Minimal travelling wave speed and explicit solutions in monostable reaction-diffusion equations

Dedicated to Professor Jeffrey R. L. Webb on the occasion of his 75th birthday

Elaine C. M. Crooks[1] and **Michael Grinfeld**[✉2]

[1]Department of Mathematics, College of Science, Swansea University,
Bay Campus, Swansea SA1 8EF, UK
[2]Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK

Abstract. We investigate the connection between the existence of an explicit travelling wave solution and the travelling wave with minimal speed in a scalar monostable reaction-diffusion equation.

Keywords: monostable reaction-diffusion equations, travelling waves, variational principles.

2020 Mathematics Subject Classification: 35C07, 35K55, 35K91.

To Jeff in appreciation and gratitude

1   Introduction

In this short paper we investigate the somewhat puzzling connection between the existence of an explicit travelling wave solution and the travelling wave with minimal speed in a monostable reaction-diffusion equation. More precisely, there are examples in the literature (see below) where the explicitly computable travelling wave solution is the solution with minimal speed. Moreover, for parameter-dependent problems with a parameter-dependent family of explicit solutions, there are many cases where in fact there is a switching between the minimal speed being given by this explicit solution for some parameters, while for others it is given by the so-called linear speed, defined as the minimal value for which the problem linearised about the unstable steady state has a suitable eigenvalue. For a particular set of equations, of a type encountered in applications, we formulate sufficient conditions for each of these phenomena to occur.

[✉]Corresponding author. Email: m.grinfeld@strath.ac.uk

The plan of the paper is as follows. In this section, we introduce scalar monostable reaction-diffusion equations, define what we mean by a minimal speed, and discuss the linear (pulled) and the non-linear (pushed) regimes.

In Section 2, we define the set of exactly solvable equations and prove a result connecting the minimal wave speed and the speed of an explicit travelling wave solution.

Finally, in Section 3 we consider conditions for the exchange of minimality between the linear minimal speed and the speed of an explicit travelling wave solution.

Our proofs exploit two main tools: the variational principle due to Hadeler and Rothe [4] and the integrability characterisations of the minimal speed proved by Lucia, Muratov and Novaga in [6].

We consider reaction-diffusion equations of the form

$$u_t = u_{xx} + f(u, \beta),\tag{1.1}$$

where $\beta \in \mathbb{R}$ is a parameter, and f is a monostable nonlinearity, *i.e.*,

$$f(0, \beta) = f(1, \beta) = 0, \quad f'(0, \beta) > 0, \quad f'(1, \beta) < 0, \quad f(u, \beta) > 0 \quad \text{for } u \in (0, 1).$$

In the travelling wave frame $z = x - ct$, $c \geq 0$, setting $U(z) = u(x, t)$, and denoting derivatives with respect to z by primes, (1.1) becomes

$$-cU' = U'' + f(U).\tag{1.2}$$

We seek **monotone fronts** connecting 1 and 0, *i.e.*, solutions $U(z)$ of (1.2) with $U'(z) < 0$ and

$$\lim_{z \to -\infty} U(z) = 1 \quad \text{and} \quad \lim_{z \to \infty} U(z) = 0.$$

Linearisation around the rest point with $U = 0$ shows that there cannot be any monotone fronts connecting 1 and 0 for $c < c_l := 2\sqrt{f'(0)}$. Phase plane analysis shows that there exists $c_{\min} \geq c_l$ such that there exists a monotone front for all $c \geq c_{\min} \geq c_l$. Determining $c_{\min}$ is often of interest in applications, see e.g. [2] for a discussion.

Definition 1.1. If $c_{\min} = c_l$, we say that we are in the case of *linear selection mechanism* ("pulled case") and if $c_{\min} > c_l$, of *nonlinear selection mechanism* ("pushed case").

