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A complimentary extended warranty: profit analysis and pricing strategy

ABSTRACT

This paper aims to investigate the emerging trend of a new warranty policy in industries, namely, complimentary extended warranty. For the warranty with complimentary extended service, customers can enjoy a free extended warranty if they register online before expiration of the base warranty. At present, with frequent technological innovations, customers are more willing to replace the product instead of repair, which makes traditional warranty less attractive. Since the warranty with complimentary extended service does not charge extra fees for the customers and the option of online registration is open to customers, it can attract a broad range of customers. Compared with traditional warranties, the proposed warranty provides flexibility for customers in their post-purchase choice of online registration. We develop a warranty model to investigate the popularity of the warranty. It turns out that the proposed warranty model exhibits advantages over the traditional warranties. The proposed warranty is profitable for risk averse customers and for products that heap benefits from customer information. We design and price the warranty for unit product when the customers have heterogeneous risk attitudes. In addition, we analyze the total profit and determine the optimal selling price considering the customer demand. Finally, a numerical example is presented to illustrate the effectiveness of the proposed warranty policy.

KEY WORDS: Extended warranty, warranty price, risk aversion, customer utility, risk attitude heterogeneity

1 Introduction

Warranty serves as a contractual obligation that a manufacturer owes to customers in company with the sold products. Warranty plays a two-fold role in driving a company's profits. On one hand, warranty serves as a guarantee signal of the product quality which contributes to increasing product sales. On the other hand, warranty claims adversely decline the company's profits as the manufacturer has to commit the expense for product repair/replacement (Wu, 2014; Wu et al., 2017; Wang et al., 2020).

For companies across all industries, warranty claims processing is believed to consume 2.5 to 4.5% of profits (Byrne, 2004). An underestimation of the true warranty costs will result in losses for a company, whilst an overestimation will lead to uncompetitive product prices. Companies are now rethinking their approach to warranty implementation and management. In particular, manufacturers are consistently considering how they can offer better warranties at minimal cost to their customers and how warranty improvement can boost their profits or even drive up the quality of their products (Thomas and Rao, 1999). By doing so, firms may see more than just cost savings as warranty improvements have been shown to boost revenues, enhance customer satisfaction and loyalty, and even drive up the quality of products (Byrne, 2004; Chu and Chintagunta, 2011).

In the existing warranty policies, warranties can be classified based on multiple metrics. According to which part pays for the repair/replacement cost, warranties policies can be classified into free-replacement warranty, free repair warranty and pro rata warranty (Jung et al., 2015; Mamer, 1982; Luo and Wu, 2018, 2019). Within the free-replacement warranty, the defective product is replaced free of charge while for the pro rata warranty, the customers will bear part of the replacement cost depending on the working age of the product. Based on the number of metrics, warranties can be divided into one-attribute and multi-attribute warranty (Huang et al., 2015; Ye and Murthy, 2016; Zhao and Xie, 2017). Most products are measured by a uniform metric, the service age. However, for some products such as automobiles, the warranty is usually measured at two dimensions, i.e., operating age and mileage (Huang et al., 2017; Wang et al., 2017). Based on the renewability of the warranty, the warranties can be classified into renewal warranty and non-renewal warranty (Bai and Pham, 2006; Wu and Longhurst, 2011; Zhou et al., 2009; Wang et al., 2019). A renewal warranty implies that the warranty length will be updated upon product failure while for a non-renewable warranty, the warranty length is fixed as stipulated (Liu et al., 2015). In addition, with the development of sensing technologies, several advanced warranties have emerged considering system degradation (Shang et al., 2018; Liu et al., 2020).

In traditional warranties, a uniform price is applied to all the customers, regardless of the heterogeneity among customers. In reality, customers are different in terms of risk attitude, usage habit and operating environment (Lee et al., 2016; Ritchken, 1985). A price-for-all warranty fails to differentiate the customers and therefore cannot provide warranties appreciated by all the customers. In general, a customer with high risk aversion and high usage rate is more willing to pay extra fees for warranty.

In the vein of extended warranty literature, several studies have discussed the effect of customer heterogeneity on warranty design. Ritchken and Tapiero (1986) developed a warranty framework for non-repairable items where both the risk preferences of sellers and buyers are incorporated. Padmanabhan and Rao (1993) characterized the manufacturer warranty design and its influence on customer behavior under risk aversion and customers' responses on warranty. Warranty policies in segmentation of consumers were further investigated in Padmanabhan (1995), where the heterogeneity of consumer moral hazard and usage rate was demonstrated to create variations in e-valuating product warranty. Chun and Tang (1995) proposed a warranty model considering the risk preferences of both the producers and consumers, where on optimal warranty price was determined by maximizing the producer's certainty profit equivalent.

Recently, various extended warranties have been developed to satisfy the customer demands and reap profits in the era of rapid technological innovations. Gallego et al. (2014a) developed a residual value warranty where a customer can redeem part of the up-front price if she/he has no or few warranty claims according to the stipulated schedule. The residual value warranty addressed the issue of risk attitude and usage rate and was shown strictly profitable than traditional warranties in either homogeneous or heterogeneous market. Gallego et al. (2014b) proposed a flexible-duration extended warranty with dynamic reliability learning, where the customers can flexibly decide the time spot to claim an extended warranty. The proposed warranty was extremely attractive to the customers who are uncertain of the product reliability and usage horizon. Jindal (2015) investigated the influence of risk preference on the high premia of extended warranty purchases.

Different from the traditional extended warranties, we proposes an alternative to cope with customer heterogeneity, namely, a warranty policy with complimentary extended service. Our policy differs the existing extended warranties in two aspects. On one hand, in the present warranty with complimentary extended service, the customers can register for extended service upon purchase or within a period since purchase, while for traditional extended warranties, the extended service is purchased at expiration of the stipulated warranty. On the other hand, with the proposed warranty, customers can enjoy the extended service for free. Implementation of traditional extended warranty may offend the customers since most customers are reluctant to pay extra money, especially when they are uncertain of the usage horizon and product reliability. At present, with the frequent technological innovations, customers would prefer to replace the product at failure rather than repair. Since the warranty with complimentary extended service does not charge extra fees for the customers, it may attract a broad range of customers.

Increasingly more manufacturers are providing complimentary warranty for customers when they register their products and the accompanying warranties online. For example, Canon and Panasonic both offer an extra 3 months' warranty on top of their standard warranty for all their products if customers register their purchased products online. For a limited time only, Nikon also offered an additional 3 months warranty for a specific range of products if customers registered online. Similarly, Sony provides a 1-year extended warranty service for their PlayStation 4 when customers register their product online. This trend is not only seen in the electrical devices market. Bugaboo, a Dutch company that sells pushchairs for infants and toddlers, offers a complimentary year of warranty on their products when customers register their Bugaboo product online within 3 months of purchase.

One reason that companies are encouraging buyers to register their product online is for consumer tracking. Consumer tracking has been long associated with consumerism and it is getting more widespread with the rapid development in technology. Retailers encourage shoppers to sign up for loyalty cards or register purchased items for warranty programs online where customers' addresses and email addresses are recorded so as to feed the retailer's mailing lists. In this way, the company can gain insights to a buyer's purchase pattern and create specific appeals that are relevant to the buyer's purchase preference. This in turn contributes to the firm's revenue.

By encouraging customers to register their product online, companies can keep track of their sold products. Should a product fail and the customer makes a claim, they can almost swiftly track which batch of production that resulted in the failed item. Manufacturers can look into the production process and rectify any issues, minimizing the occurrence of such problems in the future.

In this paper, we aim to investigate the emerging trend of warranty with complimentary extended service. A warranty model is established to characterize the customers' risk attitude as well we the manufacturer's strategy. We design and price the warranty with complimentary extended service to maximize the expected profits, taking into account the customer heterogeneity in risk preferences. The advantages of the proposed warranty policy are analytically illustrated by comparing with the traditional warranty policies in both homogeneous and heterogeneous markets.

The remainder of this paper is organized as follows. Section 2 presents the risk attitude and the post-purchase behavior of a customer. Section 3 analyzes the profits of of the manufacturer by offering the complimentary extended warranty. Advantages over traditional base warranty ARE investigated under inconsistent risk attitudes. In addition, the influence of risk attitude heterogeneity is investigated on the proposed warranty and the optimal pricing policies are developed by considering respectively market heterogeneity and customer demand. In Section 4, a numerical example is presented to illustrate the proposed warranty. Finally, conclusions and future research directions are summarized in Section 5.

2 Customers' post-purchase decisions

This section models the customers' risk preference with a utility function, and investigates customers' post-purchase decision on online registration to enjoy the complimentary extended service.

Customers' risk attitude plays an important role in their purchasing decisions. Intuitively, a customer with high risk aversion will prefer to pay more for risk premiums to counter the losses with respect to the utility function. Denote $U(v)$ as the utility function of a customer at current wealth level v, and γ as the degree of a customer's risk preference. It is usually required that a utility function should satisfy $U'(v) > 0$ and $U''(v) < 0$, where $U'(v)$ and $U''(v)$ are the first-order and second-order derivative of the utility function $U(v)$ in terms of v. $U'(v) > 0$ indicates that more wealth is preferred to less and $U''(v) < 0$ implies that risk is unwelcome (Baker, 2010). Multiple utility functions have been proposed in literature, among which the exponential form is widely used. Hence, we employ one of the commonly used exponential utility functions to characterize the customers' willingness to purchasing a warranty, which is given as (Baker, 2006)

$$U(v) = \frac{1 - exp(-\gamma v)}{\gamma} \tag{1}$$

where $\gamma > 0$ stands for risk aversion of a customer, $\gamma < 0$ indicates that the customer is risk seeking and $\gamma = 0$ implies that the customer is risk neutral. It is worth noting that it is more mathematically rigorous to express $\gamma \to 0$ instead of $\gamma = 0$. Based on the exponential utility function, it has been concluded that the initial wealth level has no effect on the customer's purchase decision and the manufacturer's selling price (Gallego et al., 2014a). Assume that the product is subject to a minimal repair upon failure, which implies that the failure rate after repair remains identical as that before repair, and occurrence of failure follows a nonhomogeneous Poisson process. Let $\lambda(t)$ denote the failure rate of the product at time t. In reality, a product usually follows an increasing failure rate, i.e., $\lambda'(t) \geq 0$. The expected number of failures from the start of the warranty to time t is given as $\Lambda(t) = \int_0^t \lambda(u)du$.

