A Stochastic Differential Equation SIS Epidemic Model with Regime Switching

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Abstract

In this paper, we combined the previous model in [1] with Gray *et al.*'s work in 2012 [2] to add telegraph noise by using Markovian switching to generate a stochastic SIS epidemic model with regime switching. Similarly, threshold value for extinction and persistence are then given and proved, followed by explanation on the stationary distribution, where the M-matrix theory elaborated in [3] is fully applied. Computer simulations are clearly illustrated with different sets of parameters, which support our theoretical results. Compared to our previous work in 2019 [1, 4], our threshold value are given based on the overall behaviour of the solution but not separately specified in every state of the Markov chain.

Keywords: SIS model, independent Brownian motions, Markovian switching, extinction, persistence, stationary distribution.

1 Introduction

Population systems are significantly influenced by the random variation from the environment [5]. The disturbance in the environment can be described as environmental noises and environmental noise can be found in all levels of biology, from molecular, sub-cell processes to the dynamics of immunity system in the human body and the whole population [6]. For example, the impact of multiple white noises in the following deterministic SIS model (1.1)

$$\mathrm{d}I(t) = [\beta(N - I(t))I(t) - (\mu + \gamma)I(t)]\mathrm{d}t, \tag{1.1}$$

has been firstly developed by Gray *et al*'s work in 2011 [7]. When some stochastic environmental factor is introduced on each individual in the population, they replace β by a random variable $\tilde{\beta}$

$$\tilde{\beta}\mathrm{d}t = \beta\mathrm{d}t + \sigma_1\mathrm{d}B_1(t) \tag{1.2}$$

Followed by this, our previous work [8, 9] considered another perturbation on $(\mu+\gamma)$ with (1.2) existing in traditional SIS model. That is, $(\mu + \gamma)$ is replaced by a random variable $(\tilde{\mu} + \tilde{\gamma})$

$$(\tilde{\mu} + \tilde{\gamma})\mathrm{d}t = (\mu + \gamma)\mathrm{d}t + \sigma_2\sqrt{N - I(t)}\mathrm{d}B_2(t) \tag{1.3}$$

Here we do not simply set $(\tilde{\mu} + \tilde{\gamma})\mathrm{d}t = (\mu + \gamma)\mathrm{d}t + \sigma_2\mathrm{d}B_2(t)$ to be the second perturbation. When susceptible population $S(t) = N - I(t)$ is large, which means there are few infected individuals, the error of estimating μ and γ will be large. Thus we suppose that the variance of estimating $\mu + \gamma$ is

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proportional to the number of susceptible population. As a result, the reduction of infections caused by medical care and death of a single infected individual in the small time interval $[t, t+\mathrm{d}t)$ is normally distributed with mean $(\mu+\gamma)\mathrm{d}t$ and variance $\sigma_2^2(N-I(t))\mathrm{d}t$. This is also a biologically reasonable model because the variance trends to 0 when $\mathrm{d}t$ goes to 0. As a result, we then have our SIS SDE model with two Brownian motions (1.4)

$$\begin{aligned}\mathrm{d}I(t) = [\beta(N-I(t))I(t) - (\mu+\gamma)I(t)]\mathrm{d}t + \sigma_1 I(t)(N-I(t))\mathrm{d}B_1(t) \\ - \sigma_2 I(t)\sqrt{N-I(t)}\mathrm{d}B_2(t).\end{aligned} \tag{1.4}$$

In our first model (1.4), B_1 and B_2 are considered initially as independent Brownian motions. And then based on Hening's work [10], we defined correlations between B_1 and B_2 to involve correlation between white noises to generalize the results from [1], which is more practical and reasonable. It is indicated from [1, 4] that considering white noise in the epidemic models can help stabilize those unstable cases in deterministic models.

However, not only white noise but also colour noise are used in classical deterministic epidemic models to describe different influence of environmental noise on population systems. For example, telegraph noise is a typical colour noise that has been studied widely in epidemic models. Telegraph noise can be illustrated as switching among different regimes, which can represent important information in the model such as change of seasons in a year, or different weathers [11, 12]. If assuming that the future switching is only based on current state and the waiting time for the next switching has exponential distribution, we can use a finite-state Markov chain to describe such behaviour. There are many previous research based on using Markovian switching in stochastic epidemic models to study the effect of telegraph noise. For instance, Greenhalgh *et al.* [12] introduce telegraph noise in SIRS model by using a two-state Markov chain to study the asymptotic behaviour of the solution. Also, based on Takeuchi's work [13], Gray *et al.* also construct a stochastic SIS model with two-state Markov chain.

$$\frac{I(t)}{\mathrm{d}t} = I(t)[\alpha_{r(t)} - \beta_{r(t)}I(t)], \tag{1.5}$$

where $\alpha_i = \beta_i N - \mu_i - \gamma_i$ and i is the Markov chain state.

Moreover, It is obvious that multiple noises can be considered in the epidemic models. For example, a finite number of independent Brownian motions can be used to introduce the disturbance of multiple independent, or correlated white noises. As a result, regime switching [14] is a more general case which includes both white noises and telegraph noise in an epidemic model. For instance, Luo and Mao [15] introduce white noise and telegraph noise in Lotka-Volterra model. However, they study the ultimate boundedness of the solution, while Li *et al.* [16] analyse the Lyapunov function in stochastic Lotka-Volterra model, which is developed by Khasminskii [17], in order to explain stochastic permanence. They clearly point out that permanence in the overall behaviour do not need permanence in every state. In some states, solution can even become extinctive. Based on [15, 16], Liu *et al.* [18] focus on the ergodic property, recurrence and the stationary distribution of the solution in a Lotka–Volterra system with pollination mutualism [19–22]. Computer simulation illustrates the fluctuation in the solution, while the integral average converges to a fixed point, which supports the recurrence and a stationary distribution in their theory. Recently, Cai et al. [8, 9, 23] introduce white noise, telegraph noise and time delay to the two-dimensional foraging arena population system. Various long-time behaviours including stochastically ultimate boundedness, extinction, pathwise estimation and ergodic property are investigated. They also study how the correlations between the Brownian motions affect the dynamical system.

A regime switching in deterministic SIS model is clearly necessary and reasonable but not common. In order to combine the two work from Gray *et al.* [2, 7] based on the model of our previous work [1], we are going to consider a finite-state Markov chain in our model (1.4) to involve the effect of telegraph noise in our model to generalize the previous work. Now firstly we need to define the Markov chain. Let $r(t)$, $t \geq 0$, be a right-continuous M-state Markov chain on the probability space. $r(t)$ only takes value in a finite state space $\mathbb{S} = \{1, 2, ..., M\}$, with generator $\Gamma = (\nu_{ij})_{M\times M}$ defined as

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \nu_{ij}\delta + o(\delta), \text{ if } i \neq j, \\ 1 + \nu_{ij}\delta + o(\delta), \text{ if } i = j, \end{cases}$$

where $\delta > 0$ and $\nu_{ij} \geq 0$ is the transition rate from state i to j for $i \neq j$. Note that $\nu_{ii} = -\Sigma_{1\leq j\leq M,\ j\neq i}\nu_{ij}$. And Almost every sample path of $r(\cdot)$ is a right-continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R} = [0, \infty)$ [24]. To be specific, there is a sequence $\{\tau_k\}_{k\geq 0}$ of finite-valued $\mathcal{F}_t$-stopping times such that $0 = \tau_0 < \tau_1 < ... < \tau_k \to \infty$ a.s. and

