ to $j = n-1$ and recalling that $e_h^0 = \mathbf{0}$ we get

$$(46) \quad \begin{aligned} & \|e_h^n\|_{0, \Omega}^2 + \Delta t \sum_{j=0}^{n-1} \left(\nu \|\nabla e_h^{j+1}\|_{0, \Omega}^2 + s_{\text{pres}}(\lambda_h^{j+1}, \lambda_h^{j+1}) \right) \\ & \leq \sum_{j=0}^{n-1} \tilde{C} \Delta t \left(1 + \|\nabla \mathbf{u}^{j+1}\|_{\infty, \Omega} + \|\mathbf{u}^{j+1}\|_{\infty, \Omega}^2 \right) \|e_h^{j+1}\|_{0, \Omega}^2 \\ & + Ch^2 \Delta t \sum_{j=0}^{n-1} \left(\|\mathbf{u}^{j+1}\|_{2, \Omega}^4 + (1 + \nu) \|\mathbf{u}^{j+1}\|_{2, \Omega}^2 + \|p^{j+1}\|_{1, \Omega}^2 \right) \\ & + C \Delta t^2 \|\partial_{tt} \mathbf{u}\|_{L^2(0, t_n; L^2(\Omega)^d)}^2 + Ch^2 \|\partial_t \mathbf{u}\|_{L^2(0, t_n; H^1(\Omega)^d)}^2. \end{aligned}$$

385 Here we recall that $\tilde{C}$ is independent of $h, \Delta t$, and ν .

386 This last bound is the main step in proving the next convergence result.

387 **THEOREM 8.** *Let $\mathbf{u}$ be the solution of (8) and $\{(\mathbf{u}_h^n, p_h^n)\}_{n=1}^N$ be the solution of*
 388 *(27). Let us suppose that the time step Δt satisfies*

$$389 \quad (47) \quad \tilde{C} \Delta t \left(1 + \|\nabla \mathbf{u}^j\|_{\infty, \Omega} + \|\mathbf{u}^j\|_{\infty, \Omega}^2 \right) \leq \frac{1}{2},$$

390 *for all $j \in \{1, \dots, N\}$, where $\tilde{C}$ is the constant from (46). Then, the following error*
 391 *estimate is valid for every $n \in \{1, \dots, N\}$*

$$(48) \quad \begin{aligned} & \|\mathbf{u}^n - \mathbf{u}_h^n\|_{0, \Omega}^2 + \Delta t \sum_{j=1}^n \left(\nu \|\nabla (\mathbf{u}^j - \mathbf{u}_h^j)\|_{0, \Omega}^2 + s_{\text{pres}}(p^j - p_h^j, p^j - p_h^j) \right) \\ & \leq Ce^{M_u} \left(K_{u,p} h^2 + \Delta t^2 \|\partial_{tt} \mathbf{u}\|_{L^2(0, t_n; L^2(\Omega)^d)}^2 + h^2 \|\partial_t \mathbf{u}\|_{L^2(0, t_n; H^1(\Omega)^d)}^2 \right), \end{aligned}$$

395 *where*

$$396 \quad \begin{aligned} M_u &= 2\tilde{C} \Delta t \sum_{j=1}^n \left(1 + \|\nabla \mathbf{u}^j\|_{\infty, \Omega} + \|\mathbf{u}^j\|_{\infty, \Omega}^2 \right), \\ K_{u,p} &= C \Delta t \sum_{j=1}^n \left(\|\mathbf{u}^j\|_{2, \Omega}^4 + (1 + \nu) \|\mathbf{u}^j\|_{2, \Omega}^2 + \|p^j\|_{1, \Omega}^2 \right). \end{aligned}$$

397 *Proof.* Standard interpolation estimates provide a bound for $(\hat{e}_h^n, \hat{\lambda}_h^n)$ in the semi-
 398 norm used in the left-hand side of (48). To bound (e_h^n, λ_h^n) in the same seminorm we
 399 use Gronwall's Lemma 7 with $a_j = \|e_h^j\|_{0, \Omega}^2, k = \Delta t, b_j = \nu \|\nabla e_h^j\|_{0, \Omega}^2 + s_{\text{pres}}(\lambda_h^j, \lambda_h^j),$

400 $\gamma_j = \tilde{C} \left(1 + \|\nabla \mathbf{u}^j\|_{\infty, \Omega} + \|\mathbf{u}^j\|_{\infty, \Omega}^2 \right)$, $c_j = Ch^2 \Delta t \left(\|\mathbf{u}^j\|_{2, \Omega}^4 + (1 + \nu) \|\mathbf{u}^j\|_{2, \Omega}^2 + \|p^j\|_{1, \Omega}^2 \right)$,
 401 and $B = \Delta t^2 \|\partial_{tt} \mathbf{u}\|_{L^2(0, t_n; L^2(\Omega)^d)}^2 + h^2 \|\partial_t \mathbf{u}\|_{L^2(0, t_n; H^1(\Omega)^d)}$. The proof follows using
 402 the triangle inequality. $\square$

403 *Remark 9.* For the semi-implicit Euler method (35) the proof of the error follows
 404 very similar arguments given in the proof of Theorem 8. The main differences are the
 405 error equation, and the fact that, if we follow similar arguments as in the proof of that
 406 result, we arrive at a bound similar to (45), but that contains the term $s_{\text{pres}}(p_h^0, p_h^0)$
 407 in its right-hand side. Thus, the need for the use of the initial condition given by
 408 (32)-(33) (satisfying (34)) appears, and leads to an error bound very similar to (46).

409 **4.2. A bound for a primitive in time of the pressure.** We now prove an
 410 error estimate on the discrete pressure, more precisely, on a discrete time integral of
 411 the pressure. We define

$$412 \quad (49) \quad \Lambda_h^n = \Delta t \sum_{j=1}^n \lambda_h^j, \quad \hat{\Lambda}_h^n = \Delta t \sum_{j=1}^n \hat{\lambda}_h^j.$$

413 The goal of this section is to prove an error bound for Λ_h^n along the lines of what was
 414 done in the previous sections for the velocity. We only detail the proof in the case of
 415 the fully implicit Euler method (27), as the semi-implicit version is totally analogous.