Although the base warranty is bundled with the product, a complimentary extended service is optional to customers. The warranty with complimentary extended service implies that a customer can enjoy a free extended warranty at the expiration of base warranty if she/he registers online within the stipulated period. The customer has the option of buying or not buying the product together with the warranty and the choice of online registration. Figure 1 depicts the customer's decisions on purchasing the product as well as online registration for extended service. In the following, we will first investigate the customers' decision and then focus on the manufacturers's perspective.

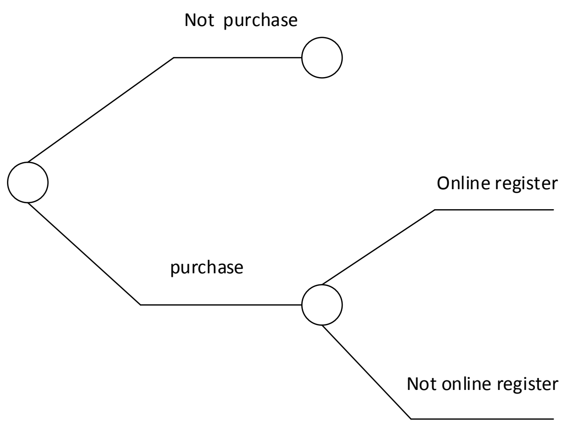

Figure 1: Description of customers' decisions

After a customer purchases the product, she/he needs to decide whether to register online to enjoy the complimentary extended service. Denote N_1 as the total number of failures by the end of base warranty and N_2 as the total number of failures upon the expiration of the extended warranty. Apparently, N_1 and N_2 are random variables following compound Poisson processes with means $\Lambda(T)$ and $\Lambda(T+s)$, where T is the length of base warranty and s is the length of complimentary extended service. Let $g(s;T)$ denote the random repair cost during the extended service period $(T,T+s]$, $g(s;T) = c(N_2 - N_1)$. $g(s;T)$ takes values in discrete set $\{ic : i = 0,1,2,...\}$, with probability

$$P\{g(s;T) = ic\} = \frac{(\Lambda(T+s)-\Lambda(T))^i}{i!} \exp(-(\Lambda(T+s)-\Lambda(T))) \tag{2}$$

where c is the cost per repair. The expected repair cost incurred during the extended service period is denoted as

$$E[g(s;T)] = c(\Lambda(T+s)-\Lambda(T))$$

Denote $g_c(s;\gamma,T)$ as the willingness to register of a customer with risk attitude γ for a free extended service covering period $(T,T+s]$. $g_c(s;\gamma,T)$ corresponds to the certainty-equivalent effort that a customer will pay in terms of utility, which is implicitly defined as

$$U\left(v-g_c\left(s;\gamma,T\right)\right)=E\left[U\left(v-g\left(s;T\right)\right)\right] \tag{3}$$

Actually, $g_c(s;\gamma,T)$ denotes the quantity that a customer is indifferent in registering and not registering online. Following Eq (3), it can be obtained that

$$g_c\left(s;\gamma,T\right)=\frac{1}{\gamma}\ln\left(E\left[\exp\left(\gamma g\left(s;T\right)\right)\right]\right) \tag{4}$$

Detailed derivation of Eq (4) is given in the Appendix. Please note that all the detailed proofs of propositions, lemmas and corollaries are provided in the Appendix.

Proposition 1. *The willingness to register $g_c(s;\gamma,T)$ is increasing with respect to the risk attitude γ. In addition, $g_c(s;\gamma,T)\to c(\Lambda(T+s)-\Lambda(T))$ when $\gamma\to 0$.*

Proposition 1 indicates that a customer prefers online registration when she/he is more risk averse. For a risk-neutral customer, the willingness to register is actually identical to the expected repair cost within the extended service period.

Let Ψ_e denote the effort that a customer pays for online registration, which is a random variable that varies with different customers. Clearly Ψ_e is related to the complexity of the registration process, wherein a customer has to pay large effort for an intricate registration process. Denote $\Psi_e=\mu \mathrm{X}$, where μ is the scale parameter and $0\le \mathrm{X}\le 1$ is assumed to follow a Beta distribution with shape parameters α and β, i.e., $\mathrm{X}\sim \mathrm{Beta}(\alpha,\beta)$.

Remark 1. *Since no previous studies have considered the complimentary extended warranty and the effort of online registration to enjoy the free service, we assume that the registration effort of a customer follows a Beta distribution. We believe that the effort of online registration for a customer should be finite, and the randomness lies in one's familiarity with the registration process. Beta distribution suits well for the case since it is able to describe random variables limited to finite intervals. Of course if survey data are available for the effort of online registration, we can obtain the distribution in a more solid way, such as using goodness-of-fit. If other distributions are applied for the effort of online registration, the conclusion of this paper is still valid, with the corresponding modifications of the online registration effort.*

A customer prefers online registration to enjoy the free extended service only when the certainty-equivalent of repair cost $g_c(s;\gamma,T)$ exceeds the corresponding effort. Each customer perceives an effort ψ_e, as a realization of the random effort Ψ_e. Denote $z \in \{0,1\}$ as the indicator of online registration, wherein $z=1$ implies that the customer prefers to enjoy the complimentary extended service and $z=0$ otherwise. We have

$$z = \begin{cases} 0, \text{ if } g_c(s;\gamma,T) \le \psi_e \\ 1, \text{ if } g_c(s;\gamma,T) > \psi_e \end{cases} \tag{5}$$

Since X is limited in $0 \le \mathrm{X} \le 1$, the customer will always prefer online registration when $g_c(s;\gamma,T) > \mu$. Denote $\upsilon = \min\{g_c(s;\gamma,T)/\mu, 1\}$. The probability that a customer prefers online registration is expressed as

$$\begin{aligned} P(z=1) &= P\{g_c(s;\gamma,T) > \mu \mathrm{X}\} \\ &= P\{\mathrm{X} < g_c(s;\gamma,T)/\mu\} \\ &= \frac{B(\upsilon;\alpha,\beta)}{B(\alpha,\beta)} = I_\upsilon(\alpha,\beta) \end{aligned} \tag{6}$$

where $B(\upsilon;\alpha,\beta)$ is the incomplete beta function, $B(\alpha,\beta)$ is the beta function, and $I_\upsilon(\alpha,\beta)$ is the regularized incomplete beta function. The benefit of the complimentary extended service is given as

$$z_e(s;\gamma,T) = \max\{0, g_c(s;\gamma,T) - \psi_e\}$$

Let $h_e(s;\gamma,T)$ represent the expected cost that a manufacturer offers to the customers, which is given as

$$\begin{aligned} h_e(s;\gamma,T) &= c\left[E[z]\Lambda(T+s) + E[1-z]\Lambda(T)\right] \\ &= c\left(I_\upsilon(\alpha,\beta)\Lambda(T+s) + (1-I_\upsilon(\alpha,\beta))\Lambda(T)\right) \end{aligned} \tag{7}$$

For simplicity, we assume that the repair cost paid by customers are identical to that by manufacturers. In fact, the manufacturers may pay less cost than the customers due to the economies of scale, which can be easily tackled by employing ηc instead of the repair cost c, for $\eta \in (0,1)$.

3 Manufacturer's profit analysis

In this section, we investigate the profit per unit from the manufacturer's perfective. We analyze the manufacturer's profit in a homogeneous and heterogeneous market. In addition, we investigate the influence of inconsistent risk attitudes on the manufacturer's profit.

3.1 Unit profit analysis in a homogeneous market

Denote $w_e(s;\gamma,T)$ as the willingness to pay of a customer with risk attitude γ for the warranty with complimentary extended service, which represents the largest quantity that a customer is willing to pay for the warranty with respect to utility. Note that in a homogeneous market, the price of a product is identical to the willingness to pay of a customer.

Proposition 2. *For any realization of the online registration effort ψ_e, the willingness to purchase of a customer with risk attitude γ is given as*

$$w_e(s;\gamma,T) = \begin{cases} \frac{\Lambda(T+s)(\exp(\gamma c)-1)}{\gamma} - \psi_e, \text{ if } g_c(s;\gamma,T) > \psi_e \\ \frac{\Lambda(T)(\exp(\gamma c)-1)}{\gamma}, \text{ if } g_c(s;\gamma,T) \le \psi_e \end{cases}$$

Proposition 2 implies that when a customer register online to enjoy the complimentary extended service, the willingness to purchase is actually the difference between the willingness to purchase of a warranty covering $T+s$ period and the online registration effort. When a customer refuses to register online, the willingness to purchase is reduced to that of the base warranty.
Since the effort Ψ_e follows a Beta distribution, the willingness to purchase can be formulated as

$$\begin{aligned} w_e(s;\gamma,T) &= E[W_e(s;\gamma,T)|z=1]P(z=1) + E[W_e(s;\gamma,T)|z=0]P(z=0) \\ &= \left(\frac{\Lambda(T+s)(\exp(\gamma c)-1)}{\gamma} - \mu\int_0^{\upsilon} x dF(x;\alpha,\beta)\right) I_\upsilon(\alpha,\beta) \\ &+ \frac{\Lambda(T)(\exp(\gamma c)-1)}{\gamma}(1-I_\upsilon(\alpha,\beta)) \end{aligned} \tag{8}$$

where $W_e(s;\gamma,T)$ is the willingness to purchase for a random effort Ψ_e. The effort of online registration mainly affects the customer's willingness of online registration to enjoy the free extended service. For illustrative purpose, we present the expression of willingness to register online in Appendix when the effort of online registration follows a uniform distribution. For different distributions, the existing results can be readily obtained by substituting the associated willingness of online registration.

Corollary 1. *The willingness to purchase of the warranty with complimentary extended service is always larger than that of the base warranty.*

Corollary 1 states that a customer prefers to pay more for the warranty with extended service coverage than the base warranty, which implies the advantage of the proposed warranty over traditional warranty. For the base warranty, the willingness to purchase is given as

$$w_{bw}(T;\gamma) = \frac{\Lambda(T)(\exp(\gamma c)-1)}{\gamma}$$

For a warranty covering $T+s$ period, the willingness to purchase is expressed as

$$w_{bw}(s+T;\gamma)=\frac{\Lambda(T+s)(\exp(\gamma c)-1)}{\gamma}$$

The willingness to register can be expressed as the difference between the willingness to purchase of a warranty with extended service and that of the basic warranty, which can be readily obtained as

$$\begin{aligned} g_c(s;\gamma,T) &= w_{bw}(s+T;\gamma)-w_{bw}(T;\gamma) \\ &= \frac{(\Lambda(T+s)-\Lambda(T))(\exp(\gamma c)-1)}{\gamma} \end{aligned} \tag{9}$$

Since the product has an increasing failure rate $\lambda(t)$, the total number of failures within the extended service period, $\Lambda(T+s)-\Lambda(T)$, increases with time t. According to Eq (8), it is straightforward to conclude that the willingness to register for complimentary extended service increases with the base warranty coverage T and the extended service coverage s. A customer is more willingness to register online to enjoy the complimentary extended service if a manufacturer offers a longer base warranty coverage and extended service coverage.