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k)\mathbf{1}_{[\tau_k,\tau_{k+1})}(t), \tag{1.6}$$

where $\mathbf{1}_A$ denotes the indicator function of set A. Also, we define $\Pi = (\pi_1, \pi_2, ..., \pi_M)$ to be the unique stationary distribution of this Markov Chain and $\sum_1^M \pi_i = 1$. Now we suppose that in the SIS epidemic model (1.4) the parameters $\mu_i,\ \beta_i,\ \gamma_i,\ \sigma_{1_i},\ \sigma_{2_i}$ are all positive numbers ($i \in \mathbb{S}$). Then we have our previous stochastic SIS SDE model (1.4) with Markovian switching given by

$$\begin{aligned} \mathrm{d}I(t) = [\beta_{r(t)}(N - I(t))I(t) - (\mu_{r(t)} + \gamma_{r(t)})I(t)]\mathrm{d}t + \sigma_{1_{r(t)}}I(t)(N - I(t))\mathrm{d}B_1(t) \\ - \sigma_{2_{r(t)}}I(t)\sqrt{N - I(t)}\mathrm{d}B_2(t), \end{aligned} \tag{1.7}$$

with $I_0 \in (0, N)$ and $r(0) = r_0 \in \mathbb{S}$. Also, B_1 and B_2 are independent Brownian motions. In the rest of the paper we examine a threshold

$$R_0^S = \frac{\sum \pi_i\beta_i N}{\sum \pi_i(\mu_i + \gamma_i)} - \frac{\sum \pi_i(\sigma_{1_i}^2 N^2 + \sigma_{2_i}^2 N)}{2\sum \pi_i(\mu_i + \gamma_i)}$$

for almost sure extinction or an invariant density of our model (1.7). We will see later that the model will die out almost surely if $R_0^S < 1$ with an additional condition while having a stationary distribution when $R_0^S > 1$. We now state a proposition which gives an equivalent condition for the threshold value in terms of the system parameters ϕ_i defined in (1.8) and the stationary distribution of the Markov chain.

Proposition 1.1. *We have the following alternative condition on the threshold value* R_0^S*.*

- $R_0^S < 1$ *if and only if* $\sum_{i=1}^N \pi_i\phi_i < 0$*;*
- $R_0^S = 1$ *if and only if* $\sum_{i=1}^N \pi_i\phi_i = 0$*;*
- $R_0^S > 1$ *if and only if* $\sum_{i=1}^N \pi_i\phi_i > 0$*,*

where

$$\phi_i = \beta_i N - \mu_i - \gamma_i - \frac{1}{2}\big(\sigma_{1_i}^2 N^2 + \sigma_{2_i}^2 N\big), \quad i \in \mathbb{S}. \tag{1.8}$$

2 Unique and bounded solution

In order for the model to make sense, we need to prove that the solution of our SDE has a unique global solution which remain within $(0, N)$, with the initial value $I_0 \in (0, N)$, $r(0) = r_0 \in \mathbb{S}$.

Theorem 2.1. *If* $\min\{\frac{2(\mu_i+\gamma_i)}{\sigma_{2_i}^2}\} \geq N$*, for all* $i \in \mathbb{S}$*, then for any given initial value* $I(0) = I_0 \in (0, N)$ *and* $r(0) = r_0 \in \mathbb{S}$*, the SDE has a unique global positive solution* $I(t) \in (0, N)$ *for all* $t \geq 0$ *with probability one, namely,*

$$\mathbb{P}\{I(t) \in (0, N),\ \forall t \geq 0\} = 1$$

Proof. It is obvious that for any $i \in \mathbb{S}$, the corresponding coefficients of our SDE (1.7) are locally Lipschitz continuous. And for any a and τ_a defined as (1.6), our solution of the equation (1.7) is uniquely determined on $t \in [\tau_a, \tau_{a+1})$, with $r(\tau_a) = i_a \in \mathbb{S}$. As a result, we see that the equation (1.7) has a unique solution on $t \in \mathbb{R}_+$.

So now for any given initial value, there is a unique maximal local solution $I(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [3]. Let $k_0 \geq 0$ be sufficient large to satisfy $\frac{1}{k_0} < I_0 < N - \frac{1}{k_0}$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) \ : I(t) \notin (1/k, N - 1/k)\}$$

In this paper, we set $\inf \emptyset = \infty$. Obviously, τ_k is increasing when $k \to \infty$. And we set $\tau_\infty = \lim_{k\to\infty} \tau_k$. It is clear that $\tau_\infty \leq \tau_e$ almost sure. So if we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $I(t) \in (0, N)$ a.s. for all $t \geq 0$.

Here we assume $\tau_\infty = \infty$ a.s. is not true. Then we can find a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

So we can find an integer $k_1 \geq k_0$ large enough, such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon \quad \forall k \geq k_1.$$

Define a function $V : (0, N) \to \mathbb{R}_+$ by

$$V(x) = -\log x - \log(N - x) + \log \frac{N^2}{4},$$

which is independent of the Markov chain $r(t)$, and

$$V_x = -\frac{1}{x} + \frac{1}{N - x}, \ V_{xx} = \frac{1}{x^2} + \frac{1}{(N - x)^2}.$$

We let $f(t) = \beta_{r(t)}(N - I(t))I(t) - (\mu_{r(t)} + \gamma_{r(t)})I(t)$, $g(t) = (\sigma_{1_{r(t)}} I(t)(N - I(t)), -\sigma_{2_{r(t)}} \sqrt{N - I(t)} I(t))$ and $\mathrm{d}B(t) = (\mathrm{d}B_1(t), \mathrm{d}B_2(t))$. Then by Itô's formula [3], we have, for any $t \in [0, T]$ and $k \geq k_1$

$$\mathbb{E}V(I(t \wedge \tau_k)) = V(I_0) + \mathbb{E}\int_0^{t\wedge\tau_k} LV(I(s))\mathrm{d}s + \mathbb{E}\int_0^{t\wedge\tau_k} V_x g(s)\mathrm{d}B(s),$$

where $\mathbb{E}\int_0^{t\wedge\tau_k} V_x g(s)\mathrm{d}B(s) = 0$. Also it is easy to show that

$$\begin{aligned}
LV(x, i) &= -\beta_i(N - x) + (\mu_i + \gamma_i) + \beta_i x - (\mu_i + \gamma_i)\frac{x}{N - x} \\
&\quad + \frac{1}{2}(\sigma_{1_i}^2 (N - x)^2 + \sigma_{1_i}^2 x^2 + \sigma_{2_i}^2 (N - x) + \sigma_{2_i}^2 \frac{x^2}{N - x}) \\
&\leq -\beta_i(N - x) + (\mu_i + \gamma_i) + \beta_i x \\
&\quad + \frac{1}{2}[\sigma_{1_i}^2 (N - x)^2 + \sigma_{1_i}^2 x^2 + \sigma_{2_i}^2 (N - x)] \\
&\leq C,
\end{aligned}$$

where C is a constant when $\mu_i + \gamma_i \geq \frac{1}{2}\sigma_{2_i}^2 N$ for all $i \in \mathbb{S}$ and $x \in (0, N)$. Hence when $\min\{\frac{2(\mu_i+\gamma_i)}{\sigma_{2_i}^2}\} \geq N$, we have

$$\begin{aligned}
\mathbb{E}V(I(t \wedge \tau_k)) &\leq V(I_0) + \mathbb{E}\int_0^{t\wedge\tau_k} C\mathrm{d}s \\
&\leq V(I_0) + Ct,
\end{aligned}$$

which yields that

$$\mathbb{E}V(I(T \wedge \tau_k)) \leq V(I_0) + CT.$$

Now set $\Omega_k = \{\tau_k \leq T\}$ for $\forall k \geq k_1$ and we have $\mathbb{P}(\Omega_k) \geq \epsilon$. For every $\omega \in \Omega_k$, $I(\tau_k, \omega)$ equals either $1/k$ or $N - 1/k$ and we have

$$V(I(\tau_k, \omega)) = \log \frac{N^2}{4(N - 1/k)1/k}.$$

Hence

$$\begin{aligned}\infty > V(I_0) + CT &\geq \mathbb{E}[\mathbf{1}_{\Omega_k}(\omega)V(I(\tau_k,\omega))] \\ &\geq \mathbb{P}(\Omega_k)\log\frac{N^2}{4(N-1/k)1/k} \\ &= \epsilon\log\frac{N^2}{4(N-1/k)1/k}.\end{aligned}$$

Let $k \to \infty$ will lead to the contradiction

$$\infty > V(I_0) + CT = \infty.$$

So the assumption is wrong and we must have $\tau_\infty = \infty$ almost sure, whence the proof is now complete. □

The result is very similar to Theorem 2.1 in [1] as we are not able to find a better substitution at this moment. However, in the following sections we will manage to give conditions by using the ergodic theory of the Markov chain [24, 25]. Those results will be stated in an average-type form which combined the parameters in each state with its corresponding Markov Chain stationary distribution π_i, $i \in \mathbb{S}$. This will let us no longer examine the solution state by state but as a whole.