416 **THEOREM 10.** *Let us assume, in addition to the hypotheses of Theorem 8, that*
 417 *$\Delta t \leq Ch$. Then, there exists a constant $\mathfrak{N}(\mathbf{u}, p)$ depending on $\mathbf{u}, p, \mathbf{u}_0$ (and their*
 418 *derivatives), and T , but independent of any negative power of ν , such that*

$$419 \quad (50) \quad \left\| \Delta t \sum_{j=1}^n (p^j - p_h^j) \right\|_{0, \Omega} \leq C [\mathfrak{N}(\mathbf{u}, p) + T \mathfrak{C}(\mathbf{u}, p) h^{1-\frac{d}{2}}] (h + \Delta t),$$

where $\mathfrak{C}(\mathbf{u}, p)$ is the constant from Theorem 8, depending on $\mathbf{u}$ and p (and their derivatives), but not in negative powers of ν , and

$$\mathfrak{N}(\mathbf{u}, p) = \mathfrak{C}(\mathbf{u}, p)^{\frac{1}{2}} \left(1 + T^{\frac{1}{2}} \left\{ \nu^{\frac{1}{2}} + 1 + \left[\Delta t \sum_{j=1}^n \|\mathbf{u}^j\|_{2, \Omega}^2 \right]^{\frac{1}{2}} \right\} \right).$$

420 *Proof.* We will base our error estimate on the inf-sup deficiency (21), from which
 421 the following bound follows

$$422 \quad (51) \quad \|\Lambda_h^n\|_{0, \Omega} \leq \beta_0 \left(\sup_{\mathbf{v}_h \in \mathbf{V}_{h,0} \setminus \{\mathbf{0}\}} \frac{(\Lambda_h^n, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{0, \Omega}} + s_{\text{pres}}(\Lambda_h^n, \Lambda_h^n)^{\frac{1}{2}} \right),$$

423 where β_0 does not depend on h or Δt . We start by bounding the first term in (51).
 424 Considering the error equation (39) for $q_h = 0$, j instead of $n+1$, and adding from
 425 $j = 1$ to n we get

$$426 \quad (52) \quad (\Lambda_h^n, \nabla \cdot \mathbf{v}_h) = (e_h^n, \mathbf{v}_h) + \Delta t \nu \sum_{j=1}^n (\nabla(\mathbf{u}^j - \mathbf{u}_h^j), \nabla \mathbf{v}_h) + (\hat{\Lambda}_h^n, \nabla \cdot \mathbf{v}_h) \\
 427 \quad + \Delta t \sum_{j=1}^n \left(b(\mathbf{u}^j, \mathbf{u}^j, \mathbf{v}_h) - b(\mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{u}_h^j, \mathbf{v}_h) \right) + \Delta t \sum_{j=1}^n \left(\partial_t \mathbf{u}^j - \frac{\hat{\mathbf{u}}_h^j - \hat{\mathbf{u}}_h^{j-1}}{\Delta t}, \mathbf{v}_h \right), \\
 428$$

429 while we also used that $\mathbf{e}_h^0 = \mathbf{0}$. We also notice that $(\hat{\Lambda}_h^n, \nabla \cdot \mathbf{v}_h) = 0$. So, applying
 430 the Cauchy-Schwarz and Poincaré inequalities we arrive at the following bound

$$\begin{aligned}
 431 \quad & \sup_{\mathbf{v}_h \in \mathbf{V}_{h,0}} \frac{(\Lambda_h^n, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{0,\Omega}} \leq C \left\{ \|\mathbf{e}_h^n\|_{0,\Omega} + \Delta t \nu \sum_{j=1}^n \|\nabla(\mathbf{u}^j - \mathbf{u}_h^j)\|_{0,\Omega} \right. \\
 432 \quad & + \Delta t \sum_{j=1}^n \sup_{\mathbf{v}_h \in \mathbf{V}_{h,0}} \frac{b(\mathbf{u}^j, \mathbf{u}^j, \mathbf{v}_h) - b(\mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{u}_h^j, \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{0,\Omega}} \\
 433 \quad (53) \quad & \left. + \Delta t \sum_{j=1}^n \left\| \partial_t \mathbf{u}^j - \frac{\hat{\mathbf{u}}_h^j - \hat{\mathbf{u}}_h^{j-1}}{\Delta t} \right\|_{0,\Omega} \right\} = C \{I + II + III + IV\}.
 \end{aligned}$$

435 The term I has already been bounded in Theorem 8. To bound II and IV we observe
 436 that for any sequence $\{\alpha_j\}_{j=1}^\infty$ of real numbers and any $n \in \{1, \dots, N\}$ Cauchy-
 437 Schwarz's inequality gives

$$438 \quad (54) \quad \Delta t \sum_{j=1}^n \alpha_j \leq T^{1/2} \left(\Delta t \sum_{j=1}^n \alpha_j^2 \right)^{\frac{1}{2}}.$$

439 So, $II \leq \nu T^{\frac{1}{2}} \left\{ \Delta t \sum_{j=1}^n \|\nabla(\mathbf{u}^j - \mathbf{u}_h^j)\|_{0,\Omega}^2 \right\}^{\frac{1}{2}}$, which, again, is bounded as in Theo-
 440 rem 8. To bound IV we use (54) and argue as in (42) to get to

$$441 \quad (55) \quad IV \leq CT^{\frac{1}{2}} \left\{ \Delta t \|\partial_{tt} \mathbf{u}\|_{L^2(0,t_n;L^2(\Omega)^d)} + h \|\partial_t \mathbf{u}\|_{L^2(0,t_n;H^1(\Omega)^d)} \right\}.$$

442 It only remains to bound III . First, integrating by parts and using that $\mathcal{L}(\mathbf{u}_h^j, p_h^j)$
 443 and $\mathbf{u}^j$ are divergence-free leads to

$$\begin{aligned}
 444 \quad & b(\mathbf{u}^j, \mathbf{u}^j, \mathbf{v}_h) - b(\mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{u}_h^j, \mathbf{v}_h) \\
 445 \quad & = b(\mathbf{u}^j - \mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{u}^j, \mathbf{v}_h) + b(\mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{u}^j - \mathbf{u}_h^j, \mathbf{v}_h) \\
 446 \quad (56) \quad & = -b(\mathbf{u}^j - \mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{v}_h, \mathbf{u}^j) - b(\mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{v}_h, \mathbf{u}^j - \mathbf{u}_h^j).
 \end{aligned}$$

448 Using Young's inequality, the embedding (1) and (29) in Lemma 4 we get

$$\begin{aligned}
 449 \quad & b(\mathbf{u}^j - \mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{v}_h, \mathbf{u}^j) \leq \|\mathbf{u}^j - \mathcal{L}(\mathbf{u}_h^j, p_h^j)\|_{0,\Omega} \|\nabla \mathbf{v}_h\|_{0,\Omega} \|\mathbf{u}^j\|_{\infty,\Omega} \\
 450 \quad (57) \quad & \leq C \left[\|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} + h s_{\text{pres}}(p_h^j, p_h^j)^{\frac{1}{2}} \right] \|\nabla \mathbf{v}_h\|_{0,\Omega} \|\mathbf{u}^j\|_{2,\Omega}.
 \end{aligned}$$

