

**OPTIMAL INVESTMENT AND PROPORTIONAL  
REINSURANCE STRATEGY UNDER THE MEAN-REVERTING  
ORNSTEIN-UHLENBECK PROCESS AND NET PROFIT  
CONDITION**

YIN LI\*

Sun Yat-sen Business School  
Sun Yat-sen University  
Guangzhou 510275, China

XUERONG MAO

Department of Mathematics and Statistics  
University of Strathclyde  
Glasgow G1 1XH, UK

YAZHI SONG

Business School  
Jiangsu Normal University  
Jiangsu 221116, China

JIAN TAO

Key Laboratory of Applied Statistics of MOE, School of Mathematics and Statistics  
Northeast Normal University  
Jilin 130024, China

(Communicated by Kok Lay Teo)

**ABSTRACT.** In this study, under the criterion of maximizing the expected exponential utility of terminal wealth, the optimal proportional reinsurance and investment strategy for an insurer is examined with the compound Poisson claim process. To make the model more realistic, the price process of the risky asset is modelled by the Brownian motion risk model with dividends and transaction costs, where the instantaneous of investment return follows as a mean-reverting Ornstein-Uhlenbeck process. At the same time, the net profit condition and variance reinsurance premium principle are also considered. Using stochastic control theory, explicit expressions for the optimal policy and value function are derived, and various numerical examples are given to further demonstrate the effectiveness of the model.

**1. Introduction.** The insurance company is an important link of the social development. They are committed to the risk protection for the public and society by providing persons and property insurance service, and get returns by using premiums in the investment of bonds, stocks, loans and other financial assets. However, profit always comes with risk. Sometimes, ruin will occur when a claim is greater

---

2020 *Mathematics Subject Classification.* Primary: 90B50, 93E20; Secondary: 91G80.

*Key words and phrases.* Proportional reinsurance, Compound Poisson process, Mean reverting process, Stochastic control, Variance reinsurance premium, Net profit condition.

\* Corresponding author: Yin Li.

than the surplus of the company. Accordingly, how to make effective investment strategies that maximize profits and minimize risk, e.g. the proportion setting of investment and reinsurance, is the core competitiveness of an insurance which restricts their vitality and development. In the past decade, lots of scholars established variety of risk models to settle down the problem in the optimization of investment and reinsurance ([1, 2, 3, 4]). By using stochastic control method and related methodologies, the minimum ruin probability or the maximum expected utility of terminal wealth were obtained (for information on these issues, see [5, 6, 7, 8, 9, 10]). However, due to the uncertainty of risks, these methods are rarely used in reality. Insurance companies still take control of risky asset investments and proportional reinsurance purchases (for information on these issues, see [11, 12, 13]).

The recent trend in the study of investment strategies for insurance companies is to design the reasonable proportion of investment and reinsurance. By building a Brownian risk model, Promislow and Young [14] derived an analytical expression for minimizing the probability of ruin and the related optimal controls. Luo et al. [15] obtained the optimal strategy with the minimum ruin probability which investment includes one risk free asset and one risky asset. Bai and Guo [16] researched the optimal reinsurance problem where investment includes multiple risky assets and no-shorting constraint. Based on their results, when we do not invest the risk-free assets, the benefit from minimizing the probability of ruin is equal to that from maximizing the expected exponential utility. Additionally, for a compound Poisson risk model with diffusion process, Liang and Guo [17] researched the optimal strategy for maximizing the adjustment coefficient.

In the simulation of price process, besides the geometric Brownian motion, other type of random processes are also used for actuarial analysis. Irgens and Paulsen [18] investigated the optimal reinsurance and investments under the return of risky assets being a jump-diffusion process. Kostadinova [19] studied on the price of the risky asset and obtained an approximation of the optimal investment strategy by using a general exponential Lévy process, which maximizes the expected terminal wealth. Baev and Bondarev [20] discussed a risk model to settle down the portfolio problem, where the stock price is not affected by the standard Brownian motion, but the Ornstein-Uhlenbeck (O-U) process. Chen et al. [21] investigated the optimal investment problem with a risky asset which the stock price is affected by the random factor, e.g. the financial crisis and the changes of policy and interest; and they derived optimization for maximizing profit by using mean-variance hedging, which is a form of variance optimal martingale measure by changing the filtration. Liang et al. [22] considered maximizing the expected exponential utility of the terminal problem, which the drift process follows an O-U process; and they obtained the closed expression of the optimal investment strategy. Wang and Peng [23] researched the problem of optimal reinsurance and investment, which the risk is measured by distortion risk measures. The premiums are calculated under the distortion premium principle, and both the upper and lower premium constraints are involved.

Analyzing the research results in the field of insurance strategy, we conclude that the existing references neglect the impact of the current operation status of insurance companies (bankruptcy or excess earnings) on reinsurance investment (also see [24, 25, 26, 27] and references therein). Specifically, Lindensjö currently use a unified approach for partial information optimal investment and consumption problems for a general Itô process model [28]. Meanwhile, they also ignore the effect

of mean reverting feature in asset price on reinsurance returns. These gaps are neither conducive to the insurance companies to choose investment strategies reasonably nor help to these companies to obtain long-term returns. In this paper, we consider a model for a stock price which can have features of bull and bear markets. By assuming that the interest rate of the underlying price is a stochastic process rather than a constant, it becomes possible to more accurately model derivatives. Mean-reverting O-U stochastic interest rate model is used in the fields of quantitative finance and financial engineering to evaluate derivative securities, such as the variance of the price returns of stock and the option price. Therefore, this paper will improve and optimize the existing studies in the following three aspects. The optimal reinsurance model analysed in this article is closely related to work by Baev and Bondarev [20] and Liang et al. [22]. There are, however, several important differences. First, the net profit condition is added in the process of decision-making, which describes the impact of the company's operation status on investment strategy. Additionally, under the condition of net profit, the reinsurance proportion is taken as the threshold to realize the classified reinsurance investment which provides more accurate investment strategy for insurance company. Second, a mean-reverting O-U process instead of a general O-U process is used in simulating the trends of asset price, which reduces the expected loss of returns from the structural risk in financial market caused by non-systematic deviation and ensures that insurance companies can obtain absolute profits in the long run. Third, on the basis of traditional average return model, the stock return and transaction are independently added on the return item, which makes the income model more reliable in fitting the actual trading of the insurance market.

The remainder of this article is organized as follows. Section 2 introduces several assumptions and the optimal reinsurance model with the mean-reverting O-U process. Section 3 introduces the solution to the proposed model and explicitly derives the optimal investment and reinsurance strategy. Section 4 provides numerical examples to illustrate the applicability of the theoretical result and the value of the optimal reinsurance strategy. Section 5 summarizes the conclusion, limitation, and future research.

**2. Model formulation.** We suppose the aggregate claim amount model  $M(t)$  is compound Poisson model. The surplus process is given by

$$M(t) = u + mt - Z(t) = u + mt - \sum_{i=1}^{N(t)} Y_i$$

or

$$dM(t) = mdt - dZ(t), \tag{1}$$

where  $u$  is the initial capital,  $m$  is the insurer premium rate. The claim arrival process  $\{N(t) : t \geq 0\}$  is a Poisson process with parameter  $\lambda$  and the claim sizes  $\{Y_i : i \geq 1\}$  which are independent and identically distributed, positive, and independent of  $\{N(t)\}$ . For convenience, let  $Y$  denote a generic random variable which has the same law as  $Y_i$ . Denote the mean and moment generating function of  $Y$  by  $\mu_1 = EY(t)$  and  $M_Y(\rho) = E(e^{\rho Y})$  respectively. In order to ensure the true of the following lemma, here we make the special hypothesis of  $Y$ . We assume that  $E(Ye^{\rho Y}) = M'_Y(\rho)$  exists for  $0 < \rho < \zeta$  with  $\lim_{\rho \rightarrow \zeta} E(Ye^{\rho Y}) = \infty$  for some  $0 < \zeta \leq +\infty$ .

