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Homotopy analysis of 1D unsteady, nonlinear groundwater flow through porous media

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ABSTRACT

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In this paper, the 1D unsteady, nonlinear groundwater flow through porous media, corresponding to flood in an aquifer between two reservoirs, is studied by mass conservation equation and Forchheimer equation instead of Darcy's law. The coupling nonlinear equations are solved by homotopy analysis method (HAM), an analytic, totally explicit mathematic method. The method uses a mapping technique to transfer the original nonlinear differential equations to a number of linear differential equations, which does not depend on any small parameters and is convenient to control the convergence region. Comparisons between the present HAM solution and the numerical results demonstrate the validity of the HAM solution. It is further revealed the strong nonlinear effects in the HAM solution at the transitional stage.

ADDITIONAL INDEX WORDS: *Homotopy analysis method, Forchheimer equation, porous media*

INTRODUCTION

Groundwater flow through porous media is traditionally described by Darcy's law and it is normally valid for low Reynolds pore-scale numbers. For moderate and high velocity flow, however, Forchheimer equation should be applied due to nonlinear effects. STARK (1972) numerically solved the Navier-Stokes laminar flow equations and tested the relations of Darcy's law, Forchheimer equation and others; INNOCENTINI *et al.* (1999) compared Darcy's law and Forchheimer equation and recommended highly the latter in order to take into account permeability; NIELD (2000) discussed the inertial effects on viscous dissipation for the case of Darcy, Forchheimer and Brinkman models.

Forchheimer equation could be derived in different approaches (e.g. AHMED and SUNADA, 1969; HASSANIZADEH and GRAY, 1987; WHITAKER, 1996) and has been proved in theoretical and experimental way (MACDONALD *et al.*, 1979; THAUVIN and MOHANTY, 1998). Extensive studies on the parameters in Forchheimer equation have been carried out. COULAUD *et al.* (1988) introduced a nonlinear term into Darcy's equation and solved it numerically. In his approach, the hydrodynamic constants in the Forchheimer equation were expressed by the expression of porosity and geometrical data. WANG and LIU (2004) investigated the scaling relations for the fluid permeability and the inertial parameter in the Forchheimer equation, by solving the Navier-Stokes equation for flow in a two-dimensional percolation porous media.

Although the application of the Forchheimer flow is very useful and practical, very limited attempts on the analytical approach have been reported in the literature. MOUTSOPOULOS and TSIHRINTZIS (2005) solved the Forchheimer flow through porous media in 1D form by perturbation method, dividing the problem into two stages and solving them by two sets of equations. For numerical method, GREENLY and JOY (1996) used one-dimensional finite element method and Forchheimer equation to investigate the groundwater flow through a valley fill. EWING *et al.* (1999) used finite difference, Galerkin finite element and mixed finite element techniques to investigate Forchheimer flow in a hydrocarbon reservoir. KIM and PARK (1999) and PARK (2005) used mixed finite element method to analyse the flow of a single-phase fluid in a porous medium governed by Forchheimer equation.

Recently, a new mathematical technique, namely homotopy analysis method (HAM) has been applied to nonlinear fluid dynamics problems (LIAO, 1995, 2004). The approach does not depend on small or large parameters and is easy to adjust the convergence region and rate of approximation series. In this paper, homotopy analysis method is applied to solve the 1D unsteady, nonlinear groundwater flow through porous media, corresponding to a flood in a long aquifer between two reservoirs, or to a flow in a laboratory column restrained by two external tanks. The coupling equations are transformed by similarity law and a global solution, which is analytical, totally explicit is obtained. The piezometric head from the present HAM solution for nonlinear flow agrees well with numerical results and the previous perturbation solutions.

THEORETICAL CONSIDERATION

Consider a one-dimensional (1D) flow in a confined porous medium shown in Figure 1. Before t=0, the piezometric head is a constant h_I. After t=0, the piezometric head at left end raises Δh, while the piezometric head at the far right end remains unchanged.

