Gravitomagnetic effects on turbulence physics

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Abstract

We indicate a limit of the post-Newtonian gravity equations with incompressible-fluid matter source in which Newtonian gravity is approximately decoupled from relativistic gravitomagnetic effects. In this gravito-magneto-hydrodynamic limit, we perform fully developed turbulence calculations. We demonstrate that gravitomagnetic effects reduce the vortical complexity and nonlinearity of turbulence, even leading to its extinction within large volumes, and generate departures from Kolmogorov turbulence scalings, that are explained via a combination of dimensional and exact analysis arguments.

INTRODUCTION

Turbulence in incompressible fluids [1] is a central problem of statistical physics and nonlinear mathematics. Although, we do not have, at present, an analytical control of key nonlinear turbulent processes (interscale energy transfer, intermittency, statistical structure), we have a plethora of numerical solutions of the Navier-Stokes equations and successful scaling theories for important physical aspects (Kolmogorov scalings of homogeneous, isotropic turbulence, log-law scaling regime in inhomogeneous turbulence in boundary layers). Indeed, there are very few other topics in physics, where so many data and models have been accumulated throughout decades of research. One of our important understandings is the fact that turbulence, when seen as a strongly out of equilibrium, nonlinear system, is dominated by linear vortical structures (quasi-defects), hence, rotating matter is the *very essence* of its physics. The turbulence cascade, for example, is a side-effect of vigorous mutual stretching of these linear structures via their nonlocal (in physical space) Biot-Savart type interactions.

Gravity on the other hand, can be described in the weak-field, slow-motion limits by the post-Newtonian formalism [2]. The latter is a gravitational version of the Maxwell theory of electromagnetism. Newtonian gravity corresponds to gravitoelectric effects which are combined with gravitomagnetic effects of relativistic origin. Although limited in their range of validity, the post-Newtonian equations provide a simple and intuitive description of relativistic gravity effects, that manifest themselves in flat spacetime, without the need of strong, spacetime bending fields, that require the mathematical machinery of differential geometry and the solution of tensor differential equations for their analysis. Although Newtonian gravity with fluid sources is a very well developed branch of fluid dynamics [3], only the most basic facts are known for the corresponding problem of gravito-magneto-hydrodynamics [4]. The key idea of gravitomagnetism is that, in contrast with Galilean invariant theory of gravity, where a rotating, massive sphere produces identical gravitational fields with a stationary one, in special relativity, the rotation of a massive sphere adds to the standard Newtonian field of a stationary sphere a gravitomagnetic element [5].

Hence, we have two important physical facts: (a) the most nonlinear physical process that we know (turbulence) is dominated by rotating masses of fluid, (b) rotating matter

creates relativistic gravitational fields. It seems that turbulence is the arena of the most intriguing gravitomagnetic effects, yet the physics of these fundamental processes are virtually unknown. The aim of this investigation is to produce gravitomagnetic turbulence physics on par with standard turbulence investigations. To achieve this goal, we need to: (a) decouple gravitoelectric from gravitomagnetic effects, i.e., formulate gravitomagnetohydrodynamics (GMHD) as a counterpart of ordinary magnetohydrodynamics (MHD) [6, 7], (b) scale the GMHD equations and identify parameter ranges where the gravitomagnetic field (GMF) has important effects on turbulence structure, (c) produce well resolved, fully developed turbulence solutions and explain the resulting scalings, including any observed departures from standard Kolmogorov scalings. Turbulence solutions are complex since they evolve over a continuum range of space-time scales, which need to be fully resolved to correctly capture their statistical structure. A key requirement is the resolution of the, all important, dissipation processes, whose efficiency peaks at high wavenumbers of the energy spectrum, where also most of the, equally important, turbulent strain resides. Next, we consider each of these tasks, in the given order.

DECOUPLING OF GRAVITOELECTRIC FROM GRAVITOMAGNETIC EFFECTS

Our starting point is the weak-field, slow-motion limit of Einstein gravity as realised by the post-Newtonian formalism. We follow, in this respect, Ciufolini and Wheeler ([4], page 327) who employed the Navier-Stokes equations as matter source of gravitational Maxwell theory, in the context of accretion disks around massive astronomical objects. Their equations are

$$\partial_i E_i^g = -4\pi G\rho, \tag{1}$$

$$\epsilon_{ijk}\partial_j E_k^g = -\partial_t B_i^g, \tag{2}$$

$$\partial_i B_i^g = 0, \tag{3}$$

$$\epsilon_{ijk}\partial_j B_k^g = -\frac{16\pi G}{c^2}\rho u_i + \frac{4}{c^2}\partial_t E_i^g, \tag{4}$$

$$\partial_i u_i = 0, \tag{5}$$

$$\partial_t u_i = E_i^g + \epsilon_{ijk} u_j B_k^g - \partial_i\left(\frac{p}{\rho} + \frac{u_j u_j}{2}\right) + \epsilon_{ijk} u_j \omega_k + \nu\partial_j\partial_j u_i. \tag{6}$$

