STABILIZATION BY INTERMITTENT CONTROL FOR HYBRID STOCHASTIC DIFFERENTIAL DELAY EQUATIONS

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Abstract. This paper is concerned with stablization of hybrid differential equations by intermittent control based on delay observations. By M-matrix theory and intermittent control strategy, we establish a sufficient stability criterion on intermittent hybrid stochastic differential equations. Meantime, we show that hybrid differential equations can be stabilized by intermittent control based on delay observations if the delay time τ is bounded by τ^*. Finally, an example is presented to illustrate our theory.

1. **Introduction.** It is well known that random noise can be used to stabilize a given unstable systems or to make a given stable systems even more stable. The study of stochastic stabilization was initiated by Hasminskii [7] who used two white noise sources to stabilize a system. After that, stabilization by random noise has been studied intensively by many authors, e.g. Arnold et al. [3], Scheutzow [19], Mao et al. [1, 14], Caraballo et al. [4], Wu et al. [20, 21]. In particular, this stabilization theory was further developed by Mao et al.[6, 13], Yuan [25], Yin [22] to a class of hybrid stochastic differential equations and by Appleby [2], Zong [27]

to functional differential equations. On the other hand, there is always a time lag between the time when the observation of the state is made and the time when the feedback control reaches the system, so it is more realistic that the feedback control depends on the past state. Mao et al. were the first to study this stabilization problem in [15] by the delay feedback control for hybrid stochastic differential equations (SDEs). Later, there have been some further developments on the delay time control problem (see, e.g.,[5, 8, 9, 16, 18, 23].)

In fact, most of stochastic stabilization theory are based on classical feedback controller which requires the continuous observation of the state $x(t)$ or $x(t-\tau)$ for all time $t \geq 0$. However, this kind of continuous control techniques will increase the control cost and reduce the life of the controller. Hence, the discontinuous control strategies have been proposed to stabilize unstable differential equations. Recently, intermittent control, which include control time and rest time, has attracted more interest from many people. For example, Zhang et al. [26] studied the stability of a class of intermittent SDEs, they utilized the intermittent stochastic noise to stablize nonlinear differential equations. Liu et al. [10, 11] investigated the stochastic stabilization based on the intermittent control strategy with discrete time feedback or time delay feedback. Ren and Yin [17] showed the quasi sure exponential stabilization of nonlinear differential equations via intermittent G-Brownian motion. Yin and Cao [24] discussed the almost sure exponential stabilization of nonlinear differential equations by intermittent stochastic perturbation with jumps.

However, to the best of our knowledge, few authors have considered the problem of stochastic stabilization for hybrid differential equations via intermittent noise control. Motivated by the above discussion, the main aim of this paper is to apply the intermittent stochastic noise with time delay to stabilize nonlinear hybrid differential equations. By using M-matrix theory and Itô formula, we give the range for the intermittent parameter θ so that the intermittently hybrid SDEs is exponentially stable in pth moment. Moreover, by comparison method and stochastic analysis, we show that hybrid differential equations can be stabilized by the intermittent stochastic noise with time delay as long as $\tau < \tau^*$.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and hypotheses concerning Eq.(1). In Section 3, we investigate the stabilization for hybrid stochastic differential delay equations (SDDEs) via intermittent noise control. While in Section 4 we give an example to illustrate our theory.

2. **Preliminaries and the global solution.** Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions. Let $w(t)$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $\tau > 0$ and $\mathbb{C}([-\tau, 0]; R^n)$ denote the family of the continuous functions ξ from $[-\tau, 0] \to R^n$ with the norm $||\xi|| = \sup_{-\tau \leq u \leq 0} |\xi(u)|$. For $p > 0$, $L^p_{\mathcal{F}_t}([-\tau, 0]; R^n)$ denote the family of all $\mathcal{F}_t$-measurable $\mathbb{C}([-\tau; 0], R^n)$-valued random variables $\xi = \{\xi(u) : -\tau \leq u \leq 0\}$ such that $E||\xi||^p < \infty$. Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \ldots N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by:

$$P(r(t+h) = j | r(t) = i) = \begin{cases} \gamma_{ij} h + \circ(h), & if \quad i \neq j, \\ 1 + \gamma_{ii} h + \circ(h), & if \quad i = j, \end{cases}$$

where $h > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while $\gamma_{ii} =$

$-\sum_{j \neq i} \gamma_{ij}$. We assume that the Markov chain $r(.)$ is independent of the Brownian motion $w(.)$.

In this paper, we consider the intermittently hybrid SDDEs

$$dx(t) = f(x(t), r(t))dt + g(x(t-\tau), r(t))I(t)dw(t), \quad t \geq t_0 \qquad (1)$$

with initial data $x(t_0 + u) = \xi(u)$ for $u \in [-\tau, 0]$, where $t_0 \geq 0$, $\xi \in L^2_{\mathcal{F}_{t_0}}$, $r(t_0) = r_0 \in S$, and

$$I(t) = \sum_{k=0}^{\infty} I_{[t_0+k\delta, t_0+k\delta+\theta\delta)}(t), \ f : R^n \times S \to R^n, \ g : R^n \times S \to R^{n \times m}.$$

Here $\delta > 0$ denotes the control period and $\theta\delta > 0$ is the working width satisfying $0 < \theta < 1$. Moreover, we define $x_t = \{x(t+u) : u \in [-\tau, 0]\}$ for $t \geq t_0$, so x_t is an $\mathcal{F}_t$-adapted $\mathbb{C}([-\tau; 0], R^n)$-valued stochastic process.

