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Peridynamic modelling of higher order functionally graded plates

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Abstract
With the development of advanced manufacturing technologies, the importance of functionally graded materials is growing as they are advantageous over widely used traditional composites. In this paper, we present a novel peridynamic model for higher order functional graded plates for various thicknesses. Moreover, the formulation eliminates the usage of shear correction factors. Euler–Lagrange equations and Taylor's expansion are utilised to derive the governing equations. The capability of the developed peridynamic model is demonstrated by considering several benchmark problems. In these benchmark cases simply supported, clamped and mixed boundary conditions are also tested. The peridynamic results are also verified by their finite element analysis counterparts. According to the comparison, peridynamic and finite element analysis results agree very well with each other.

Keywords
Peridynamics, functionally graded, higher order plate theory, non-local, Euler–Lagrange formulation

1. Introduction

Utilisation of functionally graded materials (FGMs) is increasing due to the advancement in manufacturing technologies and their advantages with respect to fibre-reinforced composites. As the properties of FGMs vary continuously along the material, the structure does not suffer from the discontinuity problem, as can be seen in fibre-reinforced composite materials, which can lead to the delamination problem. In the literature, there have been several studies related to the analysis of plates made of FGMs. Among these, Qian and Batra [1] performed thermoelastic analysis of thick plates made of

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FGMs. Their approach is based on the meshless Petrov–Galerkin method and higher order shear and normal deformable plate theory. Ferreira et al. [2] utilised a meshless method to determine static deformations of simply supported plates made of FGMs based on a third-order shear deformation theory. Cheng and Batra [3] used first-order and third-order shear deformation theories to examine plates made of FGMs. Batra [4] used the principle of virtual work and higher-order shear and normal deformable theory to examine functionally graded incompressible linear elastic plates. Belabed et al. [5] introduced a higher-order shear and normal deformation theory for functionally graded plates by considering hyperbolic variation of displacements along the thickness. Their formulation did not require the usage of shear correction factors. Xiang and Kang [6] examined the bending behaviour of plates made of FGMs. They introduced an nth-order shear deformation theory by using a meshless method based on a global collocation technique. Zhang et al. [7] introduced a semi-analytical approach to analyse in-plane deformation of functionally graded plates. Their methodology is based on scaled boundary finite element formulation.

In this paper, a novel higher-order plate (HOP) formulation for FGMs is developed by using an alternative formulation, peridynamics. A shear correction factor is not required for the current formulation. On the other hand, the developed peridynamic formulation is suitable for plates with various thicknesses, and does not require a shear correction factor. Silling [8] introduced the peridynamics method to overcome the problem of having discontinuities in the classical continuum mechanics formulation. This way, if cracks exist in the structure, the peridynamics formulation becomes suitable. Moreover, due to its length-scale parameter, horizon (interaction domain), it is possible to represent non-local deformation behaviour. Peridynamics has been used to investigate many different materials ranging from metals [9,10], composite materials [11–16], ceramics [17–19], concrete [20–23] and functionally graded materials [25–27]. Peridynamics is also suitable to perform multiphysics analysis [28,29]. Several beam [30–33] and plate formulations [34–37] are also available in the peridynamic framework. A review of benchmark experiments for the validation of peridynamic models is given in Diehl et al. [38]. However, higher-order plate formulation in peridynamic formulation for functionally graded materials is currently not available in the literature; this is the key motivation of this study. In this concept, Euler–Lagrange equation and Taylor's expansion are utilised to derive the equations of motion. To validate the developed methodology, several benchmark problems are considered. The peridynamic predictions are compared with finite element analysis (FEA) solutions.

2. Classical HOP formulation for FGMs

According to HOP formulation, the displacements of a material particle can be represented as a function of material particles located on the mid-plane of the plate. If the mid-plane is located on xy plane, the displacements can be represented by using Taylor's expansion as

$$u = u|_{z=0} + z\frac{\partial u}{\partial z}\bigg|_{z=0} + \frac{1}{2}z^2\frac{\partial^2 u}{\partial z^2}\bigg|_{z=0} + \frac{1}{3!}z^3\frac{\partial^3 u}{\partial z^3}\bigg|_{z=0} + \cdots \tag{1a}$$

$$v = v|_{z=0} + z\frac{\partial v}{\partial z}\bigg|_{z=0} + \frac{1}{2}z^2\frac{\partial^2 v}{\partial z^2}\bigg|_{z=0} + \frac{1}{3!}z^3\frac{\partial^3 v}{\partial z^3}\bigg|_{z=0} + \cdots \tag{1b}$$

$$w = w|_{z=0} + z\frac{\partial w}{\partial z}\bigg|_{z=0} + \frac{1}{2}z^2\frac{\partial^2 w}{\partial z^2}\bigg|_{z=0} + \frac{1}{3!}z^3\frac{\partial^3 w}{\partial z^3}\bigg|_{z=0} + \cdots \tag{1c}$$

where z represents the transverse coordinate along the thickness direction. Ignoring higher-order terms, the expressions given in equations (1a) to (1c) can be rewritten as

$$u(x,y,z) = \bar{u}(x,y) + z\theta_x(x,y) + z^2u^*(x,y) + z^3\theta_x^*(x,y) \tag{2a}$$

$$v(x,y,z) = \bar{v}(x,y) + z\theta_y(x,y) + z^2v^*(x,y) + z^3\theta_y^*(x,y) \tag{2b}$$

$$w(x,y,z) = \bar{w}(x,y) + z\theta_z(x,y) + z^2w^*(x,y) \tag{2c}$$

where $\bar{u}$ and $\bar{v}$ denote the membrane displacements, $\bar{w}$ denotes the transverse displacement of mid-plane, respectively, and θ_x, θ_y, θ_z, u^*, v^*, w^*, θ_x^* and θ_y^* are arisen out of the Taylor expansion and introduced as eight additional independent variables. The parameter θ_x represents the rotation of mid-plane about y-axis. On the other hand, the parameter θ_y represents the rotation of mid-plane about negative x-axis.

Next, the strain components can be obtained by using equations (2a) to (2c) as

$$\varepsilon_{11} = \frac{\partial \bar{u}_1}{\partial x_1} + z\frac{\partial \theta_1}{\partial x_1} + z^2\frac{\partial u_1^*}{\partial x_1} + z^3\frac{\partial \theta_1^*}{\partial x_1} \tag{3a}$$

$$\varepsilon_{22} = \frac{\partial \bar{u}_2}{\partial x_2} + z\frac{\partial \theta_2}{\partial x_2} + z^2\frac{\partial u_2^*}{\partial x_2} + z^3\frac{\partial \theta_2^*}{\partial x_2} \tag{3b}$$

$$\varepsilon_{33} = \theta_z + 2zw^* \tag{3c}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left[\left(\frac{\partial \bar{u}_1}{\partial x_2} + \frac{\partial \bar{u}_2}{\partial x_1}\right) + z\left(\frac{\partial \theta_1}{\partial x_2} + \frac{\partial \theta_2}{\partial x_1}\right) + z^2\left(\frac{\partial u_1^*}{\partial x_2} + \frac{\partial u_2^*}{\partial x_1}\right) + z^3\left(\frac{\partial \theta_1^*}{\partial x_2} + \frac{\partial \theta_2^*}{\partial x_1}\right)\right] \tag{3d}$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left[\left(\theta_1 + \frac{\partial \bar{w}}{\partial x_1}\right) + z\left(2u_1^* + \frac{\partial \theta_z}{\partial x_1}\right) + z^2\left(3\theta_1^* + \frac{\partial w^*}{\partial x_1}\right)\right] \tag{3e}$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left[\left(\theta_2 + \frac{\partial \bar{w}}{\partial x_2}\right) + z\left(2u_2^* + \frac{\partial \theta_z}{\partial x_2}\right) + z^2\left(3\theta_2^* + \frac{\partial w^*}{\partial x_2}\right)\right] \tag{3f}$$

The strain-displacement relationships given in equations (3a) to (3f) can be written by using indicial notation as

$$\begin{aligned}\varepsilon_{ij} = {} & \frac{1}{2}\left[\left(\frac{\partial \bar{u}_I}{\partial x_J} + \frac{\partial \bar{u}_J}{\partial x_I}\right) + z\left(\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\right) + z^2\left(\frac{\partial u_I^*}{\partial x_J} + \frac{\partial u_J^*}{\partial x_I}\right) + z^3\left(\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_J^*}{\partial x_I}\right)\right]\delta_{Ii}\delta_{Jj} \\ & + \frac{1}{2}\left[\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right) + z\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + z^2\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right]\left(\delta_{Ii}\delta_{3j} + \delta_{3i}\delta_{Ij}\right) + (\theta_z + 2zw^*)\delta_{3i}\delta_{3j}\end{aligned} \tag{4}$$

where the lower case parameters i and j range from 1 to 3 and upper case parameters I and J range from 1 to 2.

Assuming the material is planar isotropic and obeying the 3D constitutive relation, the stress components can be given by

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{5}$$

where is C_{ijkl} the elastic modulus tensor which can be described as

$$C_{ijkl} = G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \frac{2G\nu}{1-2\nu}\delta_{ij}\delta_{kl} \tag{6}$$

where G represents the shear modulus and ν represents the Poisson's ratio. These parameters are considered to vary through the thickness: $G = G(z)$ and $\nu = \nu(z)$.

Substituting equation (6) into equation (5) yields:

$$\sigma_{ij} = 2G\varepsilon_{ij} + \frac{2G\nu}{1-2\nu}\varepsilon_{ll}\delta_{ij} \tag{7}$$

The average volumetric strain energy density (SED) for an elastic plate can be written as

$$\mathrm{W} = \frac{1}{2h}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}\varepsilon_{ij}dz \tag{8}$$

where the parameter h denotes the thickness of the plate.

Plugging in equations (7) and (4) into equation (8) results in

$$
\begin{aligned}
W = \frac{1}{h} \Bigg\{ & \frac{A_0}{2}\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_I}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_J}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \bar{u}_J}{\partial x_J} + \left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right) + 2(\theta_z)^2\right] \\
& + A_1\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + \left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + 4\theta_z w^*\right] \\
& + \frac{A_2}{2}\left[\frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + 2\left(\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial u_J^*}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right)\right. \\
& \left. + \left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right)\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + 2\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right) + 8(w^*)^2\right] \\
& + A_3\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \bar{u}_J}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J} + \left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right] \\
& + \frac{A_4}{2}\left[\frac{\partial u_I^*}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial u_I^*}{\partial x_J}\frac{\partial u_J^*}{\partial x_I} + \frac{\partial u_I^*}{\partial x_I}\frac{\partial u_J^*}{\partial x_J} + 2\left(\frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right) + \left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right] \\
& + A_5\left(\frac{\partial u_I^*}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial u_J^*}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial u_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right) + \frac{A_6}{2}\left(\frac{\partial \theta_I^*}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_I^*}{\partial x_J}\frac{\partial \theta_J^*}{\partial x_I} + \frac{\partial \theta_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right)\Bigg\} \\
& + \frac{1}{h}\left[B_0\left(2\frac{\partial \bar{u}_I}{\partial x_I}\theta_z + (\theta_z)^2\right) + 2B_1\left(2\frac{\partial \bar{u}_I}{\partial x_I}w^* + \frac{\partial \theta_I}{\partial x_I}\theta_z + 2\theta_z w^*\right) + 2B_2\left(2\frac{\partial \theta_I}{\partial x_I}w^* + \frac{\partial u_I^*}{\partial x_I}\theta_z + 2(w^*)^2\right)\right. \\
& \left. + 2B_3\left(2\frac{\partial u_I^*}{\partial x_I}w^* + \frac{\partial \theta_I^*}{\partial x_I}\theta_z\right) + 4B_4\frac{\partial \theta_I^*}{\partial x_I}w^*\right] \\
& + \frac{1}{h}\Bigg\{\frac{C_0}{2}\left(\frac{\partial \bar{u}_I}{\partial x_I}\right)^2 + C_1\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + \frac{C_2}{2}\left[\left(\frac{\partial \theta_I}{\partial x_I}\right)^2 + 2\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right] + C_3\left(\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right) \\
& + \frac{C_4}{2}\left[\left(\frac{\partial u_I^*}{\partial x_I}\right)^2 + 2\frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_I}\right] + C_5\frac{\partial u_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{C_6}{2}\left(\frac{\partial \theta_I^*}{\partial x_I}\right)^2\Bigg\}
\end{aligned}
\tag{9}
$$