where the first four equations correspond to field dynamics, the last two equations are the (incompressible) Navier-Stokes (NS) equations, $i, j, k = 1, 2, 3$, ϵ_{ijk} is the Levi-Civita symbol, E^g is the gravitoelectric field (i.e., the Newtonian gravitational field), B^g is the gravitomagnetic field, u is the fluid velocity, ρ is the density of fluid mass-energy, p is the scalar pressure, G is the gravitational constant, and ν is the fluid kinematic viscosity. It is important to note here, that by employing the incompressible Navier-Stokes equation for the dynamics of matter, we essentially formulate our problem in a Galilean rather than Minkowski spacetime. In this, we follow the standard route for the formulation of electric magnetohydrodynamics ([1], page 521). Therefore, although this formulation adds gravitomagnetic, relativistic *effects* to classical Newtonian gravity, it does not constitute a fully relativistic fluid dynamics. In this sense, our work compliments fully special relativistic treatments of self-gravitating fluids ([8] and references therein).

There is observational evidence that many large scale astronomical and cosmological flows could be treated as incompressible. Indeed, it is known that compressibility effects cause deviations from Kolmogorov scalings of incompressible turbulence when turbulent eddies are in transonic or supersonic motion relative to each other [9]. The dimensionless number quantifying this notion is the Mach number of turbulence $M_t = u'/c_s$, where u' is the intensity of turbulent velocity fluctuations, and c_s is the medium's speed of sound [9]. In other words, M_t measures the ratio between turbulent kinetic energy and thermal energy, and the latter ought to be a significant proportion of the former, for compressibility effects to become important. In this respect, analysis of turbulent gas pressure maps in the intra-cluster medium revealed that pressure fluctuations are consistent with *incompressible,*

Kolmogorov turbulence [10, 11]. This follows directly from the Mach number criterion, since, for the Coma cluster [10, 11], it is estimated that $\epsilon_{turb} \geq 0.1\epsilon_{th}$, where ϵ_{turb} is the kinetic energy density of turbulence, and ϵ_{th} is the fluid's thermal energy density, hence, the turbulent fluctuations are subsonic.

We are exclusively concerned here with homogeneous turbulence, hence, our flow domain is a periodic, flat torus. These boundary conditions are consistent with the post-Newtonian approximation and with current observational evidence and theoretical models which indicate that, on the large scale, our Universe is flat and possibly closed [12–14]. It is important to note, that the flatness assumption, refers only to large scale features, and does not exclude strong gravitational effects at smaller scales (e.g., around black holes) which are not captured by our equations.

Moreover, due to our homogeneous turbulence focus, we only consider here the self-consistent dynamics of periodic (fluctuating) E^g, B^g, ρ and u fields. The more general problem of coupling turbulence with *mean field* dynamics can be tackled at a later stage. However, there are mean E^g configurations that are readily consistent with our periodic boundary conditions. These are constant E^g fields, that, despite their simplicity, are conceptually important in relativity, and their physics are discussed in specialised monographs [15]. Some nice illustrations of a flat torus topology, together with examples of constant vector fields are available in [16]. This situation, where a turbulence fluctuations problem is solved with periodic boundary conditions in the presence of a uniform mean field is reminiscent of turbulent Couette flow with *constant* pressure gradient. Pressure acts like a potential, hence the constant pressure gradient corresponds to a constant field. In the gravitational context, such a uniform mean field E^g would be equivalent to a uniform acceleration and could be added to our turbulence setting without any change in the turbulent gravitomagnetic physics (principle of equivalence [17]). Overall, our analysis is compatible with arbitrary mean field E^g profiles that are slowly varying on the length scales of turbulence fluctuations (so that E^g could approximately be quasi-constant).

To decouple, in a physically consistent way, gravitoelectric from gravitomagnetic effects in *periodic domains*, we consider a *stratified* incompressible fluid, with periodic in space, con-

stant in time, density fluctuations, that vary *arbitrarily slow* on the scale of the system. The corresponding solutions of the Poisson equation for the potential of E^g in periodic domains are easily constructed with well developed, standard methods [18, 19]. Then the *linearity* of the E^g divergence equation indicates that E^g follows the same, very slow variation of density on the scale of the system ($E^g \propto G\ell_s\rho$, where ℓ_s is the size of the system). Similarly, the very small E^g gradients in the equation for its curl indicate that $\partial_t B^g \ll 1$, and time independence of density perturbations gives for the "displacement current" the condition $\partial_t E^g = 0$. Hence, the gravitoelectric field decouples from the gravitomagnetic field, and the latter becomes enslaved to the fluid evolution, instantaneously adjusting itself to the latter.