Frequently, the basis of analysis of monotone fronts in the scalar monostable case (1.2) is the following construction: As $U(z)$ is a monotone solution, its derivative is a well-defined function of U. Set $F(U) := -U'$. Note that $F(U)$ is non-negative. Also, $F(0) = F(1) = 0$. Now,

$$F(U)' = (-U')' = -U''.$$

On the other hand, by the chain rule,

$$F(U)' = \frac{dF}{dU}U' = -\frac{dF}{dU}F.$$

Hence the problem of solving $U'' + cU' + f(U) = 0$ with the conditions that $\lim_{z \to -\infty} U(z) = 1$ and $\lim_{z \to \infty} U(z) = 0$ is equivalent to solving

$$F\frac{dF}{dU} - cF + f(U) = 0, \quad F(0) = F(1) = 0.\tag{1.3}$$

Using this construction, we have the Hadeler–Rothe variational principle [4]:

$$c_{\min} = \inf_{g \in \mathcal{G}} \sup_{0 < U < 1} \left\{ g'(U) + \frac{f(U)}{g(U)} \right\}, \tag{1.4}$$

where

$$\mathcal{G} = \left\{ g \in C^1([0,1]) \mid g(U) > 0 \text{ for } 0 < U < 1, \, g(0) = 0, \, g'(0) > 0 \right\}. \tag{1.5}$$

2 Exact solvability

We are interested in the situation when (1.2) has a solution $U(z)$ that can be determined by quadratures. A sufficient condition is:

Lemma 2.1. *The travelling wave equation of* (1.3) *with speed* $c = A(\beta)/\sqrt{B(\beta)}$ *is solvable by quadratures if* f *can be written in the form*

$$f(u, \beta) = h(u)\Big(A(\beta) - B(\beta)h'(u)\Big), \qquad h \in C^1([0,1]), \tag{2.1}$$

where $h(0) = h(1) = 0$, $h(u) \geq 0$, $h'(0) > 0$ *(without loss of generality $h'(0) = 1$), $A(\beta) > B(\beta) > 0$, and for all $u \in [0,1]$, $A(\beta) - B(\beta)h'(u) > 0$.*

Proof. In this case a solution of (1.3) is $F(U) = \gamma h(U)$ with

$$\gamma = \sqrt{B(\beta)}, \tag{2.2}$$

from which U and c can be computed by quadratures. $\square$

We introduce notation for the speeds of the explicit fronts in Lemma 2.1:

$$c_{nl}(\beta) := \frac{A(\beta)}{\sqrt{B(\beta)}}. \tag{2.3}$$

We will describe as the **solvable case** the situation in which the nonlinearity $f(u, \beta)$ satisfies the conditions of Lemma 2.1. In the solvable case, we have that

$$c_l = 2\sqrt{A(\beta) - B(\beta)}. \tag{2.4}$$

Note that the fact that $A(\beta) > B(\beta)$ follows from the conditions of Lemma 2.1.

Of course, by the definition of minimal speed, we always have that

$$c_{\min}(\beta) \leq c_{nl}(\beta) = \frac{A(\beta)}{\sqrt{B(\beta)}}. \tag{2.5}$$

3 Minimality exchange

In this section, for a nonlinearity $f(u, \beta)$ of solvable type, we investigate conditions under which there exists a value β^*, such that for values β to one side of β^*, $c_{\min}(\beta) = c_l(\beta)$, and for values of β to the other side of β^*, $c_{\min}(\beta) = c_{nl}(\beta)$, so that at β^* minimality is exchanged between $c_l(\beta)$ and $c_{nl}(\beta)$. This is what we call a **minimality exchange**. Examples, two of which we outline below, are discussed in [4, 6] and the isotropic case of [2], which is also investigated in [3, 8].

First note that for a minimality exchange, the graphs of $c_l(\beta)$ and $c_{nl}(\beta)$ must clearly intersect. Therefore the equation

$$2\sqrt{A(\beta) - B(\beta)} = \frac{A(\beta)}{\sqrt{B(\beta)}}$$

must have a solution, which is equivalent to demanding the existence of β^* such that $A(\beta^*) = 2B(\beta^*)$.

Hence, for instance, in any equation (1.1) with solvable $f(u, \beta)$ such that $A(\beta) = 2B(\beta) + 1$, there can never be a minimality exchange between the linear and the nonlinear speeds.