In this paper, the following hypothesis are imposed on the coefficients f and g.

Assumption 2.1. *Assume that there exist two nonnegative constants k_1 and k_2 such that*

$$|f(x, i) - f(y, i)| \leq k_1|x - y| \quad and \quad |g(x, i) - g(y, i)| \leq k_2|x - y|$$

for all $x, y \in R^n$ and $i \in S$. Moreover, we assume that $f(0, i) = 0$ and $g(0, i) = 0$ for all $i \in S$.

Remark 2.2. Under Assumption 2.1, it is easy to conclude that (1) has a unique global solution $x(t)$ on $t \geq t_0$ (see, Mao and Yuan [12]).

3. **Main results.** The main aim is to establish a sufficient stability criterion on intermittently hybrid SDDEs. Let us consider the auxiliary intermittently hybrid SDEs

$$dy(t) \quad = f(y(t), r(t))dt + g(y(t), r(t))I(t)dw(t), \qquad (2)$$

on $t \geq t_0$ with the initial value $y(t_0) = y_0$ and $r(t_0) = r_0$.

Similarly, under Assumption 2.1, Eq.(2) has a unique solution (see [12, 13]). Denote the unique solution by $y(t; t_0, y_0, r_0)$ on $t \geq t_0$.

Assumption 3.1. *Assume that there exist constants $\alpha_i \in R$, $\beta_i \geq 0$ and $\sigma_i \geq 0$ such that*

$$x^\top f(x, i) \leq \alpha_i|x|^2, \ |g(x, i)| \leq \beta_i|x|, \ |x^\top g(x, i)| \geq \sigma_i|x|^2$$

for all $x, y \in R^n$ and $i \in S$.

Assumption 3.2. *There exists a $p \in (0, 1)$ such that*

$$\mathcal{A}_p := \text{diag}(\rho_1(p), \cdots, \rho_N(p)) - \Gamma \qquad (3)$$

is a nonsingular M-matrix, where $\rho_i(p) = 0.5p[(2-p)\sigma_i^2 - \beta_i^2] - p\alpha_i$.

We denote by $C^2(R^n \times S; R_+)$ the family of all continuous non-negative functions $V(x, i)$ defined on $R^n \times S$ such that for each $i \in S$, they are continuously twice differentiable in x. For $V(x, i) \in C^2(R^n \times S; R_+)$, we define the function $LV : R^n \times S \times [t_0, \infty) \to R$ by

$$LV(x, i, t)$$

$$= V_x(x, i)f(x, i) + \frac{1}{2}[g^\top(x, i)I(t)V_{xx}(x, i)g(x, i)I(t)] + \sum_{j=1}^{N} \gamma_{ij}V(x, j), \quad (4)$$

where $V_x(x, i) = \left(\frac{\partial V(x,i)}{\partial x_1}, \cdots, \frac{\partial V(x,i)}{\partial x_n}\right), \quad V_{xx}(x, i) = \left(\frac{\partial^2 V(x,i)}{\partial x_i \partial x_j}\right)_{n \times n}.$

Note that $\mathcal{A}_p$ is a nonsingular M-matrix, by Theorem 2.10 of [12], it follows that $\mathcal{A}_p^{-1}$ exists and $\mathcal{A}_p^{-1} \geq 0$. Set $(\varphi_1, \cdots, \varphi_N)^\top := \mathcal{A}_p^{-1} \overrightarrow{1}$, where $\overrightarrow{1} = (1, \cdots, 1)^\top$, we can obtain that $\varphi_i > 0$, $i \in S$. In the sequel, for simplicity, we use the following notations:

$$\varphi_m = \min_{i \in S} \varphi_i, \varphi_M = \max_{i \in S} \varphi_i, \bar{\varphi} = \max_{i \in S} \frac{1}{\varphi_i}(p\varphi_i \alpha_i + \sum_{j=1}^N \gamma_{ij}\varphi_j).$$

Remark 3.3. Note that we are only interested in the case when $\bar{\varphi} \geq 0$, otherwise $dx(t)/dt = f(x(t), r(t))$ is already exponentially stable in pth moment and hence almost surely.

Theorem 3.4. *Under Assumptions 2.1, 3.1 and 3.2. If $\theta \in (\frac{\bar{\varphi}}{\varphi_M^{-1} + \bar{\varphi}}, 1)$, then the unique global solution $y(t)$ of (2) satisfies*

$$E|y(t)|^p \leq \frac{\varphi_M}{\varphi_m} E|y_0|^p e^{\gamma(t-t_0)}, \quad \forall\, t \geq t_0, \tag{5}$$

where $\gamma = \bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta$. In other words, the trivial solution of (2) is pth moment exponentially stable.

