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Robust Passivity of Coupled Cohen-Grossberg Neural Networks with Reaction-Diffusion Terms*

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Abstract— **In this paper, we deal with the robust passivity problem for coupled reaction-diffusion Cohen-Grossberg neural networks (CRDCGNNs) with spatial diffusion coupling and state coupling. First, we present the network model for CRDCGNNs with state coupling and establish some robust passivity conditions for this kind of CRDCGNNs. Then, the investigation on robust passivity for CRDCGNNs with spatial diffusion coupling is carried out similarly. At last, the feasibility of the obtained theoretical results is demonstrated by one example with simulation results.**

Keywords— Robust passivity; Cohen-Grossberg neural networks; Spatial diffusion coupling; State coupling

I. INTRODUCTION

In 1983, a kind of network was proposed by Cohen and Grossberg, which was named Cohen-Grossberg neural networks [1]. Since then it gained particular concern due to promising applications in associative memory, classification, optimization problems and parallel computing. Moreover, CGNN is very general to cover some well known networks, for instance, Hopfield neural networks, recurrent neural networks. Therefore, lots of researchers devote themselves to studying the dynamical behaviors of CGNN [2], [3], [4]. Coupled CGNNs have been found to show more sophisticated even unpredictable dynamics compared with a single CGNN. Recently, some literatures on the synchronization of coupled CGNNs have been reported [5], [6], [7]. In [6], the authors investigated synchronization problem in finite time for coupled CGNNs.

Note that the network models did not take diffusion phenomena into consideration in aforementioned works. Actually, the diffusion phenomena is very important, which cannot be ignored in networks. This motivates some researchers to study the dynamics of coupled reaction-diffusion neural networks (CRDNNs). Passivity, as one of the most significant dynamics of networks, has attracted widespread attention since it plays an indispensable part in system theory and has been applied into some research areas, e.g., induction motors and nonlinear descriptor system. Although many attention has been paid to investigating passivity of CRDNNs (see [8], [9]), only a few scholars considered the problem of passivity for coupled reaction-diffusion CGNNs (CRDCGNNs) except Chen et al. [10] obtained several passivity conditions for two types of CRDCGNNs.

In some practical cases, due to the existence of environment noises and limitations of equipment, the parameters in the networks maybe change within some bounded ranges in the network's modeling process. Thus, it is essential to study the robust passivity of CGNNs [11], [12]. In [12], Nagamani and Radhika analyzed robust passivity of Takagi-Sugeno fuzzy CGNNs. According to our knowledge, there is no research work covering on robust passivity of CRDCGNNs.

Based on the above introduction, the main content of this paper is to study the robust passivity of two kinds CRDCGNNs. The first kind is CRDCGNNs with state coupling, the second kind is CRDCGNNs with spatial diffusion coupling. Furthermore, several robust passivity conditions are derived respectively for these two kinds of CRDCGNNs. At last, we provide an example to demonstrate the feasibility for these obtained results.

II. ROBUST PASSIVITY OF CRDCGNNS WITH STATE COUPLING

A. NETWORK MODEL

In this section, we discuss the CRDCGNNs with state coupling which is composed by M identical nodes:

$$\begin{aligned}\frac{\partial z_i(\zeta,t)}{\partial t} \;=\;& D\Delta z_i(\zeta,t)+w_i(\zeta,t)-a(z_i(\zeta,t))\big[b(z_i(\zeta,t))\\ &-Cf(z_i(\zeta,t))-Eg(\overline{z_i(\zeta,t)})+J\big]\\ &+\rho\sum_{j=1}^{M}G_{ij}\Gamma z_j(\zeta,t), \qquad (1)\end{aligned}$$

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in which $i = 1, \cdots, M$, $z_i(\zeta,t) = (z_{i1}(\zeta,t), \cdots, z_{im}(\zeta,t))^T \in \mathbb{R}^m$ denotes the i-th node's state, where $\zeta = (\zeta_1, \cdots, \zeta_q)^T \in \Omega \subset \mathbb{R}^q$; $w_i(\zeta,t) = (w_{i1}(\zeta,t), \cdots, w_{im}(\zeta,t))^T \in \mathbb{R}^m$ is the control input; $\Delta = \sum_{r=1}^{q} \frac{\partial^2}{\partial \zeta_r^2}$ signifies the Laplace diffusion operator on Ω; $\overline{z_i(\zeta,t)} = (z_{i1}(\zeta, t-\tau_1(t)), \cdots, z_{im}(\zeta, t-\tau_m(t)))^T \in \mathbb{R}^m$, and $\tau_j(t)$ $(j = 1, \cdots, m)$ stands for the transmission delay, which meets $\dot{\tau}_j(t) \leqslant \delta_j < 1 (\delta_j \geqslant 0)$ and $0 \leqslant \tau_j(t) \leqslant \tau_j$; $D = \mathrm{diag}(d_1, \cdots, d_m) > 0$ and d_i signifies the transmission diffusion coefficient; $b(z_i(\zeta,t)) = (b_1(z_{i1}(\zeta,t)), \cdots, b_m(z_{im}(\zeta,t)))^T$ and $b_i(z_{ij}(\zeta,t))$ is an appropriately behaved function; $a(z_i(\zeta,t)) = \mathrm{diag}(a_1(z_{i1}(\zeta,t)), \cdots, a_m(z_{im}(\zeta,t)))$ and $a_i(z_{ij}(\zeta,t))$ presents an amplification function; $C = (c_{ij})_{m\times m}$ and $E = (e_{ij})_{m\times m}$, where the elements in them represent the connection strengths; $f(z_i(\zeta,t)) = (f_1(z_{i1}(\zeta,t)), \cdots, f_m(z_{im}(\zeta,t)))^T, g(\overline{z_i(\zeta,t)}) = (g_1(z_{i1}(\zeta, t-\tau_1(t))), \cdots, g_m(z_{im}(\zeta, t-\tau_m(t))))^T$, where $g_j(\cdot), f_j(\cdot)$ are the activation functions; $J = (J_1, \cdots, J_m)^T$ and J_j is the constant input, $\Gamma = (\gamma_{ij})_{m\times m} \in \mathbb{R}^{m\times m}$ signifies the inner coupling matrix; $\rho > 0$ denotes the coupling strength; $G = (G_{ij})_{M\times M}$ stands for the coupling weight, where $G_{ii} = -\sum_{j=1, j\neq i}^{M} G_{ij}$, $G_{ij} = G_{ji} \geqslant 0$ $(i \neq j)$. More specifically, if there is a connection between nodes i and j, then $G_{ij} > 0$; otherwise $G_{ij} = 0$ in the case of $i \neq j$. In addition, the ranges of parameters d_i, c_{ij}, e_{ij} in the matrices D, C, E of network model (1) are given by

