Passivity and Synchronization of Coupled Complex-Valued Memristive Neural Networks*

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***Abstract*—The coupled complex-valued memristive neural networks (CCVMNNs) are investigated in this study. First, we analyze the passivity of the proposed network model by designing an appropriate controller and using certain inequalities as well as Lyapunov functional method, and provide a passivity condition for the considered CCVMNNs. In addition, a criterion for guaranteeing synchronization of this kind of network is established. Finally, the effectiveness and correctness of the acquired theoretical results are verified by a numerical example.**

***Index Terms*—memristive neural networks, synchronization, passivity, state coupling**

I. Introduction

Recently, coupled neural networks (CNNs) have been widely concerned owning to their extensive application in secure communication, chaos generator design, brain science, etc. As we all know, these applications heavily depend on the dynamic behaviors of CNNs, especially the synchronization and passivity of CNNs [1]–[5]. In [2], the impulsive synchronization of Markovian jumping randomly CNNs was considered by using multiple integral approach. Some conditions for guaranteeing synchronization of CNNs were obtained in [3]. Ren et al. [5] analyzed the passivity of CNNs with directed and undirected topologies.

In 1971, Chua [6] first proposed the concept of memristor. Unlike resistor, because the memristance depends on the amount of charge passing through it, the memristor can remember its past dynamic history. Therefore, the memristor is widespread used in signal processing as well as device modeling, especially in simulating synaptic behavior [7]–[11]. Moreover, memristive neural networks (MNNs) can better present the neural processes in the human brain [12]. In recent years, coupled memristive neural networks (CMNNs) have been extensively studied and numerous interesting studies on CMNNs have been reported [13]–[15]. In [13], the robust synchronization of CMNNs with uncertain parameters was discussed.

In fact, complex-valued neural networks (CVNNs) are extensions of real-valued neural networks in which the states, connection weights as well as activation functions are all complex-valued. Certain practical problems cannot be solved by real-valued neural networks but can be better solved with CVNNs. In addition, CVNNs have a wide range of applications, including emotion analysis, analogy amplification, computer version, imaging, etc. Hence, a large number of studies have been conducted on the dynamic behavior of CVNNs [16]–[19]. Complex-valued MNNs (CVMNNs) can be also built by replacing resistors with memristor in VLSI circuits of CVNNs as described in [20], which is widely applied in image processing, engineering optimization and pattern recognition. Therefore, it is meaningful to study the passivity and synchronization in CVMNNs [20]–[23]. The authors in [20] considered the exponential stability of CVMNNs. The synchronization of uncertain fractional-order CVMNNs with multiple time delays was analyzed in [21]. To the best of our knowledge, the passivity and synchronization of coupled CVMNNs (CCVMNNs) has not yet been studied.

Accordingly, the principal goal in the present study is to investigate the passivity and synchronization problems of CCVMNNs. Firstly, a criterion for ensuring passivity of CCVMNNs is put forward by constructing an appropriate controller and using Lyapunov functional method. Secondly, we also establish a synchronization condition for the considered network. Finally, a numerical simulation is given to verify the correctness of the results.

II. Preliminaries

$\mathbb{R}^N$ and $\mathbb{C}^N$ respectively symbol the N-dimensional real vector space and the N-dimensional complex vector space. $\lambda_m(\cdot)$, $\lambda_M(\cdot)$ denote the minimal and the maximal eigenvalue of the corresponding matrix. Let $e = e^R + ie^I$ be a complex number, where i symbols the imaginary unit, which satisfies $i = \sqrt{-1}$ and $e^R, e^I \in \mathbb{R}$ are the real and imaginary part of e. The norm in $\mathbb{C}^N$ is denoted as $\|\cdot\|$. For any vector

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$e(t) \in \mathbb{C}^N$, $\|e(t)\| = \sqrt{e^H(t)e(t)}$ where H denotes the conjugate transposition. Let $e^R(t), e^I(t) \in \mathbb{R}^N$ be the real and imaginary part of $e(t) \in \mathbb{C}^N$, then one has $\|e(t)\| = \sqrt{(e^R(t))^T e^R(t) + (e^I(t))^T e^I(t)}$.

III. Passivity and synchronization of CCVMNNs

A. Network model

On the basis of the physical characteristics of memristor, a single CVMNN model can be described by

$$\dot{z}_\iota(t) = -a_\iota z_\iota(t) + \sum_{j=1}^{n} b_{\iota j}(z_\iota(t))h_j(z_j(t-\tau_j(t))) + \sum_{j=1}^{n} d_{\iota j}(z_\iota(t))f_j(z_j(t)),\ \iota = 1,2,\cdots,n \tag{1}$$

where $z_\iota(t)$ denotes the complex-valued state variable of ι-th neuron. $a_\iota > 0$ is the self-inhibition. $b_{\iota j}(z_\iota(t))$ and $d_{\iota j}(z_\iota(t))$ represent complex-valued memristors synaptic connection weights. $h_j(\cdot)$ and $f_j(\cdot)$ stand for complex-valued activation functions for the delayed configuration and non-delayed one of the j-th neuron. The time-varying delay $\tau_j(t)$ satisfies $0 \leqslant \tau_j(t) \leqslant \tau_j \leqslant \tau = \max_{j=1,2,\cdots,n}\{\tau_j\}, \dot{\tau}_j(t) \leqslant \gamma_j < 1$.