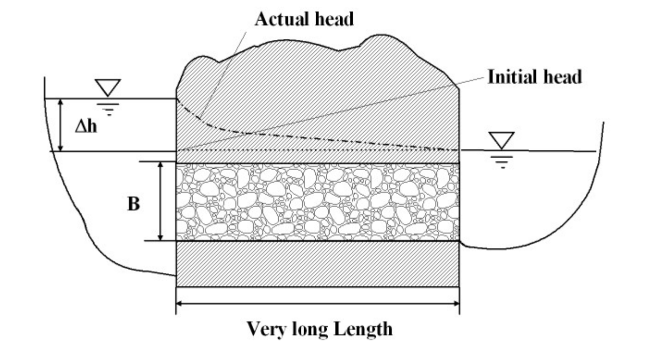

Figure 1. The sketch of the problem.

The movement of the flow satisfies the equation of mass conservation as following:

$$S\frac{\partial h}{\partial t} + \nabla(B\vec{q}) = R, \tag{1}$$

where S is the storage coefficient; h is the piezometric head; t is the time; ∇ is the 2D Nabla operator; B is the thickness of the aquifer; $\vec{q}$ is the velocity; and R is the external sink-source term, which is assigned zero in this particular problem.

Assuming that the properties of the aquifer are homogeneous, the Forchheimer or Forchheimer-Dupuit equation is:

$$-\nabla h = a\vec{q} + b\vec{q}\,|\vec{q}\,|, \tag{2}$$

where a and b are coefficients.

Equations (1) and (2) can be expressed in 1D form as

$$S\frac{\partial h}{\partial t} + B\frac{\partial q}{\partial x} = 0, \tag{3}$$

$$-\frac{\partial h}{\partial x} = aq + bq^2. \tag{4}$$

The initial condition is:

$$h = h_I \qquad \text{at} \quad t = 0. \tag{5}$$

The boundary conditions are:

$$h = h_I + \Delta h \qquad \text{for} \quad x = 0 \quad \text{at} \quad t > 0, \tag{6}$$

$$h = h_I \qquad \text{for} \quad x = +\infty \quad \text{at any} \quad t. \tag{7}$$

Introducing

$$\tilde{h} = \frac{h - h_I}{\Delta h}, \tag{8}$$

Equations (1) and (2) are transformed as:

$$\Delta h \frac{S}{B}\cdot\frac{\partial \tilde{h}}{\partial t} + \frac{\partial q}{\partial x} = 0, \tag{9}$$

$$-\Delta h\frac{\partial \tilde{h}}{\partial x} = aq + bq^2. \tag{10}$$

From Equation (10), the velocity can be expressed as

$$q = \frac{-a + \sqrt{a^2 - 4b\Delta h \frac{\partial \tilde{h}}{\partial x}}}{2b}. \tag{11}$$

If $-4b\Delta h \frac{\partial \tilde{h}}{\partial x} > a^2$, the inertial term is dominant, otherwise the Darcy (viscous) term is dominant.

Substituting Equation (11) into Equation (9), we have:

$$\left(\frac{\partial^2 \tilde{h}}{\partial x^2} / \frac{\partial \tilde{h}}{\partial t}\right)^2 = C_1\frac{\partial \tilde{h}}{\partial x} + C_2, \tag{12}$$

subject to the initial and boundary conditions:

$$\tilde{h} = 0 \qquad \text{for} \quad t = 0, \tag{13}$$

$$\tilde{h} = 1 \qquad \text{for} \quad x = 0, \tag{14}$$

$$\tilde{h} = 0 \qquad \text{for} \quad x = +\infty, \tag{15}$$

where $C_1 = -4\left(\frac{S}{B}\right)^2 b\Delta h$, $C_2 = \left(\frac{S}{B}a\right)^2$.

