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Survival signature-based sensitivity analysis of systems with epistemic uncertainties

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ABSTRACT: The survival signature provides a basis for efficient reliability assessment of systems with more than one component type. Often a perfect probabilistic modelling of the system is not possible due to limited information, vagueness and imprecision. Hence generalized probabilistic methods need to be used. These methods allow to explicitly model the uncertainties without the need of unjustified hypotheses and approximation. In this paper, a novel and efficient sensitivity approach is presented. The proposed approach is based on survival signature, allowing to identify and rank components in a system. A numerical example is used to illustrate the above methods.

1 INTRODUCTION

Systems are formed by a number of components interconnected by communication paths. It is essential to assess the importance of those components for the reliability of the system over the time of interest. Moreover, the system component often has uncertain parameters and the uncertainty can propagate to the whole system, which makes it difficult to conduct sensitivity analysis on systems.

Sensitivity analysis is the general term for a systematic study of how the inputs to a model influence the results of the model, see Ferson et al. (2006). Uncertainty exists widely in practical systems and networks, and it may be generated due to incomplete information, limited sampling data, ignorance and measure errors. Since the reliability and performance of network systems are directly affected by uncertainties, quantitative assessment of uncertainty is widely recognized as an important task in practical engineering. Moreover, it is essential to know which component is critical to the whole system when considering uncertainties. Sensitivity analyses can be group two main groups: local and global sensitivity analysis. For Local Sensitivity Analysis (LSA), Tarantola et al. (2012) extended the concepts to a regional sensitivity analysis with consideration of how the input variables influence the variance of the model output. To capture sensitivities by a differential approach, Saltelli et al. (2000) suggested a Global Sensitivity Analysis (GSA), which enlarges the scope for sensitivity analysis in computational modelling practice. Patelli et al. (2010) presented an efficient sampling-based algorithm for the estimation of the upper bounds of the total sensitivity indices. For traditional probabilistic model with generalized methods, when the model involves uncertainties, Neumaier (1990) thought the definition of global sensitivity analysis is equivalent to that of interval analysis. Probability bounds analysis, which is a marriage of probability theory and interval analysis, can also be viewed as a global sensitivity analysis by Ferson et al. (2002). Li et al. (2014) developed a technique called "contribution to failure probability plot" to detect the important aleatory and epistemic uncertain variables, and also measured the contribution of specific regions of these important input variables to the failure probability. The method of Monte Carlo simulation and probability bounds analysis is applied in environmental risk and sensitivity assessments by Tucker et al. (2003).

All the methods discussed above can be applied to sensitivity analysis on a system. However, the computational costs of the traditional sensitivity analyses on systems are normally huge, especially when the systems consist of different component types. In order to overcome the restriction, sensitivity analysis of systems based on survival signature is performed in this paper.

In recent years, system signature has been recognized as an important tool to quantify reliability of system consists of exchangeable components. The main advantage of system signature is separating the structure of the system from the probabilistic model used to describe the random failure of components. However, system signature assumes that all components in the system are of the same type. Real systems are generally formed by more than one component type, and therefore the application of system signature is not feasible.

Coolen and Coolen-Maturi (2012) presented an improved method, which called survival signature, to conduct reliability analysis of systems with more than one component type. In the case of a single component type, the survival signature is closely related to system signature, whilst in case of more than one component type, the survival signature plays its advantages of not relying on an independent and identically distributed (*iid*) assumption between the different component types. Coolen et al. (2014) have shown how the survival signature can be derived from the signatures of two subsystems in both series and parallel configuration, they developed a nonparametric-predictive inference for system reliability using the survival signature. Aslett et al. (2014) presented the use of the survival signature for systems and network reliability quantification with both a nonparametric and a parametric approach.

One of the uses of sensitivity analysis is to identify the most critical components and then allocate resources to repair the systems. For this purpose, two sensitivity measures are introduced: relative importance index and relative sensitivity index. The proposed approach allows to consider epistemic uncertainties into account, which can get bounds of survival function and relative importance index.

The next session provides an introduction of survival signature and survival function with epistemic uncertainties. The proposed relative importance index and relative sensitivity index of each component in the network system is presented in Session 3. A numerical example is illustrated in Session 4. Session 5 closed with a discussion and conclusions.