452 Moreover, using Young's inequality, the inverse inequality (20), the stability of the
 453 Scott-Zhang operator (15), and Lemma 3 once again, we get

$$\begin{aligned}
 454 \quad & b(\mathcal{L}(\mathbf{u}_h^j, p_h^j), \mathbf{v}_h, \mathbf{u}^j - \mathbf{u}_h^j) \\
 455 \quad & \leq \|\mathcal{L}(\mathbf{u}_h^j, p_h^j)\|_{\infty,\Omega} \|\nabla \mathbf{v}_h\|_{0,\Omega} \|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} \\
 456 \quad & \leq C \left(\|\mathbf{u}_h^j\|_{\infty,\Omega} + \|\mathcal{L}(\mathbf{0}, p_h^j)\|_{\infty,\Omega} \right) \|\nabla \mathbf{v}_h\|_{0,\Omega} \|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} \\
 457 \quad (58) \quad & \leq C \left(h^{-\frac{d}{2}} \|\mathbf{e}_h^j\|_{0,\Omega} + \|\mathbf{u}^j\|_{2,\Omega} + h^{1-\frac{d}{2}} s_{\text{pres}}(p_h^j, p_h^j)^{\frac{1}{2}} \right) \|\nabla \mathbf{v}_h\|_{0,\Omega} \|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega}.
 \end{aligned}$$

Thus, replacing (56) and (57) in (58), and using (54) we arrive at the following bound

$$\begin{aligned}
III &\leq C\Delta t \sum_{j=1}^n \left\{ [\|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} + h s_{\text{pres}}(p_h^j, p_h^j)^{\frac{1}{2}}] \|\mathbf{u}^j\|_{2,\Omega} \right. \\
&\quad \left. + [h^{-\frac{d}{2}} \|e_h^j\|_{0,\Omega} + \|\mathbf{u}^j\|_{2,\Omega} + h^{1-\frac{d}{2}} s_{\text{pres}}(p_h^j, p_h^j)^{\frac{1}{2}}] \|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} \right\} \\
&\leq CT^{\frac{1}{2}} \max_{1 \leq j \leq n} \|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} \left\{ \sum_{j=1}^n \Delta t \|\mathbf{u}^j\|_{2,\Omega}^2 + \Delta t h^{-d} \|e_h^j\|_{0,\Omega}^2 \right. \\
&\quad \left. + \Delta t h^{2-d} s_{\text{pres}}(p_h^j, p_h^j) \right\}^{\frac{1}{2}} + C\Delta t \sum_{j=1}^n h s_{\text{pres}}(p_h^j, p_h^j)^{\frac{1}{2}} \|\mathbf{u}^j\|_{2,\Omega} \\
&\leq \mathfrak{M} \max_{1 \leq j \leq n} \|\mathbf{u}^j - \mathbf{u}_h^j\|_{0,\Omega} + Ch \left\{ \Delta t \sum_{j=1}^n s_{\text{pres}}(p_h^j, p_h^j) \right\}^{\frac{1}{2}} \left\{ \Delta t \sum_{j=1}^n \|\mathbf{u}^j\|_{2,\Omega}^2 \right\}^{\frac{1}{2}}.
\end{aligned}$$

Since an application of Theorem 8 gives that the second term in the sum above is of higher-order, it only remains to analyse the behavior of $\mathfrak{M}$ with respect of h and Δt . The only two terms that can be problematic are the ones showing negative powers of h in the definition of $\mathfrak{M}$. First, applying Theorem 8 and $\Delta t \leq Ch$ we get

$$(59) \quad h^{-d} \Delta t \sum_{j=1}^n \|e_h^j\|_{0,\Omega}^2 \leq T\mathfrak{C}(\mathbf{u}, p) [h^{2-d} + h^{-d} \Delta t^2] \leq CT\mathfrak{C}(\mathbf{u}, p) h^{2-d},$$

where $\mathfrak{C}(\mathbf{u}, p)$ is the constant from Theorem 8, depending on $\mathbf{u}$ and p (and their derivatives), but not in negative powers of ν . The remaining term is bounded applying Theorem 8 and $\Delta t \leq Ch$ once again and the bound reads as follows

$$(60) \quad h^{2-d} \Delta t \sum_{j=1}^n s_{\text{pres}}(p_h^j, p_h^j) \leq \mathfrak{C}(\mathbf{u}, p) \{h^2 + \Delta t^2\} h^{2-d} \leq C\mathfrak{C}(\mathbf{u}, p) h^{4-d}.$$

The last step in the proof is the bound of the second term on the right-hand side of (51). An application of Cauchy-Schwarz's inequality gives

$$s_{\text{pres}}(\Lambda_h^n, \Lambda_h^n) = \sum_{j=1}^n \Delta t s_{\text{pres}}(\lambda_h^j, \Lambda_h^n) \leq \sum_{j=1}^n \Delta t s_{\text{pres}}(\lambda_h^j, \lambda_h^j)^{\frac{1}{2}} s_{\text{pres}}(\Lambda_h^n, \Lambda_h^n)^{\frac{1}{2}},$$

and consequently, using (54) we arrive at

$$(61) \quad s_{\text{pres}}(\Lambda_h^n, \Lambda_h^n)^{\frac{1}{2}} \leq \sum_{j=1}^n \Delta t s_{\text{pres}}(\lambda_h^j, \lambda_h^j)^{\frac{1}{2}} \leq T^{1/2} \left(\Delta t \sum_{j=1}^n s_{\text{pres}}(\lambda_h^j, \lambda_h^j) \right)^{\frac{1}{2}},$$

and then the bound of this term concludes by applying Theorem 8 once again. The proof is finished collecting all the above estimates. $\square$

Remark 11. Two main remarks can be made with respect to the last result:

1. The estimate shows a suboptimal order for $d = 3$. This feature can not be avoided, even if a tighter condition between h and Δt were to be assumed, since it arises from the use of inverse estimates for $d = 3$.
2. Related to what was just stated, the condition on Δt can be weakened to $\Delta t \leq Ch^{\frac{1}{2}}$ if the Crank-Nicolson scheme is used as a time integrator. In fact, for example, instead of the bound (60), if a Crank-Nicolson scheme were to be used, we would arrive at

$$h^{2-d} \Delta t \sum_{j=1}^n s_{\text{pres}}(p_h^j, p_h^j) \leq \mathfrak{C}(\mathbf{u}, p) \{h^2 + \Delta t^4\} h^{2-d} \leq C\mathfrak{C}(\mathbf{u}, p) h^{4-d},$$

under the milder condition $\Delta t \leq Ch^{\frac{1}{2}}$. The rest of the proof follows by arguments totally analogous to the ones just presented. It is worth noticing that even in this case the error estimate presents a suboptimality for $d = 3$.