As shown in previous studies, the main driver of warranty profits lies in risk aversion. Risk-averse customers are willing to pay higher premiums for warranties. On the other hand, risk-seeking customers will only pay extra money when the selling price is lower than the repair cost. The willingness to purchase for the warranty with complimentary extended service corresponds to the highest price that the manufacturer offers to a customer with risk attitude γ. For the manufacturers, however, the benefit of online registration is also an incentive to provide the warranty with complimentary extended service. Manufacturers can gain implicit benefits from the online registered customer information. For example, the manufacturer can send advertisements to the customers based on their interests or gain feedbacks from the customers so as to improve the product quality in future. Clearly the benefit is related to the effort that a customer pays and the type of products. With more effort devoted to online registration, the manufacturer can obtain more information for making retail decisions and improving product quality.

Denote the implicit benefits from customers' efforts on online registration as $b_e(s;\gamma,T)$. Intuitively, $b_e(s;\gamma,T)>0$ indicates that the customers can gain more benefits from online registration, which prompts the implementation of complimentary extended warranty. For a product with less value of customer information, the manufactures will be reluctant to offer complimentary warranty. In the following, we will focus on the effect of customers' risk preference and consider the case that the benefit of customer information equals to the effort of online registration. Under such setting, the expected implicit benefit can be formulated as

$$b_e(s;\gamma,T) = \mu \int_0^{\upsilon} x dF(x;\alpha,\beta) I_{\upsilon}(\alpha,\beta) \tag{10}$$

The expected profit of a customer with risk attitude γ is dependent on the selling price, the benefit gained from online registration and the repair cost that a manufacturer pays, which is expressed as

$$p_e(s;\gamma,T) = w_e(s;\gamma,T) + b_e(s;\gamma,T) - h_e(s;\gamma,T) \tag{11}$$

Corollary 2. *The warranty with complimentary extended service is reduced to the base warranty with coverage T when $\mu \to \infty$. It is reduced to a traditional warranty with coverage $T+s$ when $\mu \to 0$.*

Corollary 2 is intuitive. If the online registration is easy to complete, the customers are willing to register to enjoy the extended service. However, when the online registration process is extremely intricate, the customers will be reluctant to register, in spite of the potential benefit of complimentary extended service. In the following, we will investigate the profitability of the warranty policy with complimentary extended service.

Proposition 3. *In a homogeneous market, if the customers are risk averse $\gamma > 0$, the manufacturer gains profits by offering the warranty with complimentary extended service. If the customers are risk seeking $\gamma < 0$, the manufacturer loses money. If the customers are risk neutral $\gamma = 0$, the manufacturer neither gains nor loses money. In addition, the expected profit $p_e(s;\gamma,T)$ increases with the risk attitude γ.*

Proposition 3 states that the warranty with complimentary extended service is profitable only for risk averse customers, which is the same as traditional warranty. However, compared with traditional warranty which provides fixed coverage period, the proposed warranty provides more flexibility in customers' choice of online registration, which makes it more attractive for customers.

3.2 Influence of inconsistent risk attitudes

In the previous discussion, it is assumed that the customers have consistent risk attitudes before and after buying the warranty. In practice, however, customers may have inconsistent risk preferences after purchasing the warranty. In fact, risk-attitude inconsistency exists extensively in real life (Gallego et al., 2014a). Let γ_a and γ_b denote the customer's risk attitude before and after purchasing the warranty. Then the willingness to purchase is expressed as $w_e(s;\gamma_a,T)$ and the repair cost that the manufacturer pays is given as $h_e(s;\gamma_b,T)$. As can be observed from Eq (7), risk attitude exerts an impact on $h_e(s;\gamma_b,T)$ in such a way that the risk attitude influences the customer's choice on online registration to enjoy the free extended service. The profit of offering the warranty can be expressed as

$$p_e(s;\boldsymbol{\gamma},T) = w_e(s;\gamma_a,T) + b_e(s;\gamma_b,T) - h_e(s;\gamma_b,T) \tag{12}$$

where $\boldsymbol{\gamma}$ is the set of customers' risk preferences before and after purchasing the warranty, i.e., $\boldsymbol{\gamma} = \{\gamma_a, \gamma_b\}$.

Proposition 4. *The warranty with complimentary extended service is more profitable than the base warranty if the risk attitudes before and after purchasing the warranty satisfy*

$$\frac{(\exp(\gamma_a c) - 1)}{\gamma_a} I_{\upsilon_a}(\alpha,\beta) - c I_{\upsilon_b}(\alpha,\beta) > \frac{\Delta b_e(s;\gamma,T)}{\Delta\Lambda(s;T)}$$

where

$$\Delta b_e(s;\gamma,T) = \mu \int_0^{\upsilon_a} x dF(x;\alpha,\beta) I_{\upsilon_a}(\alpha,\beta) - \mu \int_0^{\upsilon_b} x dF(x;\alpha,\beta) I_{\upsilon_b}(\alpha,\beta)$$

and

$$\Delta\Lambda(s;T) = \Lambda(T+s) - \Lambda(T)$$

Based on the properties of Beta distribution, more results can be summarized, as shown in Corollary 3.

Corollary 3. *The warranty with complimentary extended service is more profitable than the base warranty when (1) the risk attitude before purchasing the warranty γ_a satisfies*

$$B(\upsilon_a;\alpha,\beta) > \frac{\gamma_a c B(\alpha,\beta)}{\exp(\gamma_a c) - 1}$$

and (2) the post-purchase risk attitude satisfies $\gamma_b \geq \gamma_a$.

Corollary 3 provides simpler indicators to measure the advantage of the complimentary extended warranty. The profitability can be easily measured by observing the risk attitude before purchase and by comparing the risk preferences before and after purchasing.

3.3 Pricing in a heterogeneous market

In a heterogeneous market, customers can vary in multiple dimensions, such as risk attitude, usage rate and repair cost. In this section, we will investigate the effect of heterogeneous risk attitudes on the warranty policy, while other factors such as failure rate and repair cost are assumed to be identical. For tractability, we assume that the market is separated into two segments: type H and type L customers, denoted as γ^H and γ^L. Without loss of generality, assume that γ^H denotes a more risk-averse customer, $\gamma^H > \gamma^L$. The proportions of the two types of customers are denoted as κ^L and κ^H, $\kappa^L + \kappa^H = 1$.

For the base warranty, the willingness to purchase $w_{bw}(T;\gamma)$ is increasing in the risk attitude γ, $w_{bw}(T;\gamma^H) > w_{bw}(T;\gamma^L)$. The optimal base warranty provides price that equals to the willingness to purchase of either the type H customers or the type L customers. If the base warranty charges price at $w_{bw}(T;\gamma^H)$, then only the type H customers will buy; the associate profit is given as

$$p_{bw}^H = \kappa^H \Lambda(T) \left(\frac{\left(\exp\left(\gamma^H c\right) - 1\right)}{\gamma^H} - c \right)$$

However, if the base warranty charges price at $w_{bw}(T;\gamma^L)$, both the type H and type L customers will buy; the corresponding profit is expressed as

$$p_{bw}^L = \Lambda(T) \left(\frac{\left(\exp\left(\gamma^L c\right) - 1\right)}{\gamma^L} - c \right)$$

For the base warranty, the optimal price of base warranty p_{rbw}^* can be obtained by solving $p_{bw}^* = \max\left\{p_{bw}^H, p_{bw}^L\right\}$.

Proposition 5. *There exists a threshold $\bar{\kappa}$ and $\bar{\gamma}$, when the proportion of the type H customers satisfies $\kappa^H > \bar{\kappa}$ or the risk attitude of the type H customers satisfies $\gamma^H > \bar{\gamma}$, it is optimal for the base warranty to offer price $w_{bw}(T;\gamma^H)$.*

For the warranty with complimentary extended service, the option of online registration is open to customers. The manufacturers need to determine the optimal price p by maximizing the expected profit over the two types of customers,

$$\max_p \left\{ \begin{array}{l} \kappa^H \left(p + b_e\left(s;\gamma^H,T\right) - h\left(s;\gamma^H,T\right)\right) \cdot 1\left(p \le w_e\left(s;\gamma^H,T\right)\right) \\ +\kappa^L \left(p + b_e\left(s;\gamma^L,T\right) - h\left(s;\gamma^L,T\right)\right) \cdot 1\left(p \le w_e\left(s;\gamma^L,T\right)\right) \end{array} \right\} \tag{13}$$

where $1(\cdot)$ is the indicator function. Direct optimization of problem (13) is difficult. The following proposition provides several structural properties with respect to the risk attitude γ, which contributes to simplifying the optimization problem.

