

ANALYSIS OF STOCHASTIC NEARLY-INTEGRABLE DYNAMICAL SYSTEMS USING POLYNOMIAL CHAOS EXPANSIONS

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In this paper we propose the use of dynamic intrusive Polynomial Chaos Expansions (dPCE) to study some properties of nearly-integrable systems in orbital mechanics, where the perturbation is stochastic; we focus on random-walk type of perturbations.

We use a simple Weiner process to model the stochastic component of the perturbation and a truncated Karhunen–Loève expansion of the Weiner process to allow the treatment with Polynomial Chaos. In particular, we use dynamic Polynomial Chaos, where the integration time is divided in segments and PCEs are restarted on each segment, to keep the number of coefficients of the Karhunen–Loève expansion contained.

We first study a stochastic version of the Hénon-Heiles system, we then consider the motion of a stochastically perturbed satellite in geostationary orbit. For both problems we show evidence of diffusion induced by the stochastic perturbation.

INTRODUCTION

Although orbital mechanics is fundamentally based on deterministic models, the position, velocity and attitude of a space object can only be known with some degree of uncertainty. Uncertainties in the dynamic model and observations concur to transform a deterministic problem into a stochastic one. A formulation of the equations of orbital mechanics as stochastic differential equations can be found in the work of Cresson et al.^{1,2} and starts from the model proposed by Sharma et al.³ Cresson derived a new set of Gauss' planetary equations starting from the hypothesis that the dynamics is affected by a Weiner type of random process. Cresson demonstrated that under certain conditions the energy is not an integral of motion, either strong or weak, and the system indeed presents a diffusion term. This diffusion term can be directly derived from Itô's lemma. Such works can be used to explain interesting phenomena like the stochastic orbital migration of small bodies in planetary rings⁴ but it has not been applied to the analysis of other problems in orbital mechanics. Furthermore, it considers only one type of stochastic process with known distribution. However, other types of processes and models can be used to study anomalous diffusions that may better represent nature. Independently of the work of Sharma and Cresson, Belbruno demonstrated that a simple random-walk can play an important role in producing an instability of stable solutions in 3-body dynamics⁵. The works of Belbruno and Cresson et al. suggest that a more complete treatment of uncertainty and random processes in orbital mechanics is missing.

This paper is a step in the direction of a complete treatment of orbital mechanics under uncertainty. Starting from the work of Cresson et al. we will present the idea of bounded constant of motion, followed by a demonstration that under the assumption of Itô's interpretation of random-walk the

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energy might not be a bounded constant of motion. This result is in line with^{1,2}. We then present a methodology based on dynamic intrusive polynomial chaos expansions to study diffusion and instability phenomena induced by stochastic perturbations to quasi-integrable systems.

After presenting the general methodology, which includes two new indicators of diffusion based on the chaos expansion, we follow with two numerical experiments on two known dynamical systems. A conclusion section closes the paper with some considerations for future work.

STOCHASTIC ORBITAL MECHANICS

The interest is in the analysis of the evolution of orbital motion under the effect of a stochastic perturbation. In particular we consider a random-walk type of perturbing process. We start from the following Langevin type of equation:

$$\begin{aligned}\dot{v} &= f(r) + \epsilon W_t \phi(r, v) \\ \dot{r} &= v\end{aligned}\tag{1}$$

governing the motion of an object in space, subject to gravity and to a perturbing term that contains a time dependent stochastic process W_t , to be specified. The constant $\epsilon \in (0, 1]$ is a small parameter and $\phi(r, v)$ is a generic perturbing function. The evolution of the system is subject to the initial conditions:

$$[r, v] \in \Sigma\tag{2}$$

where Σ is a given set. In the following we will focus only on the case in which W_t is a Wiener process and one uses the Itô interpretation of the $W_t \phi(r, v)$ term. We argue that this is a good model for the case of random impacts where the next impact cannot be anticipated and does not depend on the previous impacts that have already occurred.

Constants of Motion

In the literature a weak first integral is such that $\mathbb{E}(H(t)) = \mathbb{E}(H(0))$, where $\mathbb{E}(\cdot)$ is the expectation operator. In this paper we consider an alternative definition of first integral. We say that H is a bounded first integral, or constant, (BFI) of motion if:

$$\underline{\mathbb{E}}(H \in \eta) = 1 \quad \forall t\tag{3}$$

where η is an invariant set, bounded from below and above, and $\underline{\mathbb{E}}$ is the lower expectation:

$$\underline{\mathbb{E}}(H \in \eta) = \min_{\alpha} \int_{\Omega} (H \in \eta) p(\alpha, \xi) d\xi\tag{4}$$

with ξ an uncertain vector, $\alpha \in \mathcal{A}$ a parameter vector that defines a set of probability measures on the space Ω such that $\xi \in \Omega$ and the integral is the usual Riemann–Stieltjes integral. If the vector α assumes only a single value then the lower expectation is the ordinary expectation operator. In the following we will consider only the case in which α assumes a single well defined value, however, the definition we propose is general and will be applied to a more general case in future work. Note that if H is a strong integral of motion it is also a bounded integral of motion, but if H is a weak integral of motion it might not be a bounded integral of motion.

Lemma 1. *Given Eq.(1) subject to a perturbation governed by a Itô process $\epsilon \phi(r, v) dW_t$ the total mechanical energy is a bounded constant of motion if the time integral of the square of the perturbing function $\phi(r, v)$ and the integral of $v \epsilon dW_t \phi(r, v)$ are bounded for every time $\forall t$.*

Proof. Consider a one dimensional system governed by Eq. (1):

$$\dot{v} = f(r) + \epsilon W_t \phi(r, v) \quad (5)$$

with W_t a Weiner process. Suppose we use the Itô interpretation of the stochastic process and we write the differential of the velocity as:

$$dv = f(r)dt + dW_t \epsilon \phi(r, v) \quad (6)$$

We now multiply both sides times the velocity v :

$$\langle dv, v \rangle = \langle f(r), v \rangle + \langle \epsilon dW_t \phi(r, v), v \rangle \quad (7)$$

with $\langle \cdot, \cdot \rangle$ the usual vector scalar product. From differentiation rules in Itô stochastic calculus we have:

$$2vdv = d(v^2) - (dv)^2 \quad (8)$$

$$d(U(r)) = f(r)dr + \frac{1}{2}(d^2U(r)/dr^2)(dr)^2 \quad (9)$$

where $f(r) = dU(r)/dr$. If we make use of the relationship:

$$dr = vdt \quad (10)$$

we have:

$$d\left(\frac{1}{2}v^2 - U(r)\right) = \frac{1}{2}[(dv)^2 + (d^2U(r)/dr^2)(dr)^2] + vdW_t \epsilon \phi(r, v) \quad (11)$$

and by substituting the expressions for dr and dv :

$$d\mathcal{E} = \frac{1}{2}[(f(r)dt + dW_t \epsilon \phi(r, v))^2 + (d^2U(r)/dr^2)(vdt)^2] + vdW_t \epsilon \phi(r, v) \quad (12)$$

where we called $\mathcal{E} = \frac{1}{2}v^2 - U(r)$ the usual total mechanical energy. If we retain only first order terms:

$$d\mathcal{E} = \frac{1}{2}dW_t^2 \epsilon^2 \phi(r, v)^2 + vdW_t \epsilon \phi(r, v) \quad (13)$$

but for Wiener processes $dW_t^2 = dt$ thus:

$$d\mathcal{E} = \frac{1}{2}\epsilon^2 \phi(r, v)^2 dt + vdW_t \epsilon \phi(r, v) \quad (14)$$

We can, therefore, conclude that the classical energy \mathcal{E} is a BFI if the integral of the square of the function $\phi(r, v)$ and the integral of $vdW_t \epsilon \phi(r, v)$ are bounded for every time $\forall t$. \square

Remark 1 If $\phi(r, v)$ is a constant function then the total mechanical energy diverges as also shown in Cresson et al.³ and thus is not either a bounded constant of motion or a weak integral of motion.