We still have the presence of E^g in the Navier-Stokes equation. We can eliminate this dependence in an approximate way as follows: by construction, E^g is varying arbitrarily slow on the size of the system, whilst turbulence flow patterns vary over an enormous range of scales, that, depending on the Reynolds number, can be many orders of magnitude smaller than the system size. Hence, from the point of view of velocity fluctuations, the slowly varying E^g field is, to a very good approximation, quasi-constant. Therefore, E^g can be approximately be incorporated into an effective pressure $p + \rho E^g_i x_i \to p_e$, and the standard Navier-Stokes equations are recovered.

With the above physically plausible approximations, we can finally write our GMHD equations. Notably, gravitodynamics is more nonlinear than electrodynamics, since, in the latter, electric current J^e includes the electric charge density ρ^e which differs from the fluid density ρ, whilst, in the former, gravitational current J^g is identical to the *fluid momentum*. This identification of gravitational charge with inertial mass allows the two B^g equations to form, together with the incompressible NS, a *closed* system of differential equations that can be autonomously solved,

$$\partial_i B^g_i = 0, \tag{7}$$

$$\epsilon_{ijk}\partial_j B^g_k = -\frac{16\pi G}{c^2}\rho u_i, \tag{8}$$

$$\partial_i u_i = 0, \tag{9}$$

$$\partial_t u_i = \epsilon_{ijk} u_j B^g_k - \partial_i\left(\frac{p}{\rho} + \frac{u_j u_j}{2}\right) + \epsilon_{ijk} u_j \omega_k + \nu \partial_j \partial_j u_i. \tag{10}$$

Although the equations look similar to analogous equations in the electrodynamic magnetic limit [6], their physical motivation is *very different*. In electrodynamics, there are negative and positive charges, hence, it is possible to realize the magnetic limit, by having approximately zero charge densities, but significant currents, which are responsible for neutralizing the charges [6]. In gravity, on the other hand, there cannot be zero mass-energy densities, and we had to introduce the GMHD equations in a more careful and elaborate way. It is important to note, that the purpose of our investigation is not to provide a full special relativistic analysis of self-gravitating turbulence. We are mostly motivated by the intriguing connection between flow vorticity and gravitomagnetism, continuing the tradition of [4]. From the point of view of rigorous turbulence calculations, we also aim to reduce computational complexity, so that we fully resolve, the all important, small scale dissipation processes. Finally, it is helpful to observe a key difference in the governing equations of gravitational and electrical versions of magnetohydrodynamics. In the former, the gravitational current is the fluid momentum, hence it can appear in the Navier-Stokes equation without adding a new unknown to the system. In the latter, the electric current is an extra unknown in the Navier-Stokes equation and one needs to employ Ampere law to substitute it in terms of the magnetic field. Moreover, one needs to combine Faraday law with Ohm and Ampere laws to write a *transport* equation for the magnetic field [20]. In contrast, the gravitomagnetic field is governed by Poisson's equation which is an *elliptic* not a transport equation. Hence, despite their conceptual similarity, the phenomenologies of the two versions of magnetohydrodynamics are very different with each other.

SCALING THE GRAVITOMAGNETOHYDRODYNAMIC EQUATIONS

To obtain a scaled system of equations, we define the constant $\beta \equiv 16\pi G/c^2$, and use it to scale B^g as $\widetilde{B}^g \equiv B^g/\sqrt{\beta\rho}$, and define the scaled gravitational current $\widetilde{J}^g_i = \epsilon_{ijk}\partial_j \widetilde{B}^g_k$. Notably, $\sqrt{\beta\rho}$ has units of cm^{-1}, $\widetilde{B^g}$ has units of velocity cm s^{-1}, and $\widetilde{J}^g$ has the units s^{-1} of flow vorticity $\omega_i \equiv \epsilon_{ijk}\partial_j u_k$. Finally, by taking the *curl* of $\epsilon_{ijk}\partial_j B^g_k$, we arrive at the *scaled*

gravito-magneto-hydrodynamic (GMHD) equations

$$\partial_i \widetilde{B}^g_i = 0, \tag{11}$$

$$\partial_j \partial_j \widetilde{B}^g_i = \sqrt{\beta\rho}\omega_i, \tag{12}$$

$$\partial_i u_i = 0, \tag{13}$$

$$\partial_t u_i = -\partial_i \left(\frac{p}{\rho} + \frac{u_j u_j}{2} \right) + \epsilon_{ijk} u_j \omega_k - \epsilon_{ijk} \widetilde{B}^g_j \widetilde{J}^g_k + \nu \partial_j \partial_j u_i, \tag{14}$$

where it is instructive to compare the equation for $\widetilde{B}^g_i$ with the equation for the velocity vector potential ψ, $\partial_j \partial_j \psi_i = -\omega_i$, where the difference in signs is due to the negative sign in the left hand side of the equation for the *curl* of B^g. Notably, Lamb force $\epsilon_{ijk} u_j \omega_k$ is the vector product of velocity with its vorticity, hence, since the gravitational current is the momentum, the novelty of gravitational effects (in comparison with Lamb force effects) depends on the relative orientation of ω and B^g vectors. In this form of the NS equation, $\epsilon_{ijk} \widetilde{B}^g_j \widetilde{J}^g_k$ term encodes genuine relativistic gravitational effects on flow structures. Indeed, it is straightforward to demonstrate [5] (page 89), that by combining Newtonian gravity with Lorentz transformations, and demanding identical physical predictions for different inertial frames, one "discovers" gravitomagnetism. In other words, the latter is a direct consequence of the relativity principle. It is helpful to write the vorticity dynamics equation