$$\begin{cases} D_I := & \{D = \mathrm{diag}(d_1, \cdots, d_m) : \underline{D} \leqslant D \leqslant \overline{D}, \\ & i.e., \underline{d}_i \leqslant d_i \leqslant \overline{d}_i, i = 1, \cdots, m, \\ & \forall\, D \in D_I\}, \\ C_I := & \{C = (c_{ij})_{m\times m} : \underline{C} \leqslant C \leqslant \overline{C}, i.e., \underline{c}_{ij} \\ & \leqslant c_{ij} \leqslant \overline{c}_{ij}, i, j = 1, \cdots, m, \forall\, C \in C_I\}, \\ E_I := & \{E = (e_{ij})_{m\times m} : \underline{E} \leqslant E \leqslant \overline{E}, i.e., \underline{e}_{ij} \\ & \leqslant e_{ij} \leqslant \overline{e}_{ij}, i, j = 1, \cdots, m, \forall\, E \in E_I\}. \end{cases} \tag{2}$$

The system (1) satisfies the following conditions:

$$z_i(\zeta,t) = 0, \qquad (\zeta,t) \in \partial\Omega \times [-\tau, \infty], \tag{3}$$

$$z_i(\zeta,t) = \varphi_i(\zeta,t), \qquad (\zeta,t) \in \Omega \times [-\tau, 0], \tag{4}$$

where $\tau = \max_{j=1,\cdots,m}\{\tau_j\}$ and $\varphi_i(\zeta,t) \in \mathbb{R}^m (i = 1, \cdots, M)$ is bounded and continuous on $\Omega \times [-\tau, 0]$.

In this paper, the following assumptions need to be satisfied .

Assumption 1: f_j and g_j are continuous. Moreover, there are two scalars $F_j > 0$ and $P_j > 0$ such that

$$0 \leqslant \frac{f_j(\varpi_1) - f_j(\varpi_2)}{\varpi_1 - \varpi_2} \leqslant F_j, \quad 0 \leqslant \frac{g_j(\varpi_1) - g_j(\varpi_2)}{\varpi_1 - \varpi_2} \leqslant P_j$$

for $\forall\, \varpi_1, \varpi_2 \in \mathbb{R}$ and $\varpi_1 \neq \varpi_2$, where $j = 1, \cdots, m$.

Assumption 2: The function $b_i(\varpi)$ is continuous, and there is a constant $b_i > 0$ such that

$$\frac{b_i(\varpi_1) - b_i(\varpi_2)}{\varpi_1 - \varpi_2} \geqslant b_i,$$

for $\forall\, \varpi_1, \varpi_2 \in \mathbb{R}$ and $\varpi_1 \neq \varpi_2$, where $i = 1, \cdots, m$.

Assumption 3: The function $a_i(\varpi)$ is continuous, and

$$0 < \underline{a}_i \leqslant a_i(\varpi) \leqslant \overline{a}_i < +\infty$$

for $\forall\, \varpi \in \mathbb{R}, i = 1, \cdots, m$.

Remark 1: Note that the above assumptions are very commonly used in the existing literatures on CGNNs. Similarly, we assume that these conditions hold throughout this paper.

Suppose that an equilibrium point of CRDCGNNs (1) is $z^* = (z_1^*, \cdots, z_m^*) \in \mathbb{R}^m$, then

$$b(z^*) - Eg(z^*) - Cf(z^*) + J = 0.$$

Therefore, the error vector $\kappa_i(\zeta,t) = z_i(\zeta,t) - z^*$ can be characterized as follows:

$$\begin{aligned} \frac{\partial \kappa_i(\zeta,t)}{\partial t} &= D\Delta\kappa_i(\zeta,t) + w_i(\zeta,t) - a(z_i(\zeta,t))\big[-C(f(z_i(\zeta,t)) \\ &\quad - f(z^*)) + b(z_i(\zeta,t)) - b(z^*) - E(g(\overline{z_i(\zeta,t)}) \\ &\quad - g(z^*))\big] + \rho\sum_{j=1}^{M} G_{ij}\Gamma e_j(\zeta,t). \end{aligned} \tag{5}$$

For system (5), define the following output vector:

$$y_i(\zeta,t) = H\kappa_i(\zeta,t) + Sw_i(\zeta,t), \tag{6}$$

where $H, S \in \mathbb{R}^{m\times m}$ are known matrices.