To transfer certain risks, we suppose the insurer purchases the proportional reinsurance. Thus, the retention function is denoted by an adapted process  $q(t) \in [0, 1]$ , where  $q(t)$  is the retention level and  $\delta(q(t))$  is the premium payment of reinsurance. Throughout this paper, we assume that the reinsurance premium is calculated according to the variance principle. That is

$$\delta(q(t)) = (1 - q(t))\lambda\mu_1 + \Lambda(1 - q(t))^2\lambda\mu_2, \quad (2)$$

where  $\mu_2 = E(Y^2)$  and  $\Lambda(> 0)$  is the safety loading of the reinsurer. Because it is necessary to guarantee non-cheap reinsurance conditions and the normal operation of the insurance company, we assume that the net profit condition is not less than 0. Denoting the insurer safety loading by  $\xi = (m - \lambda\mu_1) / (\lambda\mu_2)$  which satisfies the general condition  $\Lambda > \xi \geq 0$ , the net profit condition is

$$\begin{aligned} & m - \delta(q(t)) - q(t)\lambda\mu_1 \\ &= m - (1 - q(t))\lambda\mu_1 - \Lambda(1 - q(t))^2\lambda\mu_2 - q(t)\lambda\mu_1 \\ &= m - \lambda\mu_1 - \Lambda(1 - q(t))^2\lambda\mu_2 \geq 0, \end{aligned}$$

so

$$\begin{aligned} 1 - q(t) &\leq \sqrt{\frac{m - \lambda\mu_1}{\lambda\mu_2\Lambda}}, \\ q(t) &\geq 1 - \sqrt{\frac{m - \lambda\mu_1}{\lambda\mu_2\Lambda}}, \\ q(t) &\geq 1 - \sqrt{\frac{\xi}{\Lambda}}. \end{aligned}$$

Letting  $\bar{q} = 1 - \sqrt{\frac{\xi}{\Lambda}}$ ,  $\bar{q}$  is the threshold, which plays an important role in the paper. The positive value of the net profit condition indicates the maximum or minimum value of a certain economic factor that can reach in a certain period of time. The threshold, which is calculated by the net profit condition, is defined as the limitation of the positive returns for the insurance companies in the long term. In this paper, we assume that the claim model is a jump-diffusion model, which can better describe the influence of insurer competition and other factors on the insurer's earning process. So the surplus process becomes

$$dM^q(t) = [m - \delta(q(t))]dt + \gamma dW^{(1)}(t) - q(t)dZ(t),$$

where  $\gamma \geq 0$  is a constant,  $W^{(1)}(t)$  is a standard Brownian motion independent of the  $Y_i$  ( $i \geq 1$ ) and of claim number process  $N(t)$ . The diffusion term  $\gamma dW^{(1)}(t)$  represents the additional uncertainty associated with the economic environment or the insurance market. Because the uncertainty is not necessarily related to the claims, so we only consider that  $\gamma dW^{(1)}(t)$  is not affected by reinsurance at all.

In this paper, the insurer is allowed to invest its surplus in a financial market consisting of risky asset (stock or mutual fund) and a risk-free asset (bond or bank account). According to [30], we also suppose that the  $c$  represents the dividend income of insurance companies from stock holding. The transaction cost is  $\theta$ , which consists of fees and stamp duty. So, the riskless price process is given by

$$dS_0(t) = rS_0(t)dt,$$

where  $r(> 0)$  is a risk-free interest. In the existing literature, stock price follows a geometric Brownian motion. That is, a stock price  $S(t)$  satisfies a stochastic differential equation

$$dS(t) = (\bar{a} + c - \theta) S(t) dt + \sigma S(t) dW^{(2)}(t),$$

where  $\bar{a} (> r)$  and  $\sigma$  are positive constants,  $\bar{a} (> r)$  represents the expected instantaneous return of the risky asset,  $\sigma$  is the volatility of the risky asset price and  $W^{(2)}(t)$  is a standard Brownian motion. In addition, we assume that  $\bar{a} - \theta + c > r$ . Since the process

$$(\bar{a} + c - \theta)t + \sigma W^{(2)}(t)$$

has independent stationary increments, the stock growth have the same statistical properties over any two period of equal length, which appears to rule out bull and bear markets (see Rishel [26]).

In order to make stock model more realistic to the price feature both in bull and bear market, we construct the price process which is similar as the model established by Rishel [26]

$$dS(t) = (a(t) + c - \theta) S(t) dt + \sigma S(t) dW^{(2)}(t),$$

in which  $a(t)$  is a random process. In this model, both the mean growth  $a(t)dt$  and the volatility item  $\sigma dW^{(2)}(t)$  are all random process, which  $a(t)$  is a solution of

$$da(t) = \alpha(\bar{a} - a(t))dt + \beta dW^{(3)}(t), \quad a(0) = a_0,$$

where  $W^{(3)}(t)$  and  $a_0$  indicate another standard Brownian motion and an arbitrary constant. We assume that the joint distribution of every two Brownian motions is bivariate normal, and denote by  $\rho_1$  the correlation coefficient of  $W^{(1)}(t)$  with  $W^{(2)}(t)$ , i.e.,  $E[W^{(1)}(t), W^{(2)}(t)] = \rho_1 t$ . Similarly, denote the correlation coefficient between  $W^{(2)}(t)$  and  $W^{(3)}(t)$  by  $\rho_2$ , that is,  $E[W^{(2)}(t), W^{(3)}(t)] = \rho_2 t$ . We will not discuss the uninteresting case of  $\rho_1^2 = 1$  and  $\rho_2^2 = 1$ , in such case there would only be one source of randomness in the model. Here, the quantities of  $r, \sigma, \bar{a}, b, c, \theta, \alpha$  and  $\beta$  are known constants with positive values. And we also set the values of  $\alpha$  and  $\beta$  as the precondition of the model. Accordingly, if  $\bar{a}$  is a target of the mean growth for the stock, meanwhile,  $a(t)$  is substantially larger than  $\bar{a}$ , then the price process could be considered as a bull market. Conversely, when  $a(t)$  is substantially less than  $\bar{a}$ , the price process could be considered as a bear market.

We assume the total money amount invested in the risky asset is  $A(t)$  at time  $t$ ; namely, using the investment strategy  $A(t)$ , an insurer can obtain the wealth  $V(t)$  at time  $t$ . Following the assumptions of Browne [1], we allow  $A(t) < 0$  and  $A(t) > V(t)$ . That is, we allow the company to short sell the risky asset ( $A(t) < 0$ ), and also allow the company to borrow money for investment in the risky asset  $A(t) > V(t)$ . Because we only consider investing in the risky asset and the risk-free asset, the dynamics of  $V(t)$  then takes the form

$$\begin{aligned}
dV(t) &= A(t) \frac{dS(t)}{S(t)} + (V(t) - A(t)) \frac{dS_0(t)}{S_0(t)} + dM^q(t) \\
&= [A(t)(a(t) + c - \theta - r) + rV(t)] dt \\
&\quad + [m - \delta(q(t))] dt + \gamma dW^{(1)}(t) + \sigma A(t) dW^{(2)}(t) - q(t) dZ(t) \\
&= [A(t)(a(t) + c - \theta - r) + rV(t) + m - \delta(q(t))] dt \\
&\quad + \gamma dW^{(1)}(t) + \sigma A(t) dW^{(2)}(t) - q(t) dZ(t),
\end{aligned}$$

where  $\delta(q(t))$  is given in (2).

In this paper, we assume that continuous trading is allowed and all assets are infinitely divisible. In a complete probability space  $(\Omega, \mathcal{F}, P)$  where the  $V(t)$  is well defined, the information at time  $t$  is given by complete filtration  $\mathcal{F}(t)$  generated by  $V(t)$ . If the policy  $(A(t), q(t))$  is  $\mathcal{F}(t)$ -progressively measurable, and satisfies the condition that  $E \left[ \int_0^T A^2(t) dt \right] < \infty$ , a.s., for all  $T < \infty$ , we say the strategy is admissible. The set of all admissible strategies is denoted by  $\Pi$ .

Assume that we invest to maximize the terminal wealth utility at time  $T$ , and use  $U(x)$  to present the utility function as  $U'(x) > 0$  and  $U''(x) < 0$ . Thus our optimization problem can be described by the following formulate

$$\max E[U(V(T))], \quad (3)$$

subject to

$$\begin{cases} dV(t) = [A(t)(a(t) + c - \theta - r) + rV(t) + m - \delta(q(t))] dt \\ \quad + \gamma dW^{(1)}(t) + \sigma A(t) dW^{(2)}(t) - q(t) dZ(t), \\ (A(t), q(t)) \in \Pi, \\ da(t) = \alpha(\bar{a} - a(t)) dt + \beta dW^{(3)}(t), \quad a(0) = a_0, \end{cases} \quad (4)$$

where  $\delta(q(t))$  is given in (2).