Remark 2. When the small parameter ϵ tends to 0, H reduces to the standard mechanical energy and is an integral of motion.

Remark 3 With this results, that is in line with Cresson et al. we correct also the result in⁶. In fact if $\phi = 1$ and the variance of W_t was D we would obtain that the energy diffuses with ratio D even in the Itô model.

Lemma 2. Given Eq.(1) subject to a perturbation governed by a Itô process $\epsilon\phi(r, v)dW_t$ the angular momentum is a bounded constant of motion if the time integral of the function $r \wedge \epsilon dW_t\phi(r, v)$ is bounded for every time $\forall t$.

Proof. We follow a similar process as for the total mechanical energy. We multiple both sides times r to get:

$$dv \wedge r = f(r) \wedge r + \epsilon dW_t\phi(r, v) \wedge r \quad (15)$$

where \wedge is now the usual vector product. We now introduce the usual assumption that $f(r)$ is a radial force field thus:

$$dv_t r = \epsilon dW_t\phi(r, v) \wedge r \quad (16)$$

where dv_t is the transversal component of the velocity. From the definition of angular momentum $\Gamma = r \wedge v$ one has:

$$d\Gamma = r \wedge dv + dr \wedge v \quad (17)$$

where the second term on the right hand side is zero due to the definition of dr . Thus:

$$d\Gamma = r \wedge \epsilon dW_t\phi(r, v) \quad (18)$$

□

From these two lemmas it is clear that a system that is stable without the random perturbation $\epsilon\phi(r, v)W_t$ can diffuse when this stochastic perturbation is introduced. In the following we will propose a methodology based on Polynomial Chaos Expansions to describe this diffusion process in terms of statistical moments.

Coordinate Transformation

Before presenting our proposed methodology with PCEs we recall the rule to change variables in Itô's calculus⁷. For a given stochastic system

$$dX = f(X, t)dt + g(X, t)dW_t \quad (19)$$

we are interested in writing the stochastic differential equation in a new set of variable Q , such that:

$$Q = h(X, t) \quad (20)$$

The following equality holds:

$$dQ = \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial X}dX + \frac{1}{2}dX^T \frac{\partial^2 h}{\partial X^2}dX \quad (21)$$

from which we have:

$$\begin{aligned} dQ &= \frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial X}(f dt + g dW_t) + \frac{1}{2} \frac{\partial^2 h}{\partial X^2} (f^2 dt^2 + g^2 dW_t^2 + 2gf dt dW_t) = \\ &= \left(\frac{\partial h}{\partial t} + \frac{\partial h}{\partial X}f + \frac{1}{2}g^T \frac{\partial^2 h}{\partial X^2}g \right) dt + \frac{\partial h}{\partial X}g dW_t \end{aligned} \quad (22)$$

It should be noted that, when Q is a vector, the quantity $\frac{\partial^2 h}{\partial X^2}$ is a 3-tensor, and the following holds:

$$\left(g^T \frac{\partial^2 h}{\partial X^2} g \right)_i = g^T \frac{\partial^2 h_i}{\partial X^2} g \quad (23)$$

METHODOLOGY

We are now interested in relating the evolution of system (1) with the evolution of the statistical moments associated to the stochastic variables r and v and define indicators of diffusion, chaos and instability that make use of the statistical moments rather than the stochastic variables.

Karhunen–Loève Expansion

In order to model the processes W_t in the interval $[0, T]$ we exploit the Karhunen–Loève decomposition^{8,9} :

$$W_t = \sqrt{2T} \sum_{n=0}^{\infty} z_n \frac{\sin((n + \frac{1}{2})\pi \frac{t}{T})}{(n + \frac{1}{2})\pi}, \quad \mathbb{E} \left[W_t - \sum_{n=0}^{K-1} z_n \Psi_n(t) \right]^2 \leq \frac{T}{\pi K} \quad (24)$$

where z_n are pairwise independent Gaussian variables with mean zero and unitary variance and Ψ_n are the time-dependent components of the decomposition, explicitly given in the equation on the left. While the given expansion converges in a mean square sense, the error bound is proportional to the ratio between the time interval defining the stochastic process and the number of terms considered in the expansion¹⁰ (Figure 1, Equation (24)).

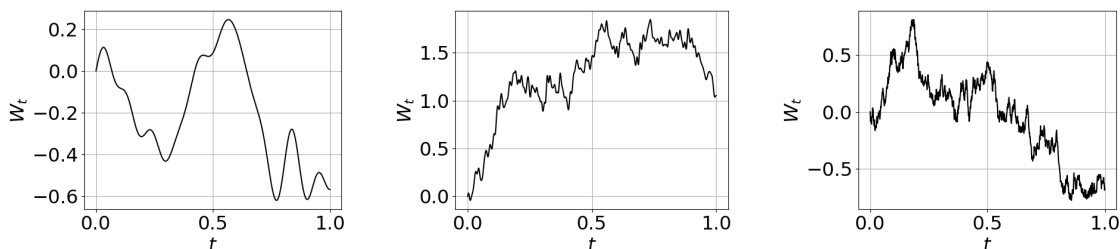


Figure 1: Realizations of the Wiener process approximation in $[0, 1]$, represented using the first 20, 200, 1000 terms of its Karhunen–Loève Expansion.

This consideration motivates the implementation of a re-initialization step that allows leveraging the Markovian nature of the process, reduces the number of uncertain variables z_n and limits the propagation error over long-term integrations. Note that, because only the first two statistical moments of the z_n variables are non-zero, a degree-one Polynomial Chaos Expansion with respect to the uncertainty space associated to the Wiener process does not lead to any truncation error, in the representation of the state.

Dynamic Polynomial Chaos Expansions

Polynomial Chaos is a computational spectral method, originated from the works of Wiener¹¹, in which the functional of the state with respect to a set of uncertain parameters is decomposed into a set of orthogonal polynomials. In this section we will briefly introduce the concept of *dynamic* generalized Polynomial Chaos, recently proposed in¹². The idea is to partition the integration domain in sub-intervals and develop a new PC expansion on each subinterval. Because the random process on each subinterval is independent of the ones in other subintervals the independence of the

uncertain variables that is required for the PC expansion is preserved. A similar restarting schemes have been also employed for deterministic dynamical systems with stochastic parameters¹³⁻¹⁶.

The method works as follows: given a set of uncertain parameters, leading to a joint probability distribution, it is possible to build a basis of polynomials, orthogonal with respect to such probability distribution, of a given degree

$$\Phi_i = \Phi_i(\mathbf{z}) \quad (25)$$

The three terms recursion relation¹⁷ can be used to create stabilized univariate orthogonal polynomials:

$$\Phi_{n+1}(q) = \Phi_n(q)(q - A_n) - \Phi_{n-1}(q)B_n, \quad A_n = \frac{\mathbb{E}[q\Phi_n^2]}{\mathbb{E}[\Phi_n^2]}, \quad B_n = \frac{\mathbb{E}[\Phi_n^2]}{\mathbb{E}[\Phi_{n-1}^2]} \quad (26)$$

It is then possible to build orthogonal multivariate polynomials using tensor product rules¹⁸, when the distributions are stochastically independent (the case of dependent variables is discussed later on in this sections). Because polynomials are orthogonal with respect to the inner product defined by the expectation operator with respect to the associated probability distribution, the following holds:

$$\langle \Phi_j, \Phi_k \rangle = \int_{\Omega} \Phi_j(\mathbf{z})\Phi_k(\mathbf{z})p(\mathbf{z})d\mathbf{z} = \mathbb{E}[\Phi_j, \Phi_k] \neq 0 \Leftrightarrow j = k \quad (27)$$

and the constant, non-trivial terms can be easily computed. Using the set of orthogonal polynomials in (26), we can write a solution, in which the randomness of the process and its time-dependence are explicitly separated, by means of:

$$\mathbf{x}(\mathbf{z}, t) = \sum_{i=1}^n \mathbf{c}_i(t)\Phi_i(\mathbf{z}) \quad (28)$$

with

$$n = \binom{K + N}{N} \quad (29)$$

In Eq. (29), K is the number of random variables characterizing the dynamics, and N is the order of the polynomial basis used to spectrally decompose the state distribution of the system: the *curse of dimensionality* arises from the fact that the number of coefficients of the PCE scales exponentially with K .