where the material property coefficients A, B can C are given as

$$
\left\{\begin{matrix} A_0 & A_1 & A_2 & A_3 \\ & A_4 & A_5 & A_6 \end{matrix}\right\} = \left\{\begin{matrix} \int_{-\frac{h}{2}}^{-\frac{h}{2}} G dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} Gz dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} Gz^2 dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} Gz^3 dz \\ & \int_{-\frac{h}{2}}^{-\frac{h}{2}} Gz^4 dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} Gz^5 dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} Gz^6 dz \end{matrix}\right\}
\tag{10a}
$$

$$
\left\{\begin{matrix} B_0 & B_1 & B_2 & B_3 \\ & B_4 & B_5 & B_6 \end{matrix}\right\} = \left\{\begin{matrix} \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^2}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^3}{1-2\nu} dz \\ & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^4}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^5}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^6}{1-2\nu} dz \end{matrix}\right\}
\tag{10b}
$$

$$
\left\{\begin{matrix} C_0 & C_1 & C_2 & C_3 \\ & C_4 & C_5 & C_6 \end{matrix}\right\} = \left\{\begin{matrix} \int_{-\frac{h}{2}}^{-\frac{h}{2}} G\frac{4\nu-1}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} G\frac{4\nu-1}{1-2\nu} z dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} G\frac{4\nu-1}{1-2\nu} z^2 dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} G\frac{4\nu-1}{1-2\nu} z^3 dz \\ & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^4}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^5}{1-2\nu} dz & \int_{-\frac{h}{2}}^{-\frac{h}{2}} \frac{G\nu z^6}{1-2\nu} dz \end{matrix}\right\}
\tag{10c}
$$

3. Peridynamic HOP Formulation for FGMs

Peridynamics is a new continuum mechanics formulation which was introduced by Silling [8] to overcome the limitations of classical continuum mechanics. Peridynamic equations of motion are in integro-differential equation form instead of the partial differential equations used in classical continuum mechanics. According to peridynamics, each material point is not only interacting with the material points located in its immediate vicinity, but is also interacting with other material points in a non-local manner located within a region of finite radius named as 'horizon', H. The original peridynamic equations of motion for a material point located at $\mathbf{x}$ can be written as [8]

$$\int_H \mathbf{f}(\mathbf{x}' - \mathbf{x}, \mathbf{u}' - \mathbf{u}, t)dV' + \mathbf{b}(\mathbf{x}, t) = \rho\ddot{\mathbf{u}}(\mathbf{x}, t) \tag{11}$$

where $\mathbf{x}'$ is the location of the material point at which the material point located at $\mathbf{x}$ is interacting, $\mathbf{f}$ is the peridynamic force between the material points located at $\mathbf{x}$ and $\mathbf{x}'$, $\ddot{\mathbf{u}}$ is the acceleration of the material point located at $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{u}'$ are the displacements of the material points located at $\mathbf{x}$ and $\mathbf{x}'$, respectively, $\mathbf{b}$ is the body force of the material point located at $\mathbf{x}$, dV' is the incremental volume of the material point located at $\mathbf{x}'$, ρ is the density and t is time. The analytical solution of equation (11) is usually not possible. Instead, numerical approaches, especially the meshless method, are widely utilised. Therefore, equation (11) can be written in a discretised form for a particular material point k as

$$\sum\nolimits_j \mathbf{f}_{(k)(j)} V_{(j)} + \mathbf{b}_{(k)} = \rho\ddot{\mathbf{u}}_{(k)} \tag{12}$$

where $\mathbf{f}_{(k)(j)}$ is the peridynamic force between the material point k and the material point j located inside the horizon of the material point k, $\mathbf{b}_{(k)}$ is the body force the material point k, $\ddot{\mathbf{u}}_{(k)}$ is the acceleration of the material point k. Note that the summation symbol in equation (12) refers to all material points inside the horizon of the material point k.

The peridynamic form of equations of motion can be obtained by utilizing Euler–Lagrange's formulation as:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_{(k)}} - \frac{\partial L}{\partial \boldsymbol{q}_{(k)}} = 0 \tag{13}$$

where the parameter L represents the Lagrangian which can be represented by using kinetic energy,T and total potential energy,U as $L = T - U$. The parameter $\boldsymbol{q}$ is the generalized displacement vector which is defined as

$$\mathbf{q} = \begin{pmatrix} \bar{u}_I & \bar{w} & \theta_I & \theta_z & u_I^* & w^* & \theta_I^* \end{pmatrix}^T \tag{14}$$

The areal kinetic energy density of the body, $\bar{T}$ is expressed as

$$\bar{T} = \frac{1}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\left(\dot{u}^2 + \dot{v}^2 + \dot{w}^2\right)dz \tag{15}$$

Substituting equation (2) into equation (15) results in

$$\bar{T} = \rho h\left\{\left(\dot{\bar{u}}_I\dot{\bar{u}}_I + \dot{\bar{w}}^2\right) + \frac{h^2}{12}\left(\dot{\theta}_I\dot{\theta}_I + 2\dot{\bar{u}}_I\dot{u}_I^* + 2\dot{\bar{w}}\dot{w}^* + \dot{\theta}_z^2\right) + \frac{h^4}{80}\left[\dot{u}_I^*\dot{u}_I^* + 2\dot{\theta}_I\dot{\theta}_I^* + (\dot{w}^*)^2\right] + \frac{h^6}{448}\dot{\theta}_I^*\dot{\theta}_I^*\right\} \tag{16}$$

By integrating equation (15) over the whole mid-plane, total kinetic energy is obtained as

$$T=\int_A \bar{T}dA=\frac{1}{2}\int_A \rho\left\{h\left(\dot{\bar{u}}_I\dot{\bar{u}}_I+\dot{\bar{w}}^2\right)+\frac{h^3}{12}\left(\dot{\theta}_I\dot{\theta}_I+2\dot{\bar{u}}_I\dot{u}_I^*+2\dot{\bar{w}}\dot{w}^*+\dot{\theta}_z^2\right)+\frac{h^5}{80}\left[\dot{u}_I^*\dot{u}_I^*+2\dot{\theta}_I\dot{\theta}_I^*+(\dot{w}^*)^2\right]+\frac{h^7}{448}\dot{\theta}_I^*\dot{\theta}_I^*\right\}dA \tag{17a}$$

Equation (17a) can be written in discretized form as

$$\begin{aligned}T=\frac{1}{2}\sum\nolimits_k \rho_{(k)}\Bigg\{&\left(\dot{\bar{u}}_I^{(k)}\dot{\bar{u}}_I^{(k)}+\dot{\bar{w}}_{(k)}^2\right)+\frac{h^2}{12}\left[\dot{\theta}_I^{(k)}\dot{\theta}_I^{(k)}+2\dot{\bar{u}}_I^{(k)}\dot{u}_I^{*(k)}+2\dot{\bar{w}}_{(k)}\dot{w}_{(k)}^*+\left(\dot{\theta}_z^{(k)}\right)^2\right]\\&+\frac{h^4}{80}\left[\dot{u}_I^{*(k)}\dot{u}_I^{*(k)}+2\dot{\theta}_I^{(k)}\dot{\theta}_I^{*(k)}+\left(\dot{w}_{(k)}^*\right)^2\right]+\frac{h^6}{448}\dot{\theta}_I^{*(k)}\dot{\theta}_I^{*(k)}\Bigg\}V_{(k)}\end{aligned} \tag{17b}$$

where $V_{(k)}$ denotes the volume of material particle k and the summation symbol in equation (17b) refers to all material points inside the body.

In peridynamics theory, the SED at each material particle has non-local character. Therefore, SED at particle k depends on the deformation fields of particle k and other material particles in its interaction domain. Therefore, in general the SED function can be demonstrated as

$$W_{(k)}=W_{(k)}(\boldsymbol{q}_{(k)},\boldsymbol{q}_{(1^k)},\boldsymbol{q}_{(2^k)},\boldsymbol{q}_{(3^k)},\ \ldots) \tag{18}$$

in which $i=1, 2, 3, \ldots$ and i^k indicates the ith family member of k.

The total potential energy, U, is the summation of strain energy of all material particles and energy of external loads. Therefore, U can be demonstrated as

$$U=\sum\nolimits_k \left(W_{(k)}(\boldsymbol{q}_{(k)},\boldsymbol{q}_{(1^k)},\boldsymbol{q}_{(2^k)},\ \ldots)-\boldsymbol{b}_{(k)}\boldsymbol{q}_{(k)}\right)V_{(k)} \tag{19}$$

where $\boldsymbol{b}$ represents external force per volume

$$\mathbf{b}=\begin{pmatrix}b_I & b_z & \hat{b}_I & 0 & 0 & 0 & 0\end{pmatrix}^T \tag{20}$$

with b_I, b_z and $\hat{b}_I$ representing external force per volume with respect to in-plane forces, transverse forces and bending moments, respectively.

Inserting equations (17b) and (19) into equation (13) yields:

$$\rho_{(k)}\begin{Bmatrix}\ddot{\bar{u}}_K^{(k)}+\frac{h^2}{12}\ddot{u}_K^{*(k)}\\ \ddot{\bar{w}}_{(k)}+\frac{h^2}{12}\ddot{w}_{(k)}^*\\ \frac{h^2}{12}\ddot{\theta}_K^{(k)}+\frac{h^4}{80}\ddot{\theta}_K^{*(k)}\\ \frac{h^2}{12}\ddot{\theta}_z^{(k)}\\ \frac{h^2}{12}\ddot{\bar{u}}_K^{(k)}+\frac{h^4}{80}\ddot{u}_K^{*(k)}\\ \frac{h^2}{12}\ddot{\bar{w}}_{(k)}+\frac{h^4}{80}\ddot{w}_{(k)}^*\\ \frac{h^4}{80}\ddot{\theta}_K^{(k)}+\frac{h^6}{448}\ddot{\theta}_K^{*(k)}\end{Bmatrix}=-\begin{Bmatrix}\frac{\partial W_{(k)}}{\partial\bar{u}_K^{(k)}}+\sum_j\frac{\partial W_{(j)}}{\partial\bar{u}_K^{(k)}}\frac{V_{(j)}}{V_{(k)}}\\ \frac{\partial W_{(k)}}{\partial\bar{w}_{(k)}}+\sum_j\frac{\partial W_{(j)}}{\partial\bar{w}_{(k)}}\frac{V_{(j)}}{V_{(k)}}\\ \frac{\partial W_{(k)}}{\partial\theta_K^{(k)}}+\sum_j\frac{\partial W_{(j)}}{\partial\theta_K^{(k)}}\frac{V_{(j)}}{V_{(k)}}\\ \frac{\partial W_{(k)}}{\partial\theta_z^{(k)}}+\sum_j\frac{\partial W_{(j)}}{\partial\theta_z^{(k)}}\frac{V_{(j)}}{V_{(k)}}\\ \frac{\partial W_{(k)}}{\partial u_K^{*(k)}}+\sum_j\frac{\partial W_{(j)}}{\partial u_K^{*(k)}}\frac{V_{(j)}}{V_{(k)}}\\ \frac{\partial W_{(k)}}{\partial w_{(k)}^*}+\sum_j\frac{\partial W_{(j)}}{\partial w_{(k)}^*}\frac{V_{(j)}}{V_{(k)}}\\ \frac{\partial W_{(k)}}{\partial\theta_K^{*(k)}}+\sum_j\frac{\partial W_{(j)}}{\partial\theta_K^{*(k)}}\frac{V_{(j)}}{V_{(k)}}\end{Bmatrix}+\begin{Bmatrix}b_K^{(k)}\\ b_z^{(k)}\\ 0\\ 0\\ \hat{b}_K^{(k)}\\ 0\\ 0\end{Bmatrix} \tag{21}$$