The equations (5) and (6) can be described in a compact form by using Kronecker product:

$$\begin{cases} \dfrac{\partial \kappa(\zeta,t)}{\partial t} = \hat{D}\Delta\kappa(\zeta,t) - a(z(\zeta,t))\big[b(z(\zeta,t)) - b(\hat{z}^*) \\ \qquad - \hat{E}(g(\overline{z(\zeta,t)}) - g(\hat{z}^*)) - \hat{C}(f(z(\zeta,t)) \\ \qquad - f(\hat{z}^*))\big] + w(\zeta,t) + \rho(G\otimes\Gamma)\kappa(\zeta,t), \\ y(\zeta,t) = \hat{H}\kappa(\zeta,t) + \hat{S}w(\zeta,t), \end{cases} \tag{7}$$

where

$$\begin{aligned} \kappa(\zeta,t) &= (\kappa_1^T(\zeta,t), \cdots, \kappa_M^T(\zeta,t))^T, \\ y(\zeta,t) &= (y_1^T(\zeta,t), \cdots, y_M^T(\zeta,t))^T, \\ w(\zeta,t) &= (w_1^T(\zeta,t), \cdots, w_M^T(\zeta,t))^T, \\ f(\hat{z}^*) &= (f^T(z^*), \cdots, f^T(z^*))^T, \\ b(\hat{z}^*) &= (b^T(z^*), \cdots, b^T(z^*))^T, \\ a(z(\zeta,t)) &= \mathrm{diag}(a(z_1(\zeta,t)), \cdots, a(z_M(\zeta,t))), \\ g(\hat{z}^*) &= (g^T(z^*), \cdots, g^T(z^*))^T, \\ g(\overline{z(\zeta,t)}) &= (g^T(\overline{z_1(\zeta,t)}), \cdots, g^T(\overline{z_M(\zeta,t)}))^T, \\ f(z(\zeta,t)) &= (f^T(z_1(\zeta,t)), \cdots, f^T(z_M(\zeta,t)))^T, \\ b(z(\zeta,t)) &= (b^T(z_1(\zeta,t)), \cdots, b^T(z_M(\zeta,t)))^T, \\ \hat{E} &= I_M \otimes E, \hat{C} = I_M \otimes C, \\ \hat{D} &= I_M \otimes D, \hat{H} = I_M \otimes H, \hat{S} = I_M \otimes S. \end{aligned}$$

Definition 2.1: If there is a constant $\alpha \in \mathbb{R}$ such that

$$\int_{t_0}^{t_p}\int_{\Omega} y^T(\zeta,t)w(\zeta,t)d\zeta dt \quad \geqslant \quad -\alpha^2$$

for all $D \in D_I, C \in C_I$ and $E \in E_I$, where $t_p, t_0 > 0$ and $t_p \geqslant t_0$, then the system (7) with the uncertain parameters given in (2) is said to be robustly passive. Especially, if there are constants $\gamma_1 \geqslant 0$ and $\gamma_2 \geqslant 0$ satisfying

$$\int_{t_0}^{t_p}\int_\Omega y^T(\zeta,t)w(\zeta,t)d\zeta dt \geqslant \gamma_1\int_{t_0}^{t_p}\int_\Omega w^T(\zeta,t)w(\zeta,t)d\zeta dt \\ -\alpha^2+\gamma_2\int_{t_0}^{t_p}\int_\Omega y^T(\zeta,t)y(\zeta,t)d\zeta dt$$

for all $D \in D_I, C \in C_I$ and $E \in E_I$, where $t_p, t_0 > 0$ and $t_p \geqslant t_0$, then the system (7) with the uncertain parameters given in (2) is called to be robustly input-strictly passive when $\gamma_1 > 0$ and robustly output-strictly passive when $\gamma_2 > 0$.

For convenience, we denote

$$\begin{aligned}
\overline{\kappa(\zeta,t)} &= (\overline{\kappa_1(\zeta,t)}^T,\cdots,\overline{\kappa_M(\zeta,t)}^T)^T,\\
\hat{c}_{ij} &= \max\{|\overline{c}_{ij}|,|\underline{c}_{ij}|\}, \quad \xi_C=\textstyle\sum_{i=1}^m\sum_{j=1}^m \hat{c}_{ij}^2,\\
\hat{e}_{ij} &= \max\{|\overline{e}_{ij}|,|\underline{e}_{ij}|\}, \quad \xi_E=\textstyle\sum_{i=1}^m\sum_{j=1}^m \hat{e}_{ij}^2,\\
\underline{\hat{D}} &= I_M\otimes \underline{D}, \quad B=I_M\otimes \mathrm{diag}(b_1,\cdots,b_m),\\
\Xi &= I_M\otimes \mathrm{diag}(\tfrac{1}{1-\delta_1},\cdots,\tfrac{1}{1-\delta_m}),\\
P &= I_M\otimes \mathrm{diag}(P_1^2,\cdots,P_m^2),\\
F &= I_M\otimes \mathrm{diag}(F_1^2,\cdots,F_m^2),\\
\underline{A} &= I_M\otimes \mathrm{diag}(\underline{a}_1,\cdots,\underline{a}_m),\\
\overline{A} &= I_M\otimes \mathrm{diag}(\overline{a}_1,\cdots,\overline{a}_m).
\end{aligned}$$

B. ROBUST PASSIVITY

Theorem 2.1: If there exists matrices $0 < L = \mathrm{diag}\,(L_1,\cdots,L_M), 0 < Q = \mathrm{diag}\,(Q_1,\cdots,Q_M)$ and a constant $\gamma_1 > 0$ such that

$$\begin{pmatrix} \Theta_1 & L-\hat{H}^T \\ L-\hat{H} & -\hat{S}^T-\hat{S}\end{pmatrix} \leqslant 0, \tag{8}$$

where $L_i = \mathrm{diag}\,(l_{i1},\cdots,l_{im}), Q_i = \mathrm{diag}\,(q_{i1},\cdots,q_{im}), \Theta_1 = -2L\underline{A}B+F-\sum_{r=1}^q \frac{L\underline{\hat{D}}+\underline{\hat{D}}L}{l_r^2}+(\xi_C+\lambda_M(Q^{-1})\xi_E)L\overline{A}^2L+P\Xi Q+\rho\big[L(G\otimes\Gamma)+(G\otimes\Gamma^T)L\big]$, then the system (7) with the uncertain parameters given in (2) achieves robust passivity.