Galerkin Method

Once the solution is expressed as in Eq. (28), the differential equation (1) becomes:

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^n c_i^{x_n}(t)\Phi_i(\mathbf{z}) &= \sum_{i=1}^n \dot{c}_i^{x_n}(t)\Phi_i(\mathbf{z}) = \\ &= f_0(\mathbf{x}) + \epsilon W_t f_1(\mathbf{x}) = f_0 \left(\sum_{i=1}^n c_i^{x_n}(t)\Phi_i(\mathbf{z}) \right) + \epsilon \left(\sum_{j=0}^{K-1} z_j \Psi_j(t) \right) f_1 \left(\sum_{i=1}^n c_i^{x_n}(t)\Phi_i(\mathbf{z}) \right) \end{aligned} \quad (30)$$

One can now exploit the orthogonality of the polynomials and project the equation on the space

defined by the polynomial basis $\Phi_k(\mathbf{z})$:

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^n \dot{c}_i^{x_n}(t) \Phi_i(\mathbf{z}) \right) \Phi_k(\mathbf{z}) p(\mathbf{z}) d\mathbf{z} = \dot{c}_k^{x_n}(t) \langle \Phi_k, \Phi_k \rangle = \\ & = \int_{\Omega} \left[f_0 \left(\sum_{i=1}^n c_i^{x_n}(t) \Phi_i(\mathbf{z}) \right) + \epsilon \left(\sum_{j=0}^{K-1} z_j \Psi_j(t) \right) f_1 \left(\sum_{i=1}^n c_i^{x_n}(t) \Phi_i(\mathbf{z}) \right) \right] \Phi_k(\mathbf{z}) p(\mathbf{z}) d\mathbf{z} \end{aligned} \quad (31)$$

The initial value of the time-dependent part of the state is obtained by means of:

$$\begin{aligned} & \sum_{i=1}^n c_i^{x_n}(0) \Phi_i(\mathbf{z}) = x_{n0} \implies \\ & \sum_{i=1}^n c_i^{x_n}(0) \langle \Phi_i, \Phi_k \rangle = \langle x_{n0}, \Phi_k \rangle \implies \\ & c_k^{x_n}(0) = \frac{1}{\langle \Phi_k, \Phi_k \rangle} \int_{\Omega} x_{n0} \Phi_k(\mathbf{z}) p(\mathbf{z}) d\mathbf{z} \end{aligned} \quad (32)$$

In the general case the integrals cannot be solved analytically and a high-dimensional numerical integration scheme is required. We here propose an approach, based on sparse¹⁹ sampling, that reduces the computational cost: we perform regression to reconstruct the polynomial approximation of any non-linear function defining the dynamics²⁰. This allows one to approximate the integrand with a polynomial, making the computation of the expectation in Equation (31) straightforward.

Re-initialization

In order to reduce the number of random variables necessary to describe the process, we decompose the time horizon into a number of sub-intervals: we hence make use of PCE on each interval, in which the stochastic perturbation can be described by a low number of variables^{12,21}. In order to do so, at the end of every time segment, we are able to forget the relation between the state distribution and the random parameters, and we perform PCE with respect to the new initial state distribution, together with a new set of random parameters; we are in fact approximating the stochastic perturbation with a sequence of Wiener processes. The initial state distribution is modelled computing the mean and variance of the PCE of each component. The most critical step of the process is the computation of a new orthogonal basis, with respect to the new probability space. This is the case because of the correlation characterizing the state components' joint distribution; in fact, the approach starting from Eq. (26) cannot be used²².

A possible way to generate orthogonal bases at each re-initialization is via the Gram-Schmidt method¹⁸. For such orthogonalization procedure, we start from a set of linearly independent functions $\{\chi_1, \chi_2, \dots\}$ which are not orthonormal with respect to the inner product defined by the expectation operator; for simplicity, we start from a monomial basis, leading to $\Phi_1 = 1.0$. For the following terms of the basis, we make use of the relation:

$$\Phi_j = \chi_j - \sum_{k=0}^{j-1} \Phi_k \frac{\langle \Phi_k, \chi_j \rangle}{\langle \Phi_k, \Phi_k \rangle} \quad (33)$$

It is therefore possible to perform projections and iteratively build terms in an expansion which are orthogonal to the previous ones.*

*Another approach, faster but both numerically unstable and not easily scalable to high number of dimensions, is

We are here using degree-one basis, both with respect to the z_n and to the initial state components. The use of Anisotropic Polynomial Expansions would nevertheless increase the representability of the method, particularly for chaotic dynamical systems. In the following we refer to this form of re-initialized PCE as dynamic PCE.

Polynomial Chaos Expansion of the Perturbative Solution

Leveraging the fact that in Eq. (1) $0 < \epsilon \ll 1$ and W_t has finite variance, we can now expand the solution in power series of ϵ up to order 2, and then use a PCE representation of the expanded solution:

$$\begin{aligned} \mathbf{x} &= \mathbf{x}_0(t) + \epsilon \mathbf{x}_1(t) + \frac{1}{2} \epsilon^2 \mathbf{x}_2(t) + O(\epsilon^3) = \\ &= \sum_{i=1}^n \left(\mathbf{c}_i^{\mathbf{x}_0}(t) + \epsilon \mathbf{c}_i^{\mathbf{x}_1}(t) + \frac{1}{2} \epsilon^2 \mathbf{c}_i^{\mathbf{x}_2}(t) \right) \Phi_i + O(\epsilon^3) \end{aligned} \quad (34)$$

In order to determine the Ordinary Differential Equations associated to the PCE terms of each component of the series, we differentiate twice with respect to ϵ and evaluate at $\epsilon = 0$ the right hand side of (1) that, written in vectorial form, leads to:

$$\begin{aligned} \dot{\mathbf{x}}_0 &= \dot{\mathbf{x}}|_{\epsilon=0} = \sum_{i=1}^n \dot{\mathbf{c}}_i^{\mathbf{x}_0}(t) \Phi_i = \mathbf{f}(\mathbf{x}_0) \\ \dot{\mathbf{x}}_1 &= \left. \frac{\partial \dot{\mathbf{x}}}{\partial \epsilon} \right|_{\epsilon=0} = \sum_{i=1}^n \dot{\mathbf{c}}_i^{\mathbf{x}_1}(t) \Phi_i = \left. \frac{\partial \mathbf{f}}{\partial \epsilon} \right|_{\epsilon=0} + W_t \phi(\mathbf{x}_0) \\ \dot{\mathbf{x}}_2 &= \left. \frac{\partial^2 \dot{\mathbf{x}}}{\partial \epsilon^2} \right|_{\epsilon=0} = \sum_{i=1}^n \dot{\mathbf{c}}_i^{\mathbf{x}_2}(t) \Phi_i = \left. \frac{\partial^2 \mathbf{f}}{\partial \epsilon^2} \right|_{\epsilon=0} + 2W_t \left. \frac{\partial \phi}{\partial \epsilon} \right|_{\epsilon=0} \end{aligned} \quad (35)$$

Diffusion Indicator

We propose here two indicators that embody the effect of the stochastic perturbation on the solution of Eq. (1). These indicators can be compute with the Polynomial Chaos Expansions presented in the previous sections. If I is a quantity of interest we first consider the expectation that the quantity of interest remains within a given set K from a given time $t > \bar{t}$ with $\bar{t} < \infty$:

$$I_{d1}(t) = \mathbb{E}(I(t > \bar{t}) \notin K) \mathbb{E}(I(0) \in \delta) \quad (36)$$

where we discount the probability that the quantity of interest starts within a given set δ . It is clear from this definition that if I is a bounded integral of motion then the indicator is 0 for all times $t > \bar{t}$.