The derivation of the peridynamic form of SED given in equation (9) is presented in Appendix A. Substituting equations (A21), (A23) and (A29) into equation (21) results in the governing equations for higher-order functionally graded plates as

$$\begin{aligned}
&\rho_{(k)}\left(\ddot{\bar{u}}_I^{(k)}+\frac{h^2}{12}\ddot{u}_I^{*(k)}\right)\\
&=\frac{24}{\pi\delta^3h^2}\Bigg[A_0\sum_j\frac{\bar{u}_J^{(j)}-\bar{u}_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+B_0\sum_j\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}n_I^{(j)(k)}V_{(j)}+A_1\sum_j\frac{\theta_J^{(j)}-\theta_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+\\
&B_1\sum_j\frac{w_{(j)}^*+w_{(k)}^*}{2}n_I^{(j)(k)}V_{(j)}+A_2\sum_j\frac{u_J^{*(j)}-u_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+A_3\sum_j\frac{\theta_J^{*(j)}-\theta_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\Bigg]\\
&+\frac{2}{\pi\delta^2h^2}\Bigg[C_0\sum_j\frac{\Theta_{(k)}+\Theta_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_1\sum_j\frac{\Phi_{(k)}+\Phi_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_2\sum_j\frac{\Theta_{(k)}^*+\Theta_{(j)}^*}{\xi_{(j)(k)}}n_I^{(j)(k)}\\
&+C_3\sum_j\frac{\Phi_{(k)}^*+\Phi_{(j)}^*}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}\Bigg]+b_I^{(k)}
\end{aligned}\tag{22a}$$

$$\begin{aligned}
&\rho_{(k)}\left(\frac{h^2}{12}\ddot{\theta}_I^{(k)}+\frac{h^4}{80}\ddot{\theta}_I^{*(k)}\right)\\
&=\frac{24}{\pi\delta^3h^2}\Bigg[A_1\sum_j\frac{\bar{u}_J^{(j)}-\bar{u}_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+B_1\sum_j\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}n_I^{(j)(k)}V_{(j)}+A_2\sum_j\frac{\theta_J^{(j)}-\theta_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\\
&+B_2\sum_j\frac{w_{(j)}^*+w_{(k)}^*}{2}n_I^{(j)(k)}V_{(j)}+A_3\sum_j\frac{u_J^{*(j)}-u_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+A_4\sum_j\frac{\theta_J^{*(j)}-\theta_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\Bigg]\\
&-\frac{3}{\pi\delta^3h^2}\Bigg[A_0\sum_j\left(\bar{w}_{(j)}-\bar{w}_{(k)}+\frac{\theta_J^{(j)}+\theta_J^{(k)}}{2}\xi_{(j)(k)}n_J^{(j)(k)}\right)n_I^{(j)(k)}V_{(j)}+\\
&A_1\sum_j\left[\theta_z^{(j)}-\theta_z^{(k)}+\left(u_J^{*(j)}+u_J^{*(k)}\right)\xi_{(j)(k)}n_J^{(j)(k)}\right]n_I^{(j)(k)}V_{(j)}\\
&+A_2\sum_j\left[w_{(j)}^*-w_{(k)}^*+\frac{3}{2}\left(\theta_J^{*(j)}+\theta_J^{*(k)}\right)\xi_{(j)(k)}n_J^{(j)(k)}\right]n_I^{(j)(k)}V_{(j)}\Bigg]\\
&+\frac{2}{\pi\delta^2h^2}\Bigg\{C_1\sum_j\frac{\Theta_{(k)}+\Theta_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_2\sum_j\frac{\Phi_{(k)}+\Phi_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}\\
&+C_3\sum_j\frac{\Theta_{(k)}^*+\Theta_{(j)}^*}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_4\sum_j\frac{\Phi_{(k)}^*+\Phi_{(j)}^*}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}\Bigg\}+\hat{b}_I^{(k)}
\end{aligned}\tag{22b}$$

$$\begin{aligned}
&\rho_{(k)}\left(\frac{h^2}{12}\ddot{\bar{u}}_I^{(k)}+\frac{h^4}{80}\ddot{u}_I^{*(k)}\right)=\\
&\frac{24}{\pi\delta^3h^2}\left[\begin{aligned}&A_2\sum_j\frac{\bar{u}_J^{(j)}-\bar{u}_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+B_2\sum_j\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}n_I^{(j)(k)}V_{(j)}+A_3\sum_j\frac{\theta_J^{(j)}-\theta_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\\&+B_3\sum_j\frac{w_{(j)}^*+w_{(k)}^*}{2}n_I^{(j)(k)}V_{(j)}+A_4\sum_j\frac{u_J^{*(j)}-u_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+A_5\sum_j\frac{\theta_J^{*(j)}-\theta_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\end{aligned}\right]\\
&-\frac{6}{\pi\delta^3h^2}\left\{\begin{aligned}&A_1\sum_j\left(\bar{w}_{(j)}-\bar{w}_{(k)}+\frac{\theta_J^{(j)}+\theta_J^{(k)}}{2}\xi_{(j)(k)}n_J^{(j)(k)}\right)n_I^{(j)(k)}V_{(j)}+A_2\sum_j\left[\theta_z^{(j)}-\theta_z^{(k)}+\left(u_J^{*(j)}+u_J^{*(k)}\right)\xi_{(j)(k)}n_J^{(j)(k)}\right]n_I^{(j)(k)}V_{(j)}\\&+A_3\sum_j\left[w_{(j)}^*-w_{(k)}^*+\frac{3}{2}\left(\theta_J^{*(j)}+\theta_J^{*(k)}\right)\xi_{(j)(k)}n_J^{(j)(k)}\right]n_I^{(j)(k)}V_{(j)}\end{aligned}\right\}\\
&+\frac{2}{\pi\delta^2h^2}\left(C_2\sum_j\frac{\Theta_{(k)}+\Theta_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_3\sum_j\frac{\Phi_{(k)}+\Phi_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_4\sum_j\frac{\Theta_{(k)}^*+\Theta_{(j)}^*}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_5\sum_j\frac{\Phi_{(k)}^*+\Phi_{(j)}^*}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}\right)
\end{aligned}\tag{22c}$$

$$\begin{aligned}
&\rho_{(k)}\left(\frac{h^4}{80}\ddot{\theta}_I^{(k)}+\frac{h^6}{448}\ddot{\theta}_I^{*(k)}\right)= \\
&\frac{24}{\pi\delta^3h^2}\left[A_3\sum_j\frac{\bar{u}_J^{(j)}-\bar{u}_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+B_3\sum_j\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}n_I^{(j)(k)}V_{(j)}+A_4\sum_j\frac{\theta_J^{(j)}-\theta_J^{(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\right. \\
&\left.+B_4\sum_j\frac{w^*_{(j)}+w^*_{(k)}}{2}n_I^{(j)(k)}V_{(j)}+A_5\sum_j\frac{u_J^{*(j)}-u_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}+A_6\sum_j\frac{\theta_J^{*(j)}-\theta_J^{*(k)}}{\xi_{(j)(k)}}n_J^{(j)(k)}n_I^{(j)(k)}V_{(j)}\right] \\
&-\frac{8}{\pi\delta^3h^2}\left[A_2\sum_j\left(\bar{w}_{(j)}-\bar{w}_{(k)}+\frac{\theta_J^{(j)}+\theta_J^{(k)}}{2}\xi_{(j)(k)}n_J^{(j)(k)}\right)n_I^{(j)(k)}V_{(j)}\right. \\
&+A_3\sum_j\left(\theta_z^{(j)}-\theta_z^{(k)}+\left(u_J^{*(j)}+u_J^{*(k)}\right)\xi_{(j)(k)}n_J^{(j)(k)}\right)n_I^{(j)(k)}V_{(j)} \\
&\left.+A_4\sum_j\left(w^*_{(j)}-w^*_{(k)}+\frac{3}{2}\left(\theta_J^{*(j)}+\theta_J^{*(k)}\right)\xi_{(j)(k)}n_J^{(j)(k)}\right)n_I^{(j)(k)}V_{(j)}\right] \\
&+\frac{2}{\pi\delta^2h^2}\left\{\begin{array}{l}C_3\sum_j\frac{\Theta_{(k)}+\Theta_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_4\sum_j\frac{\Phi_{(k)}+\Phi_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+C_5\sum_j\frac{\Theta^*_{(k)}+\Theta^*_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}+ \\ C_6\sum_j\frac{\Phi^*_{(k)}+\Phi^*_{(j)}}{\xi_{(j)(k)}}n_I^{(j)(k)}V_{(j)}\end{array}\right\}
\end{aligned} \tag{22d}$$

$$\begin{aligned}
&\rho_{(k)}\left(\ddot{\bar{w}}_{(k)}+\frac{h^2}{12}\ddot{w}^*_{(k)}\right)=\frac{6}{\pi\delta^3h^2} \\
&\left[A_0\sum_j\left(\frac{\bar{w}_{(j)}-\bar{w}_{(k)}}{\xi_{(j)(k)}}+\frac{\theta_I^{(j)}+\theta_I^{(k)}}{2}n_I^{(j)(k)}\right)V_{(j)}+A_1\sum_j\frac{\theta_z^{(j)}-\theta_z^{(k)}}{\xi_{(j)(k)}}+\left(u_I^{*(j)}+u_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}\right. \\
&\left.+A_2\sum_j\frac{w^*_{(j)}-w^*_{(k)}}{\xi_{(j)(k)}}+\frac{3}{2}\left(\theta_I^{*(j)}+\theta_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}\right]+b_z^{(k)}
\end{aligned} \tag{22e}$$

$$\begin{aligned}
&\rho_{(k)}\frac{h^2}{12}\ddot{\theta}_z^{(k)}=\frac{6}{\pi\delta^3h^2} \\
&\left\{A_1\sum_j\left(\frac{\bar{w}_{(j)}-\bar{w}_{(k)}}{\xi_{(j)(k)}}+\frac{\theta_I^{(j)}+\theta_I^{(k)}}{2}n_I^{(j)(k)}-\frac{w^*_{(j)}+w^*_{(k)}}{2}\xi_{(j)(k)}\right)V_{(j)}+A_2\sum_j\frac{\theta_z^{(j)}-\theta_z^{(k)}}{\xi_{(j)(k)}}+\left(u_I^{*(j)}+u_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}\right. \\
&+A_3\sum_j\frac{w^*_{(j)}-w^*_{(k)}}{\xi_{(j)(k)}}+\frac{3}{2}\left(\theta_I^{*(j)}+\theta_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}-A_0\sum_j\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}\xi_{(j)(k)}V_{(j)} \\
&-B_0\sum_j\left[\left(\bar{u}_I^{(j)}-\bar{u}_I^{(k)}\right)n_I^{(j)(k)}+\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}\xi_{(j)(k)}\right] \\
&V_{(j)}-B_1\sum_j\left[\left(\theta_I^{(j)}-\theta_I^{(k)}\right)n_I^{(j)(k)}+\frac{w^*_{(j)}+w^*_{(k)}}{2}\xi_{(j)(k)}\right]V_{(j)}-B_2\sum_j\left(u_I^{*(j)}-u_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)} \\
&\left.-B_3\sum_j\left(\theta_I^{*(j)}-\theta_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}\right\}
\end{aligned} \tag{22f}$$

