

## On the similarities between the simplified Phan-Thien Tanner (sPTT) and FENE-P models

R J Poole, M Davoodi, K Zografos

School of Engineering, University of Liverpool, Liverpool L69 3GH, UK

For many commonly-used single mode viscoelastic constitutive equations of differential type, it is well known that they share many features. For example, in certain parameter limits the Giesekus, Phan-Thien Tanner and FENE-type models approach the Oldroyd-B model. In this short paper we show that for homogeneous flows such as steady simple shear flow or pure extension, the response of the linear form of the simplified Phan-Thien Tanner (the ‘‘sPTT’’) model and the Finitely Extensible Nonlinear Elastic model that follows the Peterlin approximation (the ‘‘FENE-P’’) is identical under certain conditions. In effect this means in viscometric flows, any steady analytical solution derived for one of these models in a particular flow is also a solution for the other model and we demonstrate this equivalence using existing channel flow solutions for the two models from the literature.

Constitutive equations for viscoelastic fluids, and in particular those modelling polymeric melts and solutions, have a long history and numerous important contributions have been made, see for example the monographs of Bird et al [1] and Larson [2]. Here we will concentrate on two very widely-used models to represent polymeric solutions. These being the so-called PTT and FENE-P models due to Phan-Thien & Tanner [3] and Bird et al [4]. In particular, we use the simplified linear form of the PTT model (the sPTT) which can be expressed as [5]:

$$\lambda \ddot{\boldsymbol{\tau}}_p + f_S \boldsymbol{\tau}_p = \eta_p \dot{\boldsymbol{\gamma}}, \quad (1)$$

where  $\lambda$  is the relaxation time,  $\eta_p$  is the polymeric viscosity,  $\boldsymbol{\tau}_p$  the polymeric extra stress tensor,  $\dot{\boldsymbol{\gamma}} = \nabla \mathbf{u} + \nabla \mathbf{u}^T$  the strain rate tensor and  $f_S = 1 + (\lambda \varepsilon / \eta_p) Tr(\boldsymbol{\tau}_p)$  is the linear function of the PTT model in which  $\varepsilon$  is the extensionability parameter. In the above  $\ddot{\boldsymbol{\tau}}_p$  represents the upper convected derivative of the polymeric extra stress tensor (i.e.  $\ddot{\boldsymbol{\tau}}_p = \partial \boldsymbol{\tau}_p / \partial t + (\mathbf{u} \cdot \nabla) \boldsymbol{\tau}_p - (\nabla \mathbf{u}^T) \cdot \boldsymbol{\tau}_p - \boldsymbol{\tau}_p \cdot \nabla \mathbf{u}$ ) and  $Tr$  stands for the trace of the tensor  $Tr(\boldsymbol{\tau}_p) = \tau_{xx} + \tau_{yy} + \tau_{zz}$ . In contrast the FENE-P model can be expressed as [6]:

$$\left( \frac{\lambda \ddot{\boldsymbol{\tau}}_p}{f_F} \right) + \boldsymbol{\tau}_p = \frac{a \eta_p}{f_F} \dot{\boldsymbol{\gamma}} - \frac{D(a \eta_p / f_F)}{Dt} \mathbf{I}, \quad (2)$$

and then expanding the upper convected derivative term this can be rearranged to

$$\lambda \ddot{\boldsymbol{\tau}}_P + f_F \boldsymbol{\tau}_P = \left[ \frac{\lambda \boldsymbol{\tau}_P}{f_F} + \frac{a \eta_P \mathbf{I}}{f_F} \right] \frac{Df_F}{Dt} + a \eta_P \dot{\boldsymbol{\gamma}}, \quad (3)$$

where the symbols have the same meaning for the sPTT as above but now the FENE-P function is  $f_F = 1 + 3/(L^2 - 3) + (\lambda/\eta_P L^2) Tr(\boldsymbol{\tau}_P)$ ,  $a = L^2/(L^2 - 3)$  and  $L^2$  is now called the extensionalibility parameter.  $Df_F/Dt$  represents the substantial or material derivative ( $= \partial f_F/\partial t + u \cdot \partial f_F/\partial x + v \cdot \partial f_F/\partial y + w \cdot \partial f_F/\partial z$ ). Note the underlined term can also be expressed equivalently as  $-[\lambda \boldsymbol{\tau}_P + a \eta_P \mathbf{I}] f_F \frac{D(1/f_F)}{Dt}$ .