The second indicator considers the sup of the coefficients of the PCE expansion:

$$I_{d2}(t) = \frac{1}{\sup_{j>0} c_j(t)^n} \quad (37)$$

based on the Cholesky decomposition of the covariance matrix of the state, leveraging the relation between orthogonality and uncorrelation.

in which c_j is the time-dependent coefficient of the PCE of I . The idea is that if a system is diffusing we expect a diverge of the variance or higher statistical moments. This divergence is expected to be fairly fast in chaotic systems. Note that the two indicators are expected to give similar results as when the system diverges the variance increases and the expectation to leave the set K should go to 1.

NUMERICAL EXPERIMENTS

In this section we study two different dynamical systems subject to stochastic perturbations: the Hénon-Heiles system and a Geostationary Orbit subject to perturbations.

Hénon-Heiles system

Uncertainty in the initial state

Before dealing with stochastic dynamical systems, we consider a deterministic, but chaotic one. For $\lambda = 1$, the Hamiltonian associated to the Hénon-Heiles potential²³ is given by

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \left(x^2y - \frac{y^3}{3}\right) \quad (38)$$

leading to:

$$\begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2xy \\ \dot{p}_y &= -y - x^2 + y^2 \end{aligned} \quad (39)$$

with this formulation, we study the two initial conditions discussed in the referenced work. Our goal is to make use of probabilistic quantities to assess chaos, analogously to what differential objects, such as the Fast Lyapunov Indicators²⁴, Finite-time Lyapunov Exponents²⁵ and other variational indicators^{26,27} do. The initial state is identified by the value of the Hamiltonian, which for this problem is equal to the total energy of the system, of x_0, y_0, p_{y0} , from which p_{x0} , assumed to be positive, can be easily determined:

$$H_0 = 0.08333, \quad x_0 = 0.0, \quad y_0 = -0.12, \quad p_{y_0} = 0.0 \quad (40)$$

In this problem, we consider the initial position in the x-y plane as a bivariate Gaussian distribution: the two independent distributions are given, with $\sigma = 0.01$, by:

$$X_0 \sim \mathcal{N}(x_0, \sigma^2), \quad Y_0 \sim \mathcal{N}(y_0, \sigma^2) \quad (41)$$

Being interested in measuring chaos, we make use of a polynomial chaos expansion of order 1, with respect to the uncertain components of the initial position, leading to three time-dependent coefficients per position component (we are not reporting the coefficients associated to p_x, p_y for brevity), whose evolution is given in Figure 2:

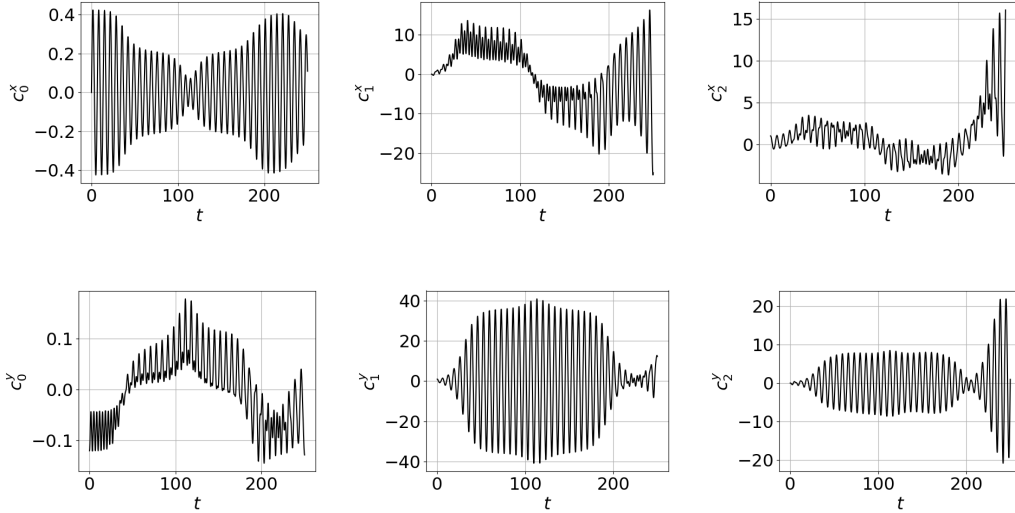


Figure 2: PCE decomposition of the position; Hénon-Heiles system.

Such coefficients are associated to the state distribution evolution, whose mean and standard deviation are represented in Figure 3:

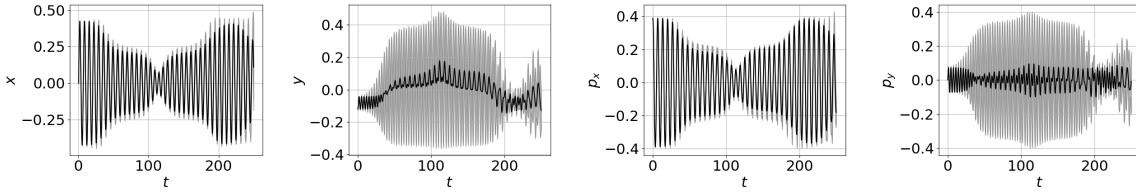


Figure 3: Mean and standard deviation of the state; Hénon-Heiles system.

As a comparison, the same computation has been performed from an initial condition distribution whose mean value, characterized by a regular behaviour, is given by:

$$H_0 = 0.08333, \quad x_0 = 0.0, \quad y_0 = 0.2, \quad p_{y0} = 0.0 \quad (42)$$

We here report only the state mean and standard deviation for brevity (Figure 4):

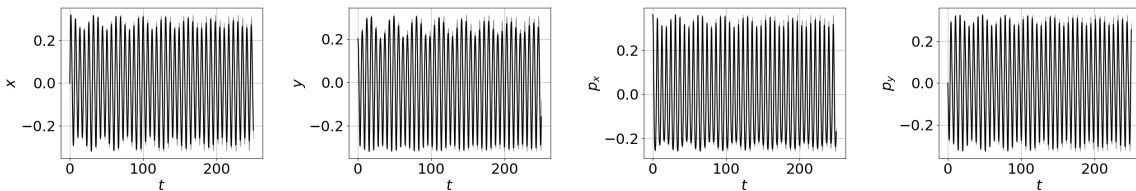


Figure 4: Mean and standard deviation of the state; Hénon-Heiles system.