Proof. For $y_0 = 0$, that is $y(t; t_0, 0, r_0) = 0$, we can deduce that the assertion is natural. Fix $y_0 \neq 0$ and $r_0 \in S$ arbitrary, write $y(t; t_0, y_0, r_0) = y(t)$. By Mao [12], we have that $y(t) \neq 0$ for all $t \geq t_0$ almost surely. Define a function $V(y, i) = \varphi_i |y|^p$. Clearly, $\varphi_m |y|^p \leq V(y, i) \leq \varphi_M |y|^p$. By (4) and Assumption 3.2, we compute the operator LV as follows:

$$
\begin{aligned}
LV(y, i, t) &\leq \left((p\alpha_i + 0.5p[\beta_i^2 - (2-p)\sigma_i^2])\varphi_i + \sum_{j=1}^N \gamma_{ij}\varphi_j\right)I(t)|y|^p \\
&\quad + \left(p\varphi_i\alpha_i + \sum_{j=1}^N \gamma_{ij}\varphi_j\right)(1 - I(t))|y|^p \\
&= \frac{1}{\varphi_i}\left((p\alpha_i + 0.5p[\beta_i^2 - (2-p)\sigma_i^2])\varphi_i + \sum_{j=1}^N \gamma_{ij}\varphi_j\right)I(t)V(y, i) \\
&\quad + \frac{1}{\varphi_i}\left(p\varphi_i\alpha_i + \sum_{j=1}^N \gamma_{ij}\varphi_j\right)(1 - I(t))V(y, i) \\
&= -\frac{1}{\varphi_i}I(t)V(y, i) + \frac{1}{\varphi_i}\left(p\varphi_i\alpha_i + \sum_{j=1}^N \gamma_{ij}\varphi_j\right)(1 - I(t))V(y, i) \\
&\leq [-\varphi_M^{-1}I(t) + \bar{\varphi}(1 - I(t))]V(y, i). \tag{6}
\end{aligned}
$$

For each integer $d \geq 1$, define a stopping time $\rho_d = \inf\{t \geq t_0 : |y(t)| \geq d\}$. Clearly, $\rho_d \to \infty$ almost surely as $d \to \infty$. For $t \geq t_0$, the generalized Itô formula shows that

$$
\begin{aligned}
&E\left[V(y(t \wedge \rho_d), r(t \wedge \rho_d))e^{-\int_{t_0}^{t \wedge \rho_d}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1 - I(s))]ds}\right] \\
&= EV(y_0, r_0) + E\int_{t_0}^{t \wedge \rho_d} e^{-\int_{t_0}^s[-\varphi_M^{-1}I(u) + \bar{\varphi}(1 - I(u))]du} \\
&\quad \times \left(LV(y(s), r(s)) - [-\varphi_M^{-1}I(s) + \bar{\varphi}(1 - I(s))]V(x(s), r(s))\right)ds.
\end{aligned}
$$

By (6), we have

$$E\left[V(y(t \wedge \rho_d), r(t \wedge \rho_d))e^{-\int_{t_0}^{t \wedge \rho_d}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1-I(s))]ds}\right] \leq EV(y_0, r_0).$$

Letting $d \to \infty$ gives

$$E\left[V(y(t), r(t))e^{-\int_{t_0}^{t}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1-I(s))]ds}\right] \leq EV(y_0, r_0).$$

This implies

$$E|y(t)|^p \leq \frac{\varphi_M}{\varphi_m}E|y_0|^p e^{\int_{t_0}^{t}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1-I(s))]ds}. \tag{7}$$

By condition $\theta \in (\frac{\bar{\varphi}}{\varphi_M^{-1} + \bar{\varphi}}, 1)$, we have

$$-\varphi_M^{-1} \leq -\varphi_M^{-1}\theta \leq -\varphi_M^{-1}\theta + \bar{\varphi}(1 - \theta) = \bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta.$$

For $t \in [t_0 + k\delta, t_0 + k\delta + \theta\delta)$, we can derive that

$$\begin{aligned}
\int_{t_0}^{t}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1 - I(s))]ds &= \bar{\varphi}(t - t_0) - (\varphi_M^{-1} + \bar{\varphi})[k\theta\delta + t - (t_0 + k\delta)] \\
&= [\bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta]k\delta - \varphi_M^{-1}[t - (t_0 + k\delta)] \\
&\leq [\bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta](t - t_0).
\end{aligned}$$

While for $t \in [t_0 + k\delta + \theta\delta, t_0 + (k+1)\delta)$, we can obtain that

$$\begin{aligned}
\int_{t_0}^{t}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1 - I(s))]ds &= \bar{\varphi}(t - t_0) - (\varphi_M^{-1} + \bar{\varphi})(k+1)\theta\delta \\
&\leq \bar{\varphi}(t - t_0) - (\varphi_M^{-1} + \bar{\varphi})(t - t_0) \\
&= [\bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta](t - t_0).
\end{aligned}$$

Consequently, for any $t \in [t_0 + k\delta, t_0 + (k+1)\delta)$, the above two inequalities show that

$$\int_{t_0}^{t}[-\varphi_M^{-1}I(s) + \bar{\varphi}(1 - I(s))]ds \leq [\bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta](t - t_0).$$

Hence, we conclude that

$$E|y(t)|^p \leq \frac{\varphi_M}{\varphi_m}E|y_0|^p e^{[\bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta](t - t_0)}.$$

The proof is therefore complete. □

Remark 3.5. Obviously, it follows from (5) that

$$\limsup_{t \to \infty} \frac{1}{t} \log E|x(t)|^p \leq \bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta. \tag{8}$$

In other words, the pth moment Lyapunov exponent of the solution (2) should be less than or equal to $\bar{\varphi} - (\varphi_M^{-1} + \bar{\varphi})\theta$.