5. An alternative using a skew-symmetric convective term. As it was mentioned in the introduction, replacing $\mathbf{u}_h$ by $\mathcal{L}(\mathbf{u}_h, p_h)$ in the convective term has as main objective to avoid rewriting the convective term in its skew-symmetric form. In this section we briefly state that qualitatively similar results can be obtained if, instead of (27) we use the following method: use $\mathbf{u}_h^0 = i_h(\mathbf{u}_0)$, and, for all $n = 0, \dots, N-1$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_{h,0} \times Q_h$ such that

$$(62) \quad \left\{ \begin{aligned} & \left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h \right) + \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) + \tilde{b}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) \\ & + (q_h, \nabla \cdot \mathbf{u}_h^{n+1}) + s_{\text{pres}}(p_h^{n+1}, q_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \end{aligned} \right.$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_{h,0} \times Q_h$, where the stabilisation bilinear form $s_{\text{pres}}(\cdot, \cdot)$ is given by (22), and $\tilde{b}(\cdot, \cdot, \cdot)$ is defined as follows

$$(63) \quad \tilde{b}(\mathbf{u}, \mathbf{v}, \mathbf{w}) := (\mathbf{u} \cdot \nabla \mathbf{v}, \mathbf{w}) + \frac{1}{2}(\nabla \cdot \mathbf{u}, \mathbf{v} \cdot \mathbf{w}).$$

The objective of this section is to prove a result analogous to Theorem 8. We will not give the details, as the proof is totally analogous to the one already given, but we only detail the main difference, namely the bound for the convective term VI in (40). For this, we start remarking that taking $\nabla \cdot \mathbf{u}_h^{n+1} \in Q_h$ as test function in the second equation of (62), applying the Cauchy-Schwarz inequality, the local trace result (16), and the interpolation bound (19) we arrive at the following bound for the solution of (62) (bound also valid for the solutions of (27) and (35))

$$(64) \quad \|\nabla \cdot \mathbf{u}_h^{n+1}\|_{0,\Omega}^2 \leq \frac{1}{8}s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1}) + Ch^2\|p^{n+1}\|_{1,\Omega}^2,$$

where the constant $C > 0$ does not depend on h , or negative powers of ν . Using this last estimate, along with the skew-symmetry of $\tilde{b}(\cdot, \cdot, \cdot)$, Hölder and Young's inequalities, interpolation inequalities, and the same type of arguments leading to the bound (44) for VI we arrive at

$$\begin{aligned} VI &\leq C\{1 + \|\nabla \mathbf{u}^{n+1}\|_{\infty,\Omega} + \|\mathbf{u}^{n+1}\|_{\infty,\Omega}^2\}\|\mathbf{e}_h^{n+1}\|_{0,\Omega}^2 + \frac{1}{8}s_{\text{pres}}(\lambda_h^{n+1}, \lambda_h^{n+1}) \\ &\quad + Ch^2\left(\|\mathbf{u}^{n+1}\|_{2,\Omega}^2 + \|\mathbf{u}^{n+1}\|_{2,\Omega}^4 + \|p^{n+1}\|_{1,\Omega}^2\right), \end{aligned}$$

and, following arguments totally analogous to the ones given in the proof of Theorem 8 a result along the lines of (48) can be proven for (62). In addition, a result like Theorem 10 can also be proven for the alternative presented in this section.

Remark 12. We finish this section by pointing out that, even if qualitatively similar error estimates can be proven for both (27) and (62), the use of the method proposed in this work presents advantages over (62). In fact, as it has been discussed, e.g., in [14], the skew-symmetric writing of the convective term does not preserve linear nor angular momentum, while, as it will be shown in the appendix, Method (27) does preserve linear momentum and presents a negligible loss of angular momentum (loss that is not negligible in the case of (62)). In addition, from a purely analytic point of view, the skew-symmetric rewriting of the convective term leads to suboptimal error

estimates for the steady-state case [47], and even to the lack of convergence of the method in some cases for time-dependent problems in non-Newtonian flow problems (see, e.g., [45]). Finally, as it has been shown in the very recent review [23], using divergence-free velocities does increase the reliability of simulations of high-Reynolds numbers flows, so we expect the present method to provide a better performance in problems with high-Reynolds numbers flows.

6. Numerical experiments. In this section we conduct a series of numerical examples that illustrate the performance of the devised stabilised scheme. The numerical experiments presented below have been carried out with the help of an in-house C++ code. All matrices have been assembled exactly and the linear systems were solved using the multifrontal massively parallel sparse direct solver (MUMPS) [3, 4]. The right hand sides and the approximation errors were computed by using quadrature formulas exact for polynomials of degree 19 for two-dimensional domains and degree 14 for three-dimensional domains.

6.1. A stationary linear fluid flow problem: inf-sup stable v/s low-order stabilised schemes. The first numerical example has for objective to show the advantages of using the lowest order pair with respect to using an inf-sup stable element with continuous approximations of the pressure. For this, we consider the following linear steady-state Oseen problem: for a given $\mathbf{b} \in W^{1,\infty}(\Omega)^d$ and $\mathbf{f} \in L^2(\Omega)^d$, find $(\mathbf{u}, p) \in H_0^1(\Omega)^d \times L_0^2(\Omega)$, such that

$$(65) \quad \begin{cases} \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\mathbf{b} \cdot \nabla \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) &= (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0^1(\Omega)^d, \\ (q, \nabla \cdot \mathbf{u}) &= 0 \quad \forall q \in L_0^2(\Omega). \end{cases}$$

As an example of inf-sup stable discretisation with continuous pressure approximations we consider the classical Taylor–Hood scheme of order 2 $\mathbb{P}_2^d \times [V_h \cap L_0^2(\Omega)]$, where $\mathbb{P}_2^d = \{\mathbf{v}_h \in C^0(\Omega)^d : \mathbf{v}_h|_K \in \mathbb{P}_2(K), \forall K \in \mathcal{T}_h\} \cap H_0^1(\Omega)^d$. We compare the Taylor–Hood solution $\mathbf{u}_h^{TH}$ to the one arising from the following low-order method: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_{h,0} \times Q_h$, such that

$$(66) \quad \begin{cases} \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\mathbf{b} \cdot \nabla \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) &= (\mathbf{f}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_{h,0}, \\ (q_h, \nabla \cdot \mathbf{u}_h) + s_{\text{pres}}(p_h, q_h) &= 0 \quad \forall q_h \in Q_h. \end{cases}$$

Since the second equation in (66) is the same one as for the Navier–Stokes equation, the field $\mathcal{L}(\mathbf{u}_h, p_h)$ can be built and it is divergence-free. So, we consider this as the solution of (66).