The diffusion of the action variables²⁸ and the energy associated to the unperturbed dynamics, given by

$$I_1 = \pi(x^2 + p_x^2), \quad I_2 = \pi(y^2 + p_y^2), \quad \mathcal{E} = \frac{1}{2\pi}(I_1 + I_2) \quad (43)$$

is represented, for both initial conditions, in Figure 5:

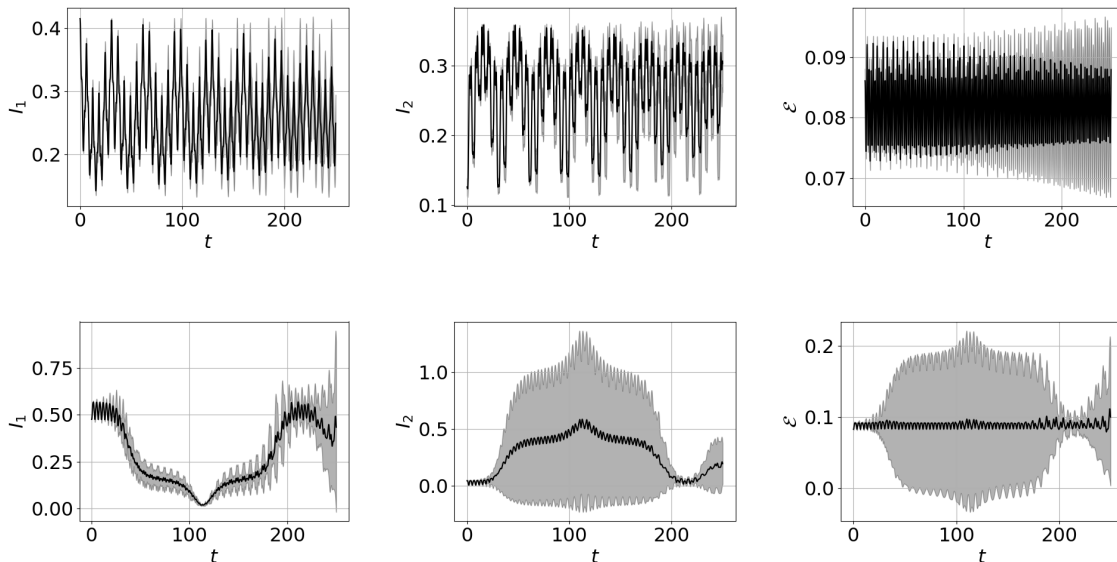


Figure 5: Mean and standard deviation of the action variables of the unperturbed system; regular and chaotic initial conditions.

It is clear that the two solutions are qualitatively different; for the regular initial condition, the energy standard deviation oscillate, while slowly increasing in time. On the other hand, the chaotic case displays a fast diffusion of the \mathcal{E} standard deviation, which quickly covers the majority of the region of the phase space inside the potential well characterizing the initial state distribution. The growth associated to the regular case is approximately linear, while the one associated to the chaotic case starts growing exponentially, in agreement with previous works^{23,29}. It is worth noticing that, sampling the distribution associated to the action variables PCE, the condition $I_i \geq 0$ is met.

We now compute the indicator in Eq. (36) to obtain a quantitative measure of chaos. To do this, we make use of the average of the initial energy (Eq. (43)), $\bar{\mathcal{E}}_0$. The behaviour of the diffusion indicators of the chaotic case are clearly different from the others (Figure 6). In the regular case, the probability distribution of the energy remains almost always bounded inside fixed intervals; in the chaotic case, the probability for the energy to be located outside the interval K quickly increases with time.

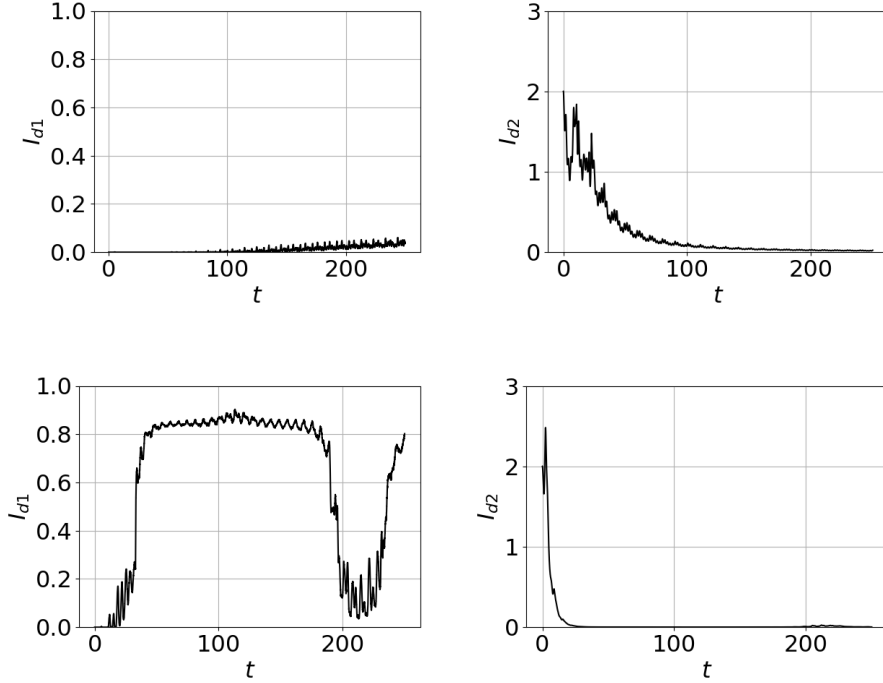


Figure 6: Diffusion indicators 1 & 2 of the Hénon-Heiles system, associated to the regular and chaotic initial conditions given above. $K = [\bar{\mathcal{E}}_0 \pm 0.02]$, 1000 samples.

While both quantities given in Figure 6 display some level of diffusion, the speed by which the second indicator associated to the chaotic condition goes to zero is much higher, particularly before $t = 100$. Its behaviour is related to the Fast Lyapunov Indicator results given in²³.

Stochastic Perturbation

We now write the Hénon-Heiles problem subject to a small stochastic perturbation. Starting from the Hamiltonian:

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \epsilon W_t \left(x^2 y - \frac{y^3}{3} \right) \quad (44)$$

we derive the system:

$$\begin{aligned} \dot{x} &= p_x \\ \dot{y} &= p_y \\ \dot{p}_x &= -x - 2\epsilon W_t x y \\ \dot{p}_y &= -y - \epsilon W_t (x^2 - y^2) \end{aligned} \quad (45)$$

With this formulation, we consider the evolution of the nominal initial condition given in Eq. (42). Moreover, we divide the time interval $[0, 2000]$ using increments of $\Delta t = 25$, and in each interval we model the stochastic perturbation using the first four terms of its Karhunen–Loève expansion. The state mean and standard deviation given in Figure 7 follow:

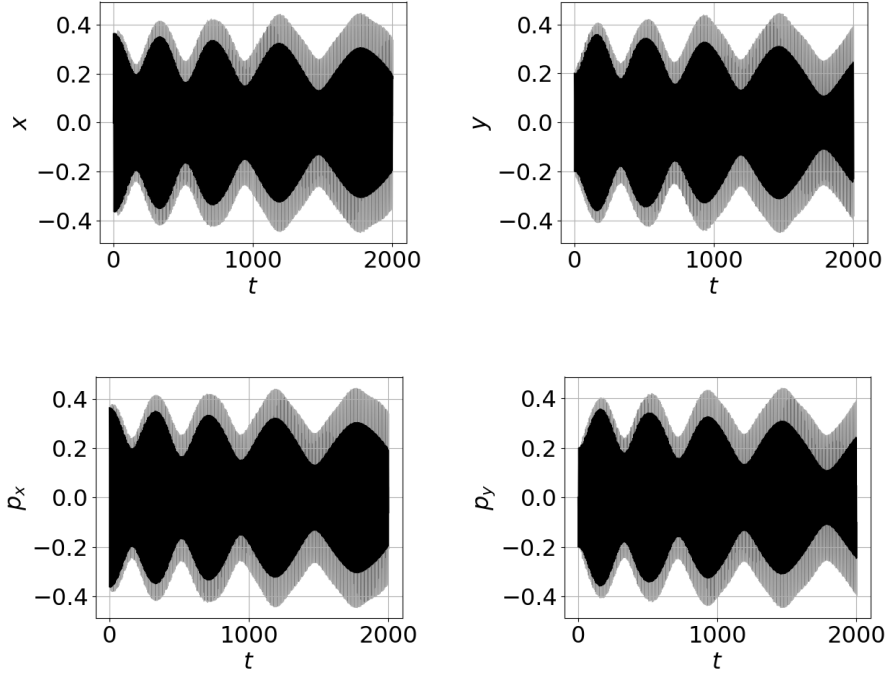


Figure 7: Mean and standard deviation of the state; Hénon-Heiles stochastic system.