$$\begin{aligned}
&\rho_{(k)}\left(\frac{h^2}{12}\ddot{\bar{w}}_{(k)}+\frac{h^4}{80}\ddot{w}^*_{(k)}\right)=\frac{6}{\pi\delta^3h^2} \\
&\left\{A_2\sum_j\left[\frac{\bar{w}_{(j)}-\bar{w}_{(k)}}{\xi_{(j)(k)}}+\frac{\theta_I^{(j)}+\theta_I^{(k)}}{2}n_I^{(j)(k)}-\left(w^*_{(j)}+w^*_{(k)}\right)\xi_{(j)(k)}\right]V_{(j)}\right. \\
&+A_3\sum_j\left[\frac{\theta_z^{(j)}-\theta_z^{(k)}}{\xi_{(j)(k)}}+\left(u_I^{*(j)}+u_I^{*(k)}\right)n_I^{(j)(k)}\right]V_{(j)}+A_4\sum_j\left[\frac{w^*_{(j)}-w^*_{(k)}}{\xi_{(j)(k)}}+\frac{3}{2}\left(\theta_I^{*(j)}+\theta_I^{*(k)}\right)n_I^{(j)(k)}\right]V_{(j)} \\
&-A_1\sum_j\frac{\theta_z^{(j)}+\theta_z^{(k)}}{2}\xi_{(j)(k)}V_{(j)}-2\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{G\nu z}{1-2\nu}dz\sum_j\left[\left(\bar{u}_I^{(j)}-\bar{u}_I^{(k)}\right)n_I^{(j)(k)}+\frac{\theta_z^{(j)}+\theta_z^{(k)}}{4}\xi_{(j)(k)}\right]V_{(j)} \\
&-2B_2\sum_j\left[\left(\theta_I^{(j)}-\theta_I^{(k)}\right)n_I^{(j)(k)}+\frac{w^*_{(j)}+w^*_{(k)}}{2}\xi_{(j)(k)}\right]V_{(j)} \\
&\left.-2B_3\sum_j\left(u_I^{*(j)}-u_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}-2B_4\sum_j\left(\theta_I^{*(j)}-\theta_I^{*(k)}\right)n_I^{(j)(k)}V_{(j)}\right\}
\end{aligned} \tag{22g}$$

where the notations Θ, Φ, Θ^* and Φ^* appeared in equations (22a) to (22d) are defined as

$$\Theta_{(k)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\bar{u}_J^{(i^k)} - \bar{u}_J^{(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \tag{23a}$$

$$\Theta_{(j)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\bar{u}_J^{(i^j)} - \bar{u}_J^{(j)}}{\xi_{(i^j)(j)}} n_J^{(i^j)(j)} V_{(i^j)} \tag{23b}$$

$$\Theta^*_{(k)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\bar{u}_J^{*(i^k)} - \bar{u}_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \tag{23c}$$

$$\Theta^*_{(j)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\bar{u}_J^{*(i^j)} - \bar{u}_J^{*(j)}}{\xi_{(i^j)(j)}} n_J^{(i^j)(j)} V_{(i^j)} \tag{23d}$$

$$\Phi_{(k)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\theta_J^{(i^k)} - \theta_J^{(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \tag{23e}$$

$$\Phi_{(j)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\theta_J^{(i^j)} - \theta_J^{(j)}}{\xi_{(i^j)(j)}} n_J^{(i^j)(j)} V_{(i^j)} \tag{23f}$$

$$\Phi^*_{(k)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\theta_J^{*(i^k)} - \theta_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \tag{23g}$$

$$\Phi^*_{(j)} = \frac{2}{\pi\delta^2 h}\sum_i \frac{\theta_J^{*(i^j)} - \theta_J^{*(j)}}{\xi_{(i^j)(j)}} n_J^{(i^j)(j)} V_{(i^j)} \tag{23h}$$

Note that if Poisson's ratio is $\nu(z) = \frac{1}{4}$, the coefficient C_0 to C_6 will equal to zero and equations (22a) to (22g) will reduce to bond-based peridynamics formulation.

4. Demonstrations of peridynamic HOP Formulation for FGMs

To demonstrate the developed HOP formulation for FGMs, the following functionally graded material properties are used:

$$E(z) = \left(E_{Top} - E_{Bottom}\right)\frac{z}{h} + \frac{1}{2}\left(E_{Top} + E_{Bottom}\right) \quad (GPa) \tag{24a}$$

$$\mu(z) = \frac{E(z)}{2(1+0.3)} \tag{24b}$$

$$\nu(z) = 0.3 \tag{24c}$$

where h denotes the plate thickness. The length, thickness and width of the plate are specified as $L = 1$m, $h = 0.15$m and $W = 0.1$ m, respectively. The Young's modulus at the top of the plate is $E_{Top} = 200\ GPa$ and at the bottom of the plate is $E_{Bottom} = 100\ GPa$. For the peridynamic model, the discretization size is specified as $\Delta x = \frac{1}{51}$ m. The interaction domain size is $\delta = 3\Delta x$.

As reference solution, the corresponding finite element (FE) models are created in ANSYS by using SHELL181 and SOLID185 elements, respectively. The FE SHELL element model is generated with 50×50 elements and divided into 30 layers. On the other hand, the SOLID element model is obtained with 50×50 elements throughout the xy plane and 30 elements along the thickness. To obtain the functionally graded character, materials properties are assigned to the layers and elements through the

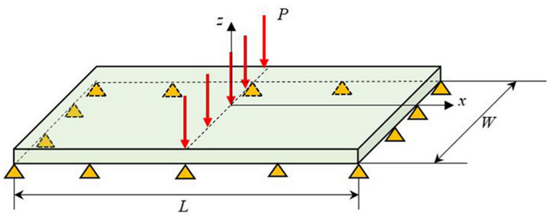

Figure 1. Simply supported FGM plate exposed to transverse line loading.

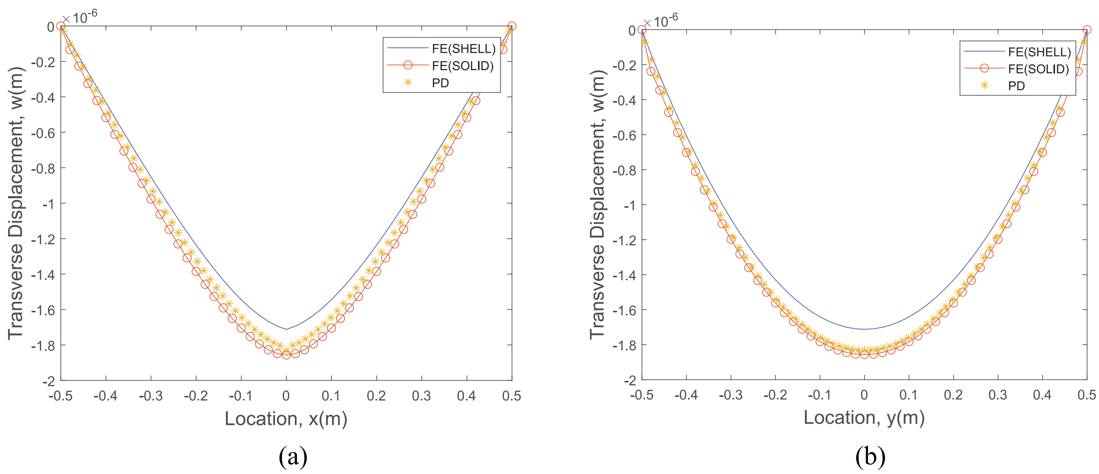

Figure 2. Variation of transverse displacement alongside (a) *x*-axis, (b) *y*-axis.

thickness direction. The Young's modulus varies gradually over the thickness from the first layer $E_1 = 101.67\ GPa$ to the last layer $E_{30} = 198.33\ GPa$. The Young's modulus of nth layer can be expressed as $E_n = 100 \times [1 + (n - 1/2)/30] GPa$. The Poisson's ratio, $\nu = 0.3$, is applied for both models in ANSYS.

4.1. Transverse loading acting on a simply supported FGM plate

In this first numerical case a FGM plate with simply supports is considered, as shown in Figure 1. The FGM plate is exposed to a line load of $p = 10000\text{N/m}$ along line at $x = 0$.

To impose simply supported boundary conditions, a fictitious region with a size of $2 \times$ horizon is introduced outside of the actual domain.

As shown in Figure 2, peridynamics predictions are compared against FEA solutions obtained by using ANSYS Solid and Shell elements. Note that ANSYS Shell element is based on classical Mindlin plate theory. Compared with ANSYS Shell element, the peridynamic higher-order formulation performs better and represents a similar deformation variation with respect to ANSYS Solid element.

4.2. Transverse loading acting on a clamped FGM plate

In this second case a FGM plate with clamped supports is considered, as shown in Figure 3. The FGM plate is exposed to a line load of $p = 10000\text{N/m}$ along line at $x = 0$.

To impose clamped boundary conditions, a fictitious region with a size of $2 \times$ horizon is introduced.

As shown in Figure 4, peridynamic predictions are compared against FEA solutions obtained by using ANSYS Solid and Shell elements. Compared with ANSYS Shell element, the peridynamic higher-order formulation performs better and represents a similar deformation variation with respect to ANSYS Solid element.

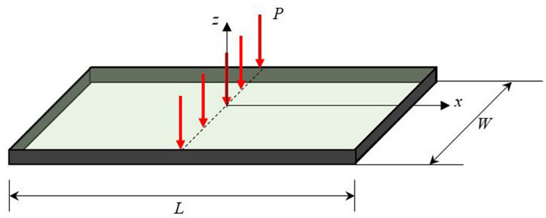

Figure 3. Clamped FGM plate exposed to transverse line loading.

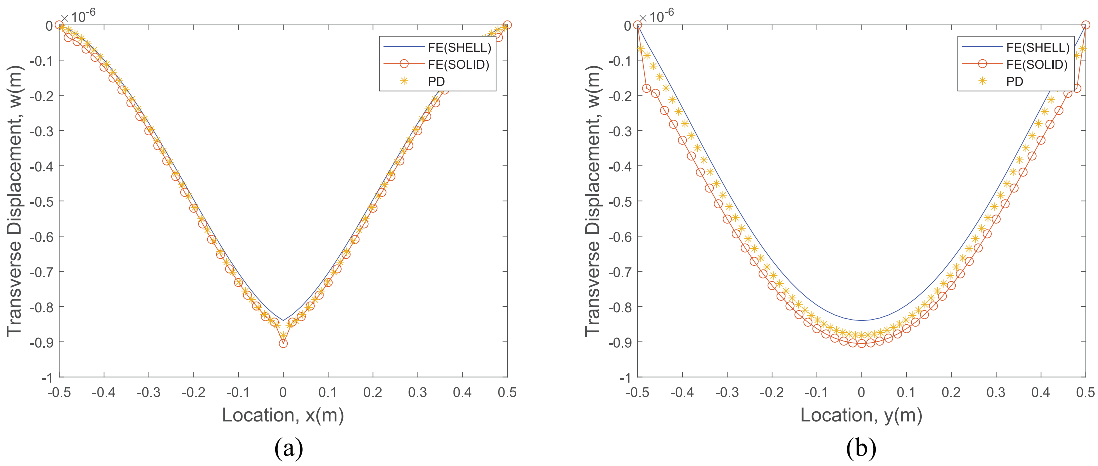

Figure 4. Variation of transverse displacement alongside (a) *x*-axis, (b) *y*-axis.