Using the similarity transformations:

$$\tilde{h}(x,t) = t^{1/2} f(\xi), \quad \xi = x^2/t, \tag{16}$$

Equation (12) becomes

$$16\cdot\left(\frac{f'(\xi) + 2\xi f''(\xi)}{f(\xi) - 2\xi f'(\xi)}\right)^2 = 2C_1\xi^{1/2} f'(\xi) + C_2, \tag{17}$$

with boundary conditions:

$$f(0) = C_3, \tag{18}$$

$$f(+\infty) = 0, \tag{19}$$

where $C_3 = 1/t^{1/2}$, which is a constant for a given time t.

Alternatively, Equation (17) can be expressed as,

$$\xi = \left\{\left[16\cdot\left(\frac{f'(\xi) + 2\xi f''(\xi)}{f(\xi) - 2\xi f'(\xi)}\right)^2 - C_2\right] / \left[2C_1 f'(\xi)\right]\right\}^2. \tag{20}$$

SOLUTION METHOD

Define

$$\tau = 1 + \lambda\xi, \tag{21}$$

where λ is a constant parameter to be determined later, Equation (20) can be transformed to Equation (22) (see Appendix), and the boundary conditions become

$$f(1) = C_3, \tag{23}$$

$$f(+\infty) = 0. \tag{24}$$

From the above boundary conditions, $f(\tau)$ can be expressed by a set of base functions

$$\{\tau^{-m} \mid m \geq 1\}. \tag{25}$$

Establish the zeroth-order deformation equation:

$$(1-p)\mathscr{L}[F(\tau;p) - f_0(\tau)] = p\hbar H(\tau)\mathscr{N}[F(\tau;p)], \tag{26}$$

subject to the boundary conditions:

$$F(1;p) = C_3, \quad F(+\infty;p) = 0, \tag{27}$$

where $p \in [0,1]$ is an embedding parameter and $\mathscr{L}$ is a linear auxiliary operator; $F(\tau;p)$ is a real function of τ and p. The auxiliary nonzero parameter $\hbar$ is used to adjust the convergence rate and region. By introducing HAM, we set up a mapping $f(\tau) \to F(\tau;p)$. It is seen from Equation (26) that the solution $F(\tau;p)$ continuously varies from the initial estimate $f_0(\tau)$ to the exact solution $f(\tau)$ as the embedding parameter p increases from 0 to 1.

The linear auxiliary operator $\mathscr{L}$ has a various choices, which will affect the convergence of the solution. We select

$$\mathscr{L}[F(\tau;p)] = \tau\frac{\partial F(\tau;p)}{\partial \tau} + F(\tau;p), \tag{28}$$

with the property

$$\mathscr{L}[C_4/\tau] = 0. \tag{29}$$

The nonlinear operator $\mathscr{N}$ is defined in Equation (30) (see Appendix).

Expand $F(\tau;p)$ in Taylor series with respect to p, we have

$$F(\tau;p) = f_0(\tau) + \sum_{m=1}^{+\infty} f_m(\tau)p^m, \tag{31}$$

where

$$f_m(\tau) = \frac{1}{m!}\frac{\partial^m F(\tau;p)}{\partial p^m}\bigg|_{p=0}. \tag{32}$$

If the auxiliary parameter $\hbar$ is properly chosen that the series are convergent at p=1, then

$$f(\tau) = f_0(\tau) + \sum_{m=1}^{+\infty} f_m(\tau) \cdot \tag{33}$$

Define

$$\vec{f}_n = \{ f_0(\tau), f_1(\tau), f_2(\tau), \cdots, f_n(\tau) \} \cdot \tag{34}$$

Differentiating Equation (26) m times with respect to p, then setting p=0, and finally dividing them by m!, the mth-order deformation equation can be obtained:

$$\mathscr{L}[f_m(\tau) - \chi_m f_{m-1}(\tau)] = \hbar H(\tau) R_m(\vec{f}_{m-1}, \tau), \tag{35}$$

subject to the boundary conditions:

$$f_m(1) = 0, \quad f_m(+\infty) = 0 \quad \text{for} \quad m \geq 1, \tag{36}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases}, \tag{37}$$

and

$$R_m(\vec{f}_{m-1}, \tau) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} \mathscr{N}[F(\tau;p)]}{\partial p^{m-1}} \right|_{p=0} \cdot \tag{38}$$

Substituting $\mathscr{N}$ and $F(\tau;p)$ in Equations (30) and (31) respectively into Equation (38), we can have the detailed form of R_m.