Proof: For system (7), construct the following candidate functional:

$$V(t)=\int_\Omega \kappa^T(\zeta,t)L\kappa(\zeta,t)d\zeta \\ +\sum_{i=1}^M\sum_{j=1}^m\frac{P_j^2}{1-\delta_j}\int_{t-\tau_j(t)}^t\int_\Omega q_{ij}\kappa_{ij}^2(\zeta,s)d\zeta ds. \tag{9}$$

Then,

$$\begin{aligned}
\dot{V}(t) \leqslant\ & 2\int_\Omega \kappa^T(\zeta,t)L\Big\{\hat{D}\Delta\kappa(\zeta,t)-a(z(\zeta,t))\big[b(z(\zeta,t))\\
&-\hat{C}(f(z(\zeta,t))-f(\hat{z}^*))-b(\hat{z}^*)-\hat{E}(g(\overline{z(\zeta,t)})\\
&-g(\hat{z}^*))\big]+w(\zeta,t)+\rho(G\otimes\Gamma)\kappa(\zeta,t)\Big\}d\zeta\\
&+\int_\Omega \kappa^T(\zeta,t)P\Xi Q\kappa(\zeta,t)d\zeta\\
&-\int_\Omega \overline{\kappa(\zeta,t)}^T PQ\overline{\kappa(\zeta,t)}d\zeta.
\end{aligned}$$

By utilizing some inequality techniques, one can obtain

$$\begin{aligned}
&-2\kappa^T(\zeta,t)La(z(\zeta,t))(b(z(\zeta,t))-b(\hat{z}^*))\\
\leqslant\ &-2\kappa^T(\zeta,t)L\underline{A}B\kappa(\zeta,t), && (10)\\
&2\kappa^T(\zeta,t)La(z(\zeta,t))\hat{C}(f(z(\zeta,t))-f(\hat{z}^*))\\
\leqslant\ &\kappa^T(\zeta,t)(\xi_C L\overline{A}^2L+F)\kappa(\zeta,t), && (11)\\
&2\kappa^T(\zeta,t)La(z(\zeta,t))\hat{E}(g(\overline{z(\zeta,t)})-g(\hat{z}^*))\\
\leqslant\ &\lambda_M(Q^{-1})\kappa^T(\zeta,t)La(z(\zeta,t))\hat{E}\hat{E}^Ta(z(\zeta,t))L\kappa(\zeta,t)\\
&+\overline{\kappa(\zeta,t)}^TPQ\overline{\kappa(\zeta,t)}\\
\leqslant\ &\lambda_M(Q^{-1})\xi_E\kappa^T(\zeta,t)L\overline{A}^2L\kappa(\zeta,t)\\
&+\overline{\kappa(\zeta,t)}^TPQ\overline{\kappa(\zeta,t)}, && (12)\\
&2\rho\int_\Omega \kappa^T(\zeta,t)L(G\otimes\Gamma)\kappa(\zeta,t)d\zeta\\
=\ &\rho\int_\Omega \kappa^T(\zeta,t)\big[L(G\otimes\Gamma)+(G\otimes\Gamma^T)L\big]\kappa(\zeta,t)d\zeta. && (13)
\end{aligned}$$

In light of Green's formula, we have

$$\begin{aligned}
&\int_\Omega \kappa^T(\zeta,t)L\hat{D}\Delta\kappa(\zeta,t)d\zeta\\
=\ & -\sum_{r=1}^q\int_\Omega\left(\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}\right)^T L\hat{D}\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}d\zeta\\
\leqslant\ & -\sum_{r=1}^q\int_\Omega\left(\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}\right)^T L\underline{\hat{D}}\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}d\zeta.
\end{aligned}$$

Similarly, one can get

$$\begin{aligned}
&\int_\Omega (\Delta\kappa(\zeta,t))^T\hat{D}L\kappa(\zeta,t)d\zeta\\
\leqslant\ & -\sum_{r=1}^q\int_\Omega\left(\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}\right)^T \underline{\hat{D}}L\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}d\zeta.
\end{aligned}$$

Obviously, there exists a matrix $\Pi \in \mathbb{R}^{mM\times mM}$ satisfying $L\underline{\hat{D}}+\underline{\hat{D}}L=\Pi^T\Pi$. Then,

$$\begin{aligned}
&\int_\Omega \big[(\Delta\kappa(\zeta,t))^T\hat{D}L\kappa(\zeta,t)+\kappa^T(\zeta,t)L\hat{D}\Delta\kappa(\zeta,t)\big]d\zeta\\
\leqslant\ & -\sum_{r=1}^q\int_\Omega\left(\frac{\partial(\Pi\kappa(\zeta,t))}{\partial\zeta_r}\right)^T\frac{\partial(\Pi\kappa(\zeta,t))}{\partial\zeta_r}d\zeta\\
\leqslant\ & -\sum_{r=1}^q\int_\Omega \kappa^T(\zeta,t)\frac{L\underline{\hat{D}}+\underline{\hat{D}}L}{l_r^2}\kappa(\zeta,t)d\zeta. && (14)
\end{aligned}$$