3 Extinction

In the study of stochastic epidemic models, the extinction of disease is usually one of the most crucial issues. So similarly, in this section, we will firstly give an almost sure extinction condition for the disease to die out.

3.1 Theorem

Lemma 3.1. *[3, p.167, Theorem 5.9] Let $p > 0$ and $\eta > 0$. For any given initial value $I_0 \in (0, N)$ and $r_0 \in \mathbb{S}$, if*

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}[I(t)]^p\right) \leq -\eta,$$

then

$$\limsup_{t\to\infty}\frac{1}{t}\log I(t) \leq -\frac{\eta}{p} \quad a.s.$$

For each $p \geq 0$, define an $M \times M$ matrix

$$\mathcal{A}(p) = \mathrm{diag}(\theta_1(p), \cdots, \theta_N(p)) - \Gamma, \tag{3.1}$$

where

$$\theta_i(p) = p\big[\mu_i + \gamma_i - \beta_i N + \frac{1}{2}(1-p)(\sigma_{1_i}^2 N^2 + \sigma_{2_i}^2 N)\big].$$

Theorem 3.2. *Let $0 < p < 1$ and $\mathcal{A}(p)$ be a nonsingular M-matrix. Suppose also that*

$$\beta_i - \big(N\sigma_{1_i}^2 + \frac{1}{2}\sigma_{2_i}^2\big) \geq -\frac{1}{2Np\lambda_i} \quad \textit{for all } i \in \mathbb{S}, \tag{3.2}$$

where λ_i will be defined later. Then for any given initial value $I_0 \in (0, N)$ and $r_0 \in \mathbb{S}$, the solution to (1.7) *obeys*

$$\limsup_{t\to\infty}\frac{1}{t}\log I(t) < -\frac{1}{2p\check{\lambda}} \quad a.s.,$$

where $\check{\lambda} = \max_{i\in\mathbb{S}}\lambda_i$.

Proof. By [3, Theorem 2.10], there exists a vector $\lambda = (\lambda_1, \cdots, \lambda_N)^T \gg 0$ such that

$$\mathcal{A}(p)\lambda = \vec{1} := (1, \cdots, 1)^T$$

i.e.

$$\theta_i(p)\lambda_i - \sum_{j=1}^{N} \nu_{ij}\lambda_j = 1 \quad \text{for all } i \in \mathbb{S}.$$

Define a function

$$V(I,t,i) = \lambda_i I^p \quad \text{for } (I,t,i) \in (0,N) \times \mathbb{R}_+ \times \mathbb{S}.$$

Applying the generalised Itô formula, we then obtain

$$\begin{aligned}
LV(I,t,i) &= \lambda_i p I^p \Big\{\beta_i(N-I) - (\mu_i+\gamma_i) - \frac{1}{2}(1-p)\big[\sigma_{1_i}^2(N-I)^2 + \sigma_{2_i}^2(N-I)\big]\Big\} + I^p \sum_{j=1}^{N} \nu_{ij}\lambda_j \\
&= -I^p \Big\{p\lambda_i\Big[\mu_i+\gamma_i-\beta_i N + \frac{1}{2}(1-p)\big(\sigma_{1_i}^2 N^2 + \sigma_{2_i}^2 N\big)\Big] - \sum_{j=1}^{N}\nu_{ij}\lambda_j \\
&\quad + Ip\lambda_i\Big(\beta_i - \frac{1}{2}(1-p)\big(2N\sigma_{1_i}^2 + \sigma_{2_i}^2\big)\Big)\Big\} \\
&\le -\frac{1}{2}I^p
\end{aligned}$$

by (3.2). We then apply the generalised Itô formula again to derive that for any $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}\Big[e^{(1/2\check{\lambda})(t\wedge\tau_k)}V\big(I(t\wedge\tau_k), t\wedge\tau_k, r(t\wedge\tau_k)\big)\Big] \\
&\le V(x_0,0,r_0) + \mathbb{E}\int_0^{t\wedge\tau_k} e^{(1/2\check{\lambda})s}\Big[\frac{1}{2\check{\lambda}}V(I(s),s,r(s)) + LV(I(s),s,r(s))\Big]ds \\
&\le \check{\lambda} I_0^p + \mathbb{E}\int_0^{t\wedge\tau_k} e^{(1/2\check{\lambda})s}\Big[\frac{1}{2\check{\lambda}}\lambda_i I^p(s) - \frac{1}{2}I^p(s)\Big]ds \\
&\le \check{\lambda} I_0^p.
\end{aligned}$$

Therefore

$$\hat{\lambda}\mathbb{E}\big[e^{(1/2\check{\lambda})(t\wedge\tau_k)}I^p(t\wedge\tau_k)\big] \le \check{\lambda} I_0^p.$$

Letting $k \to \infty$ leads to

$$\mathbb{E}[I(t)]^p \le \frac{\check{\lambda}}{\hat{\lambda}} I_0^p e^{-(1/2\check{\lambda})t}, \quad \text{i.e. } \limsup_{t\to\infty}\frac{1}{t}\log\big(\mathbb{E}[I(t)]^p\big) \le -\frac{1}{2\check{\lambda}}.$$

The required assertion then follows immediately from Lemma 3.1. □

Next we need to explore such $p \in (0,1)$ for $\mathcal{A}(p)$ to be a nonsingular M-matrix. The result is a direct use of the M-matrix theory elaborated in [3]. We would like to first impose our assumptions before presenting the final result.

Assumption 3.3. *For some $u \in \mathbb{S}, \gamma_{iu} > 0$ for all $i \in \mathbb{S}$ and $i \neq u$.*

Assumption 3.4.

$$\sum_{i=1}^{N} \pi_i \phi_i < 0, \tag{3.3}$$

where ϕ_i is defined in (1.8).

Assumption 3.5. $\phi_i < 0$ *for all* $i \in \mathbb{S}$.

Lemma 3.6. *Assumption 3.3 and 3.4 imply that there exists a constant* $p \in (0,1)$ *such that the matrix* $\mathcal{A}(p)$ *defined in* (3.1) *is a nonsingular* M*-matrix.*

Lemma 3.7. *Assumption 3.5 implies that there exists a constant* $p \in (0,1)$ *such that the matrix* $\mathcal{A}(p)$ *defined in* (3.1) *is a nonsingular* M*-matrix.*

This is a standard result from [3] and hence the proof is omited. Therefore, we have our final result as stated below.

Theorem 3.8. *Suppose that Assumption 3.3 and 3.4 and assertion* (3.2) *hold. Then system* (1.7) *will die out exponentially with probability* 1.

Remark 3.9. *In Theorem 3.8, Assumption 3.3 and 3.4 can be replaced by Assumption 3.5.*

The additional condition (3.2) looks strange in the first place, however, it essentially suggests that extinction requires smaller amplitude of external noise. Moreover, according to the proof of Lemma 3.6, a nonsingular M-matrix $\mathcal{A}(p)$ is obtained for a sufficiently small p. So the value $-\frac{1}{2Np\lambda_i}$ might be so tiny that the condition can be easily satisfied, though λ_i is dependent on p.

Alternatively, such almost sure extinction can be proved by making use of the Lyapunov function $\log I(t)$ as in e.g. [1]. However, it requires an even stronger condition than (3.2). That is, $\beta_i - \left(N\sigma_{1_i}^2 + \frac{1}{2}\sigma_{2_i}^2\right) \geq 0$. Therefore, the M-matrix theory is chosen to weaken the conditions for extinction.