Before continuing with the analysis, we present two concrete examples of minimality exchange. In [4, Eq. (27)], Hadeler and Rothe consider the nonlinearity

$$f(u, \beta) = u(1 - u)(1 + \beta u), \qquad \beta \geq -1,$$

which can be put into the framework of Lemma 2.1 by setting $h(u) = u(1 - u)$, so that

$$f(u, \beta) = h(u)(A(\beta) - B(\beta)h'(u)),$$

where

$$A(\beta) = 1 + \frac{\beta}{2}, \qquad B(\beta) = \frac{\beta}{2}.$$

The solution of $A(\beta) = 2B(\beta)$ is therefore $\beta^* = 2$, the nonlinear speed is

$$c_{nl}(\beta) = \frac{2 + \beta}{\sqrt{2\beta}},$$

and it is shown in [4] that a minimality exchange occurs at $\beta = \beta^*$, with $c_{min}(\beta) = c_l(\beta)$ for $\beta < \beta^*$ and $c_{min}(\beta) = c_{nl}(\beta)$ for $\beta > \beta^*$.

Our second example is given by the isotropic case of [2], where

$$f(u, \beta) = \frac{\sin(\pi u)}{2\pi} \left[1 - \beta \cos(\pi u)\right],$$

which fits into the framework of Lemma 2.1 by setting $h(u) = \frac{\sin(\pi u)}{\pi}$, so that

$$f(u, \beta) = h(u)(A(\beta) - B(\beta)h'(u)),$$

where

$$A(\beta) = \frac{1}{2} \quad B(\beta) = \frac{\beta}{2}.$$

The equation $A(\beta) = 2B(\beta)$ then has solution $\beta^* = \frac{1}{2}$, the nonlinear speed is

$$c_{nl}(\beta) = \frac{1}{\sqrt{2\beta}},$$

and it is proved in [2, 3] that here too, a minimality exchange occurs at $\beta = \beta^*$, again with $c_{min}(\beta) = c_l(\beta)$ for $\beta < \beta^*$ and $c_{min}(\beta) = c_{nl}(\beta)$ for $\beta > \beta^*$.

We now establish our general results, starting with a sufficient condition for nonlinear selection.

Lemma 3.1. *For all β such that $A(\beta) < 2B(\beta)$, $c_{min}(\beta) = c_{nl}(\beta)$.*

Proof. For any $c > 0$, denote by $H_c^1(\mathbb{R})$ the completion of $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\|u\|_{1,c} = \|u\|_c + \|u_x\|_c, \quad \text{where} \quad \|u\|_c^2 = \int_{\mathbb{R}} e^{cx} u^2(x) \, dx.$$

If $U(z)$ is an explicit travelling front with $-U' = F(U) = \gamma h(U)$, we have

$$\lim_{z \to \infty} \frac{U'(z)}{U(z)} = \lim_{z \to \infty} -\gamma \frac{h(U(z))}{U(z)} = -\gamma \, h'(0) = -\gamma.$$

Hence for those values of the parameter β for which $c_{nl}(\beta) < 2\gamma$, $U \in H_{c_{nl}(\beta)}^1(\mathbb{R})$ and hence for such β, by Corollary 2.7 of [6] (see also Proposition 2 of [2]), $c(\beta)$ is the (nonlinear) minimal wave speed. The claim then follows by (2.2) and (2.3). $\qquad\square$

We note that this lemma can also be obtained by the methods of [1]. To formulate our next results, we set

$$L = \max_{u \in (0,1)} h'(u) \geq 1.$$

We adapt some arguments from [2].