Proposition 6. *The profit $p_e(s;\gamma,T)$, the cost that the manufacturer offers $h_e(s;\gamma,T)$ and the willingness to purchase of a customer $w_e(s;\gamma,T)$ are increasing with the risk attitude γ.*

According to Proposition 6, both $w_e(s;\gamma,T)$ and $p_e(s;\gamma,T)$ are increasing in γ, which implies that the optimal price is selected as either $p^* = w_e\left(s;\gamma^L,T\right)$ or $p^* = w_e\left(s;\gamma^H,T\right)$, *i.e.*,

$$p^* \in \left\{w_e\left(s;\gamma^L,T\right),\, w_e\left(s;\gamma^H,T\right)\right\}$$

$p^* = w_e\left(s;\gamma^L,T\right)$ implies that the both the type H customers and type L customers are willing to purchase the warranty. The associated profit is given as

$$\begin{aligned} p_e^L = {} & w_e\left(s;\gamma^L,T\right) + \kappa^H b_e\left(s;\gamma^H,T\right) + \kappa^L b_e\left(s;\gamma^L,T\right) \\ & - \kappa^H h_e\left(s;\gamma^H,T\right) - \kappa^L h_e\left(s;\gamma^L,T\right) \end{aligned} \tag{14}$$

$p^* = w_e\left(s;\gamma^H,T\right)$ implies that only the type H customers will buy the warranty and the profit is given as

$$p_e^H = \kappa^H\left(w_e\left(s;\gamma^H,T\right) + b_e\left(s;\gamma^H,T\right) - h_e\left(s;\gamma^H,T\right)\right) \tag{15}$$

By comparing p_e^H and p_e^L, we can have the following result in terms of the optimal selling price,

$$p^* = \begin{cases} w_e\left(s;\gamma^L,T\right), \text{ if } \gamma^L < \gamma^H \leq \bar{\gamma}^H \\ w_e\left(s;\gamma^H,T\right), \text{ if } \gamma^H > \bar{\gamma}^H \end{cases} \tag{16}$$

where

$$\bar{\gamma}^H = \left\{\gamma^H | w_e\left(s;\gamma^H,T\right) = \frac{\kappa^L p_e\left(s;\gamma^L,T\right) + \kappa^H w_e\left(s;\gamma^L,T\right)}{\kappa^H}\right\}$$

$\bar{\gamma}^H$ is unique as $w_e(s;\gamma,T)$ is monotonically increasing in γ. The maximum profit is denoted as

$$p_e^* = \max\left\{p_e^H, p_e^L\right\}$$

On the other hand, the optimal price can be expressed in terms of the proportion of market segments, as the manufacturers may be interested in the proportion of type H and type L customers. The optimal price is formulated as

$$p^* = \begin{cases} w_e\left(s;\gamma^L,T\right), \text{ if } 0 < \kappa^H \leq \frac{p_e\left(s;\gamma^L,T\right)}{w_e\left(s;\gamma^H,T\right) - w_e\left(s;\gamma^L,T\right) + p_e\left(s;\gamma^L,T\right)} \\ w_e\left(s;\gamma^H,T\right), \text{ otherwise} \end{cases} \tag{17}$$

3.4 Optimal pricing considering sales volume

In this section, we investigate the manufacture's profit considering the customer demand. Based on the work of Glickman and Berger (1976) and Wu et al. (2006), customer demand can be expressed as a function of selling price p and the length of warranty period t in an exponential form, *i.e.*,

$$d(p,t) = k_1 p^{-a}(t+k_2)^b$$

where k_1 and k_2 are positive constants, denoting respectively the amplitude factor and time displacement that allows for nonzero demand when $t = 0$, a is the price elasticity, $a > 1$, and b is the displaced warranty period elasticity, $0 < b < 1$. In our study, it is assumed that the customers will buy the product only when the willingness to purchase exceeds the selling pricing. Therefore, the demand function can be formulated as

$$d(p) = \begin{cases} k_1 p^{-a}(T+s+k_2)^b, \text{ if } p \leq w_e(s;\gamma,T) \\ 0, \quad \text{otherwise} \end{cases} \tag{18}$$

Combined with the unit profit as previously discussed , the total profit can be expressed as

$$p_t(p) = \begin{cases} (p+b_e(s;\gamma,T)-h_e(s;\gamma,T))k_1 p^{-a}(T+s+k_2)^b, \text{ if } p \leq w_e(s;\gamma,T) \\ 0, \text{ otherwise} \end{cases} \tag{19}$$

Taking logarithm of Eq (19) leads to

$$\pi(p) = \log(p_t(p)) = \log(p-\vartheta) - a\log(p) + \log(k_1) + b\log(T+s+k_2) \tag{20}$$

where $\vartheta = h_e(s;\gamma,T) - b_e(s;\gamma,T)$. Taking derivative of $\pi(p)$ and setting to zero, it follows that

$$\frac{d\pi(p)}{dp} = \frac{1}{p-\vartheta} - \frac{a}{p} = 0$$

the optimal price can be obtained as $p^* = a\vartheta/(a-1)$. However, if the optimal price exceeds the willingness to purchase $w_e(s;\gamma,T)$, the optimal price is reduced to $w_e(s;\gamma,T)$. Therefore, we can express the optimal price as

$$p^* = \begin{cases} \frac{a\vartheta}{a-1}, \text{ if } \frac{a\vartheta}{a-1} \leq w_e(s;\gamma,T) \\ w_e(s;\gamma,T), \text{ otherwise} \end{cases} \tag{21}$$

4 Numerical example

Consider a simple example that a customer purchases a Single-lens reflex (SLR) camera with a 1-year base warranty and a 3-month extended service if the customer registers online before the expiration of the base warranty. The parameters are yearly measured, $T = 1$ and $s = 0.25$. Assume that the customer is risk averse, with the risk attitude $\gamma = 0.01$. The SLR camera is subject to an increasing failure rate, which is modeled as a Weibull distribution. The failure rate at time t is given as

$$\lambda(t) = \frac{\kappa_c}{\lambda_c}\left(\frac{t}{\lambda_c}\right)^{\kappa_c - 1}$$

where λ_c is the scale parameter and κ_c is shape parameter, $\kappa_c > 1$. Let $\lambda_c = 2$ and $\kappa_c = 2$. Since minimal repair is implemented upon failure, the sequence of product failures follow a non-homogeneous Poisson process. The expected total number of failures is given as $\Lambda(t) = (t/\lambda_c)^{\kappa_c}$. The repair cost is $c = 100\$$. According to Eq (9), the willingness to register can be calculated as

$$g_c(s;\gamma,T) = \frac{(\Lambda(T+s) - \Lambda(T))(\exp(\gamma c) - 1)}{\gamma} = 24.16$$

The effort of online registration follows a Beta distribution, with the scale parameter $\mu = 60$ and the shape parameters $\alpha = \beta = 2$. It follows $\upsilon = \min\{g_c(s;\gamma,T)/\mu, 1\} = 0.403$. Based on Eq (6), the probability that the customer prefers online registration to enjoy the free extended service is $I_\upsilon(\alpha,\beta) = 0.356$. It follows from Eq (7) that the expected repair cost can be obtained as

$$h_e(s;\gamma,T) = c\left(I_\upsilon(\alpha,\beta)\Lambda(T+s) + (1 - I_\upsilon(\alpha,\beta))\Lambda(T)\right) = 30$$

According to Eq (8), the willingness to purchase is given as

$$w_e(s;\gamma,T) = \left(g_c(s;\gamma,T) - \mu\int_0^\upsilon x dF(x;\alpha,\beta)\right) I_\upsilon(\alpha,\beta) + w_{bw}(T;\gamma) = 50.14$$

Following Eq (10) and Eq (11), the expected implicit benefit from online registration and the expected profit can be computed as $b_e(s;\gamma,T) = 1.41$ and $p_e(s;\gamma,T) = 21.55$.

Sensitivity analysis is conducted to investigate effect of risk attitude on the warranty model. Figure 2 shows how the willingness to register varies with respect to the risk attitude γ. As shown in Proposition 1, the willingness to register $g_c(s;\gamma,T)$ exhibits an increasing trend with γ. Moreover, for $\gamma > 0.24$, we have $g_c(s;\gamma,T) > \mu = 60$, which implies that $g_c(s;\gamma,T) > \psi_e$ for each customer and that all the customers will register online to enjoy the complimentary extended service.

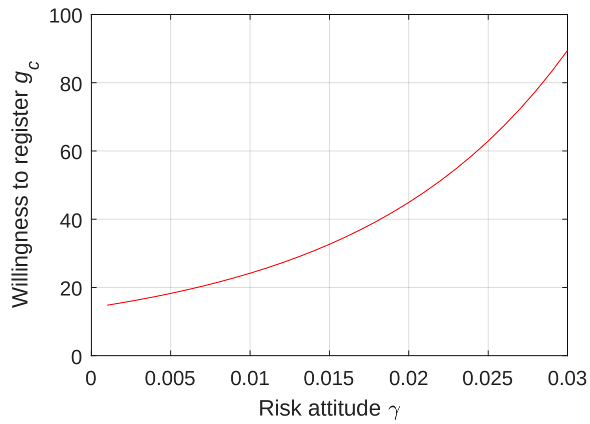

Figure 2: Variation of the willingness to register with risk attitude

Figure 3 presents variation of the expected repair cost and the expected benefit of online registration in terms of risk attitude. As can be observed, both $h_e(s;\gamma,T)$ and $b_e(s;\gamma,T)$ are increasing in γ. What is interesting is that $h_e(s;\gamma,T)$ and $b_e(s;\gamma,T)$ keep constant when the risk attitude $\gamma > 0.24$. This is due to the fact that $g_c(s;\gamma,T) > \mu$ for $\gamma > 0.24$. As can be observed from Eq (7), the risk attitude γ exerts impact on $h_e(s;\gamma,T)$ in such a way that υ increases with γ when $g_c(s;\gamma,T) < \mu$. If $g_c(s;\gamma,T) > \mu$, then $\upsilon = \min\{g_c(s;\gamma,T)/\mu, 1\} = 1$ is a constant, which implies that γ has no influence on $h_e(s;\gamma,T)$. Similar result applies to $b_e(s;\gamma,T)$.

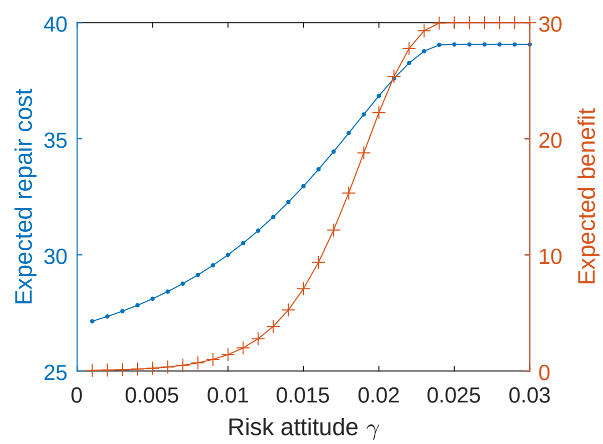

Figure 3: Variation of the expected repair cost and the expected benefit with risk attitude

Figure 4 shows how the willingness to purchase and the expected profit vary with the risk attitude. Obviously $w_e(s;\gamma,T)$ and $p_e(s;\gamma,T)$ exhibit an increasing trend with γ. In addition,

$p_e(s;\gamma,T)$ increases faster than $w_e(s;\gamma,T)$ before γ reaches 0.24. However, for $\gamma > 0.24$, the difference between $w_e(s;\gamma,T)$ and $p_e(s;\gamma,T)$ remain constant. The reason for this is similar to that of Figure 3.