Writing Eq. (14) for the Stochastic Hénon-Heiles potential, we get:

$$\mathcal{E} = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + \frac{1}{2}\epsilon^2 \int_{t_0}^t (x^2 + y^2)^2 ds + \epsilon \int_{t_0}^t (-2xyp_x + p_y(-x^2 + y^2)) dW_s \quad (46)$$

From Eq. (46), we notice how, with the introduction of a stochastic perturbation, the Hamiltonian of the system is not equal to its total mechanical energy. Moreover, recalling Eq. (43), the diffusion of the actions and of the total energy of the system can be computed (Figure 8):

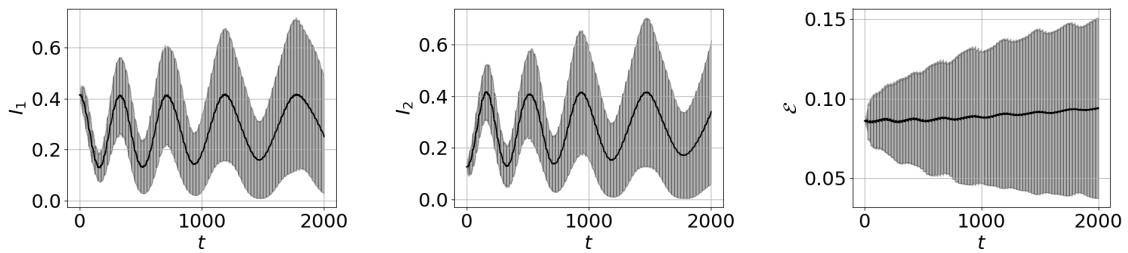


Figure 8: Mean and standard deviation of the action variables and of the Energy; Hénon-Heiles stochastic system.

This result clearly shows how the presence of a stochastic perturbation leads to a diffusion of the

action variables' probability density function. Moreover, the energy, characterized by an increasing value of the standard deviation, clearly displays an increasing value of its mean, as expected from the ϵ^2 term in Eq. (46). This implies that, even for regular (i.e., non-chaotic) orbits, the probability for the system to stay bounded around its initial orbit lowers in time: Figure 9.

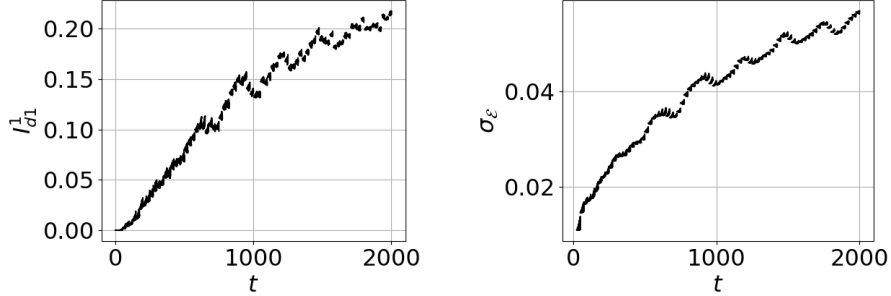


Figure 9: Diffusion indicator 1 (10000 samples, $K = [\mathcal{E}_0 \pm 0.05]$) and standard deviation of the energy of the Hénon-Heiles stochastic system.

This suggests that, in accordance with similar works³⁰, KAM tori³¹ are broken by the stochastic nature of the perturbation rather than by its increasing magnitude.

Geostationary orbit: quasi-periodic stochastic perturbation

We now study the motion of a geostationary satellite, including the main deterministic gravitational perturbations and a stochastic one³². Building on³³ (Section 5.2), we are interested in characterizing a planar system; introducing stochastic perturbations with a drag-like structure, the following equations of motion, associated to the equatorial plane, describe the system:

$$\begin{aligned}
 \dot{r} &= v_r \\
 \dot{\Lambda} &= v_\Lambda \\
 \dot{v}_r &= -\frac{\mu}{r^2} + v_\Lambda^2 r + f_r^{J_2} + f_r^{J_{22}} + v v_r \epsilon_r W_t^r \\
 \dot{v}_\Lambda &= -\frac{2v_\Lambda v_r}{r} + f_\Lambda^{J_{22}} + v v_\Lambda \epsilon_\Lambda W_t^\Lambda
 \end{aligned} \tag{47}$$

with

$$\begin{aligned}
 f_r^{J_2} &= -\frac{3}{2}\mu J_2 \frac{R^2}{r^4} \\
 f_r^{J_{22}} &= -9\mu J_{22} \frac{R^2}{r^4} \cos 2(\Lambda - \Lambda_{22}(t)), & f_\Lambda^{J_{22}} &= -6\mu J_{22} \frac{R^2}{r^4} \sin 2(\Lambda - \Lambda_{22}(t)) \\
 \Lambda_{22}(t) &= \Lambda_{220} + \omega t
 \end{aligned} \tag{48}$$

The order of magnitude of the amplitude of the two stochastic perturbations is arbitrarily set equal to the one of the J_{22} effect at GEO. This leads to the following values:

$$\epsilon_r = 9 \frac{J_{22}}{e} \frac{R^2}{r_{GEO}^3} \ll 1, \quad \epsilon_\Lambda = 6 J_{22} \frac{R^2}{r_{GEO}^3} \ll 1 \tag{49}$$

The following and initial condition has been considered³³:

$$\begin{aligned}
e &= 0.005 & i &= \omega = \Omega = 0.0^\circ & \Lambda_{22_0} &= 75.0715^\circ \\
\theta &= 0.0^\circ & a &= r_{GEO} = 42164.1696 \text{ km}
\end{aligned} \tag{50}$$

Finally, the relations used to go from the orbital, equatorial plane to spherical coordinates are:

$$r = \frac{a(1 - e^2)}{1 + e \cos \Lambda}, \quad v_r = \frac{\mu}{\Gamma} e \sin \Lambda, \quad v_\Lambda = \frac{\mu}{\Gamma} \frac{1 + e \cos \Lambda}{r}, \quad \Gamma = \sqrt{\mu a(1 - e^2)} \tag{51}$$

Polynomial Chaos Expansion of the perturbative solution

We now perform an expansion similar to the one given in Eq. (34). Here we have two small parameters (Eq. (49)) with respect to which we want to expand the expression, so we write each component of the state as:

$$x(t, \mathbf{z}) = \sum_{i=1}^n c_{i0}^x(t) \Phi_i(\mathbf{z}) + \epsilon_r \sum_{i=1}^n c_{i1}^x(t) \Phi_i(\mathbf{z}) + \epsilon_\Lambda \sum_{i=1}^n c_{i2}^x(t) \Phi_i(\mathbf{z}) \tag{52}$$