To model the utility function, prior works provided a variety of methods in investing and decision making (Merton [31] and Karatzas [32]). Here, we apply the method designed by Loubergé and Watt [33] to control the investment risks. Because in the next section we use the exponential utility function. This utility has constant absolute risk aversion parameter  $n$ .

### 3. Solution to the Model.

**3.1. General Framework.** We use dynamic programming techniques to solve above problem (see, Fleming and Soner [34]). Define the associated value function by

$$H(t, v, a) = \max_{(A(t), q(t)) \in \Pi} E[U(V(T)) | V(t) = v, a(t) = a]. \quad (5)$$

In order to solve problem (5), we use dynamic programming techniques and in particular the HJB equation. Let  $C^{1,2}$  denote the space of  $\phi(t, v, a)$  such that  $\phi$  and its partial derivatives  $\phi_t, \phi_v, \phi_a, \phi_{vv}, \phi_{aa}, \phi_{va}$  continuous on  $[0, T] \times R \times R$ . The generator for the state process (4) subject to the choice  $A(t), q(t)$  has the form

$$\begin{aligned}
 & \mathcal{A}^{A,q}\phi(t, v, a) \\
 = & \phi_t + \left[ A(a + c - \theta - r) + rv + m - (1 - q)\lambda\mu_1 - \Lambda(1 - q)^2\lambda\mu_2 \right] \phi_v \\
 & + \alpha(\bar{a} - a)\phi_a + \left[ \frac{1}{2}(A^2\sigma^2 + \gamma^2 + 2A\sigma\gamma\rho_1) \right] \phi_{vv} + \frac{1}{2}\beta^2\phi_{aa} + \sigma A\beta\rho_2\phi_{va} \\
 & + \lambda E[\phi(t, v - qY, a) - \phi(t, v, a)],
 \end{aligned}$$

where  $\phi(t, v, a) \in C^{1,2}$ . Thus, we can see that if the value function  $H(t, v, a) \in C^{1,2}$ , then  $H(t, v, a)$  satisfies the following HJB equation:

$$\sup_{A,q} \mathcal{A}^{A,q}H(t, v, a) = 0 \quad (6)$$

for  $t < T$  with the following boundary condition

$$H(T, v, a) = U(v), \quad (7)$$

where

$$\begin{aligned}
 & \mathcal{A}^{A,q}H(t, v, a) \\
 = & H_t + \left[ A(a + c - \theta - r) + rv + m - (1 - q)\lambda\mu_1 - \Lambda(1 - q)^2\lambda\mu_2 \right] H_v \\
 & + \alpha(\bar{a} - a)H_a + \left[ \frac{1}{2}(A^2\sigma^2 + \gamma^2 + 2A\sigma\gamma\rho_1) \right] H_{vv} + \frac{1}{2}\beta^2H_{aa} + \sigma A\beta\rho_2H_{va} \\
 & + \lambda E[H(t, v - qY, a) - H(t, v, a)]
 \end{aligned}$$

and  $H_t, H_v, H_a, H_{vv}, H_{aa}, H_{va}$  are the partial derivatives of  $H(t, v, a)$ .

The following verification theorem shows that the classical solution to the HJB equation yields the solution to the optimization problem (4).

**Theorem 3.1.** (*Verification Theorem*) *Let  $W \in C^{1,2}$  be concave solution to HJB (6) subject to the boundary Condition (7). Then value function  $H$  given by (5) coincides with  $W$ . That is*

$$W(t, v, a) = H(t, v, a).$$

*Proof.* Since  $\Pi$  be any admissible control system,  $(A(s), q(s))$  is  $\mathcal{F}(t)$ -progressively measurable,

$$\mathcal{A}^{A(s),q(s)}W(t, V(s), a(s)) = 0.$$

Similar method as that in Fleming and Soner [34] (Chapter III, Theorem 8.1), from (7) and the Dynkin formula

$$\begin{aligned}
 W(t, v, a) &= E \left\{ \int_t^T -\mathcal{A}^{A(s),q(s)}W(s, V(s), a(s)) ds + U(V(T)) \right\} \\
 &\leq E \{U(V(T))\}.
 \end{aligned}$$

Taking conditional expectations given  $(t, v, a)$  on both sides of above equation and taking (6) into consideration yields:

$$W(t, v, a) \leq H(t, v, a).$$

When  $\Pi = \Pi^*$ , the inequality in the above formula becomes an equality, and thus  $W(t, v, a) = H(t, v, a)$ . Then, the proof is complete.  $\square$

Furthermore, let  $(A^*, q^*)$  be such that

$$\mathcal{A}^{A^*, q^*} H(t, v, a) = 0$$

for all  $(t, v, a) \in [0, T] \times R \times R$ . If  $(A^*, q^*)$ , together with any initial data  $(t, v, a)$ , determine a Markov process  $(V^*(s), a(t))$  with backward evolution operator  $\mathcal{A}^{A^*, q^*}$ , then we can take

$$(A^*(t), q^*(t)) = (A^*(t, V^*(s), a(t)), q^*(t, V^*(s), a(t))).$$

Once the corresponding control system  $\Pi^*$  is verified to be admissible,  $\Pi^*$  is optimal. When this procedure works, we call  $(A^*(t, V^*(t), a(t)), q^*(t, V^*(t), a(t)))$  an optimal Markov control policy. Then the Markovian strategy

$$(A^*(t, V^*(t), a(t)), q^*(t, V^*(t), a(t)))$$

is optimal with

$$\max E[U(V^*(T)) | V^*(t) = v, a(t) = a] = H(t, v, a).$$

Here, because  $V^*(t)$  is a dynamic asset that is derived under the optimal strategy, which is different from  $V(t)$ , so we call  $V^*(t)$  the reserve process.

Accordingly, the function satisfies the HJB function which can use the Theory 3.1. To find the specific solution which satisfies both (6) and (7), we try to use the exponential utility to figure out the classical solution. Here, the exponential utility function is applied because it is a pure utility function when zero utility principle gives a fair premium; namely the terminal wealth defined by this function is independent against the insurance reserve level (Gerber [35]).

**3.2. Exponential Utility.** Suppose now that the investor has an exponential utility function

$$U(v) = \lambda_1 - \frac{\eta}{n} e^{-nv},$$

where the parameters  $\lambda_1, \eta, n$  are positive constants. This utility has constant absolute risk aversion (CARA) parameter  $n$ .

Following the methods of Browne [1], we conjecture a solution of the form

$$H(t, v, a) = \lambda_1 - \frac{\eta}{n} \exp[-nve^{r(T-t)} + G(t, a)], \quad (8)$$

where  $G(t, a)$  is a suitable function to be determined. And the boundary condition  $V(T, v, a) = U(x)$  implies that

$$G(T, a) = 0. \quad (9)$$

Let  $G_t, G_a, G_{aa}$  be the partial derivatives of  $G(t, a)$ . Note that

$$\begin{cases} H_t = [H(t, v, a) - \lambda_1] [nvr e^{r(T-t)} + G_t], \\ H_v = [H(t, v, a) - \lambda_1] [-ne^{r(T-t)}], \\ H_a = [H(t, v, a) - \lambda_1] [G_a], \\ H_{vv} = [H(t, v, a) - \lambda_1] [n^2 e^{2r(T-t)}], \\ H_{aa} = [H(t, v, a) - \lambda_1] [G_a^2 + G_{aa}], \\ H_{va} = [H(t, v, a) - \lambda_1] [-ne^{r(T-t)} G_a], \\ E[H(t, v - qY, a) - H(t, v, a)] = [H(t, v, a) - \lambda_1] [M_Y(nqe^{r(T-t)}) - 1], \end{cases} \quad (10)$$

where  $M_Y(nqe^{r(T-t)}) = E[e^{nqe^{r(T-t)} Y}]$ . Substituting (10) back into the HJB equation (6), since  $H(t, v, a) - \lambda_1 < 0$ , we get