4.3. Transverse loading acting on a clamped – simply support – clamped – simply support (mixed boundary conditions) FGM plate

In this third case, a FGM plate with clamped – simply support – clamped – simply support is considered, as shown in Figure 5. The FGM plate is exposed to a line load of $p = 10000\text{N/m}$ along line at $x = 0$.

As shown in Figure 6, peridynamic predictions are compared against FEA solutions obtained by using ANSYS Solid and Shell elements. Compared with ANSYS Shell element, the peridynamic higher-order formulation performs better and represents a similar deformation variation with respect to ANSYS Solid element.

4.4. Transverse loading acting on a clamped – clamped – simply support – simply support FGM plate

In this fourth case a FGM plate with Clamped – Clamped – Simply Support – Simply Support is considered, as shown in Figure 7. The FGM plate is exposed to a line load of $p = 10000\text{N/m}$ along line at $x = 0$.

As shown in Figure 8, peridynamic predictions are compared against FEA solutions obtained by using ANSYS Solid and Shell elements. Compared with ANSYS Shell element, the peridynamic higher-order formulation performs better and represents a similar deformation variation with respect to ANSYS Solid element.

5. Conclusions of peridynamic HOP formulation for FGMs

In this paper, a novel HOP Formulation for FGMs in a peridynamic framework is developed. Plates with various thicknesses can be analysed with the developed formulation. Moreover, the usage of shear correction factor is eliminated. Taylor's expansion and Euler–Lagrange formulation are used for obtaining governing equations. Four different numerical examples are considered for simply supported, clamped and mixed boundary conditions. In all these cases, variation of transverse deflections is obtained from both peridynamics and FEAs by using both solid and shell elements. In all cases, it is shown that newly developed peridynamic HOP formulation performed better with respect to Shell element that uses Mindlin plate formulation, which demonstrates the capability and accuracy of the newly developed formulation.

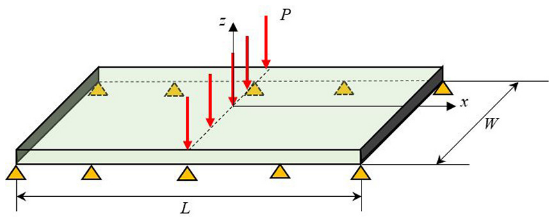

Figure 5. FGM plate with Clamped – Simply Support – Clamped – Simply Support exposed to transverse line loading.

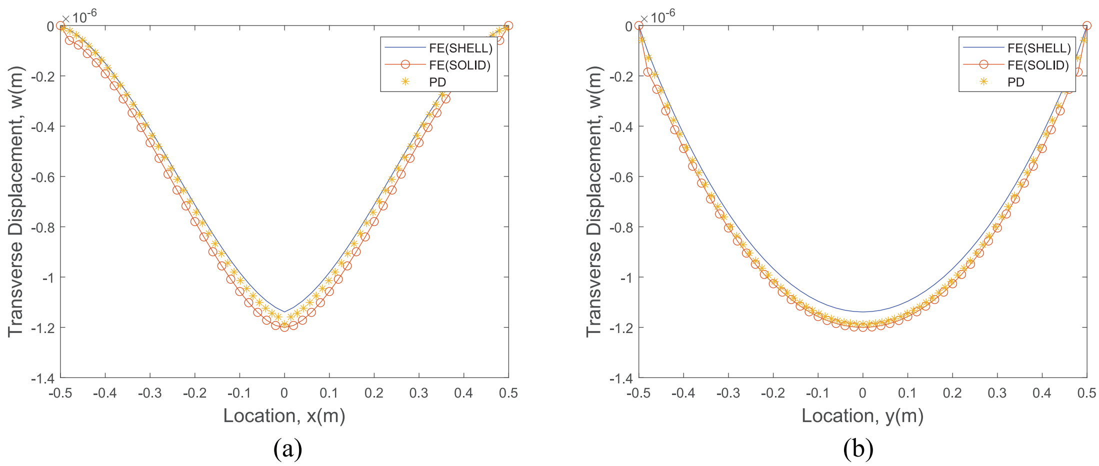

Figure 6. Variation of transverse displacement alongside (a) *x*-axis, (b) *y*-axis.

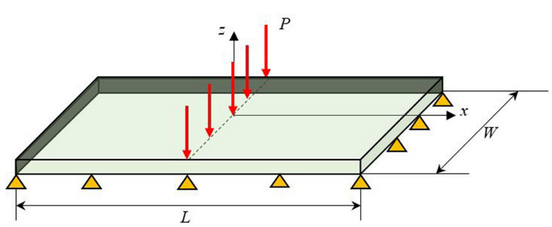

Figure 7. FGM plate with Clamped – Clamped – Simply Support – Simply Support exposed to transverse line loading.

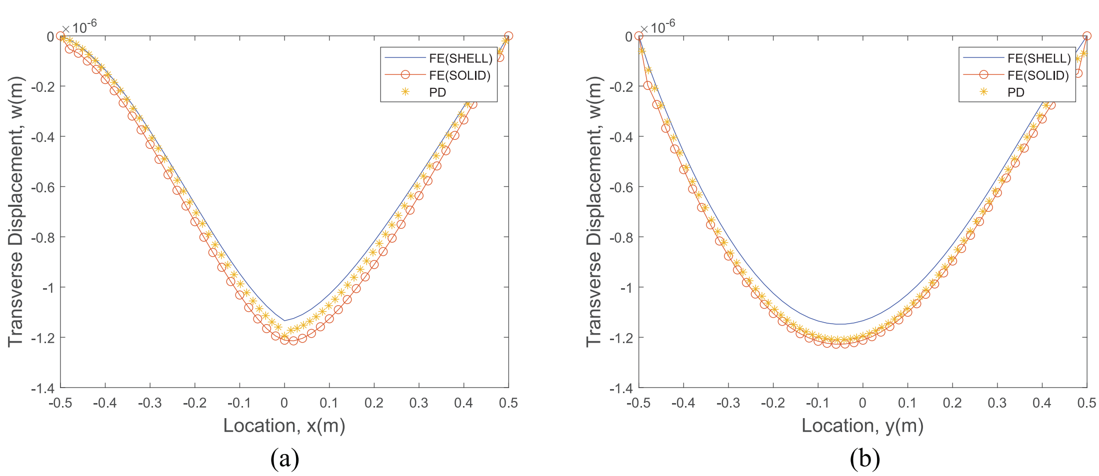

Figure 8. Variation of transverse displacement alongside (a) *x*-axis, (b) *y*-axis.

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Appendix

As explained in section 2, the classical strain energy function of the higher-order functionally graded plate can be written as

$$
\begin{aligned}
W = \frac{1}{h}\Bigg\{ & \frac{A_0}{2}\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_I}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_J}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \bar{u}_J}{\partial x_J} + \left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right) + 2(\theta_z)^2\right] \\
& + A_1\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + \left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + 4\theta_z w^*\right] \\
& + \frac{A_2}{2}\left[\frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + 2\left(\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial u_J^*}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right)\right. \\
& \left. + \left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right)\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + 2\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right) + 8(w^*)^2\right] \\
& + A_3\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \bar{u}_J}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J} + \left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right] \\
& + \frac{A_4}{2}\left[\frac{\partial u_I^*}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial u_I^*}{\partial x_J}\frac{\partial u_J^*}{\partial x_I} + \frac{\partial u_I^*}{\partial x_I}\frac{\partial u_J^*}{\partial x_J} + 2\left(\frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right) + \left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right] \\
& + A_5\left(\frac{\partial u_I^*}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial u_J^*}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial u_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right) + \frac{A_6}{2}\left(\frac{\partial \theta_I^*}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_I^*}{\partial x_J}\frac{\partial \theta_J^*}{\partial x_I} + \frac{\partial \theta_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right)\Bigg\} \\
& + \frac{1}{h}\left[B_0\left(2\frac{\partial \bar{u}_I}{\partial x_I}\theta_z + (\theta_z)^2\right) + 2B_1\left(2\frac{\partial \bar{u}_I}{\partial x_I}w^* + \frac{\partial \theta_I}{\partial x_I}\theta_z + 2\theta_z w^*\right) + 2B_2\left(2\frac{\partial \theta_I}{\partial x_I}w^* + \frac{\partial u_I^*}{\partial x_I}\theta_z + 2(w^*)^2\right)\right. \\
& \left. + 2B_3\left(2\frac{\partial u_I^*}{\partial x_I}w^* + \frac{\partial \theta_I^*}{\partial x_I}\theta_z\right) + 4B_4\frac{\partial \theta_I^*}{\partial x_I}w^*\right] \\
& + \frac{1}{h}\Bigg\{\frac{C_0}{2}\left(\frac{\partial \bar{u}_I}{\partial x_I}\right)^2 + C_1\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + \frac{C_2}{2}\left[\left(\frac{\partial \theta_I}{\partial x_I}\right)^2 + 2\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right] \\
& + C_3\left(\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right) + \frac{C_4}{2}\left[\left(\frac{\partial u_I^*}{\partial x_I}\right)^2 + 2\frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_I}\right] + C_5\frac{\partial u_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{C_6}{2}\left(\frac{\partial \theta_I^*}{\partial x_I}\right)^2\Bigg\}
\end{aligned}
\tag{A1a}
$$

Equation (A1a) can be separated into three parts for simplification as

$$W = W_I + W_{II} + W_{III} \tag{A1b}$$

where

$$\begin{aligned}
W_I = \frac{1}{h}\Bigg\{ & \frac{A_0}{2}\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_I}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_J}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \bar{u}_J}{\partial x_J} + \left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right) + 2(\theta_z)^2\right] \\
& + A_1\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + \left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + 4\theta_z w^*\right] \\
& + \frac{A_2}{2}\left[\frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + 2\left(\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial u_J^*}{\partial x_I} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right)\right. \\
& \left. + \left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right)\left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right) + 2\left(\theta_I + \frac{\partial \bar{w}}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right) + 8(w^*)^2\right] \\
& + A_3\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \bar{u}_J}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J} + \left(2u_I^* + \frac{\partial \theta_z}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right] \\
& + \frac{A_4}{2}\left[\frac{\partial u_I^*}{\partial x_J}\frac{\partial u_I^*}{\partial x_J} + \frac{\partial u_I^*}{\partial x_J}\frac{\partial u_J^*}{\partial x_I} + \frac{\partial u_I^*}{\partial x_I}\frac{\partial u_J^*}{\partial x_J} + 2\left(\frac{\partial \theta_I}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_J}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right)\right. \\
& \left. + \left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\left(3\theta_I^* + \frac{\partial w^*}{\partial x_I}\right)\right] + A_5\left(\frac{\partial u_I^*}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial u_J^*}{\partial x_I}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial u_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right) \\
& + \frac{A_6}{2}\left(\frac{\partial \theta_I^*}{\partial x_J}\frac{\partial \theta_I^*}{\partial x_J} + \frac{\partial \theta_I^*}{\partial x_J}\frac{\partial \theta_J^*}{\partial x_I} + \frac{\partial \theta_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right)\Bigg\}
\end{aligned} \tag{A2a}$$

$$\begin{aligned}
W_{II} = + \frac{1}{h}\Bigg[& B_0\left(2\frac{\partial \bar{u}_I}{\partial x_I}\theta_z + (\theta_z)^2\right) + 2B_1\left(2\frac{\partial \bar{u}_I}{\partial x_I}w^* + \frac{\partial \theta_I}{\partial x_I}\theta_z + 2\theta_z w^*\right) \\
& + 2B_2\left(2\frac{\partial \theta_I}{\partial x_I}w^* + \frac{\partial u_I^*}{\partial x_I}\theta_z + 2(w^*)^2\right) + 2B_3\left(2\frac{\partial u_I^*}{\partial x_I}w^* + \frac{\partial \theta_I^*}{\partial x_I}\theta_z\right) + 4B_4\frac{\partial \theta_I^*}{\partial x_I}w^*\Bigg]
\end{aligned} \tag{A2b}$$

and

$$\begin{aligned}
W_{III} = + \frac{1}{h}\Bigg\{ & \frac{C_0}{2}\left(\frac{\partial \bar{u}_I}{\partial x_I}\right)^2 + C_1\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J} + \frac{C_2}{2}\left[\left(\frac{\partial \theta_I}{\partial x_I}\right)^2 + 2\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right] \\
& + C_3\left(\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{\partial \theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right) + \frac{C_4}{2}\left[\left(\frac{\partial u_I^*}{\partial x_I}\right)^2 + 2\frac{\partial \theta_I}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J}\right] + C_5\frac{\partial u_I^*}{\partial x_I}\frac{\partial \theta_J^*}{\partial x_J} + \frac{C_6}{2}\left(\frac{\partial \theta_I^*}{\partial x_I}\right)^2\Bigg\}
\end{aligned} \tag{A2c}$$

Next, Taylor's expansion will be utilised to determine the corresponding peridynamic forms of equations (A2a) to (A2c).