3.2 Simulation

In this section, we use Euler-Martyrium method [7, 26] implemented in R to simulate the solutions in extinction examples. So firstly to simulate an extinction solution, we assume a simple Markov chain generator

$$\nu_{12} = 1, \nu_{21} = 2.$$

So we have stationary distribution of this Markov chain

$$\pi_1 = \frac{2}{3}, \pi_2 = \frac{1}{3}.$$

For both state we have $N = 100$ fixed. The parameters are defined as

$$\beta_1 = 0.4, \mu_1 + \gamma_1 = 45, \sigma_{1_1} = 0.03, \sigma_{2_1} = 0.6;$$

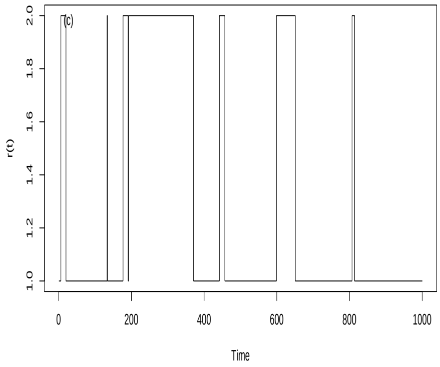
$$\beta_2 = 0.3, \mu_2 + \gamma_2 = 25, \sigma_{1_2} = 0.04, \sigma_{2_2} = 0.1.$$

According to Theorem 3.1 in [1], the disease will die out in state 1 but persist in state 2. Compute $\phi_1 = -27.5$ and $\phi_2 = 4.42$ and hence condition (3.3) is satisified. Clearly, condition (3.2) is also fulfilled. Hence by Theorem 3.2, the hybrid system (1.7) dies out ultimately.

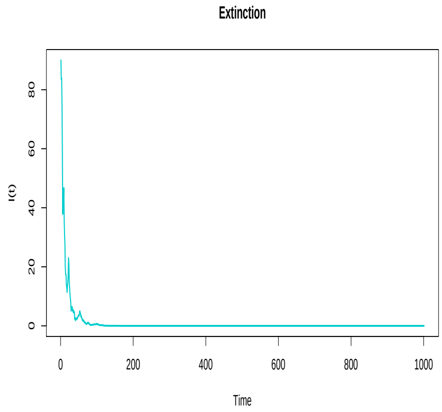
Now by using Euler-Maruyama Method in R and assuming the step size is 0.001 and $r_0 = 1$, we can see the results in Figure 1 and Figure 2 clearly show that the solution trends to 0 after 1,000 iterations, with both large and small initial values. We can also see in Figure 1 and Figure 2 that there are some decreasing and increasing behaviours early in the plots, indicating the Markovian switching between extinction state to non-extinction state with Brownian motions. The corresponding Markov Chains $r(t)$ are also illustrated.

4 Persistence

In this section, we are going to give conditions for persistence. Though there are many different definitions in persistence, we want to find a condition of oscillating around a positive level, which can be found similarly in [1, 4, 7]. Hence we give the following theorem.

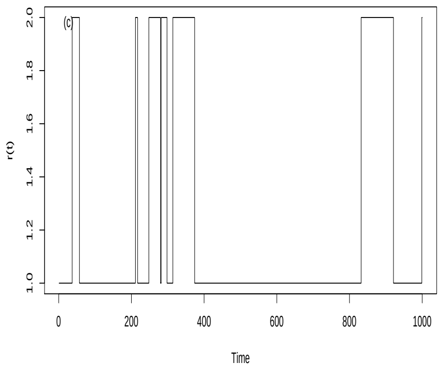


(a) $I(0) = 90$

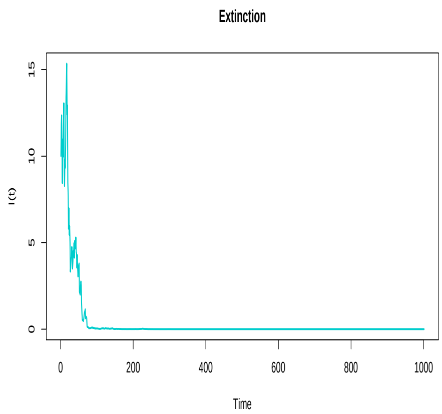


(b) $I(0) = 5$

Figure 1: Extinction with $I(0) = 90$



(a) $I(0) = 90$



(b) $I(0) = 5$

Figure 2: Extinction with $I(0) = 10$

4.1 Theorem

Assumption 4.1.

$$\sum_{i=1}^{N} \pi_i \phi_i > 0,$$

where ϕ_i is defined in (1.8).

Theorem 4.2. *Under Assumption 4.1, for any given initial value $I(0) = I_0 \in (0, N)$ and $r(0) = r_0 \in \mathbb{S}$, the solution of (1.7) follows*

$$\limsup_{t\to\infty} I(t) \geq \xi \ \text{and} \ \liminf_{t\to\infty} I(t) \leq \xi \quad a.s.,$$

where ξ is the only positive root of $\mathcal{K}(x) = 0$ in $x \in (0, N)$,

$$\mathcal{K}(x) = \sum \pi_i \phi_i + \sum \pi_i (\sigma_{1_i}^2 N + \frac{1}{2}\sigma_{2_i}^2 - \beta_i)x - \frac{1}{2}\sum \pi_i \sigma_{1_i}^2 x^2.$$

In other word, $I(t)$ will be above or below the level ξ infinitely often with probability one.

Proof. Under Assumption 4.1, we have $\mathcal{K}(0) = \sum \pi_i \phi_i > 0$ and $\mathcal{K}(N) = -\sum \pi_i(\mu_i + \gamma_i) < 0$. So as a quadratic function, $\mathcal{K}(x)$ must have only one positive root in $(0, N)$. To begin the proof, we firstly assume that $\limsup_{t\to\infty} I(t) \geq \xi$ a.s. were not true. Then we can find a small $\epsilon > 0$ for $\mathbb{P}(\Omega_1) > \epsilon$ where

$$\Omega_1 = \{\omega \in \Omega : \limsup_{t\to\infty} I(t) < \xi - 2\epsilon\}.$$

Also by the ergodic theory [3, 24], there is an Ω_2 with $\mathbb{P}(\Omega_2) = 1$, such that for any $\omega \in \Omega_2$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t L\tilde{V}(\xi - \epsilon)\mathrm{d}s = \mathcal{K}(\xi - \epsilon) > \mathcal{K}(\xi) = 0.$$

So for any $\omega \in \Omega_1 \cap \Omega_2$, there is a positive $T = T(\omega)$, such that $\forall t \geq T$

$$I(t) \leq \xi - \epsilon.$$

Then we must have

$$\begin{aligned}\liminf_{t\to\infty} \frac{1}{t}\log I(t)\mathrm{d}s &\geq \liminf_{t\to\infty} \frac{1}{t}\log(I_0)\mathrm{d}s + \liminf_{t\to\infty} \frac{1}{t}\int_0^T L\tilde{V}(I(s))\mathrm{d}s + \mathcal{K}(\xi - \epsilon)\\ &> 0.\end{aligned}$$

This implies that

$$\lim_{t\to\infty} I(t) \to \infty,$$

which contradicts our previous assumption. Therefore $\limsup_{t\to\infty} I(t) \geq \xi$ a.s. must hold.