From Eq. (52), the following ordinary differential equations can be obtained, in which the orthogonality condition can be used to separate the coefficients, as previously described:

$$\begin{aligned}
\langle \Phi_k, \Phi_k \rangle \dot{c}_{k0}^{v_r} &= \langle v_r(\epsilon_r = \epsilon_\Lambda = 0), \Phi_k \rangle \\
\left. \frac{\partial v_r}{\partial \epsilon_r} \right|_{\epsilon_r = \epsilon_\Lambda = 0} &= \sum_i \dot{c}_{i1}^{v_r} \Phi_i = \frac{2\mu}{r_0^3} r_1 + 2r_0 v_{\Lambda 0} v_{\Lambda 1} + v_{\Lambda 0}^2 r_1 + 6\mu J_2 \frac{R^2}{r_0^5} r_1 + 18\mu J_{22} R^2 \frac{\Lambda_1}{r_0^4} \sin 2(\Lambda_0 - \Lambda_{22}(t)) + \\
&\quad + 36\mu J_{22} R^2 \frac{r_1}{r_0^5} \cos 2(\Lambda_0 - \Lambda_{22}(t)) + W_t^r v_0 v_{r0} \\
\left. \frac{\partial v_r}{\partial \epsilon_\Lambda} \right|_{\epsilon_r = \epsilon_\Lambda = 0} &= \sum_i \dot{c}_{i2}^{v_r}(t) \Phi_i = 6J_2 R^2 \mu \frac{r_2}{r_0^5} + 18J_{22} R^2 \mu \sin 2(\Lambda_0 - \Lambda_{22}(t)) \frac{\Lambda_2}{r_0^4} + \\
&\quad + 36J_{22} R^2 \mu \cos 2(\Lambda_0 - \Lambda_{22}(t)) \frac{r_2}{r_0^5} + 2\mu \frac{r_2}{r_0^3} + 2r_0 v_{\Lambda 0} v_{\Lambda 2} + r_2 v_{\Lambda 0}^2
\end{aligned} \tag{53}$$

Analogously, for the second component of the acceleration:

$$\begin{aligned}
\langle \Phi_k, \Phi_k \rangle \dot{c}_{k0}^{v_\Lambda} &= \langle v_\Lambda(\epsilon_r = \epsilon_\Lambda = 0), \Phi_k \rangle \\
\left. \frac{\partial v_\Lambda}{\partial \epsilon_r} \right|_{\epsilon_r = \epsilon_\Lambda = 0} &= \sum_i \dot{c}_{i1}^{v_\Lambda} \Phi_i = -12J_{22} R^2 \mu \cos 2(\Lambda_0 - \Lambda_{22}(t)) \frac{\Lambda_1}{r_0^4} + 24J_{22} R^2 \mu \sin 2(\Lambda_0 - \Lambda_{22}(t)) \frac{r_1}{r_0^5} + \\
&\quad - 2 \frac{v_{\Lambda 0} v_{r1}}{r_0} - 2 \frac{v_{\Lambda 1} v_{r0}}{r_0} + 2 \frac{r_1}{r_0^2} v_{\Lambda 0} v_{r0} \\
\left. \frac{\partial v_\Lambda}{\partial \epsilon_\Lambda} \right|_{\epsilon_r = \epsilon_\Lambda = 0} &= \sum_i \dot{c}_{i2}^{v_\Lambda}(t) \Phi_i = -12J_{22} R^2 \mu \cos 2(\Lambda_0 - \Lambda_{22}(t)) \frac{\Lambda_2}{r_0^4} + 24J_{22} R^2 \mu \sin 2(\Lambda_0 - \Lambda_{22}(t)) \frac{r_2}{r_0^5} + \\
&\quad + W_t^\Lambda v_0 v_{\Lambda 0} - 2 \frac{v_{\Lambda 0} v_{r2}}{r_0} - 2 \frac{v_{\Lambda 2} v_{r0}}{r_0} + 2 \frac{r_2}{r_0^2} v_{\Lambda 0} v_{r0}
\end{aligned} \tag{54}$$

With this formulation, we propagate the mean and standard deviation of the system for 20 days, using time intervals of 1 day. In every time interval the two Wiener processes are modelled using 4 terms each, leading to an uncertainty space of dimension 20. The results of this are shown in Figure 10, where the time unit is the day and the length unit is the kilometer.

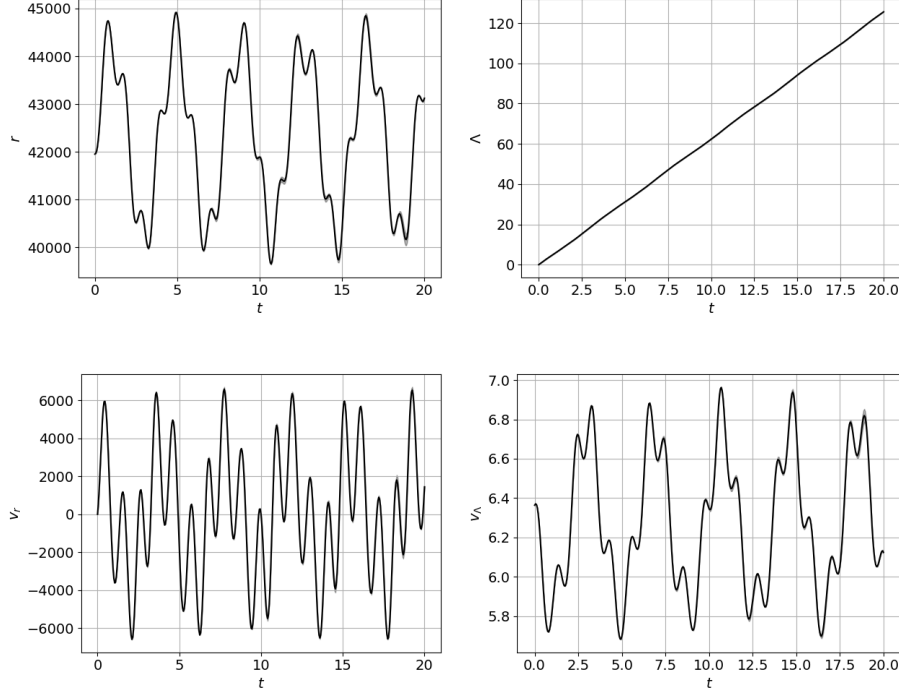


Figure 10: Mean and standard deviation of the state; GEO satellite, drag-like perturbation.

Geostationary orbit: constant stochastic perturbation

Polynomial Chaos Expansion of the perturbative solution

We now modify the stochastic perturbation given in Eq. (47):

$$\begin{aligned} \dot{v}_r &= -\frac{\mu}{r^2} + v_\Lambda^2 r + f_r^{J_2} + f_r^{J_{22}} + \epsilon W_t \\ \dot{v}_\Lambda &= -\frac{2v_\Lambda v_r}{r} + f_\Lambda^{J_2} + f_\Lambda^{J_{22}} + \epsilon W_t \end{aligned} \quad (55)$$

and perform the same expansion given in Eq. (34). The value of ϵ has been set to 0.01, leading to a stronger stochastic perturbation, compared to the previous case (all the other parameters are kept the same). The results of this are given in Figure 11.

It is clear how, in the presence of a constant* stochastic perturbation, an increasing diffusion characterizes the components of the state (Figure 12). This is in particular true for the components of the state associated to the total energy of the system, a conserved quantity in the deterministic case.

*By ‘‘constant’’ we mean that the factor multiplying the Wiener process is constant, obviously leading to a non-stationary perturbation.