A1.1 Transforming W_I into peridynamic form

Following relationships are established by means of Taylor's expansion and disregarding higher order relationships as

$$\bar{u}_I(\boldsymbol{x} + \boldsymbol{\xi}) - \bar{u}_I(\boldsymbol{x}) = \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J}\xi n_J \tag{A3a}$$

$$\bar{u}_K(\boldsymbol{x} + \boldsymbol{\xi}) - \bar{u}_K(\boldsymbol{x}) = \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_L}\xi n_L \tag{A3b}$$

where $\boldsymbol{n}$ represents the unit direction vector, as shown in Figure (A1), which is defined as

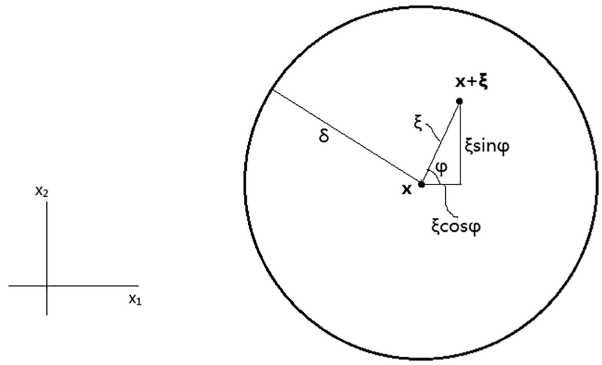

Figure A1. Peridynamic interaction between two material points.

$$\boldsymbol{n} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \cos\varphi \\ \sin\varphi \end{Bmatrix} \tag{A4}$$

If we multiply equation (A3a) and equation (A3b), and multiply both sides with $\frac{1}{\xi} n_R n_S$ yields

$$\frac{[\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_I(\boldsymbol{x})][\bar{u}_K(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_K(\boldsymbol{x})]}{\xi} n_R n_S = \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_L} \xi n_J n_L n_R n_S \tag{A5}$$

Integrating each term of equation (A5) over the interaction domain gives

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\delta} \frac{[\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_I(\boldsymbol{x})][\bar{u}_K(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_K(\boldsymbol{x})]}{\xi} n_R n_S dA = \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_L} \\ &\int_0^{2\pi}\int_0^{\delta} \xi n_J n_L n_R n_S \xi d\xi d\varphi = \frac{\pi\delta^3}{12} \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_L} (\delta_{JL}\delta_{RS} + \delta_{JR}\delta_{LS} + \delta_{JS}\delta_{RL}) \\ &= \frac{\pi\delta^3}{12} \left(\frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_J} \delta_{RS} + \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_R} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_S} + \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_S} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_R} \right) \end{aligned} \tag{A6}$$

Equation (A6) is multiplied by $\delta_{RI}\delta_{SK}$ from both sides and following relation is obtained

$$\begin{aligned} &\int_0^{2\pi}\int_0^{\delta} \frac{[\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_I(\boldsymbol{x})][\bar{u}_K(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_K(\boldsymbol{x})]}{\xi} n_I n_K dA \\ &= \frac{\pi\delta^3}{12} \left(\frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} + \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_I} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_K} + \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_K} \frac{\partial \bar{u}_K(\boldsymbol{x})}{\partial x_I} \right) \end{aligned} \tag{A7}$$

Equation (A7) can also be written as

$$\begin{aligned} &\frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} + \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_I} \frac{\partial \bar{u}_J(\boldsymbol{x})}{\partial x_J} + \frac{\partial \bar{u}_I(\boldsymbol{x})}{\partial x_J} \frac{\partial \bar{u}_J(\boldsymbol{x})}{\partial x_I} \\ &= \frac{12}{\pi\delta^3} \int_0^{2\pi}\int_0^{\delta} \frac{[\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_I(\boldsymbol{x})][\bar{u}_J(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{u}_J(\boldsymbol{x})]}{\xi} n_I n_J dA \end{aligned} \tag{A8}$$

Utilising Taylor's expansion and disregarding higher-order relationships yields the below expressions

$$\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi}) + \theta_I(\boldsymbol{x})}{2} = \theta_I(\boldsymbol{x}) + O(\xi) \tag{A9}$$

$$\bar{w}(\boldsymbol{x}+\boldsymbol{\xi}) - \bar{w}(\boldsymbol{x}) = \frac{\partial \bar{w}(\boldsymbol{x})}{\partial x_I} \xi n_I + O(\xi^2) \tag{A10}$$

Multiplying each terms of equation (A9) by ξn_I gives

$$\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi})+\theta_I(\boldsymbol{x})}{2}\xi n_I=\theta_I(\boldsymbol{x})\xi n_I+O(\xi^2) \tag{A11}$$

Combining equation (A10) with equation (A11) and ignoring higher-order terms yields

$$\bar{w}(\boldsymbol{x}+\boldsymbol{\xi})-\bar{w}(\boldsymbol{x})+\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi})+\theta_I(\boldsymbol{x})}{2}\xi n_I=\left(\theta_I(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_I}\right)\xi n_I \tag{A12a}$$

Equation (A12a) can also be written as

$$\bar{w}(\boldsymbol{x}+\boldsymbol{\xi})-\bar{w}(\boldsymbol{x})+\frac{\theta_J(\boldsymbol{x}+\boldsymbol{\xi})+\theta_J(\boldsymbol{x})}{2}\xi n_J=\left(\theta_J(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_J}\right)\xi n_J \tag{A12b}$$

Multiplying (A12a) with (A12b) and dividing each term by ξ results in

$$\frac{1}{\xi}\left(\bar{w}(\boldsymbol{x}+\boldsymbol{\xi})-\bar{w}(\boldsymbol{x})+\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi})+\theta_I(\boldsymbol{x})}{2}\xi n_I\right)^2=\left(\theta_I(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_I}\right)\left(\theta_J(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_J}\right)\xi n_I n_J \tag{A13}$$

Integrating each term of equation (A13) over the interaction domain for a material particle located at $\boldsymbol{x}$ with a radius of δ gives

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\delta}\frac{1}{\xi}\left(\bar{w}(\boldsymbol{x}+\boldsymbol{\xi})-\bar{w}(\boldsymbol{x})+\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi})+\theta_I(\boldsymbol{x})}{2}\xi n_I\right)^2\xi d\xi d\varphi=\left(\theta_I(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_I}\right)\\&\left(\theta_J(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_J}\right)\int_0^{2\pi}\int_0^{\delta}\xi n_I n_J\xi d\xi d\varphi=\frac{\pi\delta^3}{3}\left(\theta_I(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_I}\right)\left(\theta_J(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_J}\right)\delta_{IJ}\end{aligned} \tag{A14}$$

which then yields

$$\left(\theta_I(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_I}\right)\left(\theta_I(\boldsymbol{x})+\frac{\partial\bar{w}(\boldsymbol{x})}{\partial x_I}\right)=\frac{3}{\pi\delta^3}\int_0^{2\pi}\int_0^{\delta}\frac{1}{\xi}\left(\bar{w}(\boldsymbol{x}+\boldsymbol{\xi})-\bar{w}(\boldsymbol{x})+\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi})+\theta_I(\boldsymbol{x})}{2}\xi n_I\right)^2\xi d\xi d\varphi \tag{A15}$$

By using Taylor expansion, θ_z can be written as

$$\frac{\theta_z(\boldsymbol{x}+\boldsymbol{\xi})+\theta_z(\boldsymbol{x})}{2}=\theta_z(\boldsymbol{x})+O(\xi) \tag{A16}$$

If we take square of both sides of equation (A16), then multiply each term by ξ the following relation is obtained

$$\xi\left(\frac{\theta_z(\boldsymbol{x}+\boldsymbol{\xi})+\theta_z(\boldsymbol{x})}{2}\right)^2=\theta_z(\boldsymbol{x})^2\xi+O(\xi^2) \tag{A17}$$

Integrating each term of equation (A17) over the interaction domain for a material particle located at $\boldsymbol{x}$ with a radius of δ gives

$$2\theta_z(\boldsymbol{x})^2=\frac{3}{\pi\delta^3}\int_0^{2\pi}\int_0^{\delta}\xi\left(\frac{\theta_z(\boldsymbol{x}+\boldsymbol{\xi})+\theta_z(\boldsymbol{x})}{2}\right)^2\xi d\xi d\varphi \tag{A18}$$

Combining equations (A8), (A15) and (A18) yields

$$\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_I}{\partial x_J}+\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \bar{u}_J}{\partial x_J}+\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_J}{\partial x_I}+\left(\theta_I+\frac{\partial \bar{w}}{\partial x_I}\right)\left(\theta_I+\frac{\partial \bar{w}}{\partial x_I}\right)+2{\theta_z}^2\right]_{(\boldsymbol{x})}=$$
$$\frac{12}{\pi\delta^3}\int_0^{2\pi}\int_0^{\delta}\frac{[u_I(\boldsymbol{x}+\boldsymbol{\xi})-u_I(\boldsymbol{x})][u_J(\boldsymbol{x}+\boldsymbol{\xi})-u_J(\boldsymbol{x})]}{\xi}n_In_JdA+ \tag{A19a}$$
$$\frac{3}{\pi\delta^3}\int_0^{2\pi}\int_0^{\delta}\left[\frac{1}{\xi}\left(w(\boldsymbol{x}+\boldsymbol{\xi})-w(\boldsymbol{x})+\frac{\theta_I(\boldsymbol{x}+\boldsymbol{\xi})+\theta_I(\boldsymbol{x})}{2}\xi n_I\right)^2+\xi\left(\frac{\theta_z(\boldsymbol{x}+\boldsymbol{\xi})+\theta_z(\boldsymbol{x})}{2}\right)^2\right]\xi d\xi d\varphi$$

which can be demonstrated in discretized formula for a particle k as

$$\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_I}{\partial x_J}+\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \bar{u}_J}{\partial x_J}+\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \bar{u}_J}{\partial x_I}+\left(\theta_I+\frac{\partial \bar{w}}{\partial x_I}\right)\left(\theta_I+\frac{\partial \bar{w}}{\partial x_I}\right)+2\left(\theta_z\right)^2\right]_{(\boldsymbol{x})=(k)}=$$
$$\frac{12}{\pi\delta^3 h}\sum_i\frac{\left[\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)n_I^{(i^k)(k)}\right]^2}{\xi_{(i^k)(k)}}V_{(i^k)}+\frac{3}{\pi\delta^3 h}\sum_i\frac{1}{\xi_{(i^k)(k)}}\left(\bar{w}_{(i^k)}-\bar{w}_{(k)}+\frac{\theta_I^{(i^k)}+\theta_I^{(k)}}{2}\xi_{(i^k)(k)}n_I^{(i^k)(k)}\right)^2$$
$$+\left(\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\right)^2\xi_{(i^k)(k)}V_{(i^k)} \tag{A19b}$$