Similarly if we assume that $\liminf_{t\to\infty} I(t) \leq \xi$ a.s. were not true, then we can find a small $\delta > 0$ for $\mathbb{P}(\Omega_3) > \delta$ where

$$\Omega_3 = \{\omega \in \Omega : \liminf_{t\to\infty} I(t) > \xi + 2\delta\}.$$

Also by the ergodic theory we have $\mathbb{P}(\Omega_4) = 1$, where for any $\omega \in \Omega_4$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t L\tilde{V}(\xi + \delta)\mathrm{d}s = \mathcal{K}(\xi + \delta) < \mathcal{K}(\xi) = 0.$$

So for any $\omega \in \Omega_3 \cap \Omega_4$, there is a positive $T = T(\omega)$, such that $\forall t \geq T$

$$I(t) \geq \xi + \delta.$$

Then we have that

$$\limsup_{t\to\infty} \frac{1}{t} \log I(t)\mathrm{d}s \le \limsup_{t\to\infty} \frac{1}{t} \log(I_0)\mathrm{d}s + \limsup_{t\to\infty} \frac{1}{t} \int_0^T L\tilde{V}(I(s))\mathrm{d}s + \mathcal{K}(\xi+\delta)$$
$$< 0.$$

This implies that

$$\lim_{t\to\infty} I(t) \to 0,$$

which contradicts our previous assumption again. Therefore $\liminf_{t\to\infty} I(t) \le \xi$ a.s. must hold. ☐

4.2 Simulation

Similarly, for persistence examples, we want to simulate a solution of (1.7) with a simple two-state Markov chain. Firstly we still fix $N = 100$. Then we want to make the solution persist in only one of the states according to Theorem 4.1 in [1] but by the average-type result of persistence in this paper, the solution of (1.7) still have persistence in the whole behaviour. Hence we assume the parameters in the first example as

$$\beta_1 = 0.4, \mu_1 + \gamma_1 = 45, \sigma_{1_1} = 0.03, \sigma_{2_1} = 0.01.$$

So in state 1 we have

$$\phi_1 = -9.5 < 0,$$

which means $R_{0_1}^S < 1$ so by Theorem 3.1 in [1] disease in state 1 will die out. Also parameters in state 2 are

$$\beta_2 = 0.6, \mu_2 + \gamma_2 = 35, \sigma_{1_2} = 0.04, \sigma_{2_2} = 0.1.$$

Then in state 2 we have

$$\phi_2 = 16.5 > 0.$$

which means $R_{0_2}^S > 1$ so by Theorem 4.1 in [1] disease in state 2 will persist. Now we define the Markov chain generator

$$\nu_{12} = 3, \nu_{21} = 4.$$

So we have stationary distribution of this Markov chain

$$\pi_1 = \frac{4}{7}, \pi_2 = \frac{3}{7}.$$

We can easily see that Assumption 4.1 is achieved. Hence by our Theorem 4.1, our solution will be oscillating around a positive level $\xi = 4.478064$ infinitely often.

Similarly we also build a model with both state persist. Here we assume parameters as following in the second persistence example.

$$\beta_1 = 0.5, \mu_1 + \gamma_1 = 45, \sigma_{1_1} = 0.02, \sigma_{2_1} = 0.05.$$

So in state 1 we let

$$\phi_1 = 2.88 > 0,$$

which means $R_{0_1}^S > 1$. And in state 2 we have

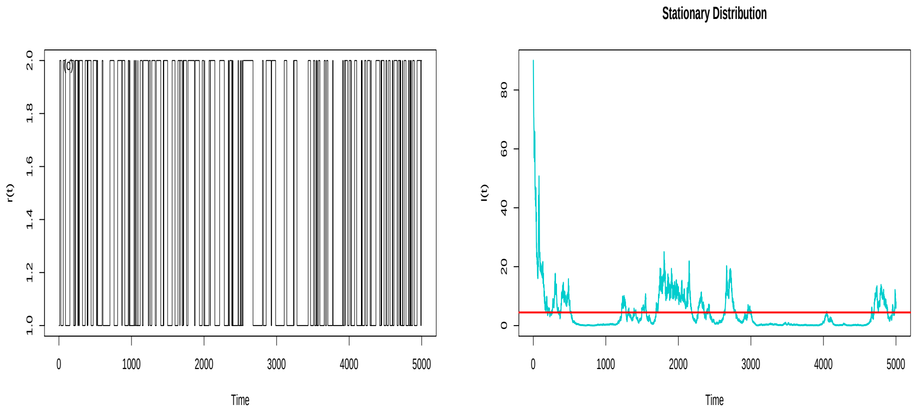
$$\beta_2 = 0.6, \mu_2 + \gamma_2 = 35, \sigma_{1_2} = 0.04, \sigma_{2_2} = 0.1.$$

Then in state 2 we have

$$\phi_2 = 16.5 > 0,$$

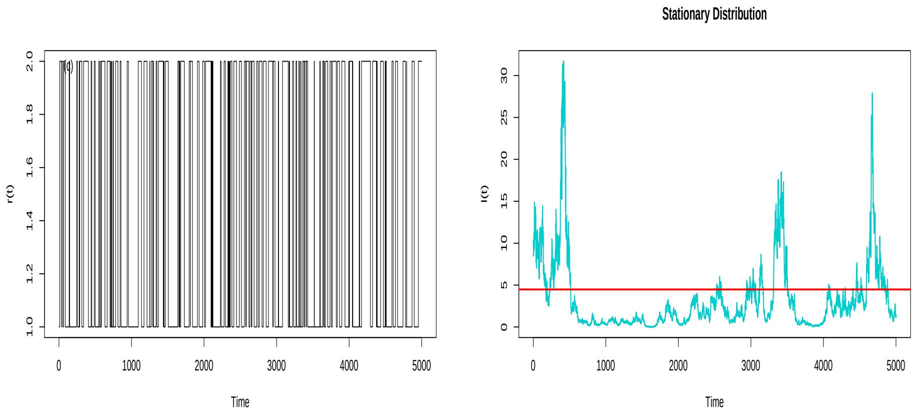
which means $R_{0_2}^S > 1$ so by Theorem 4.1 in [1] disease in both state 1 and state 2 will persist. With the same Markov chain generator in the first example

$$\nu_{12} = 3, \nu_{21} = 4,$$



(a) $I(0) = 90$ (b) $I(0) = 5$

Figure 3: Persistence Case 1 with $I(0) = 90$



(a) $I(0) = 90$ (b) $I(0) = 5$

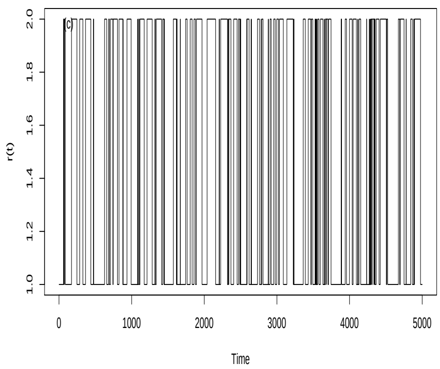
Figure 4: Persistence Case 1 with $I(0) = 10$

we can easily see that Assumption 4.1 is achieved. Hence by our Theorem 4.1, our solution will be oscillating around a positive level $\xi = 19.05665$ infinitely often.

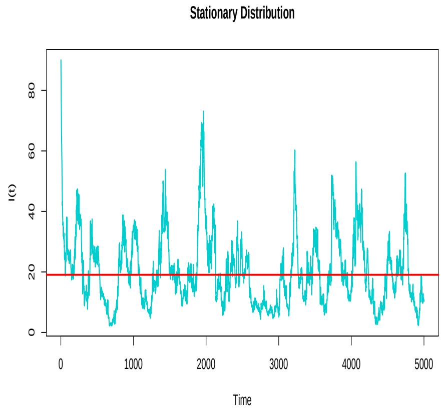
Now we use Euler-Maruyama method [2] in R with step size 0.001 and $r_0 = 1$ to simulate the solution by 5,000 iterations. Without the loss of generality, we use both large and small initial values. Clearly the solution is oscillating around the level ξ, which is marked as a red line in Figure 3, 4, 5 and 6. In Figure 3 and 4 we can see that during some iterations, the solution tends to be zero, and then goes up again to follow the fluctuation. This is clearly caused by the extinction behaviour in state 1.

5 Stationary Distribution

There are many different methods to prove the stationary distribution in a stochastic model with regime switching. For example, Zhu and Yin [27] use Lyapunov functions to develop necessary conditions for positive recurrence [28] and ergodicity in a hybrid system. Based on their results, Liu *et al.* [18] prove that their stochastic Lotka-Volterra model has a unique stationary distribution by proving the positive recurrence and ergodic property of the solution. However, these results have strong connections to Khasminskii's theory in stationary distribution. Hence in this section, we firstly recall Khasminskii's theory [17] as a lemma.