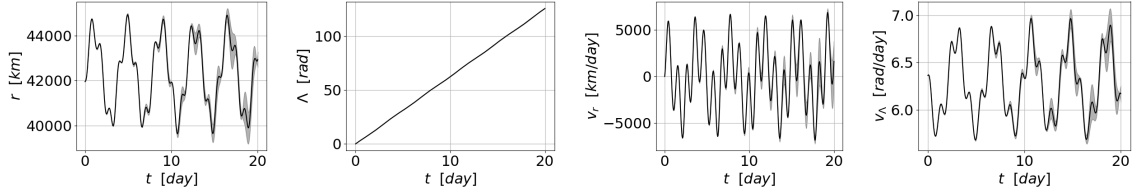


Figure 11: Mean and standard deviation of the state; GEO satellite, constant perturbation.

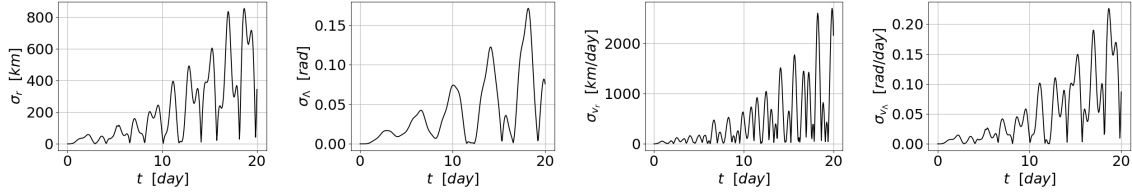


Figure 12: Standard deviation of the state components of Figure 11.

In fact, investigating the coefficients of the PCE associated to Eq. (47), a strong oscillatory behaviour can be detected (Figure 13).

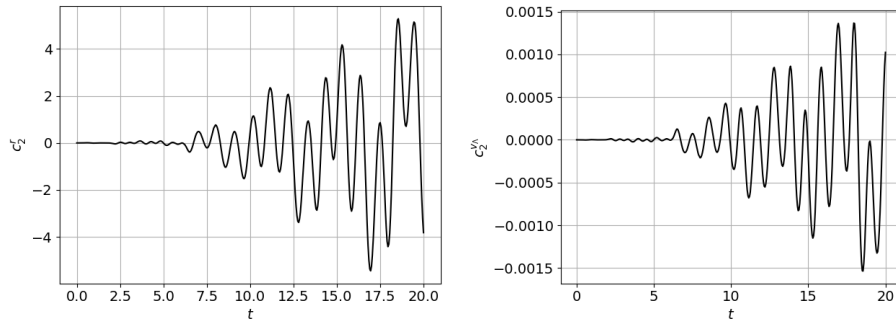


Figure 13: Two examples of PCE coefficients characterized by oscillatory behaviour, following from Eq. (47).

Stochastic Perturbation

We now perform the propagation using the original formulation of the equations of motion and the perturbation defined in Eq. (55), without using any perturbative expansion. We make use of the usual dynamic PCE with a time interval of one day, again propagating for 20 days and using the first four terms of the KL expansion. The results are given in Figure 14; the similarities between these results and the ones given in Figure 11 are due to the fact that the error of the approximation given in Eq. (34) is proportional to $\epsilon^2 = 0.0001 \ll 1$.

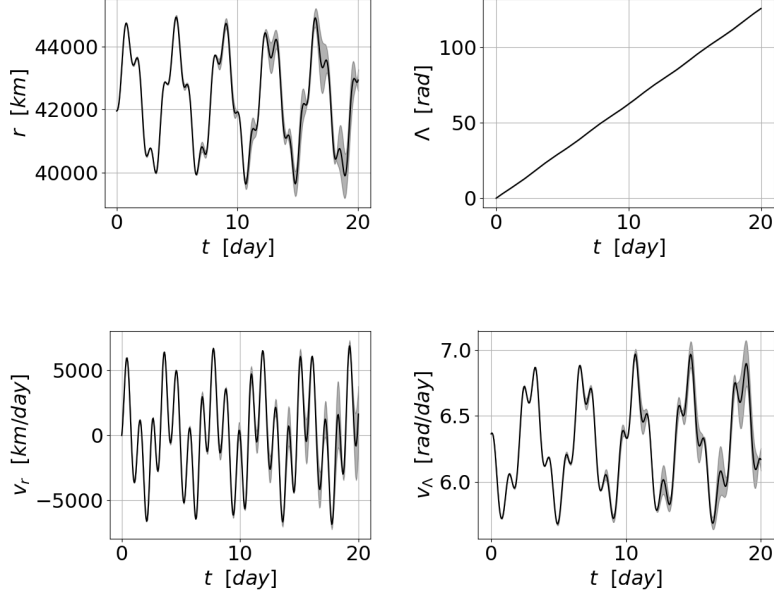


Figure 14: State evolution of a stochastically perturbed GEO satellite; original dynamic PCE.

We make use of the results obtained propagating the system for 75 days to compute the diffusion indicators associated to this problem. To do this, we first write the expression for the mechanical energy of the system, using Eq. (14), leading to Eq. (56):

$$\mathcal{E} = \frac{1}{2} ((rv_\Lambda)^2 + v_r^2) - U(r, \Lambda) + \epsilon^2(t - t_0) + \epsilon \int_{t_0}^t (v_r + v_\Lambda) dW_s \quad (56)$$

with

$$U(r, \Lambda) = \frac{\mu}{r} + \frac{1}{2}\mu J_2 \frac{R^2}{r^3} - 3\mu J_{22} \frac{R^2}{r^3} \cos(2(\Lambda - \Lambda_{22})) \quad (57)$$

With this expression, we compute the total energy distribution and the associated indicator 1, given in Figure 15; for the latter, the semi-size of the K interval, centered around \mathcal{E}_0 , has been set to $2 \cdot 10^9 \text{ km}^2/\text{day}^2$.

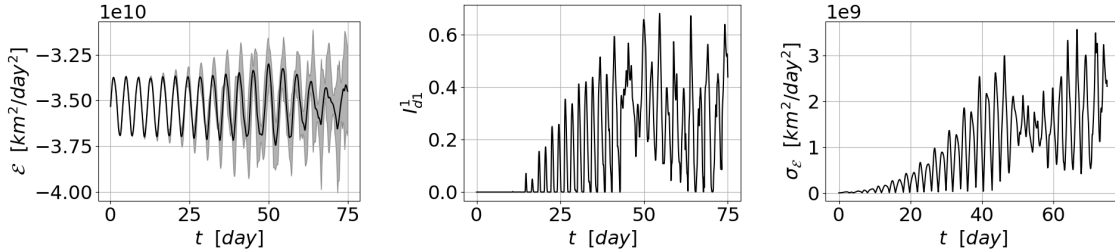


Figure 15: Energy, associated Diffusion Indicator 1 (10000 samples) and standard deviation. GEO satellite with a constant stochastic perturbation.

CONCLUSIONS

Stochastic processes driving dynamical systems are crucial, when dealing with orbital mechanics problem: from this perspective, the definition of a bounded first integral and the properties of Itô calculus have been given, showing their implications on the long-term behaviour of dynamical systems. This has also been done experimentally, making use of a Karhunen–Loève-based Polynomial Chaos Expansion, to perform the propagation. In particular, it was shown how the use of *dynamic* PCE can be used to mitigate both the course of dimensionality and long-term integration errors.

Finally, a number of results have been produced, supporting the claim that a stochastic treatment of different phenomena in orbital mechanics is necessary. Future works will in fact focus on the role of stochastic resonances in orbital mechanics and on the breakdown process of KAM tori in stochastically perturbed Hamiltonian systems; we will also investigate the applicability of the Melnikov method, supported by the Polynomial Chaos Expansion perspective, in stochastically perturbed dynamical systems.

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DATA AND SOURCE CODE

The Python Implementation developed in the context of this work will be made available, together with the data used to produce the main figures, at <https://github.com/strath-ace-labs/CASSANDRA> soon after the submission of this paper.

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