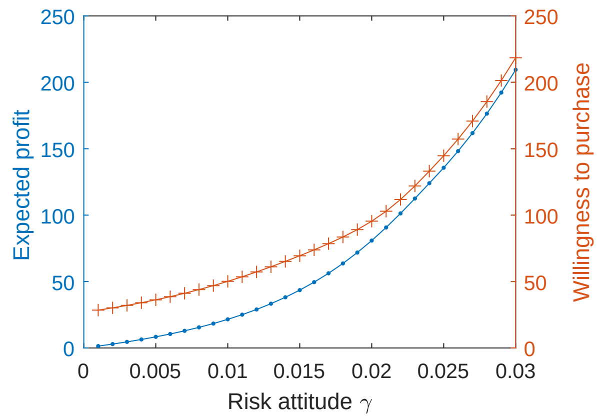

Figure 4: Variation of the willingness to purchase and the expected profit with risk attitude

4.1 Inconsistent risk preferences

If a customer has inconsistent risk attitudes before and after purchasing the warranty, she/he may exhibit different post-purchase behavior out of expectation. Following Eq (12), we can have the expected warranty profit under inconsistent risk preferences. Let $\gamma_a = 0.02$. Figure 3 plots the expected profit for different post-purchase risk attitudes.

We can observe that the expected profit shows a decreasing trend then increases with the post purchase risk attitude γ_b. This is due to the contribution of the implicit benefit of online registration and the repair cost. When γ_b is small, the benefit of online registration has little influence on the expected profit. However, for a large γ_b, the contribution overwhelms the repair cost $h_e(s;\gamma,T)$, which leads to an increasing profit. When $\gamma_b > 0.024$, the customer will register online and γ_b has no impact on the customer's post-purchase decision.

It is of interest to compare the proposed warranty with the base warranty under inconsistent risk preferences. Table 1 shows the advantage in terms of profitability, where 0 represents that the warranty with complimentary extended service is less profitable than the base warranty, and 1 otherwise. It can be observed that (i) the proposed warranty is less profitable when γ_a and γ_b are relatively small ($\gamma_a \leq 0.004$ and $\gamma_b \leq 0.012$); (ii) for a small γ_a, the boundary of being advantageous decreases with γ_b and then increases with γ_b. This is due to the fact that customers are willing to

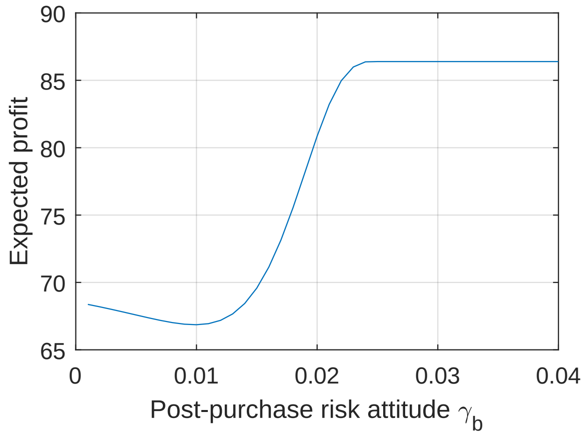

Figure 5: Expected profit under various post-purchase risk attitudes

pay more money for warranty if she/he is more risk averse before purchase. In addition, the effect of post-purchase risk attitude is twofold. On one hand, a large γ_b stimulates customers to register online, which leads to an increased repair cost. On the other hand, a large γ_b increases the implicit benefit of online registration. The influence of post-purchase risk attitude on warranty profitability is highly dependent on the contributions.

Table 1: Comparison of warranty profitability under inconsistent risk preferences

		γ_b									
		0.002	0.004	0.006	0.008	0.01	0.012	0.014	0.016	0.018	0.02
γ_a	0.001	0	0	0	0	0	0	1	1	1	1
	0.002	1	0	0	0	0	0	1	1	1	1
	0.003	1	1	0	0	0	0	1	1	1	1
	0.004	1	1	1	0	0	1	1	1	1	1
	0.005	1	1	1	1	1	1	1	1	1	1
	0.006	1	1	1	1	1	1	1	1	1	1

4.2 Heterogeneity in risk preferences

Suppose that the customers are separated into two segments in terms of the risk preference: type L customer with risk attitude $\gamma^L = 0.01$ and type H customer with risk attitude $\gamma^H = 0.02$. The proportions of the segments are $\kappa^H = 0.6$ and $\kappa^L = 0.4$. The other parameters remain identical as in the previous sections.

For the base warranty, if the manufacturer charges price at $w_{bw}(T;\gamma^L) = 43$, both the type L and type H customers will buy the base warranty, with the expected profit $p^L_{bw} = 18$. However, if the manufacturer caters for the type H customers and charges price at $w_{bw}(T;\gamma^H) = 79.9$, the expected profit is $p^H_{bw} = 32.9$.

For the warranty with complimentary extended service, if the manufacturer charges price at $w_e(T;\gamma^H) = 95.4$, the expected profit is $p^H_e = 48.5$. If the manufacturer charges price at $w_e(T;\gamma^L) = 50.1$, the expected profit is $p^L_e = 29.9$. Compared with the base warranty, the warranty with complimentary extended service improves the expected profit by 47.4%.

To generalize the advantage of the proposed warranty over the base warranty, we let γ^H vary from 0.01 to 0.03 and compare the two warranties under various risk attitudes. Table 2 presents how the optimal price and expected profit of the base warranty, p^*_{rbw} and p^*_{bw}, and that of the proposed warranty, p^* and p^*_e, vary with γ^H. As can be observed, both the optimal price and the profit of the two warranties show a non-decreasing trend of γ^H. In addition, for the base warranty, when $\gamma^H \leq 0.014$, p^*_{rbw} and p^*_{bw} remain constant, which indicates that the manufacturer charges price based on the willingness to purchase of the type L customers and both the two segments purchase the warranty. For the proposed warranty, p^* and p^*_e increase monotonically for $\gamma^H \geq 0.014$, which implies that the manufacturer offers service only for the type H customers for a large γ^H.

Table 2: Comparison of the base warranty and the proposed warranty under different γ^H

	γ^H										
	0.01	0.012	0.014	0.016	0.018	0.02	0.022	0.024	0.026	0.028	0.03
p^*_{rbw}	43.0	43.0	43.00	61.8	70.1	79.9	91.2	104.4	119.8	137.9	159.0
p^*_{bw}	18.0	18.0	18.0	22.0	27.1	32.9	39.7	47.6	56.9	67.7	80.4
p^*	50.1	50.1	65.2	73.8	83.5	95.4	111.8	133.1	157.2	185.5	218.5
p^*_e	21.6	21.6	22.9	29.7	38.2	48.5	60.8	74.4	88.9	105.8	125.7

As previously discussed, both the proportion of customer segments and the risk preference influence the warranty profit. Sensitivity analysis is performed to investigate the effect of κ^H and γ^H, as shown in Table 3 and Table 4. Table 3 shows the maximum expected profit of the base warranty under different κ^H and γ^H. When $\gamma^H \geq 0.016$, p^*_{bw} increases with κ^H and γ^H.

However, when $\gamma^H \leq 0.012$, p^*_{bw} remains constant. This is due to the fact that for a small γ^H, the manufacturer charges a low price and both the type L and type H customers purchase the warranty, which is irrelevant with κ^H. The maximum expected profit of the proposed warranty p^*_e shows a similar result, as presented in Table 4.

Table 3: Maximum expected profit of the base warranty vs γ^H and κ^H

		γ^H										
		0.01	0.012	0.014	0.016	0.018	0.02	0.022	0.024	0.026	0.028	0.03
	0.5	18.0	18.0	18.0	18.4	22.6	27.4	33.1	39.7	47.4	56.4	67.0
	0.52	18.0	18.0	18.0	19.1	23.5	28.5	34.4	41.3	49.3	58.7	69.7
	0.54	18.0	18.0	18.0	19.8	24.4	29.6	35.7	42.9	51.2	61.0	72.4
	0.56	18.0	18.0	18.0	20.6	25.3	30.7	37.1	44.5	53.1	63.2	75.1
	0.58	18.0	18.0	18.0	21.3	26.2	31.8	38.4	46.0	55.0	65.5	77.7
κ^H	0.6	18.0	18.0	18.0	22.0	27.1	32.9	39.7	47.6	56.9	67.7	80.4
	0.62	18.0	18.0	18.3	22.8	28.0	34.0	41.0	49.2	58.8	70.0	83.1
	0.64	18.0	18.0	18.9	23.5	28.9	35.1	42.6	50.8	60.7	72.2	85.8
	0.66	18.0	18.0	19.5	24.3	29.8	36.2	43.7	52.4	62.6	74.5	88.5
	0.68	18.0	18.0	20.1	25.0	30.1	37.3	45.0	54.0	64.5	76.8	91.2
	0.7	18.0	18.0	20.7	25.7	31.6	38.4	46.3	55.6	66.4	79.0	93.8

4.3 Pricing considering customer demand

When the selling price retains within the willingness to purchase, $p \leq w_e(s;\gamma,T)$, the customer demand, which depends on the selling price and length of warranty period, is assumed to follow an exponential form. Parameters of the demand function are given as $k_1 = 1$, $k_2 = 1$, $a = 3$, $b = 0.5$. Based on Eq (19) and Eq (21), the optimal price and the associated total profit are obtained as $p^* = 42.9$ and $p_t(p^*) = 2.718 \times 10^{-4}$. Fig 6 shows how the customer demand and profit vary with the selling price. It should be noted that the profit will be reduced to zero if the price exceeds the willingness to purchase, $p > 50.14$. To investigate the influence of risk attitude on the optimal price and the manufacturer's profit, in Fig 7, we plot the variation of optimal price p^* and the associated profit $p_t(p^*)$ with respect to the risk attitude γ. It is interesting to observe that p^* is divided into two segments at $\gamma = 0.0076$. This is due to the fact that $\frac{a\vartheta}{a-1} \leq w_e(s;\gamma,T)$ for $\gamma \leq 0.0076$,