Example 1: we let $\Omega = (0, 1)^2$, $\nu \in \{10^{-4}, 10^{-6}, 10^{-8}\}$, $\mathbf{b} = [1, 1]^T$, and consider as an exact solution to problem (65),

$$\mathbf{u} = [u_1, u_2]^T, \quad u_1 = \sin(\pi x)^2 \sin(2\pi y), \quad u_2 = -\sin(\pi y)^2 \sin(2\pi x), \quad p = x^2 y - 0.6.$$

In Figure 1 we present the finite element approximations for both the Taylor–Hood approximation and the divergence-free stabilised approximation. These have been computed on a structured mesh containing 545 vertices and 1024 elements, and considering $\nu \in \{10^{-4}, 10^{-6}, 10^{-8}\}$. The results depicted in Figure 1 clearly show that while the solution $(\mathbf{u}_h^{TH}, p_h^{TH})$ shows an unstable behavior with respect to the value of ν , the solution of (66) stays robust as ν decreases.

6.2. Evolutionary fluid flow problems. We let $\Delta t = \frac{T}{M}$, with $M \geq 1$ being a positive integer. Refinements in the time discretisation will always take place by

FIG. 1. *Example 1: finite element approximations $\mathbf{u}_h^{TH}$ and the stabilised approximation $\mathcal{L}(\mathbf{u}_h, p_h)$ for the Oseen problem (65), considering different values of the viscosity parameter $\nu \in \{10^{-4}, 10^{-6}, 10^{-8}\}$.*

doubling the value of M . For the spatial mesh, we consider sequence of uniformly refined quasy-uniform structured meshes. The refinement has been carried out using the longest edge bisection algorithm.

As we state in the manuscript, we consider three stabilised schemes to approximate the solution of (7), namely the implicit Euler method (27) with solution $(\mathbf{u}_h^n, p_h^n)$, considering as initial step $\mathbf{u}_h^0 = i_h(\mathbf{u}_0)$, the semi-implicit Euler method (35) with solution $(\tilde{\mathbf{u}}_h^n, \tilde{p}_h^n)$, considering as the initial step $(\tilde{\mathbf{u}}_h^0, \tilde{p}_h^0)$ the unique solution of (32)-(33), and the skew-symmetric implicit Euler method (62) with solution $(\hat{\mathbf{u}}_h^n, \hat{p}_h^n)$, as a solution to (62), considering as the initial step $\hat{\mathbf{u}}_h^0 = i_h(\mathbf{u}_0)$. The main objective of the rest of the section will be to analyse the behavior of the first two alternatives. The third one will be treated only in Example 4 below for comparison purposes.

6.2.1. Analytical solutions in two and three dimensions. In this section we focus in presenting convergence rates for the aforementioned implicit and semi-implicit schemes with different manufactured solutions.

Example 2: we consider the time-space domain $(0, 1] \times \Omega$, with $\Omega = (0, 1)^2$, the right-hand side and boundary conditions are chosen in such a way that the solution of (7) is given by

$$\mathbf{u} = [u_1, u_2]^T, \quad u_1 = \Phi(t)16 \sin^2(\pi x) y(2y - 1)(y - 1), \\ u_2 = -\Phi(t)8\pi \sin(2\pi x)(y(1 - y))^2, \quad p = \Phi(t) \sin(\pi x) \cos(\pi y), \quad \Phi(t) = 0.4 \cos(4t) + 0.6.$$

Example 3: we consider the time-space domain $(0, 1] \times \Omega$, with $\Omega = (0, 1)^3$, the right-hand side and boundary conditions are chosen in such a way that the solution

of (7), denoting by $\mathbf{e}_3 = (0, 0, 1)^T$, is given by

$$\begin{aligned}\mathbf{u} &= \text{curl} \left((e^{txyz}(1-x)(1-y)(1-z))^2 \mathbf{e}_3 \right), \\ p &= \cos(t)(z \sin(x) \cos(y) + (\cos(1) - 1) \sin(1)/2).\end{aligned}$$

Considering the following relation for the time-steeping

$$(67) \quad t_n = n\Delta t, \quad \Delta t = \frac{T}{2^m}, \quad n \in \{0, \dots, 2^m\}, \quad m \geq 1,$$

we denote the error by

$$\|\mathbf{E}\|(t_n, m, \nu) := \left\{ \|\mathbf{u}^n - \mathbf{u}_h^n\|_{0,\Omega}^2 + \Delta t \sum_{j=1}^n \left(\nu \|\nabla(\mathbf{u}^j - \mathbf{u}_h^j)\|_{0,\Omega}^2 + s_{\text{pres}}(p_h^j, p_h^j) \right) \right\}^{\frac{1}{2}},$$

where we also use the fact that, in our example, $\llbracket p^j \rrbracket_\gamma = 0$ for every internal face γ . In Figure 2 we present the experimental convergence rates for the Implicit-Euler (27) and Semi-Implicit Euler (35) methods for a final time $T = 1$ and different values of the viscosity parameter $\nu \in \{10^{-2}, 10^{-4}, 10^{-6}\}$. All the error curves show an almost ν -independent behavior of the error for the seminorm used in the error estimates. Even if it has not been explicitly proven to be so, a virtually ν -independent behavior of the (piecewise) H^1 -norm ($\|\cdot\|_{1,h}$) of the velocity error and $L^2(\Omega)$ -error of the pressure is also observed for $t = 0.5$ using $\Delta t = 2^{-10}$ in Figure 3.

FIG. 2. Examples 2 and 3: Experimental convergence rates when using the Implicit Euler scheme (27) (A-B-C-D), and the Semi-Implicit Euler scheme (35) (E-F-G-H), considering the time-steeping relation from (67) with final time $T = 1$: panels (A), (B), (E) and (F) for Example 2, and panels (C), (D), (G) and (H) for Example 3.

FIG. 3. Example 2: Experimental convergence rates when using (27) (A-B), and (35) (C-D), at time $t = 0.5$ considering $\nu \in \{1, 10^{-2}, 10^{-4}, 10^{-6}\}$.