2 SURVIVAL FUNCTION WITH EPISTEMIC UNCERTAINTY

Suppose there is one network system with m components, T_S and $T_{j:m}$ expresses the random failure time of the system and the jth order of the m random component failure times for $j = 1,2,\ldots,m$, respectively. The signature of the system is the m-vector q with jth component $q_j = P\left(T_S = T_{j:m}\right)$. So q_j denotes the probability of the network system fails at the time the jth component fails. The survival function of the system is:

$$P\left(T_S > t\right) = \sum_{j=1}^{m} q_j P(T_{j:m} > t) \tag{1}$$

Consider a network system with $K \geq 2$ types of m components, with m_k components of type $k \in \{1,2,\ldots,K\}$ and $\sum_{k=1}^{K} m_k = m$, and assume the failure times of the same component type are independently and identically distributed or exchangeable. The state vector of components $\underline{x} = (x_1, x_2, \ldots, x_m) \in \{0,1\}^m$ describes the status of the system with $x_i = 1$ if the ith component is in working state and $x_i = 0$ if not. $\varnothing = \varnothing(\underline{x}) : \{0,1\}^m \rightarrow \{0,1\}$ defines structure function, the network system status based on the state vectors $\underline{x}$. $\varnothing$ is 1 if the network system provides the expected function and 0 if it does not. We refer to the function $\varnothing(\underline{x})$ as the structure function of the network system.

The survival signature can be generalized with $K \geq 2$ types of components by $\varnothing(l_1, l_2, \ldots, l_k)$, with $l_k = 0,1,\ldots,m_k$ for $k = 1,2,\ldots,K$. Let $S_{l_1,l_2,\ldots,l_K}$ denote the set of all state vectors for the whole system for which $\sum_{i=1}^{m_k} x_i^k = l_k$, $k = 1,2,\ldots,K$. Assume that the random failure times of components of the different types are fully independent, while are exchangeable within the same component types, hence:

$$\varnothing(l_1,\ldots,l_k) = \left[\prod_{k=1}^{K} \binom{m_k}{l_k}^{-1}\right] \times \sum_{\underline{x} \in S_{l_1,\ldots,l_k}} \varnothing(\underline{x}) \tag{2}$$

Let $C_k(t) \in \{0,1,\ldots,m_k\}$ denote the number of components of type k in the system which function at time t, and assume that the components of the same type have the known CDF $F_k(t)$ for type k, then:

$$P\left(\bigcap_{k=1}^{K} \{C_k(t) = l_k\}\right) = \prod_{k=1}^{K} P\left(C_k(t) = l_k\right)$$
$$= \prod_{k=1}^{K} \binom{m_k}{l_k} \left[F_k(t)\right]^{m_k - l_k} \left[1 - F_k(t)\right]^{l_k} \tag{3}$$

Hence, the survival function of the system with K types of components is:

$$P\left(T_s > t\right)$$
$$= \sum_{l_1=0}^{m_1} \cdots \sum_{l_k=0}^{m_k} \varnothing(l_1,\ldots,l_K) P\left(\bigcap_{k=1}^{K} \{C_k(t) = l_k\}\right) \tag{4}$$

Epistemic uncertainty is uncertainty that comes from ignorance or incomplete knowledge. This lack of knowledge comes from many sources: in adequate understanding of the underlying processes, imprecise evaluation of the related characteristics, or incomplete knowledge of the phenomena. Beer et al. (2008, 2013) addressed this problem by imprecise probabilities. While Patelli et al. (2014) merged with advanced Monte Carlo simulation and further stochastic techniques and implemented into Open-Cossan software to deal with epistemic uncertainty.

Suppose $\underline{F}$ and $\overline{F}$ are non-decreasing functions from the real line $\mathfrak{R}$ into [0,1] and $\underline{F}(x) \leq \overline{F}(x)$

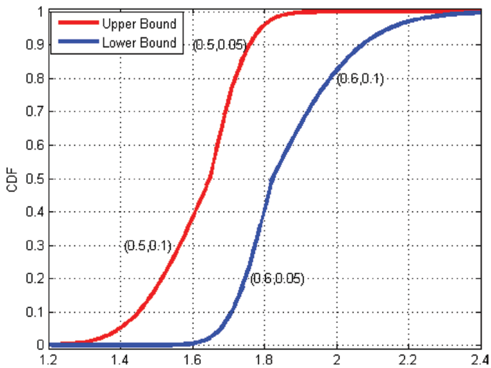

Figure 1. A p-box whose bounds arise from a lognormal distribution within parameters $\alpha = [0.5, 0.6]$ and $\beta = [0.05, 0.1]$.