Remark 3.6. As $\theta \to 0$, the intermittently hybrid SDEs (2) will degenerate into a hybrid differential equations. Due to $\bar{\varphi} > 0$, then it follow from (5) that the hybrid differential equations is not exponentially stable. As $\theta \to 1$, the intermittently hybrid SDEs (2) will become continuous hybrid SDEs

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dw(t) \tag{9}$$

which is studied by Mao and Yuan [12]. If $\varphi_M^{-1} > 0$, then by Theorem 5.8 (see, [12]), the solution of continuous hybrid SDEs (9) is pth moment exponentially stable. Therefore, our results improve and generalize the existing work [6, 12, 13].

Remark 3.7. In fact, by Theorem 3.4, we can obtain that the trivial solution of (2) is almost surely exponentially stable if $\theta \in (\frac{\bar{\varphi}}{\varphi_M^{-1}+\bar{\varphi}}, 1)$. From Theorem 3.4, it is showed that the speed at which the solution of (2) converges to the equilibrium not only depends on the coefficient $\alpha_i, \beta_i, \sigma_i$, but also on the range of the intermittent control parameter θ.

We can now state our main theorem in this paper.

Theorem 3.8. *Let Assumptions 2.1, 3.1 and 3.2 hold. Choose a free parameter* $\varepsilon \in (0,1)$ *and take* $\bar{T} = -\frac{1}{\gamma} \log\left(\frac{\varphi_M}{\varphi_m \varepsilon}\right)$. *Let* $\tau^* > 0$ *be the unique root to (18). Then for each* $\tau \in (0, \tau^*)$, *we can choose* δ *(the period of the intermittent control) such that* $\delta = (\bar{T} + 2\tau)/\bar{N}$ *for some positive integer* $\bar{N}$ *and* $\theta \in (\frac{\bar{\varphi}}{\varphi_M^{-1}+\bar{\varphi}}, 1)$ *in order for the controlled differential equations (1) to be exponentially stable in pth moment and in probablity 1.*

To prove Theorem 3.8, we present some lemmas.

Lemma 3.9. *Let Assumption 2.1 hold and* $p \in (0,1)$. *Then, for any* $t_0 \geq 0$ *and* $T \geq 0$,

$$\sup_{t_0 \leq t \leq t_0+T+\tau} E|x(t)|^p \leq H_1(p,\tau,T)(E||\xi||^2)^{\frac{p}{2}}, \qquad (10)$$

$$E\left(\sup_{t_0 \leq t \leq t_0+T+\tau} |x(t)|^p\right) \leq H_2(p,\tau,T)(E||\xi||^2)^{\frac{p}{2}}, \qquad (11)$$

$$\sup_{t_0 \leq t \leq t_0+T} E\left(\sup_{0 \leq u \leq \tau} |x(t+u)-x(t)|^p\right) \leq H_3(p,\tau,T)(E||\xi||^2)^{\frac{p}{2}}, \qquad (12)$$

where

$$H_1(p,\tau,T) = (1+k_2^2\tau)^{0.5p}e^{p(k_1+0.5k_2^2)(T+\tau)},$$

$$H_2(p,\tau,T) = \left(3 + 3(T+\tau)H_1(2,\tau,T)[(T+\tau)k_1^2 + 4k_2^2]\right)^{\frac{p}{2}}$$

and

$$H_3(p,\tau,T) = \left(H_3(2,\tau,T)\right)^{\frac{p}{2}}.$$

Proof. By the Itô formula, it is easy to show that, for $t \in [t_0, t_0+T+\tau]$,

$$|x(t)|^2 = |x(t_0)|^2 + \int_{t_0}^t [2x^\top(s)f(x(s),r(s)) + |g(x(s-\tau),r(s))I(s)|^2]ds$$

$$+ \int_{t_0}^t 2x^\top(s)g(x(s-\tau),r(s))I(s)dw(s).$$

Taking expectations on both sides and by Assumption 2.1, we get

$$E|x(t)|^2 \leq E|x(t_0)|^2 + 2k_1 \int_{t_0}^t E|x(s)|^2 ds + k_2^2 \int_{t_0}^t E|x(s-\tau)|^2 ds.$$

$$\leq (1+k_2^2\tau)E||\xi||^2 + (2k_1+k_2^2)\int_{t_0}^t \sup_{t_0 \leq u \leq s} E|x(u)|^2 ds.$$

Noting that the right-hand-side term of the above inequality is increasing in $t \in [t_0, \infty)$, we hence have

$$\sup_{t_0 \leq u \leq t} E|x(u)|^2 \leq (1+k_2^2\tau)E||\xi||^2 + (2k_1+k_2^2)\int_{t_0}^t \left(\sup_{t_0 \leq u \leq s} E|x(u)|^2\right)ds.$$