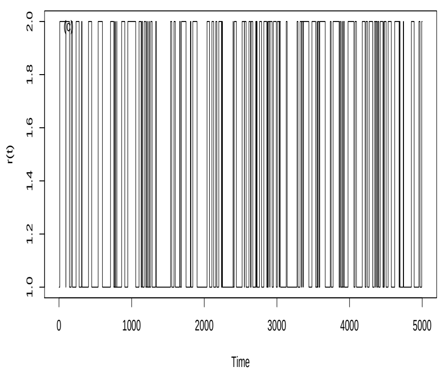


(a) $I(0) = 90$

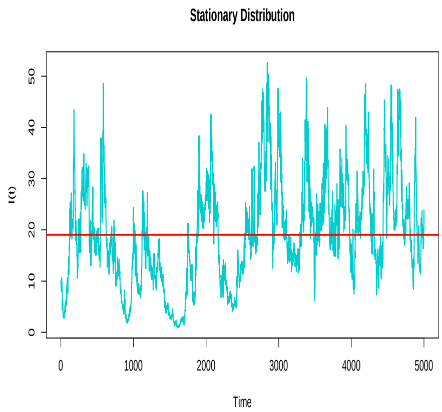


(b) $I(0) = 5$

Figure 5: Persistence Case 2 with $I(0) = 90$



(a) $I(0) = 90$



(b) $I(0) = 5$

Figure 6: Persistence Case 2 with $I(0) = 10$

5.1 Theorem

Lemma 5.1. *The SDE model* (1.7) *has a unique stationary distribution if there is a strictly proper subinterval* (a, b) *of* $(0, N)$ *such that* $\mathbb{E}(\tau) < \infty$ *for all* $I_0 \in (0, a] \cup [b, N)$*, where*

$$\tau = \inf\{t \geq 0 : I(t) \in (a, b)\}.$$

Also,

$$\sup_{I_0 \in [\bar{a}, \bar{b}]} \mathbb{E}(\tau) < \infty,$$

for every interval $[\bar{a}, \bar{b}] \subset (0, N)$*.*

Note that the other condition in Khasminskii's theory is clearly satisfied in our model (1.7). Next, for each $q \geq 0$, define an $M \times M$ matrix

$$\mathcal{C}(q) = \mathrm{diag}(\alpha_1(q), \cdots, \alpha_N(q)) - \Gamma, \tag{5.1}$$

where

$$\alpha_i(q) = q\big[\beta_i N - \mu_i - \gamma_i - \frac{1}{2}(1+q)(\sigma_{1_i}^2 N^2 + \sigma_{2_i}^2 N)\big].$$

Assumption 5.2. $\phi_i > 0$ *for all* $i \in \mathbb{S}$*.*

Theorem 5.3. *Under Assumption 3.3 and 4.1 or Assumption 5.2, for any given initial value* $I_0 \in (0, N)$ *and* $r_0 \in \mathbb{S}$*, model* (1.7) *has a unique stationary distribution.*

Proof. **Step 1.** Firstly we examine a Lyapunov function $V_1(x, i) = \log x$ with initial value $I_0 \in [b, N)$ and $r(0) = r_0$. Applying the Itô formula gives

$$LV_1(x, i) = \beta_i(N - x) - (\mu_i + \gamma_i) - \frac{1}{2}[\sigma_{1_i}^2 (N-x)^2 + \sigma_{2_i}^2 (N - x)], x \in (0, N).$$

And it is obvious that $LV_1(N, i) = -(\mu_i + \gamma_i) < 0$. So there must exists a constant b near N, for any $x \in [b, N)$ and $i \in \mathbb{S}$

$$LV_1(x, i) \leq -k_1 \quad \text{(positive constant)}.$$

Consequently, for all $t \geq 0$ and any $I_0 \in [b, N)$, we then have

$$\begin{aligned} \log b \leq \mathbb{E} \log I(t \wedge \tau) &= \mathbb{E} \log I_0 + \mathbb{E} \int_0^{t\wedge\tau} LV_1(I(s))\mathrm{d}s + 0 \\ &\leq \log I_0 - q\mathbb{E}(t \wedge \tau). \end{aligned}$$

Rearrange and we have

$$\mathbb{E}(t \wedge \tau) \leq \frac{\log \frac{I_0}{b}}{q}.$$

Let $t \to \infty$, we have

$$\mathbb{E}(\tau) \leq \frac{\log \frac{I_0}{b}}{q} < \infty, \forall I_0 \in [b, N).$$

Here we complete Step 1., which clearly indicates that we can find a positive b near the boundary N, such that the solution will proceed into $(0, b)$ in finite time with the initial value $I_0 \in [b, N)$.

Step 2. By Lemma 3.6 and 3.7, $\mathcal{C}(q)$ is a nonsingular M-matrix under Assumption 3.3 and 4.1 or Assumption 5.2. By [3, Theorem 2.10], there exists a vector $\xi = (\xi_1, \cdots, \xi_N)^T \gg 0$ such that

$$\mathcal{C}(q)\xi = \vec{1} := (1, \cdots, 1)^T,$$

i.e.

$$\alpha_i(q)\xi_i - \sum_{j=i}^{N} \nu_{ij}\xi_j = 1, \quad i \in \mathbb{S}. \tag{5.2}$$

We choose a Lyapunov function $V_2(I,t,i) = \xi_i I^{-q}$ for $(I,t,i) \in (0,N) \times \mathbb{R}_+ \times \mathbb{S}$. We want to find a positive constant a near 0, such that the time for the solution, starting in $(0,a]$, to proceed to (a,N) is finite. Using the generalised Itô formula, we have

$$\begin{aligned}
LV_2(I,i) = & -q\xi_i I^{-q}\big[\beta_i N - \mu_i - \gamma_i - \beta_i I\big] + \frac{1}{2}q(q+1)\xi_i I^{-q}\big[\sigma_{1_i}^2(N-I)^2 + \sigma_{2_i}^2(N-I)\big] \\
& + \sum_{j=1}^{N} \nu_{ij}\xi_j I^{-q} \\
& \leq -I^{-q}\Big\{q\xi_i\Big[\beta_i N - \mu_i - \gamma_i - \frac{1}{2}(q+1)\big(\sigma_{1_i}^2 N^2 + \sigma_{2_i}^2 N\big)\Big] - \sum_{j=1}^{N}\nu_{ij}\xi_j \\
& + Iq\xi_i\Big[-\beta_i + \frac{1}{2}(q+1)\big(\sigma_{1_i}^2 N + \sigma_{2_i}^2\big)\Big]\Big\} \\
& \leq -I^{-q}\Big\{1 + Iq\xi_i\Big[-\beta_i + \frac{1}{2}(q+1)\big(\sigma_{1_i}^2 N + \sigma_{2_i}^2\big)\Big]\Big\},
\end{aligned}$$

by condition (5.2). So we can find a constant a near 0, such that for all $I \in (0,a]$ and $i \in \mathbb{S}$,

$$LV_2(I,i) \leq -k_2 \text{ (a positive constant)}.$$

Consequently, for all $t \geq 0$ and any $I_0 \in (0,a]$ and $r(0) = r_0$, we then have

$$\begin{aligned}
\hat{\xi}a^{-q} \leq \mathbb{E}\big[\xi_{r(t\wedge\tau)}I^{-q}(t)\big] &= \xi_{r_0}I_0^{-q} + \mathbb{E}\int_0^{t\wedge\tau} LV_2(I(s))ds \\
&\leq \xi_{r_0}I_0^{-q} - k_2\mathbb{E}(t\wedge\tau).
\end{aligned}$$

Rearranging and letting $t \to \infty$, we have

$$\mathbb{E}(\tau) \leq \frac{\xi_{r_0}I_0^{-q} - \hat{\xi}a^{-q}}{k_2} < \infty, \quad \forall I_0 \in (0,a].$$

This indicate that we can find a positive a near 0, such that the solution will rise into (a,N) in finite time. Combining the results from Step 1 and 2 leads to the open set (a,b) we need. Hence we complete the proof. □

However, in this section, we do not intend to derive the mean and variance of this stationary distribution. Now in model (1.7), all parameters have been replaced by random variables. Thus during the deduction of mean and variance, after applying Itô formula on $I(t)$ and dividing both sides by t with $t \to \infty$, terms such as $\lim_{t\to\infty}\int_0^t \beta_i I(s)\mathrm{d}s$ are now related to the joint distributions of random variables and $I(t)$, which are very hard to compute. Hence we stop here by only provide the proof of unique stationary distribution. We will give a further discussion in the simulation section, by examining the integral average of the solution $\frac{1}{t}\int_0^t I(s)\mathrm{d}s$.