for all $x \in \Re$. Let $[\underline{F}, \overline{F}]$ denote the set of the non-decreasing functions F from the reals into [0,1] such that $\underline{F}(x) \le F(x) \le \overline{F}(x)$. When the functions $\underline{F}$ and $\overline{F}$ circumscribe an imprecisely known probability distribution, $[\underline{F}, \overline{F}]$ is called "probability box" or "p-box". Using the framework of imprecise probabilities in form of a p-box proposed by Ferson et al. (2004), the lower and upper CDF for the failure times of components of type k are denoted by $\underline{F}_k(t)$ and $\overline{F}_k(t)$, respectively. For instance, it is assumed that the failure time of one component is according to different distribution types, but there exist epistemic uncertainties within its parameters $[\underline{\alpha}, \overline{\alpha}]$ and $[\underline{\beta}, \overline{\beta}]$. The lower and upper CDF bounds can be obtained by calculating the range of all distributions that have parameters within the above intervals. For some distribution families, only two CDFs need to be computed to enclose the p-box. For most distribution families, however, four or more crossing CDFs need to be computed to define a p-box. This method can be used in cases where empirical information is available. Figure 1 depicts a p-box whose bounds arise from a lognormal distribution within parameters $\alpha = [0.5, 0.6]$ and $\beta = [0.05, 0.1]$.

Because of the *iid* assumption within failure times of components in the same type k, while full independence are assumed for components in different types, then for a monotonic system with $l_k \in \{0, 1, \ldots, m_k\}$, the lower probability for $C_k(t) = l_k$ is $\underline{P}(\bigcap_{k=1}^{K}\{C_k(t) = l_k\})$ and the corresponding upper probability is $\overline{P}(\bigcap_{k=1}^{K}\{C_k(t) = l_k\})$.

Consequently, the lower survival function of the system at time t is:

$$\underline{P}(T_s > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_k=0}^{m_k} \varnothing(l_1, \ldots, l_K) \underline{P}\left(\bigcap_{k=1}^{K}\{C_k(t) = l_k\}\right) \tag{5}$$

and the corresponding upper survival function is:

$$\overline{P}(T_s > t) = \sum_{l_1=0}^{m_1} \cdots \sum_{l_k=0}^{m_k} \varnothing(l_1, \ldots, l_K) \overline{P}\left(\bigcap_{k=1}^{K}\{C_k(t) = l_k\}\right) \tag{6}$$

That is, the bounds of the survival function of a system with epistemic uncertainty on the failure times of components can be obtained by using Eq. (5) to Eq. (6).

3 SENSITIVITY ANALYSIS

An important objective of a reliability and risk analysis is to identify those components or events that are most important (critical) from a reliability/safety point of view and that should be given priority with respect to improvements. Sensitivity analyses aim at identifying how the output of a model changes due to variations of the input. These tools allow one to study the relationship among components and the system, and to identify the most signification components affecting the reliability of the whole system. Sensitivity analysis has many manifestations in probabilistic risk analyses and there are many disparate approaches based on various measures of influence and response. Patelli et al. (2014) pointed out that due to the obvious importance of sensitivity analysis, it is essential to identify and rank the parameters that contribute mostly to the variability of the output of the system.

Relative importance index RII, is here introduced and adopted to quantify the difference between the probability that the system functions if the ith component works and the probability that the system functions if the ith component fails. The value of the difference can be defined as relative importance index $RII_i(t)$ of the ith component over the time. The RII can be expressed as follows:

$$RII_i(t) = P(T_s > t | T_i > t) - P_i(T_s > t | T_i < t) \tag{7}$$

where, $P(T_s > t | T_i > t)$ represents probability that the system functions if the ith component works; $P_i(T_s > t | T_i < t)$ represents the probability that the system functions knowing that the ith component has failed.