Consequently, the Gronwall inequality gives

$$\sup_{t_0 \leq u \leq t_0+T+\tau} E|x(t)|^2 \quad \leq \quad (1+k_2^2\tau)E||\xi||^2 e^{(2k_1+k_2^2)(T+\tau)}.$$

where $H_1(2,\tau,T) = (1+k_2^2\tau)e^{(2k_1+k_2^2)(T+\tau)}$. By the Hölder inequality, we then have

$$E|x(t)|^p \quad \leq \quad \left(E|x(t)|^2\right)^{\frac{p}{2}} \leq \left((1+k_2^2\tau)e^{(2k_1+k_2^2)(T+\tau)}\right)^{\frac{p}{2}}(E||\xi||^2)^{\frac{p}{2}}.$$

Similarly, we can show that

$$E\left(\sup_{t_0 \leq t \leq t_0+T+\tau}|x(t)|^2\right) \quad \leq \quad 3E|x(t_0)|^2 + 3E\sup_{t_0 \leq t \leq t_0+T+\tau}|\int_{t_0}^t f(x(s),r(s))ds|^2$$

$$+ \quad 3E\sup_{t_0 \leq t \leq t_0+T+\tau}|\int_{t_0}^t g(x(s-\tau),r(s))I(s)dw(s)|^2$$

$$\leq \quad 3E||\xi||^2 + 3k_1^2(T+\tau)\int_{t_0}^{t_0+T+\tau}E|x(s)|^2ds$$

$$+ \quad 12k_2^2\int_{t_0}^{t_0+T+\tau}E|x(s-\tau)|^2ds$$

$$\leq \quad \left(3+3(T+\tau)H_1(2,\tau,T)[(T+\tau)k_1^2+4k_2^2]\right)E||\xi||^2.$$

Consequently,

$$E\left(\sup_{t_0 \leq t \leq t_0+T+\tau}|x(t)|^p\right) \quad \leq \quad \left(3+3(T+\tau)H_1(2,\tau,T)[(T+\tau)k_1^2+4k_2^2]\right)^{\frac{p}{2}}(E||\xi||^2)^{\frac{p}{2}}.$$

By the Hölder inequality and Burkholder-Davis-Gundy inequality, Assumption 2.1, we derive that

$$E\left(\sup_{0 \leq u \leq \tau}|x(t+u)-x(t)|^2\right)$$

$$\leq \quad 2E\left(\sup_{0 \leq u \leq \tau}|\int_t^{t+u}f(x(s),r(s))ds|^2\right)$$

$$+2E\left(\sup_{0 \leq u \leq \tau}|\int_t^{t+u}g(x(s-\tau),r(s))dw(s)|^2\right)$$

$$\leq \quad 2\tau E\int_t^{t+\tau}|f(x(s),r(s))|^2ds + 8E\int_t^{t+\tau}|g(x(s-\tau),r(s))|^2ds$$

$$\leq \quad 2\tau k_1^2 E\int_t^{t+\tau}|x(s)|^2ds + 8k_2^2 E\int_t^{t+\tau}|x(s-\tau)|^2ds$$

$$\leq \quad H_3(2,\tau,T)E||\xi||^2.$$

where $H_3(2,\tau,T) = 2H_1(2,\tau,T)(k_1^2\tau^2+4k_2^2\tau)$. Once again, by the Hölder inequality, we have

$$E\left(\sup_{0 \leq u \leq \tau}|x(t+u)-x(t)|^p\right) \quad \leq \quad \left(H_3(2,\tau,T)\right)^{\frac{p}{2}}(E||\xi||^2)^{\frac{p}{2}}.$$

The proof is therefore complete. $\square$

Lemma 3.10. *Let Assumption 2.1 hold and $p \in (0,1)$. Fix $t_0 > \tau$ and $T \geq 0$ arbitrarily. Then, for $t \in [t_0, t_0+\tau+T]$,*

$$E|x(t)-y(t)|^p \quad \leq \quad H_4(p,\tau,T)(E||\xi||^2)^{\frac{p}{2}},$$

where $H_4(p, \tau, T) = \left(2k_2^2(T + \tau)H_3(2, \tau, T)e^{(2k_1 + 2k_2^2)(T + \tau)} \right)^{\frac{p}{2}}.$

Proof. By the Itô formula and Assumption 2.1, we can show that for $t \in [t_0, t_0 + \tau + T]$,

$$
\begin{aligned}
E|x(t) - y(t)|^2 &= E \int_{t_0}^t \Big(2[x(s) - y(s)]^\top [f(x(s), r(s)) - f(y(s), r(s))] \\
&\quad + |[g(x(s - \tau), r(s)) - g(y(s), r(s))]I(s)|^2 \Big) ds \\
&\leq (2k_1 + 2k_2^2) \int_{t_0}^t E|x(s) - y(s)|^2 ds \\
&\quad + 2k_2^2 \int_{t_0}^t E|x(s) - x(s - \tau)|^2 ds. \tag{13}
\end{aligned}
$$

Inserting (12) into (13), we get

$$
\begin{aligned}
E|x(t) - y(t)|^2 &\leq (2k_1 + 2k_2^2) \int_{t_0}^t E|x(s) - y(s)|^2 ds \\
&\quad + 2k_2^2 H_3(2, \tau, T)(T + \tau)E||\xi||^2.
\end{aligned}
$$

Then, the Gronwall inequality implies that

$$
E|x(t) - y(t)|^2 \leq 2k_2^2(T + \tau)H_3(2, \tau, T)e^{(2k_1 + 2k_2^2)(T + \tau)}E||\xi||^2.
$$