Proposition 3.2. *If $A(\beta) > 2LB(\beta)$,*

$$c_{\min}(\beta) \leq 2\sqrt{L}\sqrt{A(\beta) - LB(\beta)}, \tag{3.1}$$

and in particular,

$$c_{\min}(\beta) \neq c_{nl}(\beta).$$

Proof. Recall from Hadeler and Rothe [4] (see also [2], equation (11)) that

$$c_{\min}(\beta) = \inf_{g \in \Lambda} \sup_{U \in (0,1)} \left\{ g'(U) + \frac{f(U, \beta))}{g(U)} \right\}, \tag{3.2}$$

where

$$\Lambda = \left\{ g \in C^1([0,1]) : g(U) > 0 \text{ if } U \in (0,1), \ g(0) = 0, \ g'(0) > 0 \right\}. \tag{3.3}$$

Hence taking $g(U) = \nu h(U)$, $\nu > 0$, yields that

$$c_{\min}(\beta) \leq \inf_{\nu > 0} \sup_{U \in (0,1)} \left\{ \nu h'(U) + \frac{A(\beta)}{\nu} - \frac{B(\beta)}{\nu} h'(U) \right\}.$$

To understand

$$\sup_{U \in (0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu} \right) h'(U) + \frac{A(\beta)}{\nu} \right\},$$

there are two cases:

(i) $\nu^2 \leq B(\beta)$: Then

$$\sup_{U \in (0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu} \right) h'(U) + \frac{A(\beta)}{\nu} \right\} = \frac{A(\beta) - lB(\beta)}{\nu} + l\nu,$$

which is monotone decreasing in ν, so

$$\inf_{\nu \leq \sqrt{B(\beta)}} \sup_{U \in (0,1)} \left\{ \left(\nu - \frac{B(\beta)}{\nu} \right) h'(U) + \frac{A(\beta)}{\nu} \right\} = \frac{A(\beta)}{\sqrt{B(\beta)}}.$$

(Note that this recovers the estimate (2.5) for $c_{\min}(\beta)$.)

(ii) $v^2 \geq B$: Then

$$\sup_{U \in (0,1)} \left\{ \left(v - \frac{B(\beta)}{v} \right) h'(U) + \frac{A(\beta)}{v} \right\} = \frac{A(\beta) - LB(\beta)}{v} + Lv := q(v).$$

Since $A(\beta) - B(\beta)h'(u) > 0$ for all $u \in [0,1]$, it follows that $A(\beta) - LB(\beta) > 0$. So differentiating $q(v)$ gives that its global minimum for $v \in (0, \infty)$ occurs at

$$v_0 := \sqrt{\frac{A(\beta) - LB(\beta)}{L}}.$$

There are two possibilities: (a) If

$$\frac{A(\beta) - LB(\beta)}{L} \leq B(\beta),$$

the function $q(v)$ reaches its minimum over $[\sqrt{B(\beta)}, \infty)$ at the point $v = \sqrt{B(\beta)}$, so that

$$\inf_{v \geq \sqrt{B(\beta)}} \sup_{U \in (0,1)} \left\{ \left(v - \frac{B(\beta)}{v} \right) h'(U) + \frac{A(\beta)}{v} \right\} = \frac{A(\beta)}{\sqrt{B(\beta)}},$$

in which case we again just recover the estimate (2.5) for $c_{qmin}(\beta)$.

(b) On the other hand, if

$$\frac{A(\beta) - LB(\beta)}{L} > B(\beta),$$

that is, $A(\beta) > 2LB(\beta)$, we have that

$$c_{\min}(\beta) \leq \inf_{v > \sqrt{B}} \sup_{U \in (0,1)} \left\{ \left(v - \frac{B(\beta)}{v} \right) h'(U) + \frac{A(\beta)}{v} \right\} = q(v_0) = 2\sqrt{L}\sqrt{A(\beta) - LB(\beta)}. \quad (3.4)$$

Comparison of $q(v_0)$ in (3.4) with $c_{nl}(\beta)$ then shows that $c_{\min}(\beta) \neq c_{nl}(\beta)$ if $A(\beta) > 2LB(\beta)$. $\qquad \square$

Now we can formulate sufficient conditions for minimality exchange. Below we say that a solution β^* of the equation $A(\beta) = 2B(\beta)$ is **non-degenerate** if the graphs of the functions $A(\cdot)$ and $2B(\cdot)$ intersect transversely at β^*. The following result applies in all the examples in [2, 4] mentioned above and covers the general case when $h(u)$ is concave and there is a non-degenerate solution to $A(\beta) = 2B(\beta)$.