$$\begin{aligned} & \inf_{A,q} \left\{ G_t - \left[ A(a+c-\theta-r) + v + m - (1-q)\lambda\mu_1 - \Lambda(1-q)^2\lambda\mu_2 \right] ne^{r(T-t)} \right. \\ & + \alpha(\bar{a}-a)G_a + \frac{1}{2} (A^2\sigma^2 + \gamma^2 + 2A\sigma\gamma\rho_1) n^2 e^{2r(T-t)} + \frac{1}{2}\beta^2 [G_a^2 + G_{aa}] \\ & \left. + \sigma A\beta\rho_2 \left[ -ne^{r(T-t)}G_a \right] + \lambda \left[ M_Y \left( nqe^{r(T-t)} \right) - 1 \right] \right\} = 0 \end{aligned}$$

for  $t < T$ . Or equivalently,

$$\begin{aligned} & G_t - mne^{r(T-t)} - \lambda + \alpha(\bar{a}-a)G_a + \frac{1}{2}\beta^2 [G_a^2 + G_{aa}] + \frac{1}{2}n^2e^{2r(T-t)}\gamma^2 \\ & + \inf_A \left\{ \left[ -A(a+c-\theta-r) + \frac{1}{2}\sigma^2A^2ne^{r(T-t)} + \sigma A\gamma\rho_1ne^{r(T-t)} - \sigma A\beta\rho_2G_a \right] ne^{r(T-t)} \right\} \\ & + \inf_q \left\{ \left[ (1-q)\lambda\mu_1 - \Lambda(1-q)^2\lambda\mu_2 \right] ne^{r(T-t)} + \lambda M_Y \left( nqe^{r(T-t)} \right) \right\} = 0. \end{aligned} \quad (11)$$

Let

$$\begin{aligned} & f_1(A, t) \\ & = \left[ -A(a+c-\theta-r) + \frac{1}{2}\sigma^2A^2ne^{r(T-t)} + \sigma A\gamma\rho_1ne^{r(T-t)} - \sigma A\beta\rho_2G_a \right] ne^{r(T-t)}, \end{aligned}$$

and

$$f_2(q, t) = \left[ (1-q)\lambda\mu_1 - \Lambda(1-q)^2\lambda\mu_2 \right] ne^{r(T-t)} + \lambda M_Y \left( nqe^{r(T-t)} \right). \quad (12)$$

Differentiating  $f_1(A, t)$  with respect to  $A$  yields the minimizer

$$A^* = \frac{(a+c-\theta-r) + \sigma\beta\rho_2G_a}{\sigma^2ne^{r(T-t)}} - \frac{\gamma\rho_1}{\sigma},$$

and the value of  $f_1(A, t)$  at this minimum is

$$\begin{aligned} f_1(A^*, t) & = \frac{(a+c-\theta-r) + \sigma\beta\rho_2G_a}{\sigma} \gamma\rho_1ne^{r(T-t)} \\ & - \frac{1}{2} \left[ \frac{(a+c-\theta-r) + \sigma\beta\rho_2G_a}{\sigma^2} \right]^2 - \frac{1}{2} \left[ \gamma\rho_1ne^{r(T-t)} \right]^2. \end{aligned}$$

Similarly, differentiating  $f_2(q, t)$  with respect to  $q$ , we have

$$\frac{\partial f_2(q, t)}{\partial q} = -(\lambda\mu_1 + 2\Lambda(1-q)\lambda\mu_2) ne^{r(T-t)} + \lambda E \left[ Y e^{nqe^{r(T-t)}Y} \right] ne^{r(T-t)},$$

and

$$\frac{\partial^2 f_2(q, t)}{\partial q^2} = \lambda E \left[ n^2 e^{2r(T-t)} Y^2 e^{nqe^{r(T-t)}Y} \right] + 2\Lambda\lambda\mu_2 ne^{r(T-t)} > 0.$$

Therefore,  $f_2(q, t)$  is a convex function with respect to  $q$ , and its minimizer  $q^*$  satisfies the following equation:

$$M'_Y \left( nqe^{r(T-t)} \right) = \mu_1 + 2\Lambda(1-q)\mu_2, \quad (13)$$

where  $M'_Y \left( nqe^{r(T-t)} \right) = E \left[ Y e^{nqe^{r(T-t)}Y} \right]$ . Then, we have

**Lemma.** For any  $t \in [0, T)$ , equation (13) has a unique positive root  $q_1(t) \in (0, 1)$ .

*Proof.* Let

$$g_1(q) = M'_Y \left( nqe^{r(T-t)} \right),$$

and

$$g_2(q) = \mu_1 + 2\Lambda(1-q)\mu_2,$$

Then, we have

$$\begin{cases} g_1(0) = \mu_1, \\ g'_1(q) = E \left[ ne^{r(T-t)} Y^2 e^{nqe^{r(T-t)} Y} \right] > 0, \\ g''_1(q) = E \left[ n^2 e^{2r(T-t)} Y^3 e^{nqe^{r(T-t)} Y} \right] > 0. \end{cases} \quad (14)$$

The meaning of the (14) is that for any  $t \in [0, T]$ ,  $g_1(q)$  is an increasing convex function with  $g_1(0) = \mu_1$ . Furthermore,  $g_2(q)$  is a decreasing linear function with  $g_2(0) = \mu_1 + 2\Lambda\mu_2$  and  $g_2(1) = \mu_1$ . Since  $g_1(0) < g_2(0)$  and  $g_1(1) > g_2(1)$ , there is a unique intersection between  $g_1(q)$  and  $g_2(q)$  at some  $0 < q_1(t) < 1$ , and hence the proof is complete.  $\square$

From Lemma, we can obtain  $q_1(t)$ . For any  $0 \leq t_1 \leq t_2 \leq T$ , we have

$$E \left[ Y \exp \left( nqe^{r(T-t_2)} Y \right) \right] > E \left[ Y \exp \left( nqe^{r(T-t_1)} Y \right) \right],$$

and know  $q_1(0) < q_1(T)$  from Lemma.

Under the net profit condition with the retention level  $q_1(t) \in [\bar{q}, 1]$  and  $\bar{q} = 1 - \sqrt{\xi/\Lambda}$ , three cases can be concluded as the optimal reinsurance policy based on the different values of net profits, as

(I) If  $\bar{q} \leq q_1(0)$ , when  $t = 0$ , the result of the optimal reinsurance strategy is not lower than the net profit condition threshold. Under this condition, since the insurance company can achieve excess benefits, they will adopt the optimal strategy and ignore the net profit condition, in this case net profit conditions are negligible, therefore the optimal reinsurance policy is

$$q^*(t) = q_1(t).$$

(II) If  $\bar{q} \geq q_1(T)$ , the optimal reinsurance policy is

$$q^*(t) = \bar{q}.$$

(III) If  $q_1(0) < \bar{q} < q_1(T)$  by using the Lemma, we see that there exists  $\bar{t} \in (0, T)$  which satisfies  $q_1(\bar{t}) = \bar{q}$ . So the optimal reinsurance policy is

$$q^*(t) = \begin{cases} \bar{q}, & t \in [0, \bar{t}], \\ q_1(t), & t \in (\bar{t}, T]. \end{cases}$$

**Remark 1.** Case (II) is almost impossible in actual trading, so we just analyze this situation in the theoretical aspect.

Put  $(A, q) = (A^*, q^*)$  in (11). Then, after some algebraic simplification, it can be shown that (8) is a solution to (11) if  $G(t, a)$  is a solution to

$$\begin{aligned} & G_t - mne^{r(T-t)} - \lambda + \alpha(\bar{a} - a)G_a + \frac{1}{2}\beta^2[G_a^2 + G_{aa}] \\ & + \frac{1}{2}n^2e^{2r(T-t)}\gamma^2 + \frac{(a+c-\theta-r) + \sigma\beta\rho_2G_a}{\sigma}\gamma\rho_1ne^{r(T-t)} \\ & - \frac{1}{2}\frac{[(a+c-\theta-r) + \sigma\beta\rho_2G_a]^2}{\sigma^2} - \frac{1}{2}[\gamma\rho_1ne^{r(T-t)}]^2 + f_2(q^*(t), t) = 0, \end{aligned} \quad (15)$$

with the boundary condition of  $G(T, a) = 0$ .