By the Hölder inequality, we have

$$
E|x(t) - y(t)|^p \leq H_4(p, \tau, T)(E||\xi||^2)^{\frac{p}{2}},
$$

which is the required assertion. The proof is therefore complete. $\qquad \square$

Proof of Theorem 3.8. Fix $\tau \in (0, \tau^*)$ and the initial data $\xi \in L^2_{\mathcal{F}_{t_0}}, r_0 \in S$. For simplicity, we write $x(t; t_0, \xi, r_0) = x(t)$, $r(t; t_0, r_0) = r(t)$ for $t \geq t_0$. Likewise, we write $y(t_0 + \tau + \bar{T}; t_0 + \tau, x(t_0 + \tau), r(t_0 + \tau)) = y(t_0 + \tau + \bar{T})$. By Theorem 3.4 and Lemma 3.9, we have

$$
\begin{aligned}
E|y(t_0 + \tau + \bar{T})|^p &\leq \frac{\varphi_M}{\varphi_m} E|x(t_0 + \tau)|^p e^{\gamma \bar{T}} \\
&\leq \frac{\varphi_M}{\varphi_m}(1 + k_2^2\tau)^{0.5p}e^{p(k_1 + 0.5k_2^2)\tau}(E||\xi||^2)^{\frac{p}{2}}e^{\gamma \bar{T}}, \tag{14}
\end{aligned}
$$

where γ has been defined in Theorem 3.4. By the elementary inequality $(x + y)^p \leq x^p + y^p$ for any $x, y \geq 0$, we have

$$
E|x(t_0 + \tau + \bar{T})|^p \leq E|y(t_0 + \tau + \bar{T})|^p + E|x(t_0 + \tau + \bar{T}) - y(t_0 + \tau + \bar{T})|^p.
$$

Using (14) as well as lemma 3.10, we get

$$
\begin{aligned}
&E|x(t_0 + \tau + \bar{T})|^p \\
&\leq \left(\frac{\varphi_M}{\varphi_m}(1 + k_2^2\tau)^{0.5p}e^{p(k_1 + 0.5k_2^2)\tau}e^{\gamma \bar{T}} + H_4(p, \tau, \bar{T}) \right)(E||\xi||^2)^{\frac{p}{2}}. \tag{15}
\end{aligned}
$$

On the other hand, by lemma 3.9, we have

$$
\begin{aligned}
E||x_{t_0 + 2\tau + \bar{T}}||^p &\leq E|x(t_0 + \tau + \bar{T})|^p + E \sup_{0 \leq u \leq \tau} |x(t_0 + \tau + \bar{T}) \\
&\quad - x(t_0 + \tau + \bar{T} + u)|^p \\
&\leq E|x(t_0 + \tau + \bar{T})|^p + H_3(p, \tau, \bar{T})(E||\xi||^2)^{\frac{p}{2}}. \tag{16}
\end{aligned}
$$

Inserting (15) into (16), we obtain that

$$E||x_{t_0+2\tau+\bar{T}}||^p \leq J(\tau,\varepsilon)(E||\xi||^2)^{\frac{p}{2}}, \tag{17}$$

where

$$J(\tau,\varepsilon) = \varepsilon(1+k_2^2\tau)^{0.5p}e^{p(k_1+0.5k_2^2)\tau} + H_3(p,\tau,\bar{T}) + H_4(p,\tau,\bar{T}).$$

Noting that $J(\tau,\varepsilon)$ is a continuously increasing function of $\tau > 0$ and $J(0,\varepsilon) = \varepsilon < 1$, then the following equation

$$\varepsilon(1+k_2^2\tau)^{0.5p}e^{p(k_1+0.5k_2^2)\tau} + H_3(p,\tau,\bar{T}) + H_4(p,\tau,\bar{T}) = 1 \tag{18}$$

must have a unique root $\tau^* > 0$. Since $\tau < \tau^*$, it follows from (18) that

$$\varepsilon(1+k_2^2\tau)^{0.5p}e^{p(k_1+0.5k_2^2)\tau} + H_3(p,\tau,\bar{T}) + H_4(p,\tau,\bar{T}) < 1.$$

Therefore, we may choose $\lambda > 0$ such that

$$\varepsilon(1+k_2^2\tau)^{0.5p}e^{p(k_1+0.5k_2^2)\tau} + H_3(p,\tau,\bar{T}) + H_4(p,\tau,\bar{T}) = e^{-\lambda(2\tau+\bar{T})}.$$

Then, it follows from (17) that

$$E||x_{t_0+2\tau+\bar{T}}||^p \leq e^{-\lambda(2\tau+\bar{T})}(E||\xi||^2)^{\frac{p}{2}}. \tag{19}$$

Let us proceed to consider the solution $x(t)$ on $t \geq t_0 + 2\tau + \bar{T}$. Obviously, we can get $\bar{T} + 2\tau = \bar{N}\delta$ for some positive constant $\bar{N}$. By the flow property, this can be regarded as the solution of (1) with the initial data $x_{t_0+\bar{N}\delta}$ and $r(t_0 + \bar{N}\delta)$ at $t = t_0 + \bar{N}\delta$. In the same way as we did above, we can show

$$E||x_{t_0+2\bar{N}\delta}||^p \leq e^{-\lambda\bar{N}\delta}E||x_{t_0+\bar{N}\delta}||^p.$$