5.2 Simulation

To generate a stationary distribution we firstly use the same parameters in persistence cases. Similarly we fix $N = 100$ and we assume the parameters as following.
Case 1.

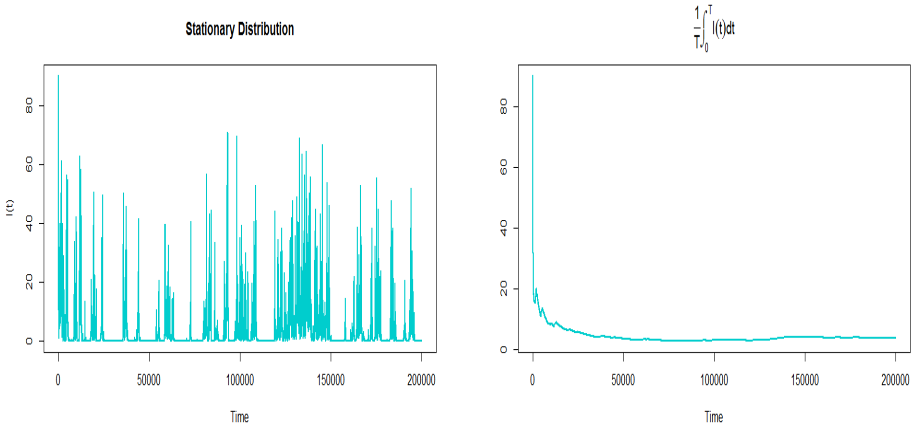
$$\beta_1 = 0.4, \mu_1 + \gamma_1 = 45, \sigma_{1_1} = 0.03, \sigma_{2_1} = 0.01;$$

$$\beta_2 = 0.6, \mu_2 + \gamma_2 = 35, \sigma_{1_2} = 0.04, \sigma_{2_2} = 0.1.$$

for state 1 and 2. Hence we have

$$\alpha_1 = \beta_1 N - (\mu_1 + \gamma_1) - \frac{1}{2}(\sigma_{1_1}^2 N^2 + \sigma_{2_1}^2 N) = -9.5 < 0;$$

$$\alpha_2 = \beta_2 N - (\mu_2 + \gamma_2) - \frac{1}{2}(\sigma_{1_2}^2 N^2 + \sigma_{2_2}^2 N) = 16.5 > 0.$$



(a) $I(0) = 90$

(b) $I(0) = 5$

Figure 7: Stationary Distribution Case 1 with $I(0) = 90$

Case 2.

$$\beta_1 = 0.5, \mu_1 + \gamma_1 = 45, \sigma_{1_1} = 0.02, \sigma_{2_1} = 0.05;$$

$$\beta_2 = 0.6, \mu_2 + \gamma_2 = 35, \sigma_{1_2} = 0.04, \sigma_{2_2} = 0.1.$$

for state 1 and 2. Hence we have

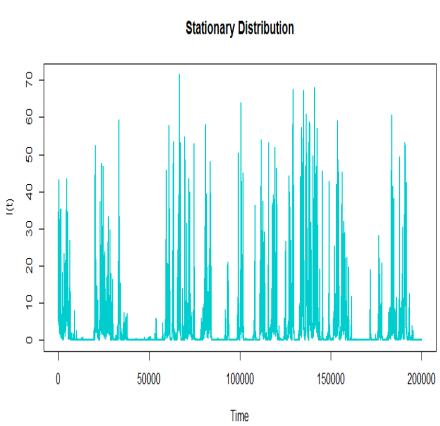
$$\alpha_1 = \beta_1 N - (\mu_1 + \gamma_1) - \frac{1}{2}(\sigma_{1_1}^2 N^2 + \sigma_{2_1}^2 N) = 2.88 > 0;$$

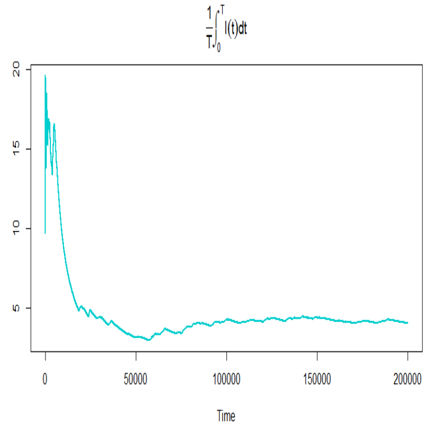
$$\alpha_2 = \beta_2 N - (\mu_2 + \gamma_2) - \frac{1}{2}(\sigma_{1_2}^2 N^2 + \sigma_{2_2}^2 N) = 16.5 > 0.$$

With the Markov Chain generator,

$$\nu_{12} = 3, \nu_{21} = 4,$$

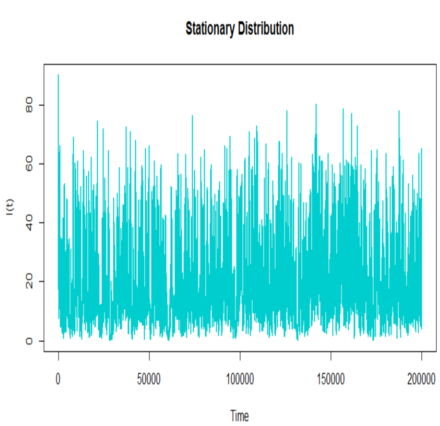
we obtain an invariant density by Theorem 5.3 in either case. Now by using Euler-Maruyama method in R, we generate a long run of 200,000 iterations with step size $\Delta = 0.001$. And we also plot the integral average of our solution, which is $\frac{1}{T}\int_0^T I(t)\mathrm{d}t$. By Khasminskii's theory [17], this integral will tend to the first moment of our solution if there is a unique stationary distribution. From Figure 7, 8, 9 and 10, we can see our solution fluctuates very intensely, which indicate the recurrence in our model (1.7). Moreover, in Figure 7 and 8, it is clearly illustrated that there are some iterations where $I(t)$ tends to zero which caused by the extinction in state 1. These results give further explanation to our persistence theory. Also, in each different cases, the integral average of $I(t)$ is also demonstrated, which clearly converges to a fixed positive level, the mean of this stationary distribution. Consequently, the numerical results support our ergodic theorem.