$$\partial_t \omega_i + u_j \partial_j \omega_i - \omega_j \partial_j u_i - \widetilde{B}^g_j \partial_j \widetilde{J}^g_i + \widetilde{J}^g_j \partial_j \widetilde{B}^g_i - \nu \partial_j \partial_j \omega_i = 0, \tag{15}$$

where the sum of the four inner terms could be succinctly written in terms of Lie derivatives as $\mathcal{L}_u \omega - \mathcal{L}_{\widetilde{B}^g} \widetilde{J}^g$.

Due to the vortical character of turbulence, performing scaling analysis in the context of vorticity dynamics is the natural choice [7, 21]. In particular, we define the gravitational interaction parameter N^g, that measures gravitomagnetic effects in units of fluid inertia, $N^g = |\mathcal{L}_{\widetilde{B}^g} \widetilde{J}^g| / |\mathcal{L}_u \omega|$. Let us employ dimensional analysis to scale N^g, by examining a typical term in its definition, for example, $\widetilde{J}^g \partial \widetilde{B}^g / u \partial \omega$, and by defining ℓ_g and ℓ_h, the length scales typical of gravitational and hydrodynamic *gradients* correspondingly. Employing the governing equation for $\widetilde{B}^g$, we have $\widehat{\widetilde{B}^g} \sim \sqrt{\beta\rho}\widehat{\omega}\ell_g^2$, where $\widehat{\widetilde{B}^g}$ is a typical gravitomagnetic field magnitude and $\widehat{\omega}$ is a typical vorticity magnitude. Similarly, $\widehat{\widetilde{J}^g} \sim \beta\rho\widehat{u}/\sqrt{\beta\rho} \sim \sqrt{\beta\rho}\widehat{u}$, where $\widehat{u}$ is a typical velocity magnitude, and so $\widetilde{J}^g \partial \widetilde{B}^g \sim (\sqrt{\beta\rho}\widehat{u}/\ell_g)\sqrt{\beta\rho}\widehat{\omega}\ell_g^2$. On the other hand, $u\partial\omega \sim \widehat{u}\widehat{\omega}/\ell_h$, hence $N^g \sim \beta\rho\ell_h\ell_g$. The validity of this scaling is *demonstrated* in

the results section, where the value of N^g is shown to be in direct correspondence with the strength of gravitomagnetic effects on the flow.

It is important to clarify a few key points. Our scaling analysis, which is later shown to describe the numerical results very well, indicates that we do not need high speeds or strong fields to achieve significant effects on turbulence structure. Instead, we can augment the strength of GMF effects on the flow by increasing the size of the system. This result is a direct consequence of the way that gravity couples to flow vorticity. As discussed before, increasing the size of the system does not increase gravitoelectric effects, since E^g is very slowly varying on the system scale. Hence, we ensure the validity of the flat spacetime condition and post-Newtonian approximation. In other words, we do not claim that we obtain strong gravitomagnetic fields. Instead, we obtain strong GMF effects on turbulence structure, which is a different thing altogether. The situation is similar to that of terrestrial flows which are significantly affected by gravity, without this implying departures from the weak field, slow motion limits. These, we demonstrate later on via a series of numerical solutions of the turbulent GMHD equations. The other important parameter, that measures *nonlinear*, inertial, nonequilibrium processes in units of linear, viscous, nonequilibrium processes, is the Reynolds number, $Re = u'l/\nu$, where $u' = \sqrt{\langle u^2 \rangle}$ is the turbulent intensity of u, and l is the *integral length scale* which measures the size of turbulence-energy containing eddies [1]. It is important to note that, similarly with N^g, the definition of Re number indicates that high velocities *are not necessary* for the onset of turbulence. One only needs an adequately large system, to have high values for both N^g and Re numbers, hence, both turbulence and relatively strong gravitomagnetic field effects altering its structure.