Similar to the result of Rishel [26], we have the following theorem.

**Theorem 3.2.** *The PDE (15) with the terminal condition (9) has a solution with the form*

$$G(t, a) = K(t)a^2 + J(t)a + L(t), \quad (16)$$

if  $K(t)$  is a solution to

$$K'(t) - 2 \left( \alpha + \frac{\beta\rho_2}{\sigma} \right) K(t) + 2\beta^2 (1 - \rho_2^2) K^2(t) - \frac{1}{2\sigma^2} = 0, \quad (17)$$

with the terminal condition

$$K(T) = 0, \quad (18)$$

defined on  $[0, T]$ ;  $J(t)$  is a solution to

$$\begin{aligned} J'(t) + \left[ 2\beta^2 (1 - \rho_2^2) K(t) - \alpha - \frac{\beta\rho_2}{\sigma} \right] J(t) \\ - \left[ \frac{(c - \theta - r)(1 + 2\sigma\beta\rho_2)}{\sigma^2} - \frac{\gamma\rho_1 n e^{r(T-t)}}{\sigma} - 2\bar{a}\alpha K(t) \right] = 0, \end{aligned} \quad (19)$$

with the terminal condition

$$J(T) = 0, \quad (20)$$

and  $L(t)$  is a solution to

$$\begin{aligned} L'(t) + N(t) + \left[ \bar{a}\alpha + \beta\rho_2\gamma\rho_1 n e^{r(T-t)} - \frac{(c - \theta - r)\sigma\beta\rho_2}{\sigma^2} \right] J(t) \\ + \frac{1}{2}\beta^2 (1 - \rho_2^2) J^2(t) + \left[ \beta^2 + 2\beta\rho_2\gamma\rho_1 n e^{r(T-t)} \right] K(t) = 0, \end{aligned} \quad (21)$$

with the terminal condition

$$L(T) = 0, \quad (22)$$

where

$$\begin{aligned} N(t) = - m n e^{r(T-t)} - \lambda + \frac{1}{2} \left[ n \gamma e^{r(T-t)} \right]^2 - \frac{(c - \theta - r)^2}{2\sigma^2} \\ + \frac{(c - \theta - r)\gamma\rho_1 n e^{r(T-t)}}{\sigma} - \frac{1}{2} \left[ \gamma\rho_1 n e^{r(T-t)} \right]^2 + f_2(q^*(t), t). \end{aligned}$$

Note that  $K'$ ,  $J'$ , and  $L'$  are the derivatives of  $K$ ,  $J$ , and  $L$ , respectively.

*Proof.* Substituting (16) into the PDE (15) gives

$$\begin{aligned} K'(t)a^2 + J'(t)a + L'(t) + (bv - m) n e^{r(T-t)} - \lambda + \alpha(\bar{a} - a) [2K(t)a + J(t)] \\ + \frac{1}{2}\beta^2 \left[ (2K(t)a + J(t))^2 + 2K(t) \right] + \frac{1}{2}n^2\gamma^2 e^{2r(T-t)} - \frac{1}{2} \left[ \gamma\rho_1 n e^{r(T-t)} \right]^2 \\ + \frac{a + (c - \theta - r) + \sigma\beta\rho_2 [2K(t)a + J(t)]}{\sigma} \gamma\rho_1 n e^{r(T-t)} \\ - \frac{\{a + (c - \theta - r) + \sigma\beta\rho_2 [2K(t)a + J(t)]\}^2}{2\sigma^2} + f_2(q^*(t), t) = 0. \end{aligned} \quad (23)$$

Multiplying out terms in (23) gives

$$\begin{aligned}
& K'(t)a^2 + J'(t)a + L'(t) + (bv - m)ne^{r(T-t)} - \lambda + 2\alpha(\bar{a} - a)K(t)a \\
& + \alpha(\bar{a} - a)J(t) + \frac{1}{2}\beta^2 [4K^2(t)a^2 + 4K(t)J(t)a + J^2(t) + 2K(t)] \\
& + \frac{1}{2}n^2\gamma^2e^{2r(T-t)} + \frac{1}{\sigma}\gamma\rho_1ne^{r(T-t)}a + \frac{\sigma\beta\rho_2}{\sigma}2K(t)\gamma\rho_1ne^{r(T-t)} \\
& + \frac{c - \theta - r}{\sigma}\gamma\rho_1ne^{r(T-t)} + \frac{\sigma\beta\rho_2}{\sigma}J(t)\gamma\rho_1ne^{r(T-t)} - \frac{1}{2}[\gamma\rho_1ne^{r(T-t)}]^2 \\
& - \frac{[2\sigma\beta\rho_2K(t) + 1]^2a^2}{2\sigma^2} - \frac{1}{2}\beta^2\rho_2^2J^2(t) - \frac{1}{\sigma}\beta\rho_2J(t)a \\
& - 2\beta\rho_2K(t) \cdot \beta\rho_2J(t)a - \frac{(c - \theta - r)^2}{2\sigma^2} - \frac{(c - \theta - r)[1 + 2\sigma\beta\rho_2K(t)]}{\sigma^2}a \\
& - \frac{(c - \theta - r)\beta\rho_2J(t)}{\sigma^2} + f_2(q^*(t), t) = 0.
\end{aligned}$$

Grouping terms according to the powers of  $a$  leads to

$$\begin{aligned}
& \left[ K'(t) - 2\left(\alpha + \frac{\beta\rho_2}{\sigma}\right)K(t) + 2\beta^2(1 - \rho_2^2)K^2(t) - \frac{1}{2\sigma^2} \right] a^2 \\
& + \left\{ J'(t) + \left[ 2\beta^2(1 - \rho_2^2)K(t) - \alpha - \frac{\beta\rho_2}{\sigma} \right] J(t) \right. \\
& \quad \left. - \left[ \frac{(c - \theta - r)(1 + 2\beta\rho_2K(t))}{\sigma^2} - \frac{\gamma\rho_1ne^{r(T-t)}}{\sigma} - 2\bar{a}\alpha K(t) \right] \right\} a \\
& + \left\{ \left[ \bar{a}\alpha + \beta\rho_2\gamma\rho_1ne^{r(T-t)} - \frac{(c - \theta - r)\sigma\beta\rho_2}{\sigma^2} \right] J(t) \right. \\
& \quad \left. + L'(t) + N(t) + \frac{1}{2}\beta^2(1 - \rho_2^2)J^2(t) + \left[ \beta^2 + 2\beta\rho_2\gamma\rho_1ne^{r(T-t)} \right] K(t) \right\} = 0.
\end{aligned}$$

where

$$\begin{aligned}
N(t) &= -mne^{r(T-t)} - \lambda + \frac{1}{2}[\gamma ne^{r(T-t)}]^2 - \frac{(c - \theta - r)^2}{2\sigma^2} \\
& \quad + \frac{(c - \theta - r)\gamma\rho_1ne^{r(T-t)}}{\sigma} - \frac{1}{2}[\gamma\rho_1ne^{r(T-t)}]^2 + f_2(q^*(t), t).
\end{aligned}$$

Thus, (16) is a solution of the PDE (15) if  $K(t), J(t), L(t)$  are solutions to the differential equations (17), (19) and (21), respectively. The boundary condition (9) implies (18), (20) and (22). Hence, the proof is complete.  $\square$

Theorem 3.2. gives a solution to the PDE (15) with the boundary condition (9), only if the ordinary differential equations (17), (19), (21) with respective boundary conditions (18), (20), (22) have solutions on  $[0, T]$ . Notice that if (17) with (18) has a solution for  $K(t)$  on  $[0, T]$ , then (19) with (20) and (21) with (22) have solutions on  $[0, T]$  for  $J(t)$  and  $L(t)$ , respectively, since (19) and (21) are linear differential equations and their solutions are just integrals. So we mainly discuss the solutions to the Riccati equation for  $K(t)$ . For notational convenience, letting

$$D = 2\beta^2(1 - \rho_2^2), \quad B = -2\left(\alpha + \frac{\beta\rho_2}{\sigma}\right), \quad C = -\frac{1}{2\sigma^2}.$$