6.2.2. Non analytical solutions in two and three dimensions. In this section we focus in presetting a series of problems with no analytical solutions, but that are standard benchmarks for the incompressible Navier-Stokes equations. Also, we go beyond our theoretical framework by considering non-homogeneous Dirichlet and Neumann boundary conditions.

Example 4 (The Driven Cavity Problem in Two Dimensions): we consider the time-space domain $(0, T] \times \Omega$, with $\Omega = (0, 1)^2$, boundary conditions as in [36, Example D.4], except for the fact that we impose $\mathbf{u}(0, 1) = \mathbf{u}(1, 1) = (1, 0)^T$. The final time is $T = 100$ where stationary behavior is observed and we consider $\nu = 1/3, 200$.

For this example we have computed the solution of the skew-symmetric implicit Euler method (62), and the implicit method (27), using $\Delta t = 2^{-7}$ and a structured uniform mesh containing 2,048 elements and 1,089 vertices. These results have been compared to a reference solution computed using the implicit method (27) considering $\Delta t = 2^{-13}$ and a spatial mesh containing 32,768 elements and 16,641 vertices (comparable also to the one shown in, e.g., [36, Fig. D.2]). The top row of Figure 4 shows the solution to (62), the middle row the solution of (27), while the reference solution is depicted at the bottom (in all the figures the streamline and isovalues of the modulus of the velocity are depicted). The results show two main features:

- comparing the finite element solutions for the skew-symmetric implicit method and the present implicit Euler method, we observe that both of the top two rows provide qualitatively similar results;
- in addition, comparing the left and right panels of the three rows we can observe that the divergence-free velocity field $\mathcal{L}(\mathbf{u}_h, p_h)$ recovers the smaller vortices near to the vertices $(0, 0)$ and $(1, 0)$ (also reported in, e.g., [13, Fig. 11]) while the conforming solution $\mathbf{u}_h$ does not. So, the non-conforming divergence-free velocity recovers physical features not present in the conforming piecewise linear one.

Additionally to the streamline pattern, in Table 1 we report the location of the center of the primary and secondary vortices, located at the center and bottom left and right corners of the domain, considering both the conforming velocity field $\mathbf{u}_h$ and its divergence-free counterpart $\mathcal{L}(\mathbf{u}_h, p_h)$. The results obtained in this coarse mesh are compared to the reference values from [25]. We can see that the computed values are in good agreement with the reference ones.

Example 5 (A Time-Dependent Flow Around an obstacle): we consider

ν	Method/Solution	Center	Bottom left	Bottom right
$\frac{1}{3,200}$	(27) ($\mathbf{u}_h$)	(0.5312, 0.5312)	(0.0892, 0.1047)	(0.8638, 0.0932)
	(27) ($\mathcal{L}(\mathbf{u}_h, p_h)$)	(0.5312, 0.5312)	(0.0887, 0.1071)	(0.8644, 0.0935)
	From [25]	(0.5165, 0.5469)	(0.0859, 0.1094)	(0.8125, 0.0859)

TABLE 1

Example 4: location of the center of the primary and secondary vortices, located at the center, bottom left, and bottom right of the domain, for $\nu = 1/3, 200$.

a fluid flow passing around a circle in two dimensions, as described in [36, Fig. D3], and around a cylinder in three dimensions described in [36, Fig. D6] (see also [35]). We fix $\nu = 10^{-3}$, $\Delta t = \frac{1}{6,400}$, $T = 8.0$, and impose as boundary conditions

$$\begin{aligned} \text{On } \Gamma_{out} : & \begin{cases} \mathbf{u} = \sin\left(\frac{\pi t}{8}\right) \left(\frac{6y(0.41-y)}{0.41^2}, 0\right)^T & \text{if } \Omega \subset \mathbb{R}^2, \\ \nu \partial_{\mathbf{n}} \mathbf{u} - p \mathbf{n} = \mathbf{0} & \text{if } \Omega \subset \mathbb{R}^3. \end{cases} \\ \text{On } \Gamma_{in} : & \begin{cases} \mathbf{u} = \sin\left(\frac{\pi t}{8}\right) \left(\frac{6y(0.41-y)}{0.41^2}, 0\right)^T & \text{if } \Omega \subset \mathbb{R}^2, \\ \mathbf{u} = \frac{36}{0.41^4} \sin\left(\pi \frac{t}{8}\right) (yz(0.41-y)(0.41-z), 0, 0)^T & \text{if } \Omega \subset \mathbb{R}^3, \end{cases} \end{aligned}$$

and $\mathbf{u} = \mathbf{0}$ everywhere else. We also present in Figure 5, the meshes in which the problem will be approximated.

As it is customary for this example we compute the drag (c_{drag}), and lift (c_{lift}) coefficients around the obstacles, which are computed as follows (see [36, Example D.9] and [35]):

$$\begin{aligned} c_{coeff}^{n+1}(\phi_{coeff}) := & -\mathcal{N} \left(\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \phi_{coeff} \right) + \nu (\nabla \mathbf{u}_h^{n+1}, \nabla \phi_{coeff}) \right. \\ & \left. + b(\mathcal{L}(\mathbf{u}_h^{n+1}, p_h^{n+1}), \mathbf{u}_h^{n+1}, \phi_{coeff}) - (p_h^{n+1}, \nabla \cdot \phi_{coeff}) \right), \end{aligned}$$

where $\mathcal{N} \in \mathbb{R}$, $\phi_{coeff} \in H^1(\Omega)^d$ ($coeff = drag$ or $coeff = lift$), and are such that

$$\mathcal{N} = \begin{cases} 20 & \text{if } \Omega \subset \mathbb{R}^2, \\ \frac{20}{0.41} & \text{if } \Omega \subset \mathbb{R}^3, \end{cases} \quad \phi_{drag} = \begin{cases} \mathbf{e}_1 & \text{on } \Gamma_{cyl}, \\ \mathbf{0} & \text{elsewhere,} \end{cases} \quad \phi_{lift} = \begin{cases} \mathbf{e}_2 & \text{on } \Gamma_{cyl}, \\ \mathbf{0} & \text{elsewhere,} \end{cases}$$

where $\mathbf{e}_i$ correspond to the i -th canonical basis vector in $\mathbb{R}^d$. Another quantity of interest is the difference of the pressure between the front and back of the obstacle

$$\Delta p = \begin{cases} p(0.15, 2) - p(0.25, 2) & \text{if } \Omega \subset \mathbb{R}^2, \\ p(0.45, 0.2, 0.205) - p(0.55, 0.2, 0.205) & \text{if } \Omega \subset \mathbb{R}^3. \end{cases}$$