Table 4: Maximum expected profit of the proposed warranty vs γ^H and κ^H

		γ^H										
		0.01	0.012	0.014	0.016	0.018	0.02	0.022	0.024	0.026	0.028	0.03
	0.5	21.6	21.7	22.4	24.8	31.8	40.4	50.6	62.0	74.1	88.2	104.7
	0.52	21.6	21.7	22.4	25.8	33.1	42.0	52.7	64.5	77.1	91.7	108.9
	0.54	21.6	21.7	22.4	26.8	34.4	43.7	54.7	67.0	80.0	95.2	113.1
	0.56	21.6	21.7	22.4	27.7	35.6	45.3	56.7	69.4	83.0	98.8	117.3
	0.58	21.6	21.7	22.5	28.7	36.9	46.9	58.8	71.9	86.0	102.3	121.5
κ^H	0.6	21.6	21.7	22.9	29.7	38.2	48.5	60.8	74.4	88.9	105.8	125.7
	0.62	21.6	21.8	23.6	30.7	39.4	50.1	62.8	76.9	91.9	109.3	129.8
	0.64	21.6	21.8	24.4	31.7	40.7	51.7	64.8	79.4	94.8	112.9	134.0
	0.66	21.6	21.8	25.2	32.7	42.0	53.4	66.8	81.8	97.8	116.4	138.2
	0.68	21.6	21.8	25.9	33.7	43.4	55.0	68.9	84.3	100.8	120.0	142.4
	0.7	21.6	21.8	26.7	34.7	44.5	56.6	70.9	86.8	103.7	123.5	146.6

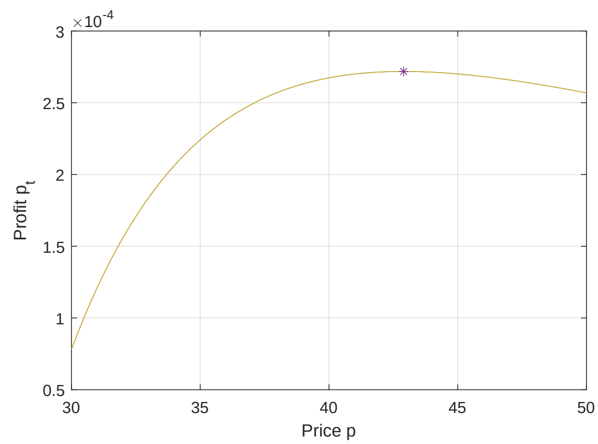

Figure 6: Variation of profit with different selling price

which indicates that p^* varies according to $p^* = \frac{a\vartheta}{a-1}$ for $\gamma \leq 0.0076$, while $p^* = w_e(s;\gamma,T)$ for $\gamma > 0.0076$. To better illustrate this observation, we plot the boundary of price variation in Fig 7(c), where

$$\delta = \begin{cases} 1, \text{ if } \frac{a\vartheta}{a-1} \leq w_e(s;\gamma,T) \\ 0, \text{ otherwise} \end{cases}$$

It is clearly shown that the optimal price exceeds the willingness to purchase for $\gamma \leq 0.0076$. In addition, we can observe that p^* and $p_t(p^*)$ remains constant for $\gamma > 0.024$. This can be explained as the fact that the customers will register online surely for a large risk preference, *i.e.*, $\gamma > 0.024$. According to Eq (21), the optimal price is highly influenced by the price elasticity a. Therefore,

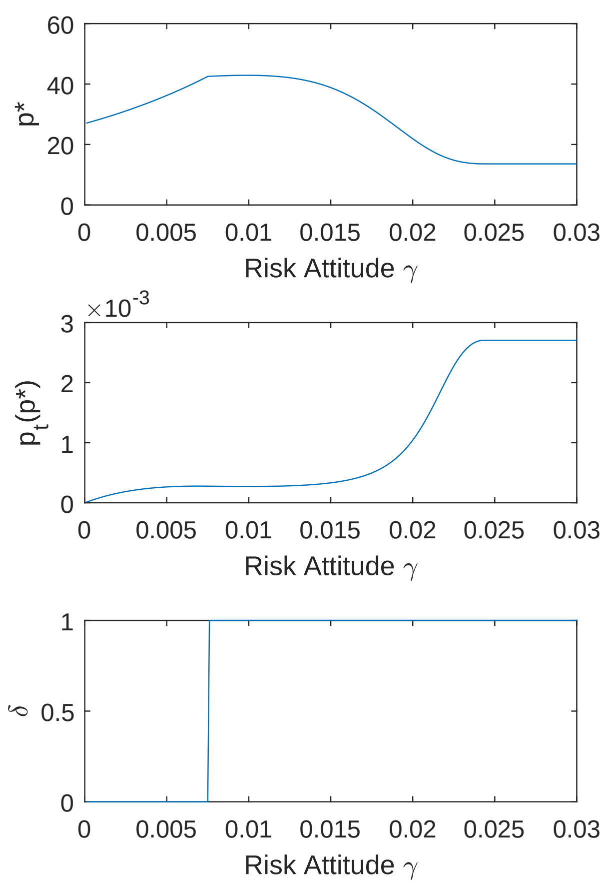

Figure 7: Influence of risk attitude on (a) p^*, (b) $p_t(p^*)$, and (c) boundary of purchase

it is of interest to investigate the impact of the parameter a on the pricing decision. Fig 8 shows how the optimal selling price varies with the price elasticity a. In correspondence to Eq (21), p^*

remains constant for a small price elasticity a, and shows a decreasing trend when a is relatively large.

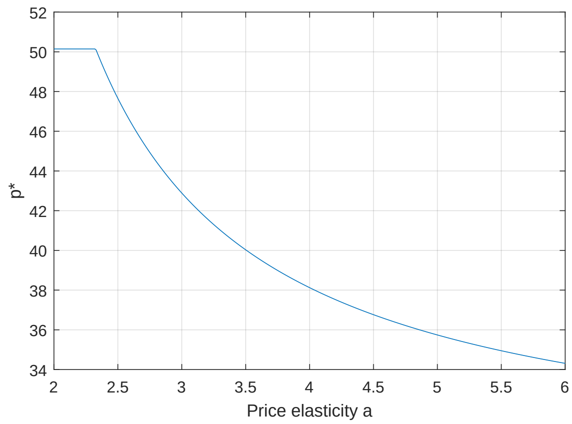

Figure 8: Variation of p^* with respect to the price elasticity a

5 Conclusions

In this paper, we develop a warranty model with a post-purchase option open to customers. If the customers register online before expiration of the base warranty, they are able to enjoy complimentary extended service. A warranty model is established to characterize the customers' post-purchase behavior and the manufacturer's strategy. We design and price the warranty with complimentary extended service by taking into account customer heterogeneity of risk attitudes. It is shown that risk attitude and the value of customer information are the drivers of the complimentary extended warranty. In addition, we investigate conditions that the proposed warranty outperforms the base warranty in either homogeneous or heterogeneous market and show that the risk attitude has a significant impact on the warranty price and profit.

In future research, the current model can be generalized by taking into account more realistic factors. For example, in practice, the customers may not be aware of the failure rate, especially for a new product. Yet the customers may have a prior of the failure rate base on their previous experience and the failure rate can be updated by observing the unexpected failures during operation. Based on the updated failure rate information, the post-purchase decision of online registration becomes a dynamic process instead of a stationary process. In addition, due to today's rapid technological innovation, customers may prefer to replace the product instead of minimal repair. If replacement is preferred upon failure, then a warranty with long coverage period is no longer attractive and a new warranty policy is desirable to appeal these customers. Other interesting extensions can be performed by considering alternative utility functions or incorporating more generalized repair actions.

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Appendix

(1) Derivation of Eq (4)

The expected utility can be obtained as

$$\begin{aligned} E\left[U\left(v-g\left(s;T\right)\right)\right] &= E\left[\frac{1-\exp\left(-\gamma\left(v-g\left(s;T\right)\right)\right)}{\gamma}\right] \\ &= \frac{1-E\left[\exp\left(-\gamma\left(v-g\left(s;T\right)\right)\right)\right]}{\gamma} \\ &= \frac{1-\exp\left(-\gamma v\right)E\left[\exp\left(\gamma g\left(s;T\right)\right)\right]}{\gamma} \end{aligned}$$

On the other hand, the utility of the certainty-equivalent is expressed as

$$\begin{aligned} U\left(v-g_c\left(s;\gamma,T\right)\right) &= \frac{1-\exp\left(-\gamma\left(v-g_c\left(s;\gamma,T\right)\right)\right)}{\gamma} \\ &= \frac{1-\exp\left(-\gamma v\right)\exp\left(\gamma g_c\left(s;\gamma,T\right)\right)}{\gamma} \end{aligned}$$

Based on Eq (3) that

$$E\left[U\left(v-g\left(s;T\right)\right)\right] = U\left(v-g_c\left(s;\gamma,T\right)\right)$$

we can readily have

$$\exp\left(\gamma g_c\left(s;\gamma,T\right)\right) = E\left[\exp\left(\gamma g\left(s;T\right)\right)\right]$$

Taking logarithm leads to

$$g_c\left(s;\gamma,T\right) = \frac{1}{\gamma}\ln\left(E\left[\exp\left(\gamma g\left(s;T\right)\right)\right]\right)$$

(2) Proof of Proposition 1

Proof. Consider the case $\gamma > 0$ first. For $\gamma > 0$, we have

$$\frac{\partial g_c(s;\gamma,T)}{\partial \gamma} = \frac{1}{\gamma E[\exp(\gamma g(s;T))]} E[g(s;T)\exp(\gamma g(s;T))] - \frac{1}{\gamma^2}\ln(E[\exp(\gamma g(s;T))])$$
$$= \frac{1}{\gamma^2 E[\exp(\gamma g(s;T))]}\left(\gamma E[g(s,t)\exp(\gamma g(s;T))] - E[\exp(\gamma g(s;T))]\ln(E[\exp(\gamma g(s;T))])\right)$$
$$\geq \frac{1}{\gamma E[\exp(\gamma g(s;T))]}(E[g(s,t)\exp(\gamma g(s;T))] - E[g(s,t)E[\exp(\gamma g(s;T))]])$$

The above inequality holds due to the Jensen's inequality:

$$\ln(E[\exp(\gamma g(s;T))]) \leq E[\ln(\exp(\gamma g(s;T)))] = \gamma E[g(s;T)]$$