NUMERICAL METHODS AND FINITE PRECISION ARITHMETIC

The model is solved with a staggered grid, fractional step, projection, finite volume, numerical method [7, 22]. Spatial partial derivatives are computed with second order accurate schemes. An implicit, second order accurate in time, Crank-Nicolson (CN) scheme is applied to the viscous/diffusion terms, whilst all other terms evolve via an explicit, third order accurate in time, low storage Runge-Kutta (RK) method. The CN scheme is incorporated

into the RK steps and the method becomes a hybrid RK/CN scheme. Flow incompressibility is enforced by projecting the velocity onto the space of divergence-free vector fields (Hodge projection). The time-steps are adaptive, limited by the Courant-Friedrichs-Lewy condition, and resolve the viscous processes in the flow. The gravitomagnetic field is computed self-consistently from the corresponding Poisson equation with Fast Fourier Transform methods. The algorithmic approximation of this numerical analysis adds finite-precision arithmetic round-off error to analytic truncation error: within the floating point number set $\mathbb{F}$ employed in the computations, the distance between 1 and the next larger floating point number is $\epsilon_m = 0.222 \times 10^{-15}$. The smallest and largest numbers that can be represented are 2.2×10^{-308} and 1.8×10^{308} correspondingly. The computational domain is a periodic cube (flat torus) discretized into 256^3 grid cells. In all calculations, we employ *periodic boundary conditions*, and the dissipation scales are fully resolved.

HOMOGENEOUS, ISOTROPIC, GRAVITOMAGNETIC TURBULENCE

First, we set up a steady-state, homogeneous, isotropic, pure NS turbulence with Taylor Reynolds number $Re_\lambda = u'\lambda/\nu \approx 80$, and then we switch on gravity. Here, λ is the Taylor microscale $\lambda^2 = 15\nu\langle u_i u_i\rangle/3\epsilon$, where $\epsilon = 2\nu\langle S_{ij}S_{ij}\rangle$ is the rate of turbulence energy dissipation, and $S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ is the *strain rate tensor* [1]. To achieve steady state, viscous dissipation action is compensated by Lundgren's linear forcing [23, 24]. λ is a good representation of the length scale where most of turbulent strain takes place, so it is a good candidate for ℓ_h, since the latter has to characterize vortex stretching. On the other hand, the solution of the Poisson equation for $\widetilde{B}^g(x) = -\frac{\sqrt{\beta\rho}}{4\pi}\int \frac{\omega(x')\mathrm{d}x'}{|x'-x|}$ indicates that $\widetilde{B}^g$ is formed by the weighted sum of neighbouring-vorticity contributions, so it could be expected that gravitomagnetic gradients would scale with the largest correlation length in the system, which in turbulence case is the integral length scale l, that measures the size of large eddies. Hence, $N^g = l\lambda\beta\rho$. This intuition is fully supported by the computational solutions. Some important time scales are the viscous time scale $\tau_d = (\Delta x)^2/(6\nu)$, where Δx is the computational grid size, the gravitational time scale $\tau_g^{N^g} = l/(\widetilde{B}^g)'$, where $(\widetilde{B}^g)' = \sqrt{\langle(\widetilde{B}^g)^2\rangle}$ is the turbulent intensity of $\widetilde{B}^g$, and the time scale of energy containing motions, $\tau_e^{N^g} = l/u'$.

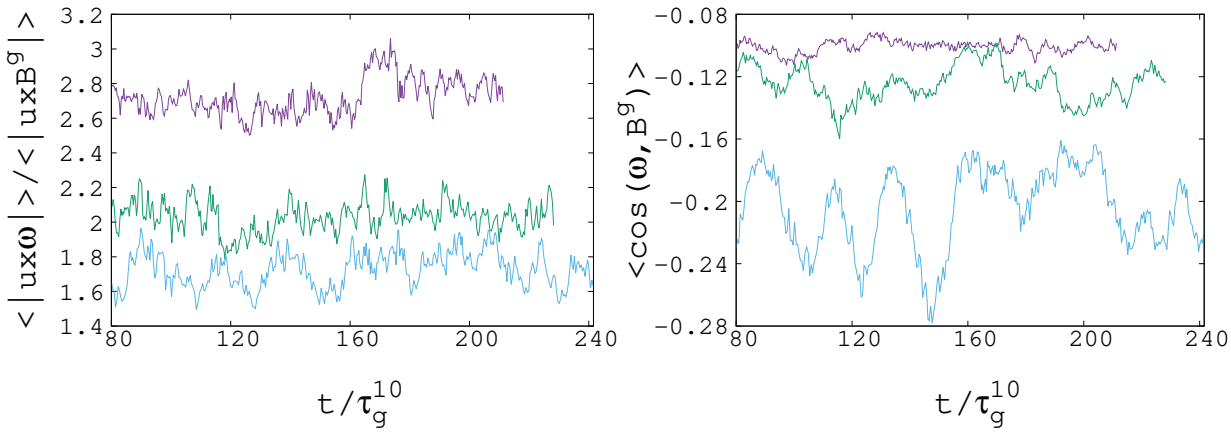

FIG. 1. Steady state turbulence averages for N^g=[10,20,40]. Left: Ratio of Lamb vector magnitude over gravitational effect magnitude. Right: Cosine of angle between ω and B^g. N^g increases from top to bottom curves. The shown time period corresponds to several dozens of $\tau_e^{N^g}$.