We start presenting the results for the two dimensional case. In Figure 6 we present the drag coefficient c_{drag} , lift coefficient c_{lift} , and pressure difference between the front and the back of the obstacle Δp , computed using the Implicit-Euler scheme (27) on the mesh depicted in Figure 5 containing 118,321 elements and 354,963 vertices. The values are compared with the reference solution presented in [37], computed based on a Rosenbrock-Wanner scheme (ROS3Pw) of order three in time and the pair $\mathbb{Q}_2/\mathbb{P}_1^{disc}$ for space discretisation (see [37, Section 5] for details). More precisely, in

FIG. 4. *Example 4: velocity finite element solutions, for (62) (top), (27) (center), and a reference solution, based on (27), obtained over a mesh with 32,768 elements and 16,641 vertices (bottom). All the results are presented at the final time $T = 100$ considering $\Delta t = 1/2^7$ (top and center), and $\Delta t = 1/2^{13}$ (bottom).*

Figure 6 we depict the values obtained with the present method alongside the reference values from [37]², where we can observe that the corresponding curves are very

²These values have been taken from V. John's online resource http://www.wias-berlin.de/people/john/ELECTRONIC_PAPERS/tnse2d_bench_ref.txt

FIG. 5. *Example 5 (2d and 3d time-dependent flow around an obstacle): mesh with 118,321 elements and 354,963 vertices, for the two dimensional case (top); mesh with 469,262 elements and 90,569 vertices, for the three dimensional case (bottom).*

close to each other. For both approximations, in Table 2 we present the reference values $c_{drag,max}$, $c_{lift,max}$ (and the time the maximum is attained $t_{.,max}$), and Δp at $t = 8$, against the reference values from [37, Table 2]. We can see that both the maximum drag and the pressure difference at $t = 8$ are very well approximated, while the lift coefficient is approximated within a 21% of the reference value. According to [37], the lift coefficient is more difficult to approximate than the other two values, which explains this considering the combined effect of using a first order method and a coarser (uniform) mesh. To stress this fact, we have also included in Table 2 the values reported in [34] where the same space discretisation as in [37] is used, but an implicit Euler scheme is used for time discretisation. We can see that the values obtained using the present approach are comparable to the latter ones. Finally, in Figure 7 we depict the finite element solution using the present method at times $t = 3.9$ and $t = 5.7$ (very near the times in which the drag and lift coefficients attain their maxima).

FIG. 6. *Example 5: fluid flow coefficients, for the two dimensional case.*

Method - Reference	Ndof	$t_{c_d,max}[s]$	$c_{drag,max}$	$t_{c_l,max}[s]$	$c_{lift,max}$	$\Delta p(t=8)$
$\mathbb{Q}_2/\mathbb{P}_1^{disc}$ - [37]	2,347,776	3.93625	2.9509	5.6925	0.47795	-0.11161567
$\mathbb{Q}_2/\mathbb{P}_1^{disc}$ -Euler - [34]	588,160	3.93375	2.9506	5.7175	0.38213	-0.11134512
$\mathbb{P}_1/\mathbb{P}_0$ Imp. Euler - (27)	828,247	3.92500	2.9425	5.7200	0.37600	-0.11497870

TABLE 2

Example 5: fluid flow coefficients for the two dimensional case.

675

FIG. 7. Example 5: the nonconforming finite element solution $\mathcal{L}(\mathbf{u}_h, p_h)$, for the Implicit Euler Method (27), at times $t = 3.9$ (top) and $t = 5.7$ (bottom), considering $\nu = 10^{-3}$, $\Delta t = \frac{1}{6,400}$ and $T = 8.0$, over a mesh with 118,321 elements and 354,963 vertices.

We conclude this section by presenting our three dimensional results. In Figure 8 we present the nonconforming divergence free finite element solution $\mathcal{L}(\mathbf{u}_h, p_h)$, based on the used of the implicit Euler method (27), on the mesh from Figure 5, containing 469,262 elements and 90,569 vertices (again, a coarse uniformly refined mesh leading to 740,969 degrees of freedom). We present the streamlines and isovalues of the module of the velocity at time $t = 4$, which is close to the time in which the maximum drag coefficient is attained. In fact, we obtain the value $c_{drag,max} = 4.3341$ at the time $t_{c_d,max}[s] = 3.96125$. Although these values are still not close enough to the reference values reported in [35, Table II], we point out that they have been obtained with a coarse uniform mesh, and a first order method (as compared to the taylored more refined meshes and higher order methods in space and time used in [35]).

7. Concluding remarks. In this work we have presented and analysed a new stabilised finite element method for the Navier-Stokes equations based on the lowest order pair $\mathbb{P}_1^2 \times \mathbb{P}_0$. The main conclusions of this paper can be summarised as follows:

- in the modified continuity equation of the stabilised method lies a *hidden* divergence-free velocity field that can be extracted at zero cost, and that enjoys the same convergence as the conforming velocity and pressure;

FIG. 8. Example 5: the nonconforming divergence free finite element solution $\mathcal{L}(\mathbf{u}_h, p_h)$, for the Implicit-Euler method, at time $t = 4.0$, over a mesh with 469,262 elements and 90,569 vertices, and streamlines of the solution at $z = 0.2$.

- this velocity field is inserted into the convective field. The resulting stabilised method has been analysed and we have proven its convergence with constants independent of negative powers of the viscosity. To achieve this, the convective term did not need to be rewritten, nor it was necessary to add any extra stabilisation to deal with the dominating convection;
- the non-conforming velocity $\mathcal{L}(\mathbf{u}_h, p_h)$ recovers features not present in the conforming velocity $\mathbf{u}_h$;
- related to the last point, the numerical results also show that there is no need to add extra stabilisation to deal with the dominating convection (at least for the examples presented in this work). In fact, satisfactory results have been obtained in standard benchmarks for the time-dependent Navier-Stokes equations. As such, replacing $\mathbf{u}_h$ by $\mathcal{L}(\mathbf{u}_h, p_h)$ in the convective term has a stabilising effect by itself.