Theorem 3.3. *Suppose there is a non-degenerate solution β^* to the equation $A(\beta) = 2B(\beta)$. Then if*

$$L = h'(0) = 1,$$

there is a minimality exchange at $\beta = \beta^*$.

Proof. Since if $A(\beta) < 2B(\beta)$ we have that $c_{\min}(\beta) = c_{nl}(\beta)$ by Lemma 3.1, and since by (3.4) with $L = 1$, for all $A(\beta) > 2B(\beta)$, $c_{\min}(\beta) = c_l(\beta)$, non-degeneracy of the solution β^* of $A(\beta) = 2B(\beta)$ implies that there is an exchange of minimality at β^*. $\qquad \square$

Theorem 3.3 fully characterises minimality exchange when $L = 1$, that is, when $h'(u)$ attains its supremum L at $u = 0$, which holds in particular when h is concave. If $L > 1$, however, the situation is less clear. Lemma 3.1 clearly still implies that $c_{\min}(\beta) = c_{nl}(\beta) > c_l(\beta)$, so in particular nonlinear selection holds, if $A(\beta) < 2B(\beta)$, and linear selection holds, with $c_{\min}(\beta) = c_{nl}(\beta) = c_l(\beta)$ if $A(\beta) = 2B(\beta)$, but whether it is possible to have again nonlinear selection for some β with $A(\beta) > 2B(\beta)$, either with the minimal speed corresponding to the explicit solution or another value, is not obvious. The estimate (3.4) only applies when $A(\beta) > 2LB(\beta)$, and even in that range, (3.4) is no longer sufficient to imply linear selection if $L > 1$.

In Theorem 3.6 below, we present a result complementary to Theorem 3.3 that makes no assumption on h beyond the hypotheses in Lemma 2.1, but instead imposes monotonicity conditions on the dependence of A and B on β. This yields a partial answer to what happens when $L > 1$ and $A(\beta) > 2B(\beta)$. We begin with the following preliminary result, based on [6, Theorem 2.8], which forms the basis for the alternative sufficient condition for minimality exchange in Theorem 3.6.

Lemma 3.4. *Suppose that $A(\beta)$ and $B(\beta)$ are each non-decreasing in β, and $A(\beta) - B(\beta)$ is non-increasing in β. If $c_{\min}(\beta_1) > c_l(\beta_1)$ and $\beta_2 > \beta_1$, then*

$$c_{\min}(\beta_2) > c_l(\beta_2).$$

that is, if nonlinear selection holds for some β_1, nonlinear selection also holds for any $\beta_2 > \beta_1$,

Proof. We draw on Theorem 2.8 of Lucia, Muratov and Novaga [6], which says that $c_{\min}(\beta) > c_l(\beta)$ if and only if there exists $c > c_l(\beta)$ and $u \in H_c^1(\mathbb{R})$ such that

$$\Phi_c^\beta[u] := \int_\mathbb{R} e^{cx} \left(\frac{1}{2} u_x^2 - \int_0^u f(s, \beta)\, ds \right)\, dx \ \leq\ 0, \tag{3.5}$$

where $H_c^1(\mathbb{R})$ is as defined in the proof of Lemma 3.1.

First note that it follows from [6, Theorem 2.8] that since $c_{\min}(\beta_1) > c_l(\beta_1)$, there exists $c > c_l(\beta_1)$ and $u \in H_c^1(\mathbb{R})$ such that $\Phi_c^{\beta_1}[u] \leq 0$. Then

$$\begin{aligned}
\Phi_c^{\beta_1}[u] &= \int_\mathbb{R} e^{cx} \left(\frac{1}{2} u_x^2 - \int_0^u f(s, \beta_1) ds \right) dx \\
&= \int_\mathbb{R} e^{cx} \left(\frac{1}{2} u_x^2 - \int_0^u h(s)(A(\beta_1) - B(\beta_1)h'(s)) ds \right) dx \\
&= \int_\mathbb{R} e^{cx} \left(\frac{1}{2} u_x^2 - A(\beta_1) \int_0^u h(s)\, ds - \frac{B(\beta_1)}{2} h(u)^2 \right) dx \\
&\leq 0,
\end{aligned}$$