The measure $RII_i(t)$ expresses the system reliability potential improvement that can be obtained improving the reliability of the ith component. In other words, it represents the reliability loss if the component i fails. This measure can be used for all types of definitions of reliability, and it can be used for repairable and non-repairable system as well.

Moore et al. (1966, 1979) described the use of interval arithmetic to evaluate the ranges of functions taking interval arguments. This approach is to generalize the definitions of the binary operations out of which the function is compared to handle interval inputs. For instance, for all real numbers a, b, c and d such that $0 \le a \le b \le 1$ and $0 \le c \le d \le 1$, then

$$[a,b]-[c,d]=[a-d,b-c] \tag{8}$$

According to the interval arithmetic, the relative importance index can also be adopted in presence of epistemic uncertainties in the failure times of components, the lower bound RII of the network system at time t is:

$$\underline{RII_i}(t)=\left|\underline{P}\left(T_s>t\middle|T_i>t\right)-\overline{P_i}\left(T_s>t\middle|T_i<t\right)\right| \tag{9}$$

and the corresponding upper bound RII is:

$$\overline{RII_i}(t)=\overline{P}\left(T_s>t\middle|T_i>t\right)-\underline{P_i}\left(T_s>t\middle|T_i<t\right) \tag{10}$$

The overall effect of the epistemic uncertainty can be measured by:

$$RSI_i(t)=\overline{RII_i}(t)-\underline{RII_i}(t) \tag{11}$$

From the equation above, the relative sensitivity index RSI is a positive value and it quantifies the effect degree of each component contains parameters uncertainties, i.e., the bigger value of $RSI_i(t)$, the bigger influence of the ith component with uncertainties to the whole system at a specific time t, and vice versa. Moreover, the relative sensitivity index can be introduced to conduct sensitivity analysis on different systems.

4 NUMERICAL EXAMPLE

Now illustrate the above approaches using the system layout given in Figure 2, with $K = 6$ types of components.

It is assumed that components with the same type have the same failure time distribution. There are so many influential factors affecting the value range of parameters in the real world, which causes epistemic uncertainties. If the system under an ideal condition, the component has higher parameters, while if it is affected by temperature, acid rain, man-made sabotage, different working environment or the other influence factors, the component has lower parameters. Therefore, it is

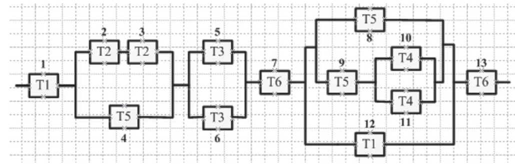

Figure 2. A thirteen components system with six different component types, while the component type is inside the rectangle.

Table 1. Components failure time distribution parameters bounds of the system.

Component name	Distribution type	Parameters (α,β) or λ	
T1	Beta	[1.2,1.5]	[1.5,2.1]
T2	Weibull	[1.0,1.6]	[2.1,2.5]
T3	Exponential	[0.4,1.2]	
T4	Beta	[1.3,1.8]	[2.3,2.9]
T5	Gamma	[1.2,1.4]	[2.8,3.3]
T6	Exponential	[0.8,1.3]	

necessary to take the epistemic uncertainties into account when considering the failure time distribution parameters of each component in the system. Now let suppose only intervals of the distribution parameters of the failure times are known. These parameters are shown in Table 1.

Since the upper and lower bounds of parameters reflect the ideal and bad work conditions of the system respectively. As a result, this leads to upper and lower survival functions of the system with epistemic uncertainties, see Figure 3. It is obvious that the interval inputs can get interval outputs.

Now taking the upper and lower bounds of each component failure time parameters in Table 1 into account, it is easy to get the range relative importance index values of each component, just as revealed in Figure 4 and Figure 5.

From the above two pictures, it is cleat relative importance index values are difficult to express the sensitivity degree of component with epistemic uncertainties. So relative sensitivity index values can be calculated and plotted, as Figure 6 depicts.