Next, we will prove $E[g(s;T)\exp(\gamma g(s;T))] \geq E[g(s;T)]E[\exp(\gamma g(s;T))]$. Let β_i be the probability that $g(s;T)$ takes values in ic, it follows that

$$\begin{aligned}
&E[g(s;T)\exp(\gamma g(s;T))] - E[g(s;T)]E[\exp(\gamma g(s;T))] \\
&= \sum_{i=0}^{\infty}\beta_i ic\exp(\gamma ic) - \sum_{i=0}^{\infty}\beta_i ic\sum_{i=0}^{\infty}\beta_i\exp(\gamma ic) \\
&= \sum_{i=0}^{\infty}\beta_i ic\left(\exp(\gamma ic) - \sum_{j=0}^{\infty}\beta_j\exp(\gamma jc)\right) \\
&= \sum_{i=0}^{\infty}\beta_i ic\sum_{j=0}^{\infty}\beta_j(\exp(\gamma ic) - \exp(\gamma jc)) \\
&= \sum_{(i,j)}\beta_i\beta_j(ic - jc)(\exp(\gamma ic) - \exp(\gamma jc)) \geq 0
\end{aligned}$$

where (i,j) and (j,i) are considered as identical. The last equality holds because of $\sum_{i=0}^{\infty}\beta_i = 1$. The last inequality holds as

$$(ic - jc)(\exp(\gamma ic) - \exp(\gamma jc)) \geq 0$$

Thus for $\gamma > 0$, we have

$$\frac{\partial g_c(s;\gamma,T)}{\partial \gamma} \geq 0$$

The inequality also holds for $\gamma < 0$, which concludes that $g_c(s;\gamma,T)$ is increasing in the risk attitude γ. When $\gamma \to 0$, it follows that

$$\begin{aligned}
\lim_{\gamma\to 0} g_c(s;\gamma,T) &= \lim_{\gamma\to 0}\frac{\partial\ln(E[\exp(\gamma g(s;T))])//\partial\gamma}{\partial\gamma/\partial\gamma} \\
&= \lim_{\gamma\to 0}\frac{E[g(s;T)\exp(\gamma g(s;T))]}{E[\exp(\gamma g(s;T))]} \\
&= E[g(s;T)] = c(\Lambda(T+s) - \Lambda(T))
\end{aligned}$$

$\square$

(3) Proof of Proposition 2

Proof. Denote $w_f(t)$ as the certainty-equivalent benefits that a customer will gain from a warranty covers a period t. Consider the case that $g_c(s;\gamma,T) \leq \psi_e$ first, *i.e.*, a customer is reluctant to register online to get the free extended warranty. Since the failure process follows a nonhomogeneous Poisson process, the probability that one failure occurs within a small interval $[t, t+\delta)$ can be approximated by $\lambda(t)\delta$, while the probability of two or more failures is negligible, denoted by $o(\delta)$. We can have

$$\frac{1-\exp\left(-\gamma\left(v+w_f(t)\right)\right)}{\gamma} = \lambda(t)\delta\frac{1-\exp\left(-\gamma\left(v+w_f(t+\delta)-c\right)\right)}{\gamma}$$
$$+(1-\lambda(t)\delta)\frac{1-\exp\left(-\gamma\left(v+w_f(t+\delta)\right)\right)}{\gamma}+o(\delta)$$

Rearranging the equation leads to

$$\frac{\exp\left(-\gamma\left(v+w_f(t+\delta)\right)\right)-\exp\left(-\gamma\left(v+w_f(t)\right)\right)}{\delta} = \lambda(t)\exp\left(-\gamma\left(v+w_f(t+\delta)\right)\right)(1-\exp(\gamma c))$$

It follows that

$$\frac{\exp\left(-\gamma w_f(t+\delta)\right)-\exp\left(-\gamma w_f(t)\right)}{\delta} = \lambda(t)\exp\left(-\gamma w_f(t+\delta)\right)(1-\exp(\gamma c))$$

which indicates that

$$\frac{d\exp\left(-\gamma w_f(t)\right)}{dt} = \lambda(t)\exp\left(-\gamma w_f(t)\right)(1-\exp(\gamma c))$$

Since

$$\frac{d\exp\left(-\gamma w_f(t)\right)}{dt} = -\gamma\exp\left(-\gamma w_f(t)\right)\frac{dw_f(t)}{dt}$$

Simple algebra leads to the following differential equation,

$$\frac{dw_f(t)}{dt} = \frac{1}{\gamma}\lambda(t)\left(\exp(\gamma c)-1\right)$$

With the boundary condition

$$w_f(0) = 0$$

we can have the willingness to pay for a warranty that covers a period t as

$$w_f(t) = \frac{1}{\gamma}\left(\exp(\gamma c)-1\right)\Lambda(t)$$

which implies that

$$w_e(s;\gamma,T) = \frac{1}{\gamma}(\exp(\gamma c)-1)\Lambda(T)$$

For the case that $g_c(s;\gamma,T) > \psi_e$, *i.e.*, a customer prefers online registration, the willingness to purchase can be obtained in a similar way, *i.e.*,

$$w_e(s;\gamma,T) = \frac{1}{\gamma}(\exp(\gamma c)-1)\Lambda(T+s) - \psi_e$$

□

(4) Willingness of online registration for uniform distribution

Suppose the effort that a customer pays for online registration follows a uniform distribution, i.e., $\Psi_e = \mu \mathrm{X}$, where $\mathrm{X} \sim \mathrm{U}(0,1)$. The probability that a customer prefers online registration is given as

$$P(z=1) = P\{g_c(s;\gamma,T) > \mu \mathrm{X}\} = \upsilon$$

Then the associated willingness of online registration can be obtained as

$$\begin{aligned} w_e(s;\gamma,T) &= E\left[W_e(s;\gamma,T)\,|z=1\right]P(z=1) + E\left[W_e(s;\gamma,T)\,|z=0\right]P(z=0) \\ &= \left(\frac{\Lambda(T+s)(\exp(\gamma c)-1)}{\gamma} - \frac{\mu\upsilon^2}{2}\right)\upsilon + \frac{\Lambda(T)(\exp(\gamma c)-1)}{\gamma}(1-\upsilon) \end{aligned}$$

(5) Proof of Corollary 1

Proof. The difference between the willingness to purchase of the warranty with complimentary extended service and that of the base warranty is expressed as

$$\begin{aligned} & w_e(s;\gamma,T) - w_{bw}(T;\gamma) \\ &= I_\upsilon(\alpha,\beta)(\Lambda(T+s)-\Lambda(T))\left(\frac{(\exp(\gamma c)-1)}{\gamma}\right) \\ &- \mu\int_0^\upsilon x dF(x;\alpha,\beta) I_\upsilon(\alpha,\beta) \end{aligned}$$

If $g_c(s;\gamma,T) > \mu$, we have $\upsilon = \min\{g_c(s;\gamma,T)/\mu, 1\} = 1$ and $I_\upsilon(\alpha,\beta) = 1$. It follows

$$\begin{aligned} & w_e(s;\gamma,T) - w_{bw}(T;\gamma) \\ &= (\Lambda(T+s)-\Lambda(T))\left(\frac{(\exp(\gamma c)-1)}{\gamma}\right) - \mu\int_0^1 x dF(x;\alpha,\beta) \\ &> (\Lambda(T+s)-\Lambda(T))\left(\frac{(\exp(\gamma c)-1)}{\gamma}\right) - \mu > 0 \end{aligned}$$

For the case where $g_c(s;\gamma,T) \le \mu$,

$$\mu \int_0^{\upsilon} x dF(x;\alpha,\beta) = \mu \int_0^{g_c(s;\gamma,T)/\mu} x dF(x;\alpha,\beta)$$
$$< \mu \int_0^{g_c(s;\gamma,T)/\mu} dF(x;\alpha,\beta)$$
$$< g_c(s;\gamma,T) = (\Lambda(T+s) - \Lambda(T)) \left(\frac{(\exp(\gamma c) - 1)}{\gamma} \right)$$

Combining the two cases completes the proof. □

(6) Proof of Corollary 2

Proof. When $\mu \to \infty$, $\upsilon = \min\{g_c(s;\gamma,T)/\mu, 1\} \to 0$ and $I_\upsilon(\alpha,\beta) \to 0$. Moreover, we have the implicit benefit as

$$\lim_{\mu\to\infty} b_e(s;\gamma,T) = \lim_{\mu\to\infty} \mu \int_0^{\upsilon} x dF(x;\alpha,\beta) \cdot I_\upsilon(\alpha,\beta)$$
$$< \lim_{\mu\to\infty} \mu \cdot g_c(s;\gamma,T)/\mu \cdot I_\upsilon(\alpha,\beta) = 0$$

the willingness to purchase as

$$\lim_{\mu\to\infty} w_e(s;\gamma,T) = \frac{\Lambda(T)(\exp(\gamma c) - 1)}{\gamma}$$
$$+ \lim_{\mu\to\infty} I_\upsilon(\alpha,\beta)(\Lambda(T+s) - \Lambda(T)) \left(\frac{(\exp(\gamma c) - 1)}{\gamma} \right)$$
$$- \lim_{\mu\to\infty} \mu \int_0^{\upsilon} x dF(x;\alpha,\beta) I_\upsilon(\alpha,\beta)$$
$$= \frac{\Lambda(T)(\exp(\gamma c) - 1)}{\gamma}$$

and the repair cost as

$$\lim_{\mu\to\infty} h_e(s;\gamma,T) = \lim_{\mu\to\infty} c\left[I_\upsilon(\alpha,\beta)(\Lambda(T+s) - \Lambda(T)) + \Lambda(T)\right] = c\Lambda(T)$$

which implies that the warranty with complimentary extended service is reduced to the base warranty with coverage T when $\mu \to \infty$. Similarly, we can prove that the present warranty is reduced to a traditional warranty with coverage $T+S$ when $\mu \to 0$. □

(7) Proof of Proposition 3

Proof. By substituting Eq (7), Eq (8) and Eq (10) into Eq (11), we can have

$$
\begin{aligned}
p_e(s;\gamma,T) &= \frac{\Lambda(T+s)(\exp(\gamma c)-1)}{\gamma} I_\upsilon(\alpha,\beta) \\
&+ \frac{\Lambda(T)(\exp(\gamma c)-1)}{\gamma}(1-I_\upsilon(\alpha,\beta)) \\
&- c\left[I_\upsilon(\alpha,\beta)\Lambda(T+s)+(1-I_\upsilon(\alpha,\beta))\Lambda(T)\right] \\
&= I_\upsilon(\alpha,\beta)\Lambda(T+s)\left(\frac{(\exp(\gamma c)-1)}{\gamma}-c\right) \\
&+ (1-I_\upsilon(\alpha,\beta))\Lambda(T)\left(\frac{(\exp(\gamma c)-1)}{\gamma}-c\right) \\
&= (I_\upsilon(\alpha,\beta)\Lambda(T+s)+(1-I_\upsilon(\alpha,\beta))\Lambda(T))\left(\frac{(\exp(\gamma c)-1)}{\gamma}-c\right)
\end{aligned}
$$

It is easy to obtain that $(\exp(\gamma c)-1)/\gamma - c > 0$ for $\gamma > 0$. In addition,

$$(I_\upsilon(\alpha,\beta)\Lambda(T+s)+(1-I_\upsilon(\alpha,\beta))\Lambda(T)) > 0$$

for any $\gamma \in \mathbb{R}$. We can conclude that $p_e(s;\gamma,T) > 0$ for $\gamma > 0$. Similarly, we can have $p_e(s;\gamma,T) = 0$ for $\gamma = 0$ and $p_e(s;\gamma,T) < 0$ for $\gamma < 0$.