Then, equation (17) becomes

$$K'(t) + DK^2(t) + BK(t) + C = 0, \quad (24)$$

which is a normal Riccati equation with condition

$$B^2 - 4DC > 0.$$

Using standard methods, we obtain the following solution to the Riccati equation (24) with the boundary condition  $K(T) = 0$  :

$$K(t) = C_1 + \frac{e^{t\sqrt{B^2-4DC}}}{\frac{D}{\sqrt{B^2-4DC}} \left( e^{t\sqrt{B^2-4DC}} - e^{T\sqrt{B^2-4DC}} \right) - \frac{1}{C_1} e^{T\sqrt{B^2-4DC}}}, \quad (25)$$

where

$$C_1 = \frac{-B - \sqrt{B^2 - 4DC}}{2D}.$$

Based on (25), one can show that the solution to the linear ordinary differential equation (19) with the boundary condition (20) has the form

$$J(t) = e^{\int_t^T p(s)ds} \left[ - \int_t^T q(s) e^{-\int_s^T p(u)du} ds \right], \quad (26)$$

where

$$p(s) = 2\beta^2 (1 - \rho_2^2) K(s) - \alpha - \frac{\beta\rho_2}{\sigma},$$

and

$$q(s) = \frac{(c - \theta - r)(1 + 2\sigma\beta\rho_2)}{\sigma^2} - \frac{\gamma\rho_1 ne^{r(T-t)}}{\sigma} - 2\bar{a}\alpha K(s).$$

Moreover, from (21), (25) and (26), integration yields

$$\begin{aligned} L(t) &= - \int_t^T \left\{ N(s) + \left[ \bar{a}\alpha + \beta\rho_2\gamma\rho_1 ne^{r(T-s)} - \frac{(c - \theta - r)\sigma\beta\rho_2}{\sigma^2} \right] J(s) \right. \\ &\quad \left. + \frac{1}{2}\beta^2 (1 - \rho_2^2) J^2(s) + \left[ \beta^2 + 2\beta\rho_2\gamma\rho_1 ne^{r(T-s)} \right] K(s) \right\} ds \\ &= -n \left[ -\frac{c}{r} + \frac{(c - \theta - r)\gamma\rho_1}{\sigma} \right] \left[ e^{r(T-t)} - 1 \right] - \int_t^T f_2(q^*(s), s) ds \\ &\quad + \left[ \lambda + \frac{(c - \theta - r)^2}{2\sigma^2} \right] (T - t) - \frac{1}{4}\gamma^2 n^2 \left[ \frac{1}{r} - \rho_1^2 \right] \left[ e^{2r(T-t)} - 1 \right] \\ &\quad - \int_t^T \left\{ \left[ \bar{a}\alpha + \beta\rho_2\gamma\rho_1 ne^{r(T-s)} - \frac{(c - \theta - r)\sigma\beta\rho_2}{\sigma^2} \right] J(s) \right. \\ &\quad \left. + \frac{1}{2}\beta^2 (1 - \rho_2^2) J^2(s) + \left[ \beta^2 + 2\beta\rho_2\gamma\rho_1 ne^{r(T-s)} \right] K(s) \right\} ds. \end{aligned}$$

Letting

$$\begin{aligned}
L_1(t) = & -n \left[ -\frac{c}{r} + \frac{(c-\theta-r)\gamma\rho_1}{\sigma} \right] \left[ e^{r(T-t)} - 1 \right] \\
& + \left[ \lambda + \frac{(c-\theta-r)^2}{2\sigma^2} \right] (T-t) - \gamma^2 n^2 \left[ \frac{1}{r} - \frac{1}{4}\rho_1^2 \right] \left[ e^{2r(T-t)} - 1 \right] \\
& - \int_t^T \left\{ \left[ \bar{a}\alpha + \beta\rho_2\gamma\rho_1 n e^{r(T-s)} - \frac{(c-\theta-r)\sigma\beta\rho_2}{\sigma^2} \right] J(s) \right. \\
& \left. + \frac{1}{2}\beta^2 (1-\rho_2^2) J^2(s) + \left[ \beta^2 + 2\beta\rho_2\gamma\rho_1 n e^{r(T-s)} \right] K(s) \right\} ds.
\end{aligned} \tag{27}$$

Parallel to Cases (I)-(III) discussed earlier, we get

$$L(t) = \begin{cases} L_1(t) - \int_t^T f_2(q^*(s), s) ds, & \text{if } \bar{q} \leq q_1(0), \\ L_1(t) - \int_t^T f_2(\bar{q}, s) ds, & \text{if } \bar{q} \geq q_1(T), \end{cases}$$

and if  $q_1(0) < \bar{q} < q_1(T)$ ,

$$L(t) = \begin{cases} L_1(t) - \int_t^T f_2(\bar{q}, s) ds + k, & \text{if } t \in [0, \bar{t}], \\ L_1(t) - \int_t^T f_2(q^*(s), s) ds, & \text{if } t \in (\bar{t}, T], \end{cases}$$

where  $f_2(q, t)$  is determined by (12),  $q^*(t)$  is determined by Cases (I)-(III),  $\bar{t}$  is given by equation  $q^*(\bar{t}) = \bar{q}$ . The constant  $k = \int_t^T f_2(\bar{q}, s) ds - \int_t^T f_2(q^*(s), s) ds$ , so  $L(t)$  is continuous differential function for all  $t \in [0, T]$ .

To end the section, we summarize the result in the following theorem.

**Theorem 3.3.** *Assume that the investor has an exponential utility function*

$$U(v) = \lambda_1 - \frac{\xi}{n} e^{-nv},$$

and  $q_1(t)$  is a unique positive root to equation (13) for any  $t \in [0, T]$ . Then, the optimal strategy for the optimization problem (3) subject to (4) is

$$A^*(t) = \frac{[a(t) + c - \theta - r] + \sigma\beta\rho_2 [2K(t)a(t) + J(t)a(t)]}{\sigma^2 n e^{r(T-t)}} - \frac{\gamma\rho_1}{\sigma},$$

for any  $t \in [0, T]$ . Furthermore,

(I) For any  $t \in [0, T]$ , if  $\bar{q} \leq q_1(0)$ , the optimal reinsurance strategy  $q^*(t) = q_1(t)$ , and the corresponding value function has the form

$$H(t, v, a) = \lambda_1 - \frac{\eta}{n} \exp \left[ -nve^{r(T-t)} + G_1(t, a) \right],$$

where

$$G_1(t, a) = K(t)a^2 + J(t)a + L_1(t) - \int_t^T f_2(q^*(s), s) ds.$$

(II) For any  $t \in [0, T]$ , if  $\bar{q} \geq q_1(T)$ , the optimal reinsurance strategy  $q^*(t) = \bar{q}$ , and the corresponding value function has the form

$$H(t, v, a) = \lambda_1 - \frac{\eta}{n} \exp \left[ -nve^{r(T-t)} + G_2(t, a) \right],$$

where

$$G_2(t, a) = K(t)a^2 + J(t)a + L_1(t) - \int_t^T f_2(\bar{q}, s) ds.$$

(III) For any  $t \in [0, T]$ , if  $q_1(0) < \bar{q} < q_1(T)$ , the optimal reinsurance strategy

$$q^*(t) = \begin{cases} \bar{q}, & t \in (0, \bar{t}), \\ q_1(t), & t \in (\bar{t}, T] \end{cases}$$

and the corresponding value function has the form

$$H(t, v, a) = \begin{cases} \lambda_1 - \frac{\eta}{n} \exp \left[ -nve^{r(T-t)} + G_3(t, a) \right], & t \in [0, \bar{t}], \\ \lambda_1 - \frac{\eta}{n} \exp \left[ -nve^{r(T-t)} + G_1(t, a) \right], & t \in (\bar{t}, T], \end{cases}$$

where

$$G_3(t, a) = K(t)a^2 + J(t)a + L_1(t) - \int_t^T f_2(\bar{q}, s) ds + k,$$

$f_2, q_1(t), K(t), J(t)$  and  $L_1(t)$  are given in (12), (13), (25), (26) and (27) respectively.

**Remark 2.** In this paper, because of the choice of exponential utility, the optimal strategies do not depend on wealth, but on time to maturity  $T$  and  $a(t)$ . If other utility functions are selected, this will not be the case.