It is clear from the above figure that which component with uncertain parameters has bigger influence degree to the system at different time. At the beginning time, the relative sensitivity index values of C5 and C6 are much bigger than other components, and then the influence degree of uncertainties within these components decreases. As time goes on, the relative sensitivity index values of components C1, C7 and C13 are the biggest, which

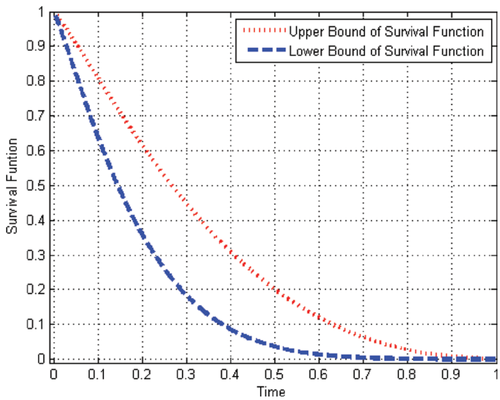

Figure 3. Upper and lower survival functions of the system.

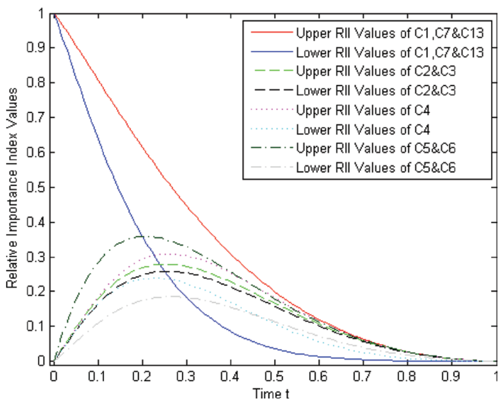

Figure 4. Upper and lower relative importance index values of components C1, C2, C3, C4, C5, C6, C7 and C13.

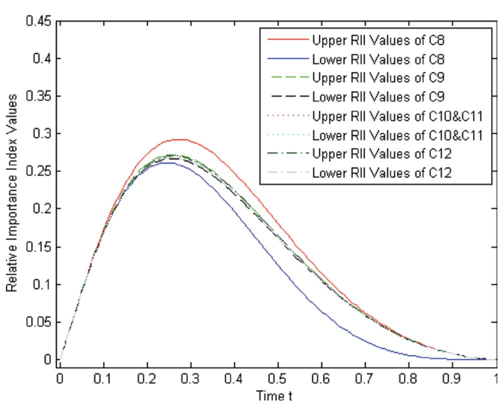

Figure 5. Upper and lower relative importance index values of components C8, C9, C10, C11 and C12.

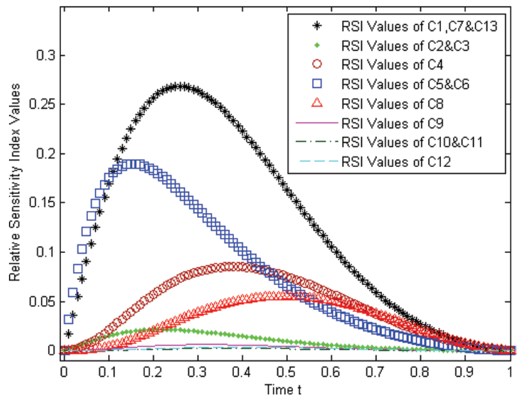

Figure 6. Relative sensitivity index values of components C1 to C13.

means the uncertain parameters have the biggest influence degree to the whole system during that period. While for components C9, C10, C11 and C12, the relative sensitivity index values are always lower than other components, and the values are close to 0. However, C12 is more sensitive than C10 and C11, while the RSI values of C12 are always less than C9. Although C8 always has smaller values than C4, they have the same trend. For C2 and C3, RSI values are between C4 and C8 at first time and then decrease between C8 and C9.

5 CONCLUSIONS

In this paper an efficient approach for sensitivity analysis has been presented. The method is based on survival signature. Survival signature has been proven to be an efficient method to estimate the survival function of systems with multiple component types. Conducting sensitivity analysis on systems by introducing survival signature is a novel way.

In addition, here the effect of imprecision as incomplete data has been taken into account. As a consequence, lower and upper survival functions of the network system can be obtained. In this paper, both relative importance index and relative sensitivity index of the ith component is introduced to identify the most "critical" network system component at a specific time t. This allows an optimal allocation of resources for repair, maintenance and inspection. When taking the epistemic uncertainties into account, the upper and lower relative importance index of the ith component in the system are obtained. This is a novel and efficient way to see the sensitivity of each component at different time, so it is easy to know which component is the most critical one to the whole system.

Further research may focus on survival signature used in repairable systems.

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