In addition, $p_e(s;\gamma,T)$ can be rewritten as

$$p_e(s;\gamma,T) = (I_\upsilon(\alpha,\beta)(\Lambda(T+s)-\Lambda(T))+\Lambda(T))\left(\frac{(\exp(\gamma c)-1)}{\gamma}-c\right)$$

The regularized incomplete beta function, $I_\upsilon(\alpha,\beta) = B(\upsilon;\alpha,\beta)/B(\alpha,\beta)$, is increasing with v. As Proposition 1 shows that $g_c(s;\gamma,T)$ increases with the risk attitude γ, it can be obtained that $(I_\upsilon(\alpha,\beta)(\Lambda(T+s)-\Lambda(T))+\Lambda(T))$ is increasing with γ. In addition, it is easy to obtain that $(e^{\gamma c}-1)/\gamma$ increases with γ. We can conclude that $p_e(s;\gamma,T)$ is increasing in the risk attitude γ. □

(8) Proof of Proposition 4

Proof. The profit can be rewritten as

$$
\begin{aligned}
p_e(s;\gamma,T) &= \frac{\Lambda(T)(\exp(\gamma_a c)-1)}{\gamma_a} - c\Lambda(T) \\
&+ (\Lambda(T+s)-\Lambda(T))\left(\frac{(\exp(\gamma_a c)-1)}{\gamma_a} I_{\upsilon_a}(\alpha,\beta) - cI_{\upsilon_b}(\alpha,\beta)\right) \\
&+ \mu\int_0^{\upsilon_b} xdF(x;\alpha,\beta) I_{\upsilon_b}(\alpha,\beta) - \mu\int_0^{\upsilon_a} xdF(x;\alpha,\beta) I_{\upsilon_a}(\alpha,\beta)
\end{aligned}
$$

where

$$I_{\upsilon_a}(\alpha,\beta)=\frac{B(\upsilon_a;\alpha,\beta)}{B(\alpha,\beta)},\ \upsilon_a=\min\left\{\frac{g_c(s;\gamma_a,T)}{\mu},1\right\}$$

and

$$I_{\upsilon_b}(\alpha,\beta)=\frac{B(\upsilon_b;\alpha,\beta)}{B(\alpha,\beta)},\ \upsilon_b=\min\left\{\frac{g_c(s;\gamma_b,T)}{\mu},1\right\}$$

For the base warranty, the profit can be obtained as

$$p_{bw}(s;\gamma_a,T)=\frac{\Lambda(T)(\exp(\gamma_a c)-1)}{\gamma_a}-c\Lambda(T)$$

For the case where the warranty with complimentary extended service is more profitable than the base warranty,

$$p_e(s;\gamma,T)>p_{bw}(s;\gamma_a,T)$$

we have

$$\begin{aligned}&(\Lambda(T+s)-\Lambda(T))\left(\frac{(\exp(\gamma_a c)-1)}{\gamma_a}I_{\upsilon_a}(\alpha,\beta)-cI_{\upsilon_b}(\alpha,\beta)\right)\\&+\mu\int_0^{\upsilon_b}xdF(x;\alpha,\beta)I_{\upsilon_b}(\alpha,\beta)-\mu\int_0^{\upsilon_a}xdF(x;\alpha,\beta)I_{\upsilon_a}(\alpha,\beta)>0\end{aligned}$$

Simple algebra leads to the conclusion. □

(9) Proof of Corollary 3

Proof. Since $I_\upsilon(\alpha,\beta)$ is increasing in the risk attitude γ, it holds that

$$\mu\int_0^{\upsilon_b}xdF(x;\alpha,\beta)I_{\upsilon_b}(\alpha,\beta)>\mu\int_0^{\upsilon_a}xdF(x;\alpha,\beta)I_{\upsilon_a}(\alpha,\beta)$$

for $\gamma_b>\gamma_a$. On the other hand,

$$\frac{(\exp(\gamma_a c)-1)}{\gamma_a}I_{\upsilon_a}(\alpha,\beta)-cI_{\upsilon_b}(\alpha,\beta)>0$$

equals to

$$I_{\upsilon_a}(\alpha,\beta)>\frac{\gamma_a c}{(\exp(\gamma_a c)-1)}I_{\upsilon_b}(\alpha,\beta)$$

Due to the property of incomplete beta function, $I_{v_b}(\alpha,\beta)\in(0,1]$, $\forall\gamma_b\in\mathbb{R}$, we have

$$I_{\upsilon_a}(\alpha,\beta)>\frac{\gamma_a c}{(\exp(\gamma_a c)-1)}$$

According to Proposition 4, we can conclude that the warranty with complimentary extended service is more profitable than the base warranty if the two conditions hold. □

(10) Proof of Proposition 5

Proof. Let

$$p_{bw}^{H} = p_{bw}^{L}$$

Simple algebra leads to

$$\kappa^{H} = \frac{\gamma^{H}\left(\exp\left(\gamma^{L}c\right)-1\right)-c\gamma^{H}\gamma^{L}}{\gamma^{L}\left(\exp\left(\gamma^{H}c\right)-1\right)-c\gamma^{H}\gamma^{L}}$$

Obviously p_{bw}^{H} increases with κ^{H}. Thus, when

$$\kappa^{H} > \bar{\kappa} = \frac{\gamma^{H}\left(\exp\left(\gamma^{L}c\right)-1\right)-c\gamma^{H}\gamma^{L}}{\gamma^{L}\left(\exp\left(\gamma^{H}c\right)-1\right)-c\gamma^{H}\gamma^{L}}$$

it is optimal for the base warranty to offer price $w_{bw}\left(T;\gamma^{H}\right)$. On the other hand,

$$p_{bw}^{H} = p_{bw}^{L}$$

implies

$$\frac{\left(\exp\left(\gamma^{H}c\right)-1\right)}{\gamma^{H}} = \frac{\frac{\left(\exp\left(\gamma^{L}c\right)-1\right)}{\gamma^{L}}+\left(\kappa^{H}-1\right)c}{\kappa^{H}}$$

Obviously, the left term $\left(\exp\left(\gamma^{H}c\right)-1\right)/\gamma^{H}$ is increasing with γ^{H}. Thus, There exists a threshold $\bar{\gamma}$, when $\gamma^{H} > \bar{\gamma}$, it is optimal to offer price $w_{bw}\left(T;\gamma^{H}\right)$. □

(11) Proof of Proposition 6

Proof. Based on Eq (11), we can rewrite $p_{e}(s;\gamma,T)$ in the following form,

$$p_{e}(s;\gamma,T) = \left(I_{\upsilon}(\alpha,\beta)\left(\Lambda(T+s)-\Lambda(T)\right)+\Lambda(T)\right)\left(\frac{\left(\exp(\gamma c)-1\right)}{\gamma}-c\right)$$

By definition, $I_{\upsilon}(\alpha,\beta)$ is increasing in γ and $\left(\exp\left(\gamma^{H}c\right)-1\right)/\gamma^{H}$ is increasing in γ. Thus , we can conclude that $p_{e}(s;\gamma,T)$ is increasing in the risk attitude γ. Similarly, we can prove that $h_{e}(s;\gamma,T)$ is increasing in γ. The willingness to purchase $w_{e}(s;\gamma,T)$ can be rewritten as

$$w_{e}(s;\gamma,T) = I_{\upsilon}(\alpha,\beta)\left(g_{c}(s;\gamma,T)-\mu\int_{0}^{\upsilon}xdF(x;\alpha,\beta)\right)+w_{bw}(T;\gamma)$$

The monotonicity of $w_{e}(s;\gamma,T)$ can be proved in two separate cases: $g_{c}(s;\gamma,T) > \mu$ and $g_{c}(s;\gamma,T) \leq \mu$. When $g_{c}(s;\gamma,T) > \mu$, , all the customers will register online to enjoy the free extended service. Then the present warranty with complimentary extended service is reduced to the traditional warranty covering $(T+s)$ period. $w_e(s;\gamma,T)$ is then expressed as

$$w_e(s;\gamma,T) = g_c(s;\gamma,T) + w_{bw}(T;\gamma) - \mu\int_0^1 xdF(x;\alpha,\beta)$$
$$= \frac{(\exp(\gamma c)-1)\Lambda(T+s)}{\gamma} - \frac{\mu\alpha}{\alpha+\beta}$$

which is increasing in γ. When $g_c(s;\gamma,T) \le \mu$, the customers may have to decide whether to register online to enjoy the free extended service. Let

$$\psi_e(\gamma) = g_c(s;\gamma,T) - \mu\int_0^{g_c(s;\gamma,T)/\mu} xdF(x;\alpha,\beta)$$

For any two different risk preferences, γ_1 and γ_2 $(\gamma_1 > \gamma_2)$, we have

$$\psi_e(\gamma_1) - \psi_e(\gamma_2) = g_c(s;\gamma_1,T) - g_c(s;\gamma_2,T)$$
$$-\left(\mu\int_0^{g_c(s;\gamma_1,T)/\mu} xdF(x;\alpha,\beta) - \mu\int_0^{g_c(s;\gamma_2,T)/\mu} xdF(x;\alpha,\beta)\right)$$
$$> g_c(s;\gamma_1,T) - g_c(s;\gamma_2,T) - \mu\left(g_c(s;\gamma_1,T)/\mu - g_c(s;\gamma_2,T)/\mu\right) = 0$$

The inequality holds due to $\int_0^{g_c(s;\gamma_2,T)/\mu} xdF(x;\alpha,\beta) < g_c(s;\gamma_2,T)/\mu$. Thus, $\psi_e(\gamma)$ is increasing in γ. Since $I_\upsilon(\alpha,\beta)$ and $w_{bw}(T;\gamma)$ are increasing in γ, we can conclude that $w_e(s;\gamma,T)$ increases with γ for $g_c(s;\gamma,T) \le \mu$. Combining the two cases completes the proof. □