The remaining local terms of W_I is similarly stated in peridynamic formulation as

$$\left[\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_I}{\partial x_J}+\frac{\partial \bar{u}_I}{\partial x_J}\frac{\partial \theta_J}{\partial x_I}+\frac{\partial \bar{u}_I}{\partial x_I}\frac{\partial \theta_J}{\partial x_J}+\left(\theta_I+\frac{\partial \bar{w}}{\partial x_I}\right)\left(2u_I^*+\frac{\partial \theta_z}{\partial x_I}\right)+4\theta_z w^*\right]_{(\boldsymbol{x})=(k)}=$$
$$\frac{12}{\pi\delta^3 h}\sum_i\frac{\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)\left(\theta_J^{(i^k)}-\theta_J^{(k)}\right)n_I^{(i^k)(k)}n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}}V_{(i^k)}+ \tag{A20a}$$
$$\frac{3}{\pi\delta^3 h}\sum_i\left[\left(\bar{w}_{(i^k)}-\bar{w}_{(k)}+\frac{\theta_I^{(i^k)}+\theta_I^{(k)}}{2}\xi_{(i^k)(k)}n_I^{(i^k)(k)}\right)\left(\frac{\theta_z^{(i^k)}-\theta_z^{(k)}}{\xi_{(i^k)(k)}}+\left(u_J^{*(i^k)}+u_J^{*(k)}\right)n_J^{(i^k)(k)}\right)\right.$$
$$\left.+\left(\theta_z^{(i^k)}+\theta_z^{(k)}\right)\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\xi_{(i^k)(k)}\right]V_{(i^k)}$$

$$\left[\frac{\partial\theta_I}{\partial x_J}\frac{\partial\theta_I}{\partial x_J}+\frac{\partial\theta_I}{\partial x_J}\frac{\partial\theta_J}{\partial x_I}+\frac{\partial\theta_I}{\partial x_I}\frac{\partial\theta_J}{\partial x_J}+2\left(\frac{\partial\bar{u}_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J}+\frac{\partial\bar{u}_I}{\partial x_J}\frac{\partial u_J^*}{\partial x_I}+\frac{\partial\bar{u}_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}\right)\right.$$
$$\left.+\left(2u_I^*+\frac{\partial\theta_z}{\partial x_I}\right)\left(2u_I^*+\frac{\partial\theta_z}{\partial x_I}\right)+2\left(\theta_I+\frac{\partial\bar{w}}{\partial x_I}\right)\left(3\theta_I^*+\frac{\partial w^*}{\partial x_I}\right)+8\left(w^*\right)^2\right]_{(x)=(k)}=$$
$$\frac{12}{\pi\delta^3 h}\sum_i\frac{\left[\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)n_I^{(i^k)(k)}\right]^2+2\left(u_I^{(i^k)}-u_I^{(k)}\right)\left(u_J^{*(i^k)}-u_J^{*(k)}\right)n_I^{(i^k)(k)}n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}}V_{(i^k)}$$
$$+\frac{3}{\pi\delta^3 h}\sum_i\left\{\frac{1}{\xi_{(i^k)(k)}}\left[\theta_z^{(i^k)}-\theta_z^{(k)}+\left(u_I^{*(i^k)}+u_I^{*(k)}\right)\xi_{(i^k)(k)}n_I^{(i^k)(k)}\right]^2\right.$$
$$+2\left(\frac{w_{(i^k)}-w_{(k)}}{\xi_{(i^k)(k)}}+\frac{\theta_I^{(i^k)}+\theta_I^{(k)}}{2}n_I^{(i^k)(k)}\right)\left(w^*_{(i^k)}-w^*_{(k)}+3\frac{\theta_J^{*(i^k)}+\theta_J^{*(k)}}{2}\xi_{(i^k)(k)}n_J^{(i^k)(k)}\right)$$
$$\left.+\left(w^*_{(i^k)}+w^*_{(k)}\right)^2\xi_{(i^k)(k)}\right\}V_{(i^k)} \tag{A20b}$$

$$\left[\frac{\partial\bar{u}_I}{\partial x_J}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial\bar{u}_J}{\partial x_I}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial\bar{u}_I}{\partial x_I}\frac{\partial\theta_J^*}{\partial x_J}+\frac{\partial\theta_I}{\partial x_J}\frac{\partial u_I^*}{\partial x_J}+\frac{\partial\theta_J}{\partial x_I}\frac{\partial u_I^*}{\partial x_J}+\frac{\partial\theta_I}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}+\left(2u_I^*+\frac{\partial\theta_z}{\partial x_I}\right)\left(3\theta_I^*+\frac{\partial w^*}{\partial x_I}\right)\right]_{(\boldsymbol{x})=(k)}=$$
$$\frac{12}{\pi\delta^3 h}\sum_i\frac{\left[\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)\left(\theta_J^{*(i^k)}-\theta_J^{*(k)}\right)+\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)\left(u_J^{*(i^k)}-u_J^{*(k)}\right)\right]n_I^{(i^k)(k)}n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}}V_{(i^k)}$$
$$+\frac{3}{\pi\delta^3 h}\sum_i\left[w^*_{(i^k)}-w^*_{(k)}+\frac{3}{2}\left(\theta_I^{*(i^k)}+\theta_I^{*(k)}\right)\xi_{(i^k)(k)}n_I^{(i^k)(k)}\right]\left[\frac{\theta_z^{(i^k)}-\theta_z^{(k)}}{\xi_{(i^k)(k)}}+\left(u_J^{*(i^k)}+u_J^{*(k)}\right)n_J^{(i^k)(k)}\right]V_{(i^k)} \tag{A20c}$$

$$\left[\frac{\partial u_I^*}{\partial x_J}\frac{\partial u_I^*}{\partial x_J}+\frac{\partial u_I^*}{\partial x_J}\frac{\partial u_J^*}{\partial x_I}+\frac{\partial u_I^*}{\partial x_I}\frac{\partial u_J^*}{\partial x_J}+2\left(\frac{\partial\theta_I}{\partial x_J}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial\theta_J}{\partial x_I}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial\theta_I}{\partial x_I}\frac{\partial\theta_J^*}{\partial x_J}\right)+\left(3\theta_I^*+\frac{\partial w^*}{\partial x_I}\right)\left(3\theta_I^*+\frac{\partial w^*}{\partial x_I}\right)\right]_{(\boldsymbol{x})=(k)}=$$
$$\frac{12}{\pi\delta^3 h}\sum_i\frac{\left[\left(u_I^{*(i^k)}-u_I^{*(k)}\right)n_I^{(i^k)(k)}\right]^2+2\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)\left(\theta_J^{*(i^k)}-\theta_J^{*(k)}\right)n_I^{(i^k)(k)}n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}}V_{(i^k)}$$
$$+\frac{3}{\pi\delta^3 h}\sum_i\frac{\left(w^*_{(i^k)}-w^*_{(k)}+\frac{3}{2}\left(\theta_I^{*(i^k)}+\theta_I^{*(k)}\right)\xi_{(i^k)(k)}n_I^{(i^k)(k)}\right)^2}{\xi_{(i^k)(k)}}V_{(i^k)} \tag{A20d}$$

$$\left[\frac{\partial u_I^*}{\partial x_J}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial u_J^*}{\partial x_I}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial u_I^*}{\partial x_I}\frac{\partial\theta_J^*}{\partial x_J}\right]_{(\boldsymbol{x})=(k)}=\frac{12}{\pi\delta^3 h}\sum_i\frac{\left(u_I^{*(i^k)}-u_I^{*(k)}\right)\left(\theta_J^{*(i^k)}-\theta_J^{*(k)}\right)n_I^{(i^k)(k)}n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}}V_{(i^k)} \tag{A20e}$$

$$\left[\frac{\partial\theta_I^*}{\partial x_J}\frac{\partial\theta_I^*}{\partial x_J}+\frac{\partial\theta_I^*}{\partial x_J}\frac{\partial\theta_J^*}{\partial x_I}+\frac{\partial\theta_I^*}{\partial x_I}\frac{\partial\theta_J^*}{\partial x_J}\right]_{(\boldsymbol{x})=(k)}=\frac{12}{\pi\delta^3 h}\sum_i\frac{\left[\left(\theta_I^{*(i^k)}-\theta_I^{*(k)}\right)n_I^{(i^k)(k)}\right]^2}{\xi_{(i^k)(k)}}V_{(i^k)} \tag{A20f}$$

Substituting equations (A10b) and (A20a) to (A20f) into equation (A2a) results in the first part of the SED function in peridynamic formulation as

$$
\begin{aligned}
& W_I^{(k)} = \\
& \frac{3}{\pi\delta^3 h^2}\Bigg\{ \frac{A_0}{2}\Bigg[4\sum_i \frac{\left[\left(\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}\right) n_I^{(i^k)(k)}\right]^2}{\xi_{(i^k)(k)}} V_{(i^k)} \\
& + \sum_i \Bigg[\frac{1}{\xi_{(i^k)(k)}} \left(\bar{w}_{(i^k)} - \bar{w}_{(k)} + \frac{\theta_I^{(i^k)} + \theta_I^{(k)}}{2} \xi_{(i^k)(k)} n_I^{(i^k)(k)}\right)^2 + \left(\frac{\theta_z^{(i^k)} + \theta_z^{(k)}}{2}\right)^2 \xi_{(i^k)(k)}\Bigg] V_{(i^k)}\Bigg] \\
& + A_1 \Bigg[4\sum_i \frac{\left(\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}\right)\left(\theta_J^{(i^k)} - \theta_J^{(k)}\right) n_I^{(i^k)(k)} n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}} V_{(i^k)} \\
& + \sum_i \Bigg[\left(\bar{w}_{(i^k)} - \bar{w}_{(k)} + \frac{\theta_I^{(i^k)} + \theta_I^{(k)}}{2} \xi_{(i^k)(k)} n_I^{(i^k)(k)}\right)\left(\frac{\theta_z^{(i^k)} - \theta_z^{(k)}}{\xi_{(i^k)(k)}} + \left(u_I^{*(i^k)} + u_I^{*(k)}\right) n_I^{(i^k)(k)}\right) + \left(\theta_z^{(i^k)} + \theta_z^{(k)}\right) \frac{w^*_{(i^k)} + w^*_{(k)}}{2} \xi_{(i^k)(k)}\Bigg] V_{(i^k)}\Bigg] \\
& + \frac{A_2}{2}\Bigg[4\sum_i \frac{\left[\left(\theta_I^{(i^k)} - \theta_I^{(k)}\right) n_I^{(i^k)(k)}\right]^2 + 2\left(\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}\right)\left(u_J^{*(i^k)} - u_J^{*(k)}\right) n_I^{(i^k)(k)} n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}} V_{(i^k)} \\
& + \sum_i \Bigg(\frac{1}{\xi_{(i^k)(k)}}\left(\theta_z^{(i^k)} - \theta_z^{(k)} + \left(u_I^{*(i^k)} + u_I^{*(k)}\right) \xi_{(i^k)(k)} n_I^{(i^k)(k)}\right)^2 \\
& + 2\left(\frac{\bar{w}_{(i^k)} - \bar{w}_{(k)}}{\xi_{(i^k)(k)}} + \frac{\theta_I^{(i^k)} + \theta_I^{(k)}}{2} n_I^{(i^k)(k)}\right) r \left(w^*_{(i^k)} - w^*_{(k)} + 3\frac{\theta_I^{*(i^k)} + \theta_I^{*(k)}}{2} \xi_{(i^k)(k)} n_I^{(i^k)(k)}\right) + \left(w^*_{(i^k)} + w^*_{(k)}\right)^2 \xi_{(i^k)(k)}\Bigg) V_{(i^k)}\Bigg] \\
& + A_3 \Bigg[4\sum_i \frac{\left[\left(\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}\right)\left(\theta_J^{*(i^k)} - \theta_J^{*(k)}\right) + \left(\theta_I^{(i^k)} - \theta_I^{(k)}\right)\left(u_J^{*(i^k)} - u_J^{*(k)}\right)\right] n_I^{(i^k)(k)} n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}} V_{(i^k)} \\
& + \sum_i \Bigg[w^*_{(i^k)} - w^*_{(k)} + \frac{3}{2}\left(\theta_I^{*(i^k)} + \theta_I^{*(k)}\right) \xi_{(i^k)(k)} n_I^{(i^k)(k)}\Bigg]\Bigg[\frac{\theta_z^{(i^k)} - \theta_z^{(k)}}{\xi_{(i^k)(k)}} + \left(u_J^{*(i^k)} + u_J^{*(k)}\right) n_J^{(i^k)(k)}\Bigg] V_{(i^k)}\Bigg] \\
& + \frac{A_4}{2}\Bigg[4\sum_i \frac{\left[\left(u_I^{*(i^k)} - u_I^{*(k)}\right) n_I^{(i^k)(k)}\right]^2 + 2\left(\theta_I^{(i^k)} - \theta_I^{(k)}\right)\left(\theta_J^{*(i^k)} - \theta_J^{*(k)}\right) n_I^{(i^k)(k)} n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}} \\
& V_{(i^k)} + \sum_i \frac{\left(w^*_{(i^k)} - w^*_{(k)} + \frac{3}{2}\left(\theta_I^{*(i^k)} + \theta_I^{*(k)}\right) \xi_{(i^k)(k)} n_I^{(i^k)(k)}\right)^2}{\xi_{(i^k)(k)}} V_{(i^k)}\Bigg] + 4A_5 \sum_i \frac{\left(u_I^{*(i^k)} - u_I^{*(k)}\right)\left(\theta_J^{*(i^k)} - \theta_J^{*(k)}\right) n_I^{(i^k)(k)} n_J^{(i^k)(k)}}{\xi_{(i^k)(k)}} V_{(i^k)} + \\
& 2A_6 \sum_i \frac{\left[\left(\theta_I^{*(i^k)} - \theta_I^{*(k)}\right) n_I^{(i^k)(k)}\right]^2}{\xi_{(i^k)(k)}} V_{(i^k)}\Bigg\}
\end{aligned}
\tag{A21}
$$