Suppose

$$f_0(\tau) = C_3 / \tau, \tag{39}$$

then the whole problem can be solved by iteration:

$$f_m(\tau) = \chi_m f_{m-1}(\tau) + \frac{\hbar}{\tau} \int_1^{\tau} H(\alpha) R_m(\vec{f}_{m-1}, \tau) d\alpha + \frac{C_4}{\tau}, \tag{40}$$

where $H(\tau)$ can be chosen as $H(\tau) = \tau^2$.

From the first few orders of the solution, it can be concluded that $f_m(\tau)$ can be expressed as

$$f_m(\tau) = \sum_{n=1}^{9m+1} \beta_{m,n} \tau^{-n}, \tag{41}$$

where $\beta_{m,n}$ is a coefficient.

Substituting Equation (41) into the high-order deformation equation (35) and equating the same power of τ, the recurrence formulae of $\beta_{m,n}$ can be obtained, which is very long and omitted here.

From Equation (36), $\beta_{m,1}$ can be determined uniquely.

$$\beta_{m,1} = -\sum_{n=2}^{9m+1} \beta_{m,n} \qquad m \geq 1 \cdot \tag{42}$$

If the series (33) is convergent, it must be an exact solution of Equation (20), since the following equation stands when the series (33) is convergent:

$$\lim_{m \to +\infty} f_m(\tau) = 0 \cdot \tag{43}$$

Using Equations (26), (35) and (37), we have

$$\hbar H(\tau) \sum_{m=1}^{+\infty} R_m(\vec{f}_{m-1}, \tau) = \lim_{n \to +\infty} \sum_{m=1}^{n} \mathscr{L}[f_m(\tau) - \chi_m f_{m-1}(\tau)] \tag{44}$$

$$= \mathscr{L}\left\{ \lim_{n \to +\infty} \sum_{m=1}^{n} [f_m(\tau) - \chi_m f_{m-1}(\tau)] \right\} = \mathscr{L}\left[\lim_{n \to +\infty} f_n(\tau) \right] = 0,$$

which gives

$$\sum_{m=1}^{+\infty} R_m(\vec{f}_{m-1}, \tau) = 0 \cdot \tag{45}$$

for any $\tau \geq 1$.

Under the definition (38), it is easy to prove that Equation (20) holds. From Equations (36) and (39), the boundary conditions (18) and (19) also hold. Therefore, if $\hbar$ and λ are properly chosen to ensure the series convergence, $f(\xi)$ is an exact solution of the similarity questions.

RESULTS AND DISCUSSION

In this section, the physical problem of groundwater flow through porous media is solved by homotopy analysis method described above and the HAM solutions are compared to the perturbation solution of MOUTSOPOULOS and TSIHRINTZIS (2005) in Figure 2. The principle data of the first three figures are a=0.05s/m, b=15s^2/m^2, S_0=S/B=0.02m^{-1} and Δh=1m while the data of the last two figures are a=0.634s/m, b=30.8s^2/m^2, S_0= 0.02m^{-1} and Δh=1m. The time instances corresponding to the results shown in the five figures are (a) 42s, (b) 99s, (c) 504s, (d) 2970s and (e) 4425s respectively. As can be seen in the figures, the first three figures are nonlinear dominated and the rest two are quasi-Darcy. The numerical results from Matlab are calculated by *pdepe* function with Δx=3m and Δt=0.2s for the distance and time intervals, assuming x=3000m is sufficiently large. For homotopy analysis method, we choose $\hbar$ = -1 and λ=1/50 respectively.