By (19), this implies

$$E||x_{t_0+2\bar{N}\delta}||^p \leq e^{-2\lambda\bar{N}\delta}(E||\xi||^2)^{\frac{p}{2}}.$$

Repeating this procedure, we have

$$E||x_{t_0+k\bar{N}\delta}||^p \leq e^{-k\lambda\bar{N}\delta}(E||\xi||^2)^{\frac{p}{2}},$$

for all $k = 1, 2, \cdots$. Now, by lemma 3.9, we have

$$E\Big(\sup_{t_0+k\bar{N}\delta \leq t \leq t_0+(k+1)\bar{N}\delta} E|x(t)|^p\Big) \leq H_2(p,\tau,\bar{N}\delta-\tau)e^{-k\lambda\bar{N}\delta}(E||\xi||^2)^{\frac{p}{2}}, \tag{20}$$

for all $k = 0, 1, 2, \cdots$. Hence, for $t \in [t_0 + k\bar{N}\delta, t_0 + (k+1)\bar{N}\delta]$,

$$\frac{1}{t}\log(E|x(t)|^p) \leq \frac{\log(H_2(p,\tau,\bar{N}\delta-\tau)(E||\xi||^2)^{\frac{p}{2}}) - k\lambda\bar{N}\delta}{t_0 + k\bar{N}\delta}.$$

This implies

$$\limsup_{t\to\infty} \frac{1}{t}\log E|x(t)|^p \leq -\lambda.$$

Using Markov inequality and (20), we get

$$P\Big(\sup_{t_0+k\bar{N}\delta \leq t \leq t_0+(k+1)\bar{N}\delta} |x(t)|^p \geq e^{-0.5k\lambda\bar{N}\delta}\Big) \leq H_2(p,\tau,\bar{N}\delta-\tau)e^{-0.5k\lambda\bar{N}\delta}(E||\xi||^2)^{\frac{p}{2}},$$

for all $k \geq 0$. By the Borel-Cantelli lemma, we can obtain that for almost all $\omega \in \Omega$, there exists an integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_0+k\bar{N}\delta \leq t \leq t_0+(k+1)\bar{N}\delta} |x(t)|^p < e^{-0.5k\lambda\bar{N}\delta}$$

for any $k > k_0(\omega)$. This implies that

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t,\omega)|) < -\frac{\lambda}{2p}$$

for almost all $\omega \in \Omega$. The proof is therefore complete. $\square$

Remark 3.11. In [10], T_1 of (41) is defined by $T_1 > \max\{\frac{2}{\gamma_p}\log\left(\frac{4^p}{\varepsilon E||\xi||^p}\right), T\}$, where T is an uncertain positive number. So even if ε is chosen, T_1 is difficult to determine, thus the upper bound of τ cannot be determined. While in this paper, once ε is chosen, $\bar{T}$ of (18) is determined, then we can obtain an estimate on τ^*.

Remark 3.12. In fact, by using the existing work [8], we can not prove Theorem 3.8 dirctly. Since the intermittent stochastic noise is introduced in this paper, we do not only need to determine the upper bound of the time delay τ, but also require some constraints on the control period δ. From the proof of Theorem 3.8, we see that the controlled differential equations (1) can be stabilized if the delay term τ is chosen in $(0, \tau^*)$ and the period of the intermittent control δ is equal to $(\bar{T}+2\tau)/\bar{N}$.

4. **An example.**

Example 4.1. Let us consider the following linear hybrid differential equations

$$\frac{dx(t)}{dt} = \mu(r(t))x(t), \tag{21}$$

where $\mu(1) = 0.5$, $\mu(2) = -1$ and $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix} -1.9 & 1.9 \\ 4 & -4 \end{bmatrix}.$$

It is obvious that $\dot{x}(t) = 0.5x(t)$ is unstable, but $\dot{x}(t) = -x(t)$ is stable. However, hybrid differential equations (21) is unstable. Figure 1 also shows this instability.

Let us now apply our theory to design the periodic intermittent control to stabilize the hybrid differential equations (21). Consider $\beta(r(t))x(t)dw(t)$ as the stochastic perturbation, then the intermittently hybrid SDEs can be described by

$$dx(t) = \mu(r(t))x(t)dt + \beta(r(t))x(t)I(t)dw(t), \tag{22}$$

where $I(t) = \sum_{k=0}^{\infty} I_{[k\delta, k\delta+\theta\delta)}(t)$, $\theta \in (0,1)$, $\beta(1) = 1$, $\beta(2) = \sqrt{2}$. We assume $w(t)$ and $r(t)$ are assumed to be independent. Choose $p = 0.5$, then $\rho_1(0.5) = -0.125$ and $\rho_2(0.5) = 0.75$. The matrix $\mathcal{A}_{0.5}$ defined in (3) becomes

$$\mathcal{A}_{0.5} = \begin{bmatrix} 1.775 & -1.9 \\ -4 & 4.75 \end{bmatrix}$$

which is a nonsingular M-matrix. Then, we can determine $\varphi_1 = 6.917, \varphi_2 = 5.864$ and hence, $\varphi_M^{-1} = 0.1445$ and $\bar{\varphi} = 0.218$. By Theorem 3.4, we can conclude that if $\theta \in (0.61, 1)$, then intermittently hybrid SDEs (22) has the property that

$$\limsup_{t\to\infty} \frac{1}{t}\log E|x(t)|^p \leq 0.218 - 0.3625\theta < 0.$$