A1.2 Transforming W_{II} into PD form

If a similar procedure is followed as described above, the following local terms can be transformed into peridynamic form as

$$\left[2\frac{\partial\bar{u}_I}{\partial x_I}\theta_z+(\theta_z)^2\right]_{(\boldsymbol{x})=(k)} = \frac{3}{\pi\delta^3 h}\sum_i\left[\left(\theta_z^{(i^k)}+\theta_z^{(k)}\right)\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{1}{2}\left(\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\right)^2\xi_{(i^k)(k)}\right]V_{(i^k)} \tag{A22a}$$

$$\left[2\frac{\partial\bar{u}_I}{\partial x_I}w^*+\frac{\partial\theta_I}{\partial x_I}\theta_z+2\theta_z w^*\right]_{(\boldsymbol{x})=(k)} = \frac{3}{\pi\delta^3 h}\sum_i\left[\left(w^*_{(i^k)}+w^*_{(k)}\right)\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\xi_{(i^k)(k)}\right]V_{(i^k)} \tag{A22b}$$

$$\left[2\frac{\partial\theta_I}{\partial x_I}w^*+\frac{\partial u_I^*}{\partial x_I}\theta_z+2(w^*)^2\right]_{(\boldsymbol{x})=(k)} = \frac{3}{\pi\delta^3 h}\sum_i\left[\left(w^*_{(i^k)}+w^*_{(k)}\right)\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\left(u_I^{*(i^k)}-u_I^{*(k)}\right)n_I^{(i^k)(k)}+\left(\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\right)^2\xi_{(i^k)(k)}\right]V_{(i^k)} \tag{A22c}$$

$$\left[2\frac{\partial u_I^*}{\partial x_I}w^*+\frac{\partial\theta_I^*}{\partial x_I}\theta_z\right]_{(\boldsymbol{x})=(k)} = \frac{3}{\pi\delta^3 h}\sum_i\left[\left(w^*_{(i^k)}+w^*_{(k)}\right)\left(u_I^{*(i^k)}-u_I^{*(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\left(\theta_I^{*(i^k)}-\theta_I^{*(k)}\right)n_I^{(i^k)(k)}\right]V_{(i^k)} \tag{A22d}$$

$$\left[\frac{\partial\theta_I^*}{\partial x_I}w^*\right]_{(\boldsymbol{x})=(k)} = \frac{3}{\pi\delta^3 h}\sum_i\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\left(\theta_I^{*(i^k)}-\theta_I^{*(k)}\right)n_I^{(i^k)(k)}V_{(i^k)} \tag{A22e}$$

Inserting equations (A22a) to (A22e) into equation (A2b) gives the second part of the SED function in peridynamic formulation as

$$\begin{aligned}
W_{II}^{(k)} = \frac{3}{\pi\delta^3 h^2}
&\left\{B_0\sum_i\left[\left(\theta_z^{(i^k)}+\theta_z^{(k)}\right)\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)n_I^{(i^k)(k)}+\tfrac{1}{2}\left(\tfrac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\right)^2\xi_{(i^k)(k)}\right]V_{(i^k)}\right.\\
&+2B_1\sum_i\left[\left(w^*_{(i^k)}+w^*_{(k)}\right)\left(\bar{u}_I^{(i^k)}-\bar{u}_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\xi_{(i^k)(k)}\right]V_{(i^k)}\\
&+2B_2\sum_i\left[\left(w^*_{(i^k)}+w^*_{(k)}\right)\left(\theta_I^{(i^k)}-\theta_I^{(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\left(u_I^{*(i^k)}-u_I^{*(k)}\right)n_I^{(i^k)(k)}+\left(\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\right)^2\xi_{(i^k)(k)}\right]V_{(i^k)}\\
&+2B_3\sum_i\left[\left(w^*_{(i^k)}+w^*_{(k)}\right)\left(u_I^{*(i^k)}-u_I^{*(k)}\right)n_I^{(i^k)(k)}+\frac{\theta_z^{(i^k)}+\theta_z^{(k)}}{2}\left(\theta_I^{*(i^k)}-\theta_I^{*(k)}\right)n_I^{(i^k)(k)}\right]V_{(i^k)}\\
&\left.+4B_4\sum_i\frac{w^*_{(i^k)}+w^*_{(k)}}{2}\left(\theta_I^{*(i^k)}-\theta_I^{*(k)}\right)n_I^{(i^k)(k)}V_{(i^k)}\right\}
\end{aligned} \tag{A23}$$

A1.3 Transforming W_{III} into peridynamic form

Multiplying both sides of equation (A3) by the direction vector and $\frac{1}{\xi}$ gives

$$\frac{\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi})-\bar{u}_I(\boldsymbol{x})}{\xi}n_K=\frac{\partial\bar{u}_I(\boldsymbol{x})}{\partial x_J}n_K n_J \tag{A25}$$

Integrating each term of equation (A25) over the interaction domain of the material particle $\boldsymbol{x}$ with a radius of δ gives

$$\int_0^{2\pi}\int_0^{\delta}\frac{\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi})-\bar{u}_I(\boldsymbol{x})}{\xi}n_K\xi d\xi d\varphi=\frac{\pi\delta^2}{2}\frac{\partial\bar{u}_I(\boldsymbol{x})}{\partial x_J}\delta_{JK} \tag{A26}$$

Multiplying δ_{IK} with both sides of equation (A26) results in

$$\frac{\partial\bar{u}_I(\boldsymbol{x})}{\partial x_I}=\frac{2}{\pi\delta^2}\int_0^{2\pi}\int_0^{\delta}\frac{\bar{u}_I(\boldsymbol{x}+\boldsymbol{\xi})-\bar{u}_I(\boldsymbol{x})}{\xi}n_I\xi d\xi d\varphi \tag{A27a}$$

which is denoted in discrete formulation for the material particle k as

$$\frac{\partial\bar{u}_I^{(k)}}{\partial x_I}=\frac{2}{\pi\delta^2 h}\sum_i\frac{\bar{u}_I^{\left(i^k\right)}-\bar{u}_I^{(k)}}{\xi_{(i^k)(k)}}n_I^{\left(i^k\right)(k)}V_{(i^k)} \tag{A27b}$$

Other local terms can be similarly transformed as

$$\frac{\partial\theta_I^{(k)}}{\partial x_I}=\frac{2}{\pi\delta^2 h}\sum_i\frac{\theta_I^{\left(i^k\right)}-\theta_I^{(k)}}{\xi_{(i^k)(k)}}n_I^{\left(i^k\right)(k)}V_{(i^k)} \tag{A28a}$$

$$\frac{\partial u_I^{*(k)}}{\partial x_I}=\frac{2}{\pi\delta^2 h}\sum_i\frac{u_I^{*\left(i^k\right)}-u_I^{*(k)}}{\xi_{(i^k)(k)}}n_I^{\left(i^k\right)(k)}V_{(i^k)} \tag{A28b}$$

$$\frac{\partial\theta_I^{*(k)}}{\partial x_I}=\frac{2}{\pi\delta^2 h}\sum_i\frac{\theta_I^{*\left(i^k\right)}-\theta_I^{*(k)}}{\xi_{(i^k)(k)}}n_I^{\left(i^k\right)(k)}V_{(i^k)} \tag{A28c}$$

Inserting equations (A27b) and (A28a) to (A28c) into equation (A2c) gives the third part of SED function in peridynamic formulation as

$$W_{III}^{(k)} = \frac{1}{h}\left(\frac{2}{\pi\delta^2 h}\right)^2$$

$$\left\{\begin{array}{l} \frac{c_0}{2}\left(\sum_i \frac{\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)}\right)^2 + c_1 \sum_i \frac{\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{\theta_J^{(i^k)} - \theta_J^{(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} \\ + \frac{c_2}{2}\left[\left(\sum_i \frac{\theta_I^{(i^k)} - \theta_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)}\right)^2 + 2\sum_i \frac{\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{u_J^{*(i^k)} - u_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)}\right] \\ c_3\left[\sum_i \frac{\bar{u}_I^{(i^k)} - \bar{u}_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{\theta_J^{*(i^k)} - \theta_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} + \sum_i \frac{\theta_I^{(i^k)} - \theta_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{u_J^{*(i^k)} - u_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)}\right] \\ + \frac{c_4}{2}\left[\left(\sum_i \frac{u_I^{*(i^k)} - u_I^{*(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)}\right)^2 + 2\sum_i \frac{\theta_I^{(i^k)} - \theta_I^{(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{\theta_J^{*(i^k)} - \theta_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)}\right] \\ + c_5 \sum_i \frac{u_I^{*(i^k)} - u_I^{*(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)} \sum_i \frac{\theta_J^{*(i^k)} - \theta_J^{*(k)}}{\xi_{(i^k)(k)}} n_J^{(i^k)(k)} V_{(i^k)} + \frac{c_6}{2}\left(\sum_i \frac{\theta_I^{*(i^k)} - \theta_I^{*(k)}}{\xi_{(i^k)(k)}} n_I^{(i^k)(k)} V_{(i^k)}\right)^2 \end{array}\right\} \tag{A29}$$

The total SED function in peridynamic formulation of the material particle k is acquired by combining equations (A21), (A23) and (A29). The total SED function in peridynamic formulation of the material particle j is also acquired similarly.