as $h(0) = 0$, and since $\beta_2 > \beta_1$ and $A(\cdot)$ and $B(\cdot)$ are non-decreasing, we have $A(\beta_2) \geq A(\beta_1)$ and $B(\beta_2) \geq B(\beta_1)$, so that

$$\Phi_c^{\beta_2}[u] \ \leq\ \Phi_c^{\beta_1}[u] \ \leq\ 0,$$

since $h(s) > 0$ for $0 < s < 1$. Moreover, $A(\cdot) - B(\cdot)$ is non-increasing, so

$$c_l(\beta_2) = 2\sqrt{A(\beta_2) - B(\beta_2)} \leq 2\sqrt{A(\beta_1) - B(\beta_1)} = c_l(\beta_1),$$

and hence

$$c > c_l(\beta_1) \geq c_l(\beta_2).$$

Thus $c > c_l(\beta_2)$ and $\Phi_c^{\beta_2}[u] \leq 0$, and hence [6, Theorem 2.8] implies that $c_{\min}(\beta_2) > c_l(\beta_2)$. $\square$

The following is an immediate consequence of Lemma 3.4.

Corollary 3.5. *Suppose that $A(\beta)$ and $B(\beta)$ are each non-decreasing in β, and that $A(\beta) - B(\beta)$ is non-increasing in β. If $c_{\min}(\beta_2) = c_l(\beta_2)$ for some β_2 and $\beta_1 < \beta_2$, then $c_{\min}(\beta_1) = c_l(\beta_1)$.*

We can now prove our second set of sufficient conditions for minimality exchange.

Theorem 3.6. *Suppose that $A(\beta)$ and $B(\beta)$ are each non-decreasing in β, and $A(\beta) - B(\beta)$ is non-increasing in β. If there is a non-degenerate solution β^* to the equation $A(\beta) = 2B(\beta)$, then there is a minimality exchange at $\beta = \beta^*$, with $c_{\min}(\beta) = c_l(\beta)$ for $\beta \leq \beta^*$ and $c_{\min}(\beta) = c_{nl}(\beta) > c_l(\beta)$ for $\beta > \beta^*$.*

Proof. Note first that $A(\beta) - 2B(\beta = [A(\beta) - B(\beta)] - B(\beta)$ is non-increasing in β, so since the graphs of $A(\cdot)$ and $2B(\cdot)$ intersect transversally at β^*, it follows that $A(\beta) > 2B(\beta)$ when $\beta < \beta^*$, whereas $A(\beta) < 2B(\beta)$ when $\beta > \beta^*$. Lemma 3.1 then implies that $c_{\min}(\beta) = c_{nl}(\beta)$ when $\beta > \beta^*$, whereas Corollary 3.5 implies that linear selection holds when $\beta < \beta^*$. ☐

Note that for the two concrete examples of minimality exchange discussed in Section 3, both Theorem 3.3 and Theorem 3.6 apply.

An example of a solvable problem for which Theorem 3.6 applies but Theorem 3.3 does not, is given by taking $A = 1$, $B = \beta/2$ and $h(u) = e^{2u}u(1 - u)$, which is not concave. Then $L = 1.52218$, $c_l = \sqrt{4 - 2\beta}$, $c_{nl} = \sqrt{2/\beta}$, $c_l(\beta) = c_{nl}(\beta)$ at $\beta^* = 1$, and Theorem 3.6 ensures that there is minimality exchange at $\beta^* = 1$.