It can be seen that the results from homotopy analysis method agree well with the numerical method and MOUTSOPOULOS and TSIHRINTZIS (2005) in nonlinear dominated cases, but less accurate in quasi-linear cases. If the solution contains a linear function, the results should be more reasonable for quasi-linear flow. In the present study, nonlinear base functions were employed, so that the phenomena of unsteady, nonlinear flow through porous media could be described accurately. Without the component of a linear function, the present HAM solution is more suitable for the initial unsteady, nonlinear stage, which is highly concerned by scientists and engineers. For quasi-linear flow, Darcy's law could be applied directly. Thus the problem could be simplified to a linear problem and easy to be solved.

Table 1 is an example of the convergence of the series at t=42s and x=30m. A rapid convergence rate of the series is evident.

CONCLUSIONS

In this paper, a new analytic method, namely homotopy analysis method (HAM) has been applied to give an analytic, totally explicit and uniform valid solution to the problem of unsteady, nonlinear groundwater flow through porous media. The HAM analytic solution agrees well with numerical results and previous perturbation method for nonlinear flow. The present approach shows a great potential to other unsteady nonlinear problems.

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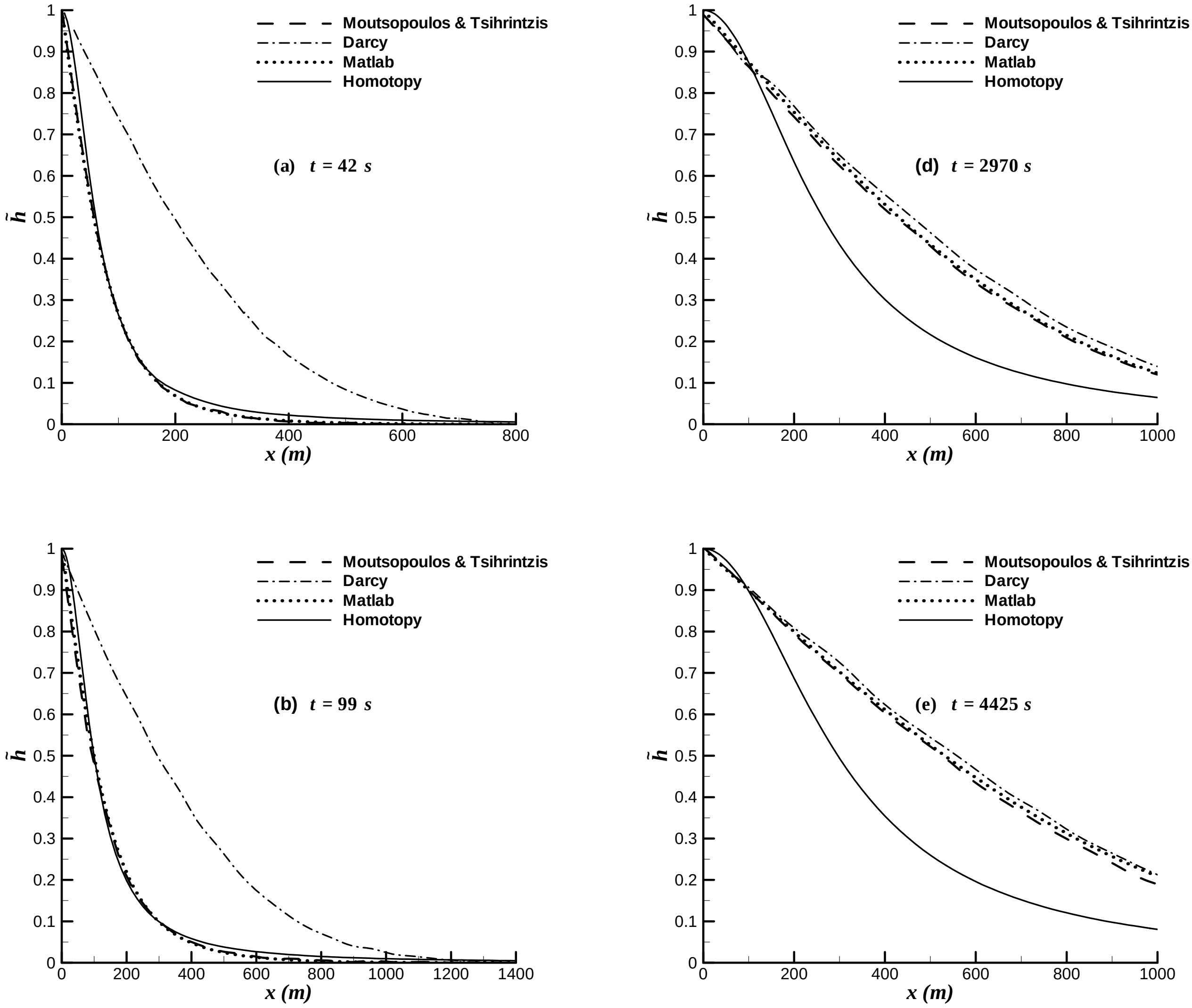
Table 1: The convergence of the series at t=42s and x=30m.