For any homogeneous steady flow at a constant deformation rate, such as steady-state simple shear flow or extension planar/uniaxial,  $Df_F/Dt = 0$  and thus, the underlined term on the right-hand side of Eq. (2) will vanish with the FENE-P model obtaining a similar expression to Eq. (1) of the sPTT model. For values of  $L^2 \gg 3$ ,  $a$  tends to one and the function  $f_F$  therefore simplifies to:

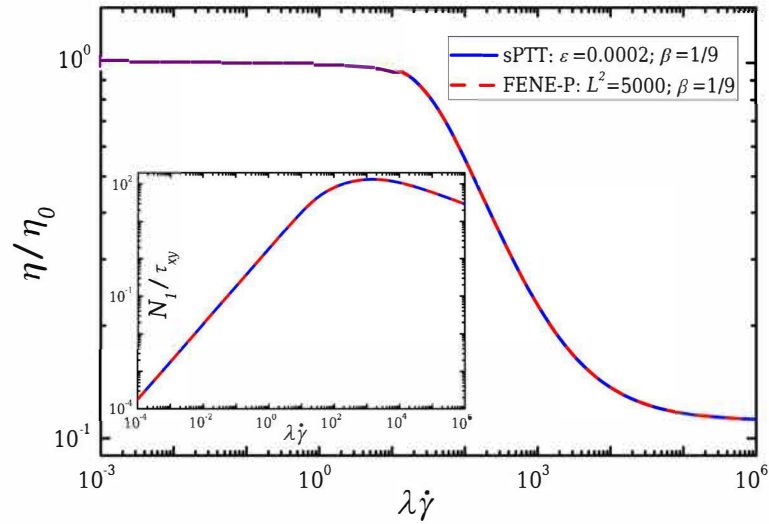
$$f_F = 1 + (\lambda/L^2 \eta_P) Tr(\boldsymbol{\tau}_P) \quad (3)$$

where now the FENE-P and sPTT models can be seen to be identical under the transformation  $\varepsilon = 1/L^2$ . Thus, in the limit that  $L^2 \gg 3$ , the FENE-P and sPTT material properties (using  $\varepsilon = 1/L^2$ ) should be identical. We show this in Figure 1 for  $L^2 = 5000$  ( $\varepsilon = 0.0002$ ) and including a solvent viscosity contribution of  $\beta = \eta / \eta_0 = 1/9$  where  $\eta_s$  is the solvent viscosity contribution and  $\eta_0 = \eta_s + \eta_P$  is the total viscosity (although as this is a linear addition, this precise value of solvent contribution is insignificant in demonstrating this equivalence).

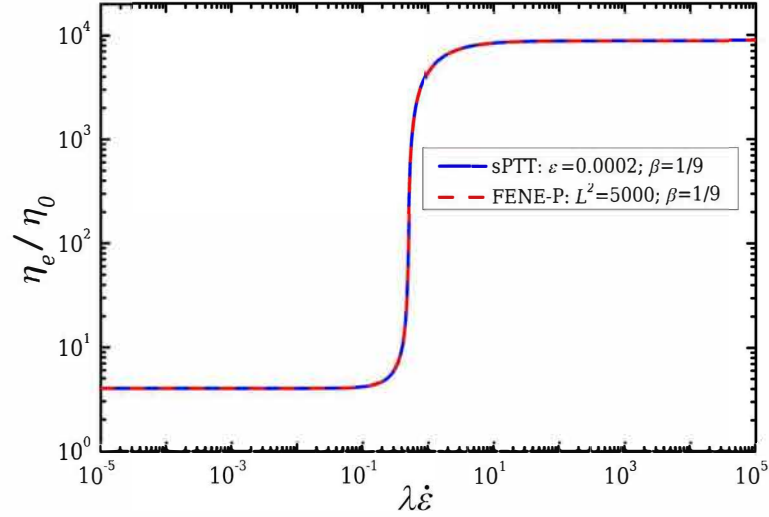
In Figure 2 we also show that this equivalence holds in planar extensional flow under the same conditions.

In fact, closer examination of the reduced equations for both models in pure-shear flow shows (not shown here for conciseness but the interested reader is welcome to confirm for themselves) that the analogy between the models can be pushed further. Doing so, demonstrates that the equivalence  $\varepsilon = 1/a^2 L^2$  provides shear viscosity expressions which are identical between the two models for all values of the respective extensionability parameters ( $\varepsilon, L^2$ ). Unfortunately, this exact equivalence is observed only for the shear viscosity whilst, for the other material properties,  $\varepsilon = 1/L^2$  remains a better approximation for all values of the extensionability parameters.

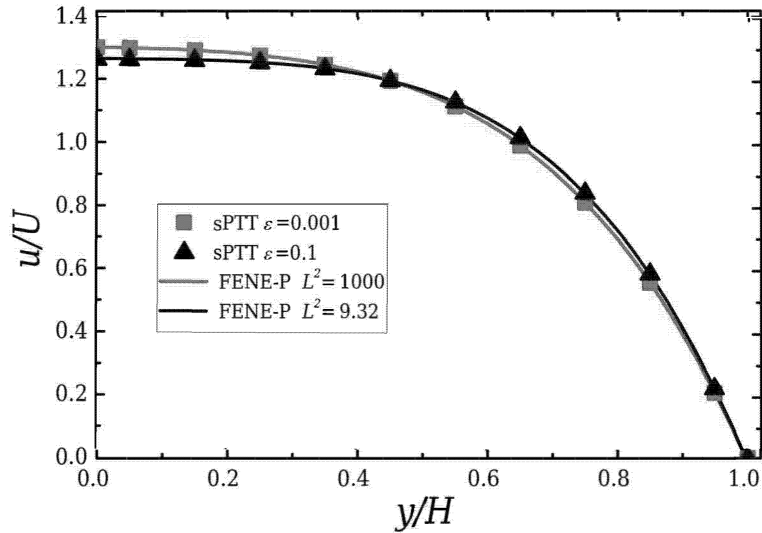
Once a model's shear viscosity response is known, this information is then sufficient [2] to obtain analytical (or semi analytical) solutions to any steady fully-developed -shear flow such as Couette, pipe, channel or annuli. To test our hypothesis that