To achieve a statistical steady state of turbulent gravity-matter interactions, we continue compensating viscous dissipation after enabling gravity. Starting from $N^g = 0.01$ and increasing the strength of gravitational effects, we find important effects close to $N^g = 10$, so we performed three extended-time calculations for $N^g = [10, 20, 40]$. Notably, N^g is indicative of the coupling-strength between matter and gravity, which is a dynamical quality not to be confused with the strength of the resulting gravitational fields. The relations between the various time scales are $\tau_g^{10} = 9.8\tau_d$, $\tau_g^{10} = 0.41\tau_e^{10}$, $\tau_g^{20} = 1.52\tau_g^{10}$, and $\tau_g^{40} = 2.4\tau_g^{10}$, $\tau_e^{20} = 1.37\tau_e^{10}$, $\tau_e^{40} = 1.81\tau_e^{10}$. All shown results correspond to steady states which are established after a transient period whose duration is *inversely proportional* to the coupling strength. In full support of our scalings, the solutions inform that, as N^g increases, so does the strength of gravitational effects relative to Lamb-force effects (Fig.1, left). Moreover, B^g and ω tend to be *antiparallel*, and the intensity of this effect is proportional to N^g (Fig.1, right). This indicates that gravity tends to *neutralise* the Lamb force, hence, to reduce the vortical complexity of turbulence, and make it *less nonlinear*. Indeed, the *highest* entstrophy and turbulence intensity values are associated with the *weakest* coupling (Fig.2, left and centre). In addition, due to reduction of the gravitational source (vorticity) levels, strong coupling leads to smaller values for the mean square of $\widetilde{B}^g$ (Fig.2, right).

Of key importance are the velocity E_k^u, vorticity E_k^ω, and gravitational-field $E_k^{\widetilde{B}^g}$ spectra.

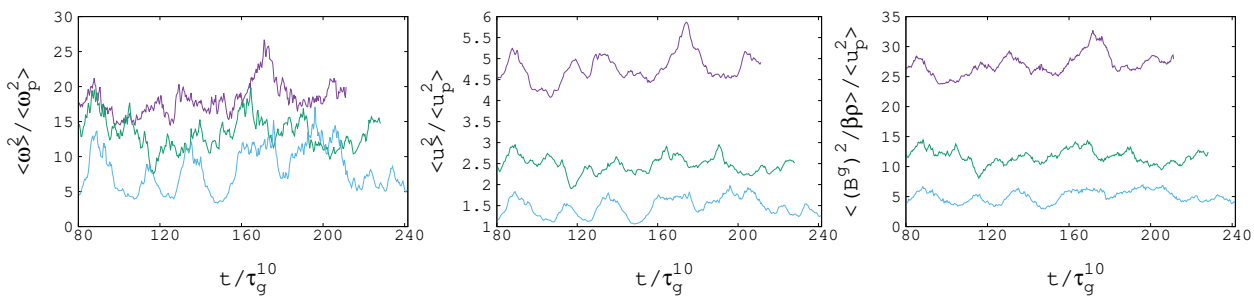

FIG. 2. Steady state turbulence averages for N^g=[10,20,40]. From left to right: vorticity, velocity and gravitomagnetic field mean squares. N^g increases from top to bottom curves. The shown time period corresponds to several dozens of $\tau_e^{N^g}$. The shown quantities are scaled with the corresponding *pure* turbulence values.

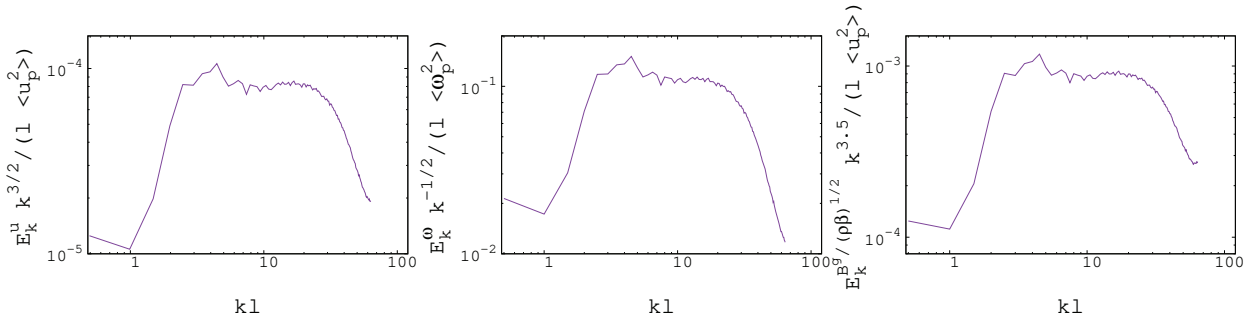

FIG. 3. Steady state turbulence for $N^g = 20$. From left to right: velocity, vorticity and gravitomagnetic field compensated spectra, exemplifying the corresponding $k^{-3/2}$, $k^{1/2}$ and $k^{-3.5}$ scalings.