**Remark 3.** The results in Theorem 3.3 show that the investment strategy and the value function are all affected by the above parameters. Specifically, the higher the dividend income is, the greater the number of risk assets will be invested. Conversely, the higher the transaction costs and risk-free returns are, the less the number of risk assets will be invested.

**4. Numerical Examples for the Optimal Reinsurance Policy.** In this section, we present two numerical examples, to illustrate the results we obtained in previous sections. Suppose that claim size  $\{Y_i\}$  are uniform distributed in the interval  $[0, 2]$ , denoted by  $Y_i \sim U[0, 2]$ , then we have  $\mu_1 = 1, \mu_2 = \frac{4}{3}$ , and

$$M'_Y \left( nqe^{r(T-t)} \right) = \frac{\exp \{2mqe^{r(T-t)}\} [2mqe^{r(T-t)} - 1] + 1}{2[mqe^{r(T-t)}]^2}. \quad (28)$$

Putting equation (28) back into equation (13) yields

$$\mu_1 + 2\Lambda(1-q)\lambda\mu_2 = \frac{\exp \{2mqe^{r(T-t)}\} [2mqe^{r(T-t)} - 1] + 1}{2[mqe^{r(T-t)}]^2}. \quad (29)$$

**Example 1.** (Reinsurance proportion under different levels of risk aversion) In this example, we assume that  $\lambda = 1, m = 2, \Lambda = 1, T = 10$  and  $r = 0.3$ . The result is shown in Figure 1.

A simple calculation shows that  $\theta = (m - \lambda\mu_1) / \lambda\mu_2 = \frac{3}{4}$ , and  $\bar{q} = 1 - \sqrt{\frac{\theta}{\Lambda}} \approx 0.134$ . So equation (29) becomes

$$1 + \frac{8}{3}(1 - q(t)) = \frac{\exp \{q(t)e^{0.3(10-t)}\} [q(t)e^{0.3(10-t)} - 1] + 1}{2[nq(t)e^{0.3(10-t)}]^2}.$$

(1) When  $n = 0.2$ , from Figures 1 we notice that  $\bar{q} < q_1(0)$ , in this case, the net profit condition becomes trivial, thus the optimal reinsurance policy is

$$q^*(t) = q_1(t).$$

(2) When  $n = 0.5$ , we can see  $\bar{q} > q_1(0)$  from Figures 1, in this case, we need calculating  $\bar{t}$ , that is

$$1 + \frac{8}{3}(1 - 0.134) = \frac{\exp\{0.134e^{0.3(10-\bar{t})}\} [0.134e^{0.3(10-\bar{t})} - 1] + 1}{2[0.067e^{0.3(10-\bar{t})}]^2},$$

because  $\bar{t}$  is positive, we obtain  $\bar{t} = 1.559$ . So the optimal reinsurance policy is

$$q^*(t) = \begin{cases} 0.134, & t \in [0, 1.559], \\ q_1(t), & t \in (1.559, 10]. \end{cases}$$

(3) When  $n = 1$ , similar to case (2), we obtain  $\bar{t} = 3.870$ . So the optimal reinsurance policy is

$$q^*(t) = \begin{cases} 0.134, & t \in [0, 3.870], \\ q_1(t), & t \in (3.870, 10]. \end{cases}$$

Figure 1 shows that the optimal reinsurance strategy  $q^*$  decrease w.r.t. the CARA parameter  $n$ . It is well-known that larger  $n$  means that the insurer is more risk averse. Hence, if the insurer is more risk averse, she would like to purchase reinsurance to cede part of their losses to reinsurance.

**Example 2.** (*Reinsurance proportion under different safety loadings*) For  $\lambda = 1, m = 2, n = 0.5, T = 10$  and  $r = 0.3$ , the optimal result is displayed in Figure 2.

A simple calculation shows that  $\theta = (m - \lambda\mu_1) / \lambda\mu_2 = \frac{3}{4}$ , and equation (13) becomes

$$1 + \frac{8}{3}\Lambda(1 - q(t)) = \frac{\exp\{q(t)e^{0.3(10-t)}\} [q(t)e^{0.3(10-t)} - 1] + 1}{2[0.5q(t)e^{0.3(10-t)}]^2}.$$

Similar to example 1, when  $\Lambda = 1, 1.2, 1.5$ , putting them into  $1 - \sqrt{\frac{\theta}{\Lambda}}$ , we obtain  $\bar{q} = 0.1340, 0.2094, 0.2929$ , respectively. From Figures 2 we notice that  $\bar{q} < q_1(0)$  in these three cases. So in this example, the net profit condition becomes trivial, thus their the optimal reinsurance policies are

$$q^*(t) = q_1(t).$$

In Figure 2, we can also see that, a larger value of  $\Lambda$  will yield a higher retention level of optimal reinsurance. This simply states that as the price of reinsurance increases, the insurer should retain a greater share of each claim.

**Remark 4.** From Figures 1 and 2, we find that the optimal reinsurance strategy increases when the value of  $t$  increases, which means that the larger value of  $t$  is, the larger insurers share in the risk will be. Moreover, with the increase of  $\Lambda$ , the insurer would take more insurance business risk, and so the insurer should raise the retention level of reinsurance to decrease the expensive payment to reinsurance.

**5. Conclusion.** To solve the problem of optimizing the investment and reinsurance strategy under different operation states of insurance company, this paper designs a new actuarial strategy selection model from the perspective of net profits to provide differentiated investment and reinsurance schemes. Specifically, we use mean reverting O-U process to show the trend of price, which describes the conversion of bull and bear market. Additionally, according to the standard of net profit, we set the threshold to simulate the operation state of enterprises by applying segmental



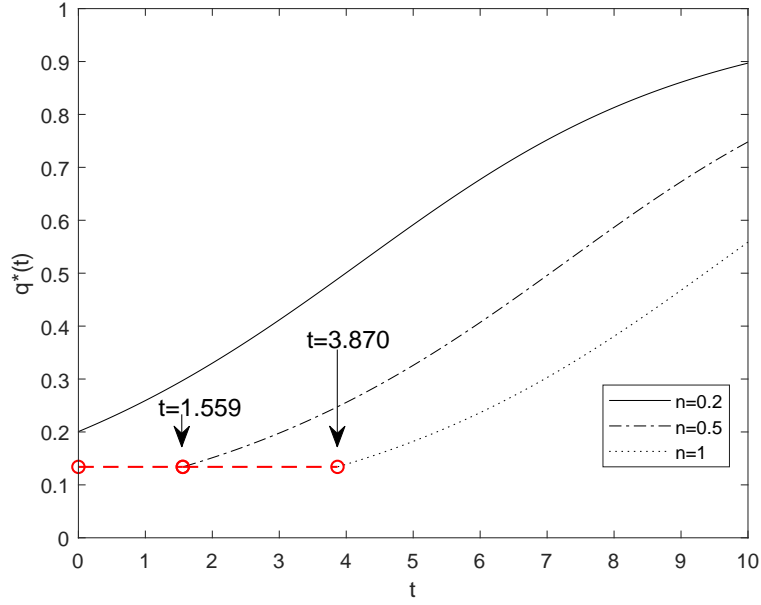


FIGURE 1. The optimal proportional level  $q^*(t)$  with  $n = 0.2, n = 0.5, n = 1$ , respectively