From the above equations (10)-(14), we have

$$\begin{aligned}
\dot{V}(t) \leqslant\ & \int_\Omega \kappa^T(\zeta,t)\Big\{-\sum_{r=1}^q\frac{L\underline{\hat{D}}+\underline{\hat{D}}L}{l_r^2}-2L\underline{A}B+F\\
&+(\xi_C+\lambda_M(Q^{-1})\xi_E)L\overline{A}^2L+\rho\big[L(G\otimes\Gamma)\\
&+(G\otimes\Gamma^T)L\big]+P\Xi Q\Big\}\kappa(\zeta,t)d\zeta\\
&+2\int_\Omega \kappa^T(\zeta,t)Lw(\zeta,t)d\zeta. && (15)
\end{aligned}$$

Then,

$$
\begin{aligned}
& \dot{V}(t) - 2\int_\Omega y^T(\zeta,t)w(\zeta,t)d\zeta \\
\leqslant & \int_\Omega \kappa^T(\zeta,t)\Big\{-\sum_{r=1}^{q}\frac{L\underline{\hat{D}}+\underline{\hat{D}}L}{l_r^2} - 2L\underline{A}B + F \\
& +(\xi_C+\lambda_M(Q^{-1})\xi_E)L\overline{A}^2L + \rho\big[L(G\otimes\Gamma) \\
& +(G\otimes\Gamma^T)L\big] + P\Xi Q\Big\}\kappa(\zeta,t)d\zeta \\
& +2\int_\Omega \kappa^T(\zeta,t)(L-\hat{H}^T)w(\zeta,t)d\zeta \\
& +\int_\Omega w^T(\zeta,t)(-\hat{S}^T-\hat{S})w(\zeta,t)d\zeta \\
= & \int_\Omega \varepsilon^T(\zeta,t)\begin{pmatrix}\Theta_1 & L-\hat{H}^T \\ L-\hat{H} & -\hat{S}^T-\hat{S}\end{pmatrix}\varepsilon(\zeta,t)d\zeta,
\end{aligned}
$$

where $\varepsilon(\zeta,t) = (\kappa^T(\zeta,t), w^T(\zeta,t))^T$. From (8), one gets

$$
\dot{V}(t) \leqslant 2\int_\Omega y^T(\zeta,t)w(\zeta,t)d\zeta. \tag{16}
$$

By integrating (16) about t from t_0 to t_p, it is easy to know

$$
2\int_{t_0}^{t_p}\int_\Omega y^T(\zeta,t)w(\zeta,t)d\zeta dt \geqslant V(t_p)-V(t_0) \geqslant -V(t_0).
$$

Therefore,

$$
\int_{t_0}^{t_p}\int_\Omega y^T(\zeta,t)w(\zeta,t)d\zeta dt \geqslant -\eta_1^2,
$$

where $\eta_1 = \sqrt{\frac{V(t_0)}{2}}$, $t_p, t_0 \in \mathbb{R}^+$ and $t_p \geqslant t_0$. ∎

Similarly, the following results can be derived.

Theorem 2.2: If there are matrices $0 < L = \text{diag}\,(L_1,\cdots,L_M), 0 < Q = \text{diag}\,(Q_1,\cdots,Q_M)$ and a constant $\gamma_1 > 0$ such that

$$
\begin{pmatrix}\Theta_1 & L-\hat{H}^T \\ L-\hat{H} & \gamma_1 I_{mM}-\hat{S}^T-\hat{S}\end{pmatrix} \leqslant 0, \tag{17}
$$

where $L_i = \text{diag}\,(l_{i1},\cdots,l_{im}), Q_i = \text{diag}\,(q_{i1},\cdots,q_{im}), \Theta_1 = -2L\underline{A}B + F - \sum_{r=1}^{q}\frac{L\underline{\hat{D}}+\underline{\hat{D}}L}{l_r^2} + (\xi_C + \lambda_M(Q^{-1})\xi_E)L\overline{A}^2L + P\Xi Q + \rho\big[L(G\otimes\Gamma)+(G\otimes\Gamma^T)L\big]$, then the system (7) with the uncertain parameters given in (2) achieves robust input-strict passivity.

Theorem 2.3: If there are matrices $0 < Q = \text{diag}\,(Q_1,\cdots,Q_M), 0 < L = \text{diag}\,(L_1,\cdots,L_M)$ and a constant $\gamma_2 > 0$ such that

$$
\begin{pmatrix}\Theta_2 & L-\hat{H}^T+\gamma_2\hat{H}^T\hat{S} \\ L-\hat{H}+\gamma_2\hat{S}^T\hat{H} & \gamma_2\hat{S}^T\hat{S}-\hat{S}^T-\hat{S}\end{pmatrix} \leqslant 0, \tag{18}
$$

where $L_i = \text{diag}\,(l_{i1},\cdots,l_{im}), Q_i = \text{diag}\,(q_{i1},\cdots,q_{im}), \Theta_2 = -2L\underline{A}B + F - \sum_{r=1}^{q}\frac{L\underline{\hat{D}}+\underline{\hat{D}}L}{l_r^2} + (\xi_C + \lambda_M(Q^{-1})\xi_E)L\overline{A}^2L + \rho\big[L(G\otimes\Gamma)+(G\otimes\Gamma^T)L\big] + \gamma_2\hat{H}^T\hat{H} + P\Xi Q$, then the system (7) with the uncertain parameters given in (2) achieves robust output-strict passivity.