Before enabling gravity, we verified the Kolmogorov scalings, $E_k^u \sim k^{-5/3}$ and $E_k^\omega \sim k^{1/3}$ for our pure turbulence calculation. Gravity induces new, GMHD scalings: $E_k^u \sim k^{-3/2}$, $E_k^\omega \sim k^{1/2}$, and $E_k^{\widetilde{B}^g} \sim k^{-3.5}$ (Fig.3). The spectra shown are for $N^g = 20$, since $N^g = 10$ turbulence is not equally representative of strong coupling effects, and $N^g = 40$ turbulence is not equally well resolved (albeit still satisfactorily). The computed scalings can be understood via dimensional and exact analysis arguments. For E_k^u, we can use a dimensional analytic relation due to Kraichnan [6] for the rate of kinetic energy dissipation (equal to the energy-flux in wavenumber space) $\epsilon = \tau_k (E_k^u)^2 k^4$. Here, τ_k is a time scale, which is associated with wavenumber k, and is characteristic of energy transfer processes from k to higher wavenumbers. In Kolmogorov turbulence, $\tau_k = (E_k^M k^3)^{-1/2}$, i.e., the time scale of local in k space turbulence eddies, which is indicative of their lifetime or the time-interval over which fluid motions of length scale k^{-1} remain correlated. However, for a self-gravitating fluid with high N^g, τ_k ought to scale with $\widetilde{B}^g$, since the latter would deter-

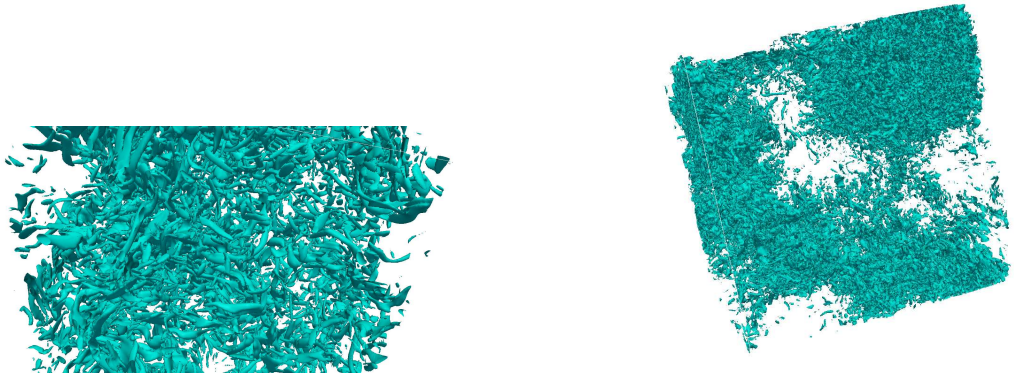

FIG. 4. Vorticity isosurfaces for pure turbulence (left) and gravitational turbulence for $N^g = 20$ (right).

mine the decorrelation time via the gravitational forcing term in the NS equation. Hence, $\tau_k = (\widetilde{B}^g k)^{-1}$, and inserting this into the expression for ϵ, we obtain $E_k^u = (\epsilon \widetilde{B}^g)^{1/2} k^{-3/2}$, which agrees with the computational result. Noting that E_k^u units are $[E_k^u] = L^3 T^{-2}$, and that $[E_k^\omega] = LT^{-2}$, we obtain $E_k^\omega \sim E_k^u k^2$, hence, $E_k^\omega \sim k^{1/2}$, which also agrees with the results. Next, it is straightforward to predict $\widetilde{B}^g$ scaling: take the Fourier transform of $\widetilde{B}^g$ equation, and square to obtain: $k^4 |\widehat{\widetilde{B}^g}(k)|^2 = \beta\rho |\widehat{\omega}(k)|^2$. Employing the definition $\int_0^\infty E_k^\omega dk = \iiint |\widehat{\omega}(k)|^2 d^3k$, we deduce $|\widehat{\omega}(k)|^2 \sim k^{-3/2}$. This gives $|\widehat{\widetilde{B}^g}(k)|^2 \sim k^{-11/2}$, and via the definition $\int_0^\infty E_k^{\widetilde{B}^g} dk = \iiint |\widehat{\widetilde{B}^g}(k)|^2 d^3k$, we obtain $E_k^{\widetilde{B}^g} \sim k^{-3.5}$, which accurately matches the computed value.

The morphologies of vorticity and gravitomagnetic fields present both surprising and deducible features (Figs.4-5). Pure turbulence vorticity isosurfaces drawn at 15% of maximum value (Fig.4, left) indicate the standard, predominantly linear, NS structures, spreading homogeneously over the whole domain. However, in the gravitational case, the vortex size is smaller, and, even when isosurfaces spanning the whole range of vorticity levels are shown simultaneously (Fig.4, right), large volumes devoid of any vorticity are observed. This is explained by the tendency of the gravitomagnetic field to neutralize the Lamb force, and suppress turbulence. On the other hand, due to the vorticity source in the $\widetilde{B}^g$ equation, $\widetilde{B}^g$ and ω coexist in space, and, since the former involves nonlocal space averages of the latter, its isosurfaces form extended structures spanning the system's domain (Fig.5). Here, vorticity isosurfaces are drawn at 15% of the maximum value, and magnetic field isosurfaces