**Figure 1.** Comparison of rheological properties in steady shear flow between sPTT ( $\varepsilon = 0.0002$ ) and FENE-P ( $L^2 = 5000$ ) models with  $\beta = 1/9$ : normalized shear viscosity ( $\eta$ ) and the normalized first normal stress difference ( $N_1 = \tau_{xx} - \tau_{yy}$ ).



**Figure 2.** Comparison of the normalized extensional viscosity ( $\eta_e$ ) in planar extensional flow between sPTT ( $\varepsilon = 0.0002$ ) and FENE-P ( $L^2 = 5000$ ) models with  $\beta = 1/9$ .



**Figure 3.** Channel flow velocity profiles along the transverse direction at  $Wi=100$ , obtained from analytical solutions for the sPTT [7] and FENE-P [8] models for  $\epsilon = 0.001$ ,  $L^2 = 1000$  ( $a^2L^2 \sim 1000$ ) and also  $\epsilon = 0.1$  and  $L^2 = 9.32$  such that  $a^2L^2 = 10$ .

the sPTT and FENE-P models are identical in such flows we thus compare in Figure 3 the independent analytical solutions derived for channel flow for the sPTT model [7] and the FENE-P model [8] for  $\epsilon = 0.001$ ,  $L^2 = 1000$  ( $a^2L^2 \sim 1000$ ) and also  $\epsilon = 0.1$  and  $L^2 = 9.32$  such that  $a^2L^2 = 10$  at Weissenberg =100 ( $Wi = \dot{\lambda}\dot{\gamma}$ ). As can be seen, a perfect agreement is obtained.

Thus, solutions obtained for the sPTT model in such flows, but not yet independently for the FENE-P, do not now need to be derived. Examples for the sPTT include annular flow solutions due to cylinder rotation in a concentric annulus [9], or channel flows with slip [10], electro-osmotic Poiseuille flows [11] or thin-film flows [12]. For the FENE-P there is a nice solution for capillary thinning in uniaxial extension [13] and a boundary layer approximation [14] which should be equally valid for the sPTT.

Others have previously discussed similarities between the PTT model and the extended pom pom (XPP) model [15] and also between a modified FENE-P equation and the sPTT [6] but the rather general connection we discuss here does not seem to have been explicitly stated in the literature before and was, indeed, unknown to one of the originators of the PTT model [16].

Although the response of the two models in steady viscometric flows is essentially identical for  $\varepsilon = 1/L^2$  when  $L^2 \gg 3$ , this does not mean the results in more complex flows – where the underlined term in Eqn. 3 is not identically zero – will necessarily be the same. In such flows, the effects of non-homogenous flow/Lagrangian unsteadiness may result in differences between the models and indeed cause the functions  $f_S$  and  $f_F$  to differ for example if complexity in the flow/flow history causes  $Tr(\boldsymbol{\tau}_P)$  to differ between the two models. We have investigated some complex flows and observed that in the cross-slot flow for example the differences between the two models - for a case of high  $L^2$  and low  $\varepsilon$  - is small and the same critical Weissenberg number for bifurcation to a steady asymmetric solution is observed [17]. In contrast for flow through a hyperbolic contraction-expansion geometry [18] (the “eVROC” geometry) the differences between the models can be more substantial with the FENE-P model even predicting a small enhanced pressure-drop compared to the sPTT model’s well-known inability to do so [5].

In this short paper we have identified some situations in which the sPTT model, derived from network theory, and the FENE-P model, derived from kinetic theory, are identical. Although in complex, Lagrangian unsteady flows this equivalence between the two models no longer formally holds, the differences – at least in Eulerian steady flows we have examined – are not large. Thus, for practical purposes, we posit that quantities of interest from complex flows such as the steady drag on a sphere or cylinder in a confined flow for one model may be reasonably approximated, at a given Deborah or Weissenberg number, from the other and vice versa. In the limit of small Deborah number, as this is a measure of Lagrangian unsteadiness, they should converge.

Finally, we should end on a cautionary note. As the results shown here attest, simply matching the steady material properties of a model to measured values for a viscoelastic liquid – even if the matching includes both shear and extensional (in all modes) and the normal-stress differences – is no guarantee that the model will then be “accurate” (in the sense of predicting the experimental response) in complex steady or time-dependent flows.

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