III. ROBUST PASSIVITY OF CRDCGNNs WITH SPATIAL DIFFUSION COUPLING

A. NETWORK MODEL

A spatial diffusion CRDCGNNs consisting of M identical nodes can be presented by:

$$
\begin{aligned}
\frac{\partial z_i(\zeta,t)}{\partial t} = & D\Delta z_i(\zeta,t) + w_i(\zeta,t) - a(z_i(\zeta,t))\big[b(z_i(\zeta,t)) \\
& -Cf(z_i(\zeta,t)) - Eg(\overline{z_i(\zeta,t)}) + J\big] \\
& +\hat{\rho}\sum_{j=1}^{M}\hat{G}_{ij}\hat{\Gamma}\Delta z_j(\zeta,t), i = 1,\cdots,M,
\end{aligned} \tag{19}
$$

where $z_i(\zeta,t), \Delta, b(\cdot), a(\cdot), g(\cdot), f(\cdot), J, w_i(\zeta,t)$ have the same meanings as those in Section II, the quantities in matrices D, C, E are belong to the parameter ranges defined by (2), $\hat{\rho}$, $\hat{\Gamma}$ and $\hat{G}_{ij}$ meet the similar conditions as in Section II.

Then, the error vector $\kappa_i(\zeta,t) = z_i(\zeta,t) - z^*$ can be described by

$$
\begin{aligned}
\frac{\partial \kappa_i(\zeta,t)}{\partial t} = & D\Delta\kappa_i(\zeta,t) + w_i(\zeta,t) - a(z_i(\zeta,t))\big[b(z_i(\zeta,t)) \\
& - C(f(z_i(\zeta,t)) - f(z^*)) - b(z^*) - E(g(\overline{z_i(\zeta,t)}) \\
& - g(z^*))\big] + \hat{\rho}\sum_{j=1}^{M}\hat{G}_{ij}\hat{\Gamma}\Delta e_j(\zeta,t),
\end{aligned} \tag{20}
$$

where $i = 1,\cdots,M$.

Define the same $y_i(\zeta,t)$ as in (6) for system (20). Then, one has

$$
\begin{cases}
\frac{\partial \kappa(\zeta,t)}{\partial t} = \hat{D}\Delta\kappa(\zeta,t) - a(z(\zeta,t))\big[b(z(\zeta,t)) - b(\hat{z}^*) \\
\quad - \hat{E}(g(\overline{z(\zeta,t)}) - g(\hat{z}^*)) - \hat{C}(f(z(\zeta,t)) \\
\quad - f(\hat{z}^*))\big] + w(\zeta,t) + \hat{\rho}(\hat{G}\otimes\hat{\Gamma})\Delta\kappa(\zeta,t), \\
y(\zeta,t) = \hat{H}\kappa(\zeta,t) + \hat{S}w(\zeta,t).
\end{cases} \tag{21}
$$

B. ROBUST PASSIVITY

Theorem 3.1: If there exist two matrices $0 < L = \text{diag}\,(L_1,\cdots,L_M)$ and $0 < Q = \text{diag}\,(Q_1,\cdots,Q_M)$ such that

$$
\Lambda \geqslant 0, \tag{22}
$$

$$
\begin{pmatrix}\Theta_3 & L-\hat{H}^T \\ L-\hat{H} & -\hat{S}^T-\hat{S}\end{pmatrix} \leqslant 0, \tag{23}
$$

where $L_i = \text{diag}\,(l_{i1},\cdots,l_{im}), Q_i = \text{diag}\,(q_{i1},\cdots,q_{im}), \Lambda = L\underline{\hat{D}} + \underline{\hat{D}}L + \hat{\rho}L(\hat{G}\otimes\hat{\Gamma}) + \hat{\rho}(\hat{G}\otimes\hat{\Gamma}^T)L, \Theta_3 = -\sum_{r=1}^{q}\frac{\Lambda}{l_r^2} + F + P\Xi Q + (\xi_C + \lambda_M(Q^{-1})\xi_E)L\overline{A}^2L - 2L\underline{A}B$, then the network (21) with the uncertain parameters given in (2) reaches robust passivity.

Proof: Choose the same $V(t)$ as (9) for error system

(21). Then,

$$\begin{aligned}\dot{V}(t) \leqslant\ & 2\int_{\Omega}\kappa^T(\zeta,t)L\Big\{\hat{D}\Delta\kappa(\zeta,t)-a(z(\zeta,t))\Big[b(z(\zeta,t))\\ &-\hat{C}(f(z(\zeta,t))-f(\hat{z}^*))-b(\hat{z}^*)-\hat{E}(g(\overline{z(\zeta,t)})\\ &-g(\hat{z}^*))\Big]+w(\zeta,t)+\hat{\rho}(\hat{G}\otimes\hat{\Gamma})\Delta\kappa(\zeta,t)\Big\}d\zeta\\ &+\int_{\Omega}\kappa^T(\zeta,t)P\Xi Q\kappa(\zeta,t)d\zeta\\ &-\int_{\Omega}\overline{\kappa(\zeta,t)}^TPQ\overline{\kappa(\zeta,t)}d\zeta.\end{aligned}$$

According to Green's formula, one knows

$$\begin{aligned}&\int_{\Omega}(\Delta\kappa(\zeta,t))^T(\hat{G}\otimes\hat{\Gamma}^T)L\kappa(\zeta,t)d\zeta\\ =&-\sum_{r=1}^{q}\int_{\Omega}\left(\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}\right)^T(\hat{G}\otimes\hat{\Gamma}^T)L\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}d\zeta,\\ &\int_{\Omega}\kappa^T(\zeta,t)L(\hat{G}\otimes\hat{\Gamma})\Delta\kappa(\zeta,t)d\zeta\\ =&-\sum_{r=1}^{q}\int_{\Omega}\left(\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}\right)^TL(\hat{G}\otimes\hat{\Gamma})\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}d\zeta.\end{aligned}$$