Order	$\tilde{h}$
0	0.007200822998231
1	0.700000106609584
2	0.700000213217835
3	0.700000319824756
4	0.700000426430345
5	0.700000533034602
6	0.700000639637528



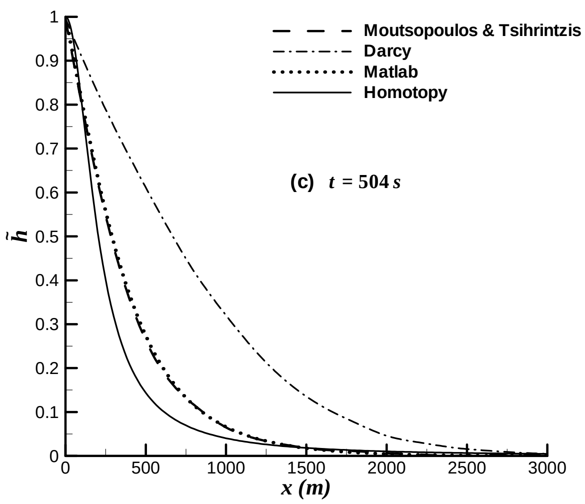


Figure 2. Comparison of dimensionless piezometric head in MOUTSOPOULOS and TSIHRINTZIS (2005).

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APPENDIX

$$\begin{aligned}
& C_2^2 f(\tau)^4 - 8C_2^2(\tau-1) f(\tau)^3 f'(\tau) + [24C_2^2(\tau-1)^2 - 32C_2\lambda^2] f(\tau)^2 f'(\tau)^2 \\
& +[128C_2(\tau-1)\lambda^2 - 32C_2^2(\tau-1)^3] f(\tau) f'(\tau)^3 \\
& +[256\lambda^4 - 128C_2(\tau-1)^2\lambda^2 + 16C_2^2(\tau-1)^4] f'(\tau)^4 - 128C_2(\tau-1)\lambda^2 f(\tau)^2 f'(\tau) f''(\tau) \\
& +512C_2(\tau-1)^2\lambda^2 f(\tau) f'(\tau)^2 f''(\tau) + [2048(\tau-1)\lambda^4 - 512C_2(\tau-1)^3\lambda^2] f'(\tau)^3 f''(\tau) \\
& -128C_2(\tau-1)^2\lambda^2 f(\tau)^2 f''(\tau)^2 + 512C_2(\tau-1)^3\lambda^2 f(\tau) f'(\tau) f''(\tau)^2 \\
& +[6144(\tau-1)^2\lambda^4 - 512C_2(\tau-1)^4\lambda^2] f'(\tau)^2 f''(\tau)^2 + 8192(\tau-1)^3\lambda^4 f'(\tau) f''(\tau)^3 \\
& +4096(\tau-1)^4\lambda^4 f''(\tau)^4 - 4C_1^2(\tau-1)\lambda f(\tau)^4 f'(\tau)^2 + 32C_1^2(\tau-1)^2\lambda f(\tau)^3 f'(\tau)^3 \\
& -96C_1^2(\tau-1)^3\lambda f(\tau)^2 f'(\tau)^4 + 128C_1^2(\tau-1)^4\lambda f(\tau) f'(\tau)^5 - 64C_1^2(\tau-1)^5\lambda f'(\tau)^6 = 0
\end{aligned} \tag{22}$$