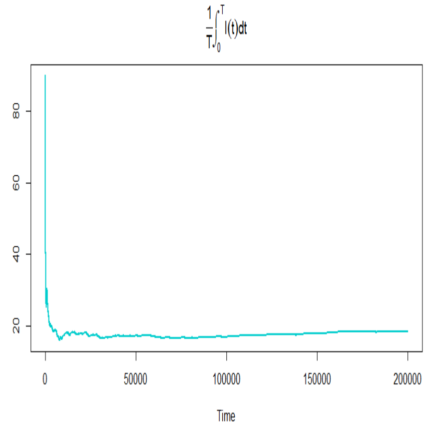




(a) $I(0) = 90$ (b) $I(0) = 5$

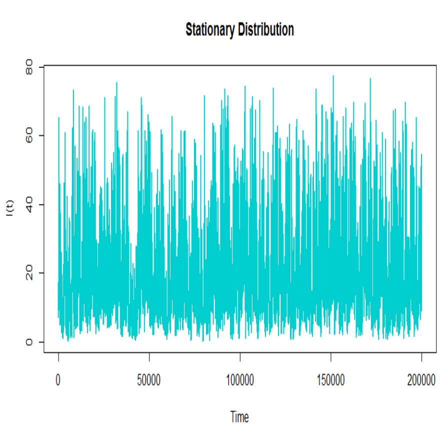
Figure 8: Stationary Distribution Case 1 with $I(0) = 10$

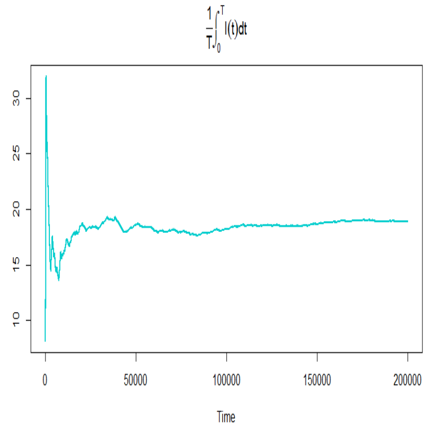




(a) $I(0) = 90$ (b) $I(0) = 5$

Figure 9: Stationary Distribution Case 2 with $I(0) = 90$





(a) $I(0) = 90$ (b) $I(0) = 5$

Figure 10: Stationary Distribution Case 2 with $I(0) = 10$

6 Conclusion

In this paper, we have discussed telegraph noise in SIS epidemic model based on Gray *et al.*'s work [2]. A finite-state Markov chain is used to describe the switching between different environments in our previous model (1.4), which formulates a stochastic SIS model with two independent Brownian motions and Markovian switching. It is obviously a generalized model of (1.4), which can be applied to more complicated cases in epidemic study. From our results in each section, we discover very interesting facts that can be related to the disturbance of telegraph noise. From the overall behaviour of the solutions, we can conclude that if we need to eliminate the disease, we do not need to have consistent extinction in every state. In some certain states, disease can even persist in the population. Similar result is also obtained in persistence analysis. This is the contribution of considering telegraph noise in our model, which indicates the expansion of extinction conditions. Moreover, while Gray *et al.* did not examine the stationary distribution in their research [2], we regard this as a very important property of the solution and in order to keep up with our previous work in [1, 4], we again prove the existence of a unique stationary distribution by using Khasminskii's theory, where the M-matrix theory plays a key role.

There is no doubt that introducing telegraph noise in model (1.4) makes our new model (1.7) more practical and complicated. However, the results in our model is also weakened and incomplete in some aspects due to the impact of telegraph noise. For example, although the simulation illustrates the integral average as the mean of the stationary distribution, we are not able to compute the explicit expression for the mean and variance; to prove the boundedness of the solution, we still require every state of the solution to be bounded within $(0, N)$ instead of replacing this condition by another one based on the overall behaviour of the solution. If such a condition can be found, then it is possible to conclude that we do not need every state to be bounded to have overall boundedness. This means, in some of the states, solution may proceed beyond N if it is examined individually. However, the disturbance of telegraph noise will always pull the solution back to $(0, N)$ by switching to other bounded states. These are the problems that we are not able to answer now but can be left as a future work.

Conflict of Interest

The authors declare that they have no conflict of interest.

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References

[1] Siyang Cai, Yongmei Cai, and Xuerong Mao. A stochastic differential equation SIS epidemic model with two independent Brownian motions. *Journal of Mathematical Analysis and Applications*, 474(2):1536–1550, 2019.

[2] Alison Gray, David Greenhalgh, Xuerong Mao, and Jiafeng Pan. The SIS epidemic model with Markovian switching. *Journal of Mathematical Analysis and Applications*, 394(2):496–516, 2012.

[3] Xuerong Mao and Chenggui Yuan. *Stochastic differential equations with Markovian switching.* Imperial college press, 2006.

[4] Siyang Cai, Yongmei Cai, and Xuerong Mao. A stochastic differential equation SIS epidemic model with two correlated Brownian motions. *Nonlinear Dynamics*, 97(4):2175–2187, Sep 2019.

[5] David A Vasseur and Peter Yodzis. The color of environmental noise. *Ecology*, 85(4):1146–1152, 2004.

[6] Lev S Tsimring. Noise in biology. *Reports on Progress in Physics*, 77(2):026601, 2014.

[7] Alison Gray, David Greenhalgh, Liangjian Hu, Xuerong Mao, and Jiafeng Pan. A stochastic differential equation SIS epidemic model. *SIAM Journal on Applied Mathematics*, 71(3):876–902, 2011.

[8] Yongmei Cai and Xuerong Mao. Stochastic prey-predator system with foraging arena scheme. *Applied Mathematical Modelling*, 64:357 – 371, 2018.

[9] Yongmei Cai, Siyang Cai, and Xuerong Mao. Stochastic delay foraging arena predator–prey system with Markov switching. *Stochastic Analysis and Applications*, 0(0):1–22, 2019.

[10] Alexandru Hening and Dang H Nguyen. Stochastic Lotka–Volterra food chains. *Journal of mathematical biology*, 77(1):135–163, 2018.

[11] Montgomery Slatkin. The dynamics of a population in a Markovian environment. *Ecology*, 59(2):249–256, 1978.

[12] D Greenhalgh, Y Liang, and X Mao. Modelling the effect of telegraph noise in the SIRS epidemic model using Markovian switching. *Physica A: Statistical Mechanics and its Applications*, 462:684–704, 2016.

[13] Y Takeuchi, NH Du, NT Hieu, and K Sato. Evolution of predator–prey systems described by a Lotka–Volterra equation under random environment. *Journal of Mathematical Analysis and applications*, 323(2):938–957, 2006.

[14] James D Hamilton. Regime switching models. *The new palgrave dictionary of economics*, pages 1–7, 2016.

[15] Qi Luo and Xuerong Mao. Stochastic population dynamics under regime switching. *Journal of Mathematical Analysis and applications*, 334(1):69–84, 2007.

[16] Xiaoyue Li, Daqing Jiang, and Xuerong Mao. Population dynamical behavior of Lotka–Volterra system under regime switching. *Journal of Computational and Applied Mathematics*, 232(2):427–448, 2009.

[17] Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66. Springer Science & Business Media, 2011.

[18] Hong Liu, Xiaoxia Li, and Qingshan Yang. The ergodic property and positive recurrence of a multi-group Lotka–Volterra mutualistic system with regime switching. *Systems & Control Letters*, 62(10):805–810, 2013.

[19] Theodore H Fleming and J Nathaniel Holland. The evolution of obligate pollination mutualisms: senita cactus and senita moth. *Oecologia*, 114(3):368–375, 1998.

[20] J Nathaniel Holland and Theodore H Fleming. Geographic and population variation in pollinating seed-consuming interactions between senita cacti (lophocereus schottii) and senita moths (upiga virescens). *Oecologia*, 121(3):405–410, 1999.

[21] J Nathaniel Holland, Donald L DeAngelis, and Judith L Bronstein. Population dynamics and mutualism: functional responses of benefits and costs. *The American Naturalist*, 159(3):231–244, 2002.

[22] Sulin Pang, Feiqi Deng, and Xuerong Mao. Asymptotic properties of stochastic population dynamics. *Dynamics of Continuous Discrete and Impulsive Systems Series A: Mathematical Analysis*, 15(5a):603–620, 2008.

[23] Yongmei Cai, Siyang Cai, and Xuerong Mao. Analysis of a stochastic predator-prey system with foraging arena scheme. *Stochastics*, pages 1–30, 2019.

[24] William J Anderson. *Continuous-time Markov chains: An applications-oriented approach*. Springer Science & Business Media, 2012.

[25] George G Yin and Qing Zhang. *Continuous-time Markov chains and applications: a singular perturbation approach*, volume 37. Springer, 2012.

[26] Desmond J Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM review*, 43(3):525–546, 2001.

[27] Chao Zhu and George Yin. Asymptotic properties of hybrid diffusion systems. *SIAM Journal on Control and Optimization*, 46(4):1155–1179, 2007.

[28] James R Norris and James Robert Norris. *Markov chains*. Cambridge university press, 1998.