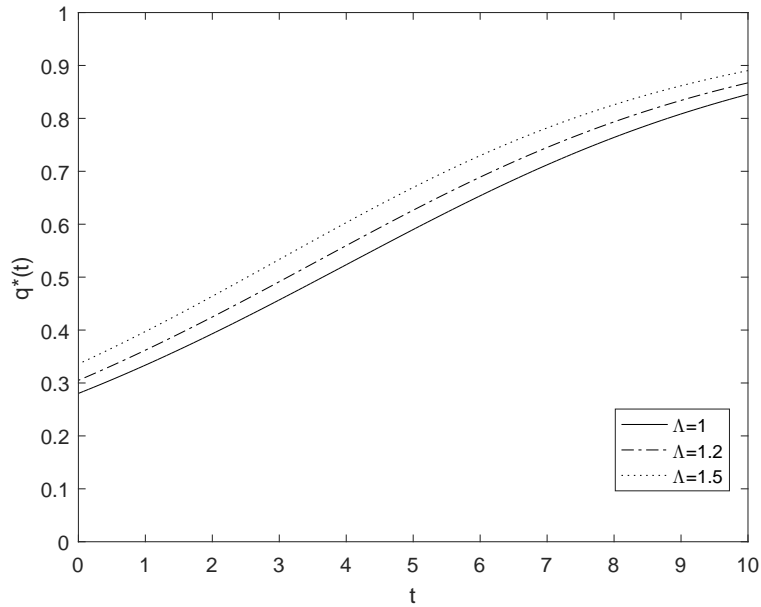


FIGURE 2. The optimal proportional level  $q^*(t)$  with  $\Lambda = 1, \Lambda = 1.2, \Lambda = 1.5$ , respectively

function; and further account for the influence of stock income and cost on the return of investment and reinsurance. Using the method of stochastic optimal control and HJB equation, the numerical results of this paper reveal that the proposed model can maximize the expected exponential utility of terminal wealth. Meanwhile, based on the net profit, the sectional investment strategy can also effectively reduce the lagging risk of investment strategy adjustment caused by the change of insurance company's operating state. Therefore, in the actual trading, insurance companies can maximize the long-term return of investment and reinsurance, while reduce the operational risk in investing by using our proposed scheme. However, this paper is not without limitation. Due to the difficulty in solving the explicit solution, the proposed model fails to obtain the optimal proportion of investment and reinsurance when considering the ruin probability. In future research, it would be interesting to extend our model by relaxing these assumptions and considering more general cases. Additionally, based on the basis of stochastic interest rate and stochastic volatility mode, we will further increase the discussion of setting the optimal investment and reinsurance strategies under the condition of ruin probability.

**Acknowledgments.** This work was supported by the National Natural Science Foundation of China (Grants No. 71971096), the the Fundamental Research Funds for the Central Universities (Grant No. 31610529) and the Ministry of Education of Humanities and Social Science Project (Grant No. 20YJC630123).

#### REFERENCES

- [1] S. Browne, Optimal investment policies for a firm with random risk process: exponential utility and minimizing the probability of ruin, *Mathematics of Operations Research*, **20** (1995), 937–958.
- [2] C. Hipp and M. Plum, Optimal investment for insurers, *Insurance: Mathematics and Economics*, **27** (2000), 215–228.
- [3] C. Hipp and M. Plum, Optimal investment for investors with state dependent income, and for insurers, *Finance and Stochastics*, **7** (2003), 299–321.
- [4] H. Schmidli, Optimal proportional reinsurance policies in a dynamic setting, *Scandinavian Actuarial Journal*, **1** (2001), 55–68.
- [5] J. Gaier, P. Grandits and W. Schachermeyer, Asymptotic ruin probabilities and optimal investment, *The Annals of Applied Probability*, **13** (2003), 1054–1076.
- [6] C. Hipp and H. Schmidli, Asymptotics of ruin probabilities for controlled risk processes in the small claims case, *Scandinavian Actuarial Journal*, **5** (2004), 321–335.
- [7] H. Yang and L. Zhang, Optimal investment for insurer with jump-diffusion risk process, *Insurance: Mathematics and Economics*, **37** (2005), 615–634.
- [8] Z. Liang, Optimal proportional reinsurance for controlled risk process which is perturbed by diffusion, *Acta Mathematicae Applicatae Sinica (English Series)*, **23** (2007), 477–488.
- [9] Z. Liang, L. Bai and J. Guo, Optimal investment and proportional reinsurance with constrained control variables, *Optimal Control Applications and Methods*, **32** (2010), 587–608.
- [10] Q. Zhao, J. Zhuo and J. Wei, Optimal investment and dividend payment strategies with debt management and reinsurance, *Journal of Industrial and Management Optimization*, **14** (2018), 1323–1348.
- [11] G. Guan, Z. Liang and J. Feng, Time-consistent proportional reinsurance and investment strategies under ambiguous environment, *Insurance: Mathematics and Economics*, **83** (2018), 122–133.
- [12] A. Chen, T. Nguyen and M. Stadje, Optimal investment and dividend payment strategies with debt management and reinsurance, *Insurance: Mathematics and Economics*, **79** (2018), 194–209.
- [13] J. Bi and J. Cai, Optimal investment-reinsurance strategies with state dependent risk aversion and VaR constraints in correlated markets, *Insurance: Mathematics and Economics*, **85** (2019), 1–14.

- [14] D. Promislow and V. Young, Minimizing the probability of ruin when claims follow Brownian motion with drift, *North American Actuarial Journal*, **9** (2005), 109–128.
- [15] S. Luo, M. Taksar and A. Tsoi, On reinsurance and investment for large insurance portfolios, *Insurance: Mathematics and Economics*, **42** (2008), 434–444.
- [16] L. Bai and J. Guo, Optimal proportional reinsurance and investment with multiple risky assets and no-shorting constraint, *Insurance: Mathematics and Economics*, **42** (2008), 968–975.
- [17] Z. Liang and J. Guo, Upper bound for ruin probabilities under optimal investment and proportional reinsurance, *Applied Stochastic Models in Business and Industry*, **24** (2008), 109–128.
- [18] C. Irgens and J. Paulsen, Optimal control of risk exposure, reinsurance and investments for insurance portfolios, *Insurance: Mathematics and Economics*, **35** (2004), 21–51.
- [19] R. Kostadinova, Optimal investment for insurers when the stock price follows an exponential Lévy process, *Insurance: Mathematics and Economics*, **41** (2007), 250–2631.
- [20] A.V. Baev and B.V. Bondarev, On the ruin probability of an insurance company dealing in a BS-market, *Theory of Probability and Mathematical Statistics*, **74** (2007), 11–23.
- [21] W. Chen, D. Xiong and Z. Ye, Investment with sequence losses in an uncertain environment and mean-variance hedging, *Stochastic Analysis and Applications*, **25** (2007), 55–71.
- [22] Z. Liang, K. Yuen and J. Guo, Optimal proportional reinsurance and investment in a stock market with Ornstein-Uhlenbeck process, *Insurance: Mathematics and Economics*, **49** (2011), 207–215.
- [23] W. Wang and X. Peng, Reinsurers optimal reinsurance strategy with upper and lower premium constraints under distortion risk measures, *Journal of Computational and Applied Mathematics*, **315** (2017), 142–160.
- [24] P. Lakner, Utility maximization with partial information, *Stochastic Processes and their Applications*, **56** (1995), 247–273.
- [25] P. Lakner, Optimal trading strategy for an investor: the case of partial information, *Stochastic Processes and their Applications*, **76** (1998), 77–97.
- [26] R. Rishel, Optimal portfolio management with partial observation and power utility function, *Stochastic Analysis, Control, Optimization and Applications*, (1999), 605–619.
- [27] L. Bai and J. Guo, Utility maximization with partial information: the HJB equation approach, *Frontiers of Mathematics in China*, **2** (2007), 527–538.
- [28] K. Lindensjö, Optimal investment and consumption under partial information, *Mathematical Methods of Operations Research*, **83** (2016), 87–107.
- [29] R. Kaas, M. Goovaerts, J. Dhaene and M. Denuit, *Modern actuarial risk theory using R*, Springer-Verlag Berlin Heidelberg, 2008.
- [30] D. Li, X. Rong and H. Zhao, Optimal investment problem with taxes, dividends and transaction costs under the constant elasticity of variance (CEV) model, *Wseas Transactions on Mathematics*, **12** (2013), 243–255.
- [31] R.C. Merton, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory*, **3** (1971), 373–413.
- [32] I. Karatzas, Optimization problems in the theory of continuous trading, *SIAM Journal on Control and Optimization*, **27** (1989), 1221–1259.
- [33] H. Loubergé and R. Watt, Insuring a risky investment project, *Insurance: Mathematics and Economics*, **42** (2008), 301–310.
- [34] W. Fleming and H. Soner, *Controlled Markov Process and Viscosity Solutions*, Spring-Verlag, New York, 1993.
- [35] H. Gerber, *An Introduction to Mathematical Risk Theory*, Heubner Foundation Monograph, 1979.

Received September 2018; revised February 2020.

*E-mail address:* liyin57@mail.sysu.edu.cn

*E-mail address:* x.mao@strath.ac.uk

*E-mail address:* 15B910046@hit.edu.cn

*E-mail address:* taojiannenu@126.com