Then, one can derive from (22) that

$$\begin{aligned}&2\int_{\Omega}\kappa^T(\zeta,t)\big[\hat{\rho}L(\hat{G}\otimes\hat{\Gamma})+L\hat{D}\big]\Delta\kappa(\zeta,t)d\zeta\\ \leqslant&-\sum_{r=1}^{q}\int_{\Omega}\left(\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}\right)^T\Lambda\frac{\partial\kappa(\zeta,t)}{\partial\zeta_r}d\zeta\\ \leqslant&-\sum_{r=1}^{q}\int_{\Omega}\kappa^T(\zeta,t)\frac{\Lambda}{l_r^2}\kappa(\zeta,t)d\zeta.\end{aligned}\tag{24}$$

From equations (10)-(12) and (24), it is easy to obtain

$$\begin{aligned}\dot{V}(t)\leqslant&\int_{\Omega}\kappa^T(\zeta,t)\Big\{-\sum_{r=1}^{q}\frac{\Lambda}{l_r^2}-2L\underline{A}B+F+P\Xi Q\\ &+(\xi_C+\lambda_M(Q^{-1})\xi_E)L\overline{A}^2L\Big\}\kappa(\zeta,t)d\zeta\\ &+2\int_{\Omega}\kappa^T(\zeta,t)Lw(\zeta,t)d\zeta.\end{aligned}$$

Then,

$$\begin{aligned}&\dot{V}(t)-2\int_{\Omega}y^T(\zeta,t)w(\zeta,t)d\zeta\\ \leqslant&\int_{\Omega}\varepsilon^T(\zeta,t)\begin{pmatrix}\Theta_3 & L-\hat{H}^T\\ L-\hat{H} & -\hat{S}^T-\hat{S}\end{pmatrix}\varepsilon(\zeta,t)d\zeta,\end{aligned}$$

where $\varepsilon(\zeta,t)=(\kappa^T(\zeta,t),w^T(\zeta,t))^T$. From (23), one gets

$$\dot{V}(t)\leqslant 2\int_{\Omega}y^T(\zeta,t)w(\zeta,t)d\zeta.\tag{25}$$

By integrating (25) about t from t_0 to t_p, we have

$$2\int_{t_0}^{t_p}\int_{\Omega}y^T(\zeta,t)w(\zeta,t)d\zeta dt\geqslant V(t_p)-V(t_0)\geqslant -V(t_0).$$

Hence,

$$\int_{t_0}^{t_p}\int_{\Omega}y^T(\zeta,t)w(\zeta,t)d\zeta dt \ \geqslant\ -\eta_2^2$$

where $\eta_2=\sqrt{\frac{V(t_0)}{2}}$, $t_p,t_0\in\mathbb{R}^+$ and $t_p\geqslant t_0$. ■

Similarly, the following results can be deduced.

Theorem 3.2: If there exist two matrices $0<L=\text{diag}\,(L_1,\cdots,L_M)$, $0<Q=\text{diag}\,(Q_1,\cdots,Q_M)$ and a constant $\gamma_1>0$ such that

$$\Lambda \ \geqslant\ 0,\tag{26}$$

$$\begin{pmatrix}\Theta_3 & L-\hat{H}^T\\ L-\hat{H} & \gamma_1I_{mM}-\hat{S}^T-\hat{S}\end{pmatrix}\ \leqslant\ 0,\tag{27}$$

where $L_i=\text{diag}\,(l_{i1},\cdots,l_{im}), Q_i=\text{diag}\,(q_{i1},\cdots,q_{im}), \Lambda=L\underline{\hat{D}}+\underline{\hat{D}}L+\hat{\rho}L(\hat{G}\otimes\hat{\Gamma})+\hat{\rho}(\hat{G}\otimes\hat{\Gamma}^T)L, \Theta_3=-\sum_{r=1}^{q}\frac{\Lambda}{l_r^2}-2L\underline{A}B+F+P\Xi Q+(\xi_C+\lambda_M(Q^{-1})\xi_E)L\overline{A}^2L$, then the system (21) with the uncertain parameters given in (2) reaches robust input-strict passivity.

Theorem 3.3: If there exist two matrices $0<L=\text{diag}\,(L_1,\cdots,L_M)$, $0<Q=\text{diag}\,(Q_1,\cdots,Q_M)$ and a constant $\gamma_2>0$ such that

$$\Lambda\geqslant 0,\tag{28}$$

$$\begin{pmatrix}\Theta_4 & L-\hat{H}^T+\gamma_2\hat{H}^T\hat{S}\\ L-\hat{H}+\gamma_1\hat{S}^T\hat{H} & \gamma_2\hat{S}^T\hat{S}-\hat{S}^T-\hat{S}\end{pmatrix}\leqslant 0,\tag{29}$$

where $L_i=\text{diag}\,(l_{i1},\cdots,l_{im}), Q_i=\text{diag}\,(q_{i1},\cdots,q_{im}), \Lambda=L\underline{\hat{D}}+\underline{\hat{D}}L+\hat{\rho}L(\hat{G}\otimes\hat{\Gamma})+\hat{\rho}(\hat{G}\otimes\hat{\Gamma}^T)L, \Theta_4=-\sum_{r=1}^{q}\frac{\Lambda}{l_r^2}+P\Xi Q+(\xi_C+\lambda_M(Q^{-1})\xi_E)L\overline{A}^2L+\gamma_2\hat{H}^T\hat{H}-2L\underline{A}B+F$, then the network (21) with the parameters given in (2) reaches robust output-strict passivity.