Appendix A : Energy and linear momentum conservation. Basing us in the discussion carried out in [14], we now briefly review the preservation (or lack thereof) of kinetic energy and momentum of the method presented in this paper. In [14] it is stated that if the discrete velocity is divergence-free, then the convective formulation (as used in this manuscript) preserves energy and momentum. The difference with that setting is that in the current paper we have replaced $\mathbf{u}_h$ by $\mathcal{L}(\mathbf{u}_h, p_h)$ in the convective term, so the setting of [14] is not directly applicable here. In order to simplify the presentation (and to avoid the discussion about the time integration

method) we will just briefly state the results for the following semi-discrete method:
Find $(\mathbf{u}_h, p_h)$ such that

$$(68) \begin{cases} \frac{d}{dt}(\mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + b(\mathcal{L}(\mathbf{u}_h, p_h), \mathbf{u}_h, \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ (q_h, \nabla \cdot \mathbf{u}_h) + s_{\text{pres}}(p_h, q_h) = 0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_{h,0} \times Q_h$. We now briefly outline the arguments for conservation of kinematic energy and linear momentum.

Kinematic Energy : Taking $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$ in (68) and using that

$b(\mathcal{L}(\mathbf{u}_h, p_h), \mathbf{u}_h, \mathbf{u}_h) = 0$ we get $\frac{d}{dt} \|\mathbf{u}_h\|_{0,\Omega}^2 + \nu \|\nabla \mathbf{u}_h\|_{0,\Omega}^2 + s_{\text{pres}}(p_h, p_h) = (\mathbf{f}, \mathbf{u}_h)$.
Then, in the absence of forcing terms ($\mathbf{f} = \mathbf{0}$) and of diffusion ($\nu = 0$), we get

$$(69) \quad \frac{d}{dt} \|\mathbf{u}_h\|_{0,\Omega}^2 = -s_{\text{pres}}(p_h, p_h) \leq 0.$$

Strictly speaking, this last result shows a loss of kinetic energy, but this loss is unavoidable if stabilised methods are used, or, at least, if stabilised methods with symmetric pressure stabilisation are used. This loss of kinetic energy is, in any case, of the order of the method, at most (and in many situations significantly smaller than the global error of the method).

Linear momentum : We will set ourselves in the setting given by [14]. This is, we will suppose that there is a strip of elements $S = \Omega \setminus \hat{\Omega}$ such that $\mathbf{u}_h$, p_h , and $\mathbf{f}$ vanish in S (so, in particular, we assume that $\mathbf{u}_h = \mathbf{0}$ on $\partial \hat{\Omega}$). For a function g defined in $\hat{\Omega}$ we will consider $\xi(g)$ as the extension of g to the whole of Ω that decreases to zero on $\partial \Omega$ and is piecewise linear in S . In particular, for a function g in V_h , we suppose that $\xi(g) \in V_{h,0}$. Denoting by $\mathbf{e}_i, i = 1, \dots, d$ the i^{th} canonical basis vector in $\mathbb{R}^d$ we consider $\mathbf{v}_h = \xi(\mathbf{e}_i)$ and $q_h = 0$ in (68) and arrive at

$$(70) \quad \frac{d}{dt}(\mathbf{u}_h, \mathbf{e}_i)_{\hat{\Omega}} + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{e}_i)_{\hat{\Omega}} + (\mathcal{L}(\mathbf{u}_h, p_h) \cdot \nabla \mathbf{u}_h, \mathbf{e}_i)_{\hat{\Omega}} = (\mathbf{f}, \mathbf{e}_i)_{\hat{\Omega}}.$$

First, clearly $\nu(\nabla \mathbf{u}_h, \nabla \mathbf{e}_i)_{\hat{\Omega}} = 0$. Next, integrating by parts and using the fact that $\mathcal{L}(\mathbf{u}_h, p_h)$ is solenoidal and that $\mathbf{u}_h$ vanishes on $\partial \hat{\Omega}$ we get $(\mathcal{L}(\mathbf{u}_h, p_h) \cdot \nabla \mathbf{u}_h, \mathbf{e}_i)_{\hat{\Omega}} = -(\mathcal{L}(\mathbf{u}_h, p_h) \cdot \nabla \mathbf{e}_i, \mathbf{u}_h)_{\hat{\Omega}} = 0$. So, if $\mathbf{f} = \mathbf{0}$, then (70) leads to

$$(71) \quad \frac{d}{dt}(\mathbf{u}_h, \mathbf{e}_i)_{\hat{\Omega}} = 0.$$

In other words, (68) preserves linear momentum.

Angular momentum : We set ourselves in the setting of the last paragraph, and use the same notation. We define $\phi_i = \mathbf{x} \times \mathbf{e}_i, i = 1, \dots, d$. Taking $\mathbf{v}_h = \xi(\phi_i), q_h = 0$ in (68) and noticing that ϕ_i is solenoidal in $\hat{\Omega}$ we get

$$(72) \quad \frac{d}{dt}(\mathbf{u}_h, \phi_i)_{\hat{\Omega}} + \nu(\nabla \mathbf{u}_h, \nabla \phi_i)_{\hat{\Omega}} + (\mathcal{L}(\mathbf{u}_h, p_h) \cdot \nabla \mathbf{u}_h, \phi_i)_{\hat{\Omega}} = 0.$$

First, since $\Delta \phi_i = 0$ and $\mathbf{u}_h$ vanishes on $\partial \hat{\Omega}$, then $\nu(\nabla \mathbf{u}_h, \nabla \phi_i)_{\hat{\Omega}} = 0$. In addition, arguing as before we get $(\mathcal{L}(\mathbf{u}_h, p_h) \cdot \nabla \mathbf{u}_h, \phi_i)_{\hat{\Omega}} = -(\mathbf{u}_h \cdot \nabla \phi_i, \mathbf{u}_h)_{\hat{\Omega}} - (\mathcal{L}(\mathbf{0}, p_h) \cdot \nabla \phi_i, \mathbf{u}_h)_{\hat{\Omega}}$. A simple algebraic manipulation gives $(\mathbf{u}_h \cdot \nabla \phi_i) \cdot \mathbf{u}_h = 0$, and then the first term vanishes. To deal with the second one we observe that $\|\nabla \phi_i\|_{\infty, \hat{\Omega}} = 1$ and that (69) implies $\|\mathbf{u}_h\|_{0,\Omega} \leq C \|\mathbf{u}_0\|_{0,\Omega}$. Thus, using Young's inequality and Lemma 3 we arrive at

$$(73) \quad \frac{d}{dt}(\mathbf{u}_h, \phi_i)_{\hat{\Omega}} = (\mathcal{L}(\mathbf{0}, p_h) \cdot \nabla \phi_i, \mathbf{u}_h)_{\hat{\Omega}} \leq Ch s_{\text{pres}}(p_h, p_h)^{\frac{1}{2}} \|\mathbf{u}_0\|_{0,\Omega}.$$

This is a loss of angular momentum, but, thanks to the properties of $\mathcal{L}$, this loss is negligible with respect to the order of the method.

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