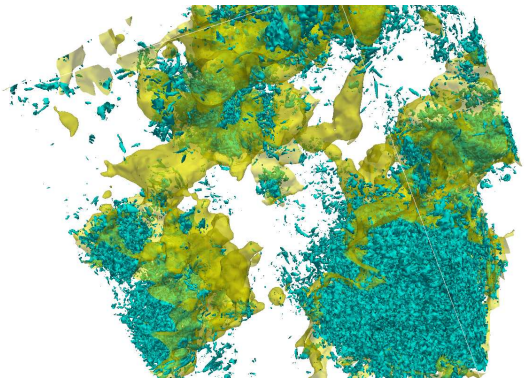

FIG. 5. Gravitational field (yellow) and vorticity (turquoise) isosurfaces in gravitational turbulence for $N^g = 20$.

at 55% of maximum value. The $\widetilde{B}^g$ morphology remains the same from the smallest isolevels up to 70% of maximum value, and its structures become localized only when, at sufficiently high field values, the corresponding high-vorticity source has very small support.

CONCLUSION

Rotating matter (vortices) is the source of gravitomagnetism, and the most complicated vortical patterns are associated with turbulent flows. In other words, the sources of GMF are the very structures that dominate turbulence physics. Hence, the combination of these two phenomena is a problem of significant conceptual importance. In general, turbulence induced gravitomagnetic phenomena have three nonlinearity sources: (1) the nonlinearity of general relativistic gravity, (2) the nonlinearity of the Navier-Stokes equation, and (3) the standard nonlinearity of interacting field theory emanating from matter-field coupling. In this work, we have shown that it is physically meaningful to couple GMF and turbulence in the context of the post-Newtonian approximation, and in this way, to avoid the first nonlinearity. We have derived the relevant gravitomagnetohydrodynamic equations and, by scaling them, we identified parameter ranges where significant GMF effects on turbulence structure are expected.

By means of numerical computations, we have shown, that, within the identified parameter ranges, the third nonlinearity tames the second. Indeed, turbulent enstrophy is intensified

by vortex stretching and peaks at high wavenumbers, as a result of Lamb-force driven turbulence kinetic energy cascade. But as the cascade intensifies small-scale enstrophy, the latter generates a strong gravitomagnetic field whose action on the fluid counterbalances the Lamb force driving the cascade. In other words, *gravitomagnetism tends to linearize and damp out turbulence*. At statistical steady state, gravitomagnetic field levels are consistent with the amount of turbulence enstrophy intensification which is allowed by the degree of flow nonlinearity reduction due to these gravitomagnetic field levels. The latter are inversely proportional to the strength of field-matter coupling. Moreover, our results indicate that GMHD turbulence presents, new, non-Kolmogorov scalings, that we explained with dimensional and analytical arguments.

It is important to specify the limits of applicability of our investigation. Gravitomagnetism is a force of relativistic origin, hence our computation includes by default post-Newtonian features. In fact, since the gravitoelectric field is not dynamically important in our formulation, all observed effects on fluid motion in our results are of relativistic origin. On the other hand, similarly with electric magnetohydrodynamics, the model is formulated in Galilean spacetime, hence it is not proper relativistic fluid dynamics in Minkowski spacetimes. It can only be applied to fluids moving with velocities much smaller than the speed of light. Based on our scaling of the GMHD equations, we have indicated that, even with Galilean relativity, we can retain important relativistic effects in such slow flows by increasing the size of the system. This was not evident before a close examination of the way gravitomagnetism couples with fluid motion. There is a similarity here with the effects of electrodynamic magnetic fields on fluid motion captured by the traditional, Galilean invariant MHD equations.

Future elaboration of GMHD vortex dynamics and detailed probing of gravity mediated vortex interactions in turbulence could be informative. Gravitomagnetic effects would generate novel coherent-structure formation mechanisms, and alter strain-rate tensor related statistics. Finally, the employment of advanced geometrical [25] and topological [26, 27] methods for the characterization of GMF and vorticity field structures would help indicate in a more precise, quantitative manner the differences between pure and gravitomagnetic turbulence flow patterns.

The results could, possibly, benefit astrophysical and cosmological investigations. Indeed, in *neutron stars*, gravitomagnetic effects were shown to affect precession rates by about 10% [28], and there is need to improve the quality of employed phenomenological *turbulence models* in *relativistic hydrodynamics* investigations of their differential rotation and mergers [29, 30]. Moreover, the provided insight into the structure of gravitomagnetic turbulence could inform the physics of zero-curvature cosmologies [31, 32]. In particular, the feature of our flat-torus shaped computational domain could help investigations concerning the large scale topology of the Universe [33] (pg 14). Finally, although the results assume weak gravitational fields, they could provide hints for strong field physics as is the case with other applications of post-Newtonian theory [34].

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