$$\begin{aligned}
& \mathscr{N}[F(\tau;p)] = C_2^2 F(\tau;p)^4 - 8C_2^2(\tau-1) F(\tau;p)^3 \frac{\partial F(\tau;p)}{\partial p} \\
& +[24C_2^2(\tau-1)^2 - 32C_2\lambda^2] F(\tau;p)^2 [\frac{\partial F(\tau;p)}{\partial p}]^2 + [128C_2(\tau-1)\lambda^2 - 32C_2^2(\tau-1)^3] F(\tau;p) [\frac{\partial F(\tau;p)}{\partial p}]^3 \\
& +[256\lambda^4 - 128C_2(\tau-1)^2\lambda^2 + 16C_2^2(\tau-1)^4][\frac{\partial F(\tau;p)}{\partial p}]^4 - 128C_2(\tau-1)\lambda^2 F(\tau;p)^2 \frac{\partial F(\tau;p)}{\partial p}\frac{\partial^2 F(\tau;p)}{\partial p^2} \\
& +512C_2(\tau-1)^2\lambda^2 F(\tau;p)[\frac{\partial F(\tau;p)}{\partial p}]^2 \frac{\partial^2 F(\tau;p)}{\partial p^2} + [2048(\tau-1)\lambda^4 - 512C_2(\tau-1)^3\lambda^2][\frac{\partial F(\tau;p)}{\partial p}]^3 \frac{\partial^2 F(\tau;p)}{\partial p^2} \\
& -128C_2(\tau-1)^2\lambda^2 F(\tau;p)^2 [\frac{\partial^2 F(\tau;p)}{\partial p^2}]^2 + 512C_2(\tau-1)^3\lambda^2 F(\tau;p) \frac{\partial F(\tau;p)}{\partial p}[\frac{\partial^2 F(\tau;p)}{\partial p^2}]^2 \\
& +[6144(\tau-1)^2\lambda^4 - 512C_2(\tau-1)^4\lambda^2][\frac{\partial F(\tau;p)}{\partial p}]^2[\frac{\partial^2 F(\tau;p)}{\partial p^2}]^2 + 8192(\tau-1)^3\lambda^4 \frac{\partial F(\tau;p)}{\partial p}[\frac{\partial^2 F(\tau;p)}{\partial p^2}]^3 \\
& +4096(\tau-1)^4\lambda^4[\frac{\partial^2 F(\tau;p)}{\partial p^2}]^4 - 4C_1^2(\tau-1)\lambda F(\tau;p)^4[\frac{\partial F(\tau;p)}{\partial p}]^2 + 32C_1^2(\tau-1)^2\lambda F(\tau;p)^3[\frac{\partial F(\tau;p)}{\partial p}]^3 \\
& -96C_1^2(\tau-1)^3\lambda F(\tau;p)^2[\frac{\partial F(\tau;p)}{\partial p}]^4 + 128C_1^2(\tau-1)^4\lambda F(\tau;p)[\frac{\partial F(\tau;p)}{\partial p}]^5 - 64C_1^2(\tau-1)^5\lambda[\frac{\partial F(\tau;p)}{\partial p}]^6
\end{aligned} \tag{30}$$