That is, hybrid differential equations (21) can be stabilized by intermittent stochastic perturbation $\beta(r(t))x(t)I(t)dw(t)$. Figure 2 show that the differential equations (22) is stable if the intermittent parameters $\theta = 0.95$.
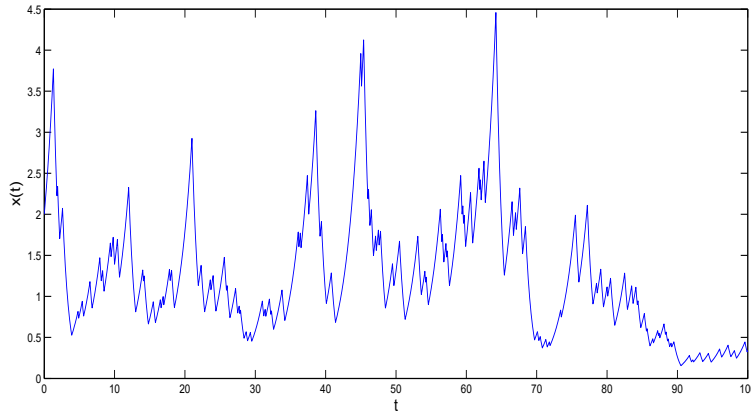
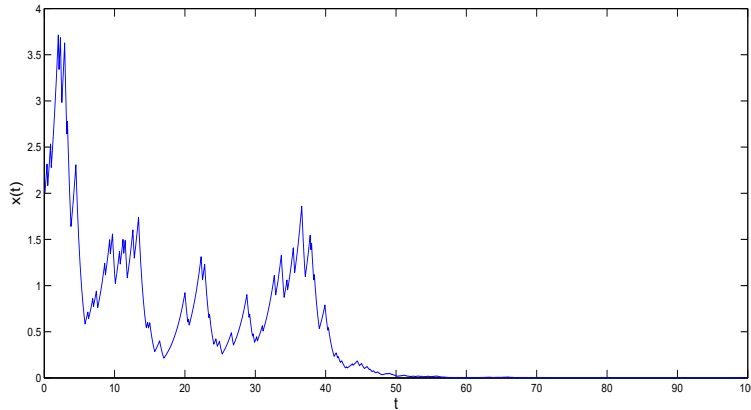FIGURE 1. The sample paths of the hybrid differential equations (21).

FIGURE 2. The sample paths of the intermittently hybrid SDEs (22) with $\theta = 0.95$.

Now, we will use the delay feedback control to stablize the hybrid differential equations (21)

$$dx(t) \quad = \mu(r(t))x(t)dt + \beta(r(t))x(t - \tau)I(t)dw(t). \tag{23}$$

We further choose $\theta = 0.95$, $\varepsilon = 0.9$, then Eq.(18) becomes

$$0.9(1 + 2\tau)^{0.25}e^\tau + H_3(0.5, \tau, 0.9293) + H_4(0.5, \tau, 0.9293) = 1$$

which has the unique positive root $\tau^* = 8.01 \times 10^{-11}$ (which is about 2.6 mi-croseconds if the time unit is of year). By Theorem 3.8, we can conclude that the intermittently hybrid SDDEs (23) is almost surely exponentially stable provided $\tau < 8.01 \times 10^{-11}$ and $\delta = 0.00273$ (which is about one day if the time unit is of year). The computer simulation (see Figure 3) supports this results clearly.
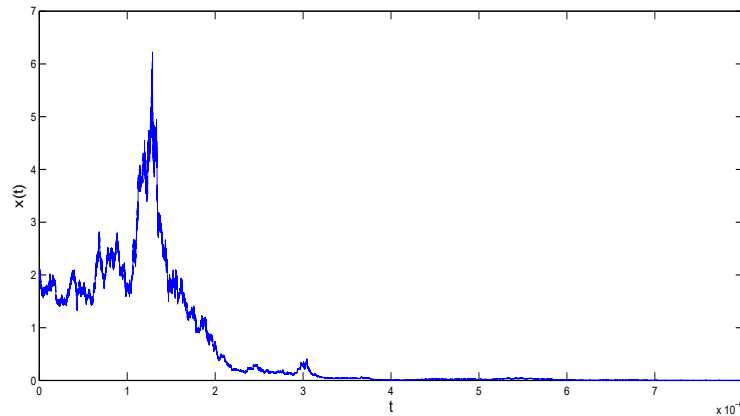
FIGURE 3. The sample paths of the intermittently hybrid SDDEs (23).

Conclusions. This paper is devoted to the stablization of hybrid differential equations by intermittent control based on delay observations. The M-matrix theory and intermittent control strategy are used to establish sufficient stability criterion on intermittent hybrid SDEs. Moreover, by comparison method and stochastic analysis, we obtain that hybrid differential equations can be stabilized by the intermittent stochastic noise with time delay as long as $\tau < \tau^*$.

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