IV. EXAMPLE

Example 1: Given the following spatial diffusion CRD-CGNNs which is composed by six identical nodes:

$$\begin{aligned}\frac{\partial z_i(\zeta,t)}{\partial t}=&\ D\Delta z_i(\zeta,t)+w_i(\zeta,t)-a(z_i(\zeta,t))\big[b(z_i(\zeta,t))\\ &-Cf(z_i(\zeta,t))-Eg(\overline{z_i(\zeta,t)})+J\big]\\ &+0.2\sum_{j=1}^{6}\hat{G}_{ij}\hat{\Gamma}\Delta z_j(\zeta,t), i=1,\cdots,6,\end{aligned}\tag{30}$$

where $a(\mu_i)=\text{diag}\,(1.2,1.2,1.2), \Omega=\{x||x|<0.4\}, J=(0,0,0)^T, b(\mu_i)=(3\mu_{i1},3\mu_{i2},3\mu_{i3})^T, \tau_j(t)=\frac{j}{3}-\frac{1}{j+2}e^{-t}, \delta_j=\frac{1}{j+1}, \tau=1, f_j(\mu)=g_j(\mu)=\frac{|\mu+1|-|\mu-1|}{10}, j=1,2,3, w_{i1}=0.2i\sqrt{t}\sin(\pi x), w_{i2}=0.3i\sqrt{t}\sin(\pi x), w_{i3}=0.8i\sqrt{t}\sin(\pi x)$. Obviously, $g_j(\cdot)$ and $f_j(\cdot)$ satisfy Assumption 1 with $F_j=P_j=0.2$; $z^*=(0,0,0)^T\in\mathbb{R}^3$. Moreover, the parameters D,C,E are given in the following ranges:

$$\begin{cases} D_I := & \{D = \mathrm{diag}(d_1, \cdots, d_m) : \underline{D} \leqslant D \leqslant \overline{D}, \\ & i.e., 0 < 0.3i \leqslant d_i \leqslant 0.5i, i = 1, \cdots, m, \\ & \forall\ D \in D_I\}, \\ C_I := & \{C = (c_{ij})_{m\times m} : \underline{C} \leqslant C \leqslant \overline{C}, i.e., \frac{0.1j}{2i+j} \\ & \leqslant c_{ij} \leqslant \frac{0.2j}{2i+j}, i, j = 1, \cdots, m, \forall\ C \in C_I\}, \\ E_I := & \{E = (e_{ij})_{m\times m} : \underline{E} \leqslant E \leqslant \overline{E}, i.e., \frac{0.2j}{i+j} \\ & \leqslant e_{ij} \leqslant \frac{0.4j}{i+j}, i, j = 1, \cdots, m, \forall\ E \in E_I\}. \end{cases} \tag{31}$$

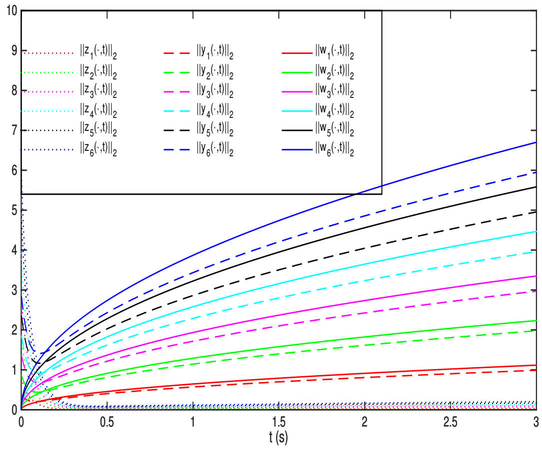

Fig. 1. The dynamics of the state, output and input

In addition, $\hat{\Gamma} = \mathrm{diag}(0.2, 0.9, 0.7)$, and $H, S, \hat{G} = (\hat{G}_{ij})_{6\times 6}$ are taken as:

$$\hat{G} = \begin{pmatrix} -0.7 & 0.3 & 0.2 & 0.1 & 0.1 & 0 \\ 0.3 & -1.0 & 0.4 & 0 & 0.1 & 0.2 \\ 0.2 & 0.4 & -1.1 & 0 & 0.3 & 0.2 \\ 0.1 & 0 & 0 & -0.5 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.3 & 0.2 & -1.0 & 0.3 \\ 0 & 0.2 & 0.2 & 0.2 & 0.3 & -0.9 \end{pmatrix},$$

$$H = \begin{pmatrix} 0.3 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0 \\ 0 & 0.3 & 0.2 \end{pmatrix}, S = \begin{pmatrix} 0.5 & 0 & 0.3 \\ 0 & 0.6 & 0.2 \\ 0.3 & 0.2 & 0.6 \end{pmatrix}.$$

By calculation using the MATLAB, we can obtain the following matrices:

$$L = I_M \otimes \begin{pmatrix} 0.1419 & 0 & 0 \\ 0 & 0.1172 & 0 \\ 0 & 0 & 0.0919 \end{pmatrix},$$

$$Q = I_M \otimes \begin{pmatrix} 0.4325 & 0 & 0 \\ 0 & 0.7654 & 0 \\ 0 & 0 & 0.9382 \end{pmatrix},$$

which satisfy (22) and (23). From Theorem 3.1, the system (30) with the uncertain parameters given by (31) is robustly passive according to Definition 2.1. The change processes of input, output and state are shown in Fig. 1.

V. CONCLUSION

In this paper, the robust passivity of two types CRDCGNNs has been studied. Based on some inequality techniques, several conditions are established respectively for ensuring the robust passivity of CRDCGNNs with state coupling and spatial diffusion coupling. In addition, the obtained results has been corroborated by an example with some simulation results.

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