4   Conclusions

In this article we have focussed on a class of parameter-dependent monostable reaction-diffusion equations with explicit travelling-wave solutions and used this class to explore the phenomenon of minimality exchange, when the minimal wave speed switches from a linearly determined value to the speed of the explicitly determined front as a parameter changes. Two alternative sets of sufficient conditions for minimality exchange are proved, in Theorems 3.3 and 3.6. Why there should be such an exchange, not only from linear selection to nonlinear selection, but to nonlinear selection given by an *explicit* solution, is quite puzzling at first sight. Our framework here provides insight into why minimality exchange of this type occurs, and includes concrete examples from [2–4, 6]. The proofs draw on various tools for determining whether there is linear or nonlinear selection - in particular, ideas developed previously in the special case of an isotropic liquid-crystal model [2], as well as general results from [4, 6]. Some additional interesting material about minimal wave speeds is given in [3, Section 10.1.1], including Theorem 10.12, which provides sufficient criteria that can be used to identify cases when a given explicit solution has the minimal wave speed, and the examples that follow.

As suggested by the anonymous referee, instead of considering in (2.1) a nonlinearity parameterised by β, as was also done in [4, 6, 8] and in many examples in [3], our methods could have been used to treat a two-parameter system $f(u, A, B) = h(u)(A - Bh'(u))$ to map out domains of linear and nonlinear speed selection in the (A, B) plane.

We have treated one class of parameter-dependent solvable equations that includes important special cases, but clearly there are many further solvability results for explicit travelling-wave solutions in the literature. See, for instance, [3, Chapter 13] and [7]. In addition, the change of variables $G := 1/F$ converts (1.3) into an Abel equation, for which certain classes of explicit solutions can be found using tools such as the Chiellini integrability condition and

the Lemke transformation (see, for example, [5] and the references therein). It would be interesting to expand and develop the approach introduced here to cover a larger range of explicit solutions to obtain further insight into the mechanisms for minimality exchange.

Acknowledgements

References

[1] R. D. Benguria, M. C. Depassier, Exact fronts for the nonlinear diffusion equation with quintic nonlinearities, *Phys. Rev. E* **50**(1994), No. 1, 3701–3704. https://doi.org/10.1103/PhysRevE.50.3701

[2] E. C. M. Crooks, M. Grinfeld, G. McKay, Minimality conditions for wave speed in anisotropic smectic C^* liquid crystals, *Math. Meth. Appl. Sci.* **41**(2018), No. 1, 88–102. https://doi.org/10.1002/mma.4596; MR374358; Zbl 1173.35545

[3] B. H. Gilding, R. Kersner, *Travelling waves in nonlinear diffusion-convection-reaction*, Progress in Nonlinear Differential Equations and their Applications, Vol. 60, Birkhäuser Verlag, Basel, 2004. MR2081104; Zbl 1073.35002

[4] K. P. Hadeler, F. Rothe, Travelling fronts in nonlinear diffusion equations, *J. Math. Biol.* **2**(1975), No. 3, 251–263. https://doi.org/10.1007/BF00277154; MR411693; Zbl 0343.92009

[5] T. Harko, M. K. Mak, Exact travelling wave solutions of nonlinear reaction-convection-diffusion equations: an Abel equation based approach, *J. Math. Phys.* **56**(2015), No. 11, 111501. https://doi.org/10.1063/1.4935299; MR3421799; Zbl 1328.35096

[6] M. Lucia, C. B. Muratov, M. Novaga, Linear vs. nonlinear selection for the propagation speed of the solutions of scalar reaction-diffusion equations invading an unstable equilibrium, *Comm. Pure Appl. Math.* **57**(2004), No. 3, 616–636. https://doi.org/10.1002/cpa.20014; MR2032915; Zbl 1053.35065

[7] S. Petrovskii, B.-L. Li, *Exactly solvable models of biological invasion*, Chapman and Hall/CRC Mathematical Biology and Medicine Series, Chapman and Hall/CRC, Boca Raton, FL., 2006. MR2167891; Zbl 1151.92034

[8] W. van Saarloos, M. van Hecke, R. Hołyst, Front propagation into unstable and metastable states in smectic C^* liquid crystals: Linear and nonlinear marginal-stability analysis, *Phys. Rev. E* **52**(1995), 1773–1777. https://doi.org/10.1103/PhysRevE.52.1773