Let $z_\iota(t)$, $b_{\iota j}(z_\iota(t))$, $d_{\iota j}(z_\iota(t))$, $h_j(\cdot)$, and $f_j(\cdot)$ be the following:

$$\begin{aligned}
z_\iota(t) &= z_\iota^R(t) + iz_\iota^I(t),\\
b_{\iota j}(z_\iota(t)) &= b_{\iota j}^R(z_\iota^R(t)) + ib_{\iota j}^I(z_\iota^I(t)),\\
d_{\iota j}(z_\iota(t)) &= d_{\iota j}^R(z_\iota^R(t)) + id_{\iota j}^I(z_\iota^I(t)),\\
f_j(z_j(t)) &= f_j^R(z_j^R(t)) + if_j^I(z_j^I(t)),\\
h_j(z_j(t-\tau_j(t))) &= h_j^R(z_j^R(t-\tau_j(t))) + ih_j^I(z_j^I(t-\tau_j(t))),
\end{aligned}$$

where $z_\iota^R(t)$, $b_{\iota j}^R(z_\iota^R(t))$, $d_{\iota j}^R(z_\iota^R(t))$, $f_j^R(z_j^R(t))$, $h_j^R(z_j^R(t-\tau_j(t)))$ are the real parts of $z_\iota(t)$, $b_{\iota j}(z_\iota(t))$, $d_{\iota j}(z_\iota(t)), f_j(z_j(t)), h_j(z_j(t-\tau_j(t)))$, respectively. i is the imaginary unit which satisfies $i = \sqrt{-1}$. $z_\iota^I(t)$, $b_{\iota j}^I(z_\iota^I(t))$, $d_{\iota j}^I(z_\iota^I(t))$, $f_j^I(z_j^I(t))$, $h_j^I(z_j^I(t-\tau_j(t)))$ are the imaginary parts of $z_\iota(t)$, $b_{\iota j}(z_\iota(t))$, $d_{\iota j}(z_\iota(t)), f_j(z_j(t)), h_j(z_j(t-\tau_j(t)))$, respectively.

In accordance with the voltage-current characteristic of memristor, one has

$$\begin{aligned}
b_{\iota j}^R(z_\iota^R(t)) &= \begin{cases} \hat{b}_{\iota j}^R, & |z_\iota^R(t)| \leqslant \Gamma_\iota,\\ \check{b}_{\iota j}^R, & |z_\iota^R(t)| > \Gamma_\iota,\end{cases}\\
b_{\iota j}^I(z_\iota^I(t)) &= \begin{cases} \hat{b}_{\iota j}^I, & |z_\iota^I(t)| \leqslant \Gamma_\iota,\\ \check{b}_{\iota j}^I, & |z_\iota^I(t)| > \Gamma_\iota,\end{cases}\\
d_{\iota j}^R(z_\iota^R(t)) &= \begin{cases} \hat{d}_{\iota j}^R, & |z_\iota^R(t)| \leqslant \Gamma_\iota,\\ \check{d}_{\iota j}^R, & |z_\iota^R(t)| > \Gamma_\iota,\end{cases}\\
d_{\iota j}^I(z_\iota^I(t)) &= \begin{cases} \hat{d}_{\iota j}^I, & |z_\iota^I(t)| \leqslant \Gamma_\iota,\\ \check{d}_{\iota j}^I, & |z_\iota^I(t)| > \Gamma_\iota,\end{cases}
\end{aligned}$$

where $\iota, j \in \{1,2,\cdots,n\}$; $\hat{b}_{\iota j}^R$, $\hat{b}_{\iota j}^I$, $\check{b}_{\iota j}^R$, $\check{b}_{\iota j}^I$, $\hat{d}_{\iota j}^R$, $\hat{d}_{\iota j}^I$, $\check{d}_{\iota j}^R$, $\check{d}_{\iota j}^I$ are all constants. $\Gamma_\iota > 0$ represents the threshold level. Let $\tilde{b}_{\iota j}^R = \max\{|\hat{b}_{\iota j}^R|, |\check{b}_{\iota j}^R|\}$, $\tilde{b}_{\iota j}^I = \max\{|\hat{b}_{\iota j}^I|, |\check{b}_{\iota j}^I|\}$, $\tilde{d}_{\iota j}^R = \max\{|\hat{d}_{\iota j}^R|, |\check{d}_{\iota j}^R|\}$, $\tilde{d}_{\iota j}^I = \max\{|\hat{d}_{\iota j}^I|, |\check{d}_{\iota j}^I|\}$, $\bar{b}_{\iota j}^R = |\hat{b}_{\iota j}^R - \check{b}_{\iota j}^R|$, $\bar{b}_{\iota j}^I = |\hat{b}_{\iota j}^I - \check{b}_{\iota j}^I|$, $\bar{d}_{\iota j}^R = |\hat{d}_{\iota j}^R - \check{d}_{\iota j}^R|$, $\bar{d}_{\iota j}^I = |\hat{d}_{\iota j}^I - \check{d}_{\iota j}^I|$, $\bar{B}^R = (\bar{b}_{\iota j}^R)_{n\times n}$, $\bar{B}^I = (\bar{b}_{\iota j}^I)_{n\times n}$, $\tilde{B}^R = \text{diag}(\sum_{j=1}^n(\tilde{b}_{1j}^R)^2, \sum_{j=1}^n(\tilde{b}_{2j}^R)^2, \cdots, \sum_{j=1}^n(\tilde{b}_{nj}^R)^2)$, $\tilde{B}^I = \text{diag}(\sum_{j=1}^n(\tilde{b}_{1j}^I)^2, \sum_{j=1}^n(\tilde{b}_{2j}^I)^2, \cdots, \sum_{j=1}^n(\tilde{b}_{nj}^I)^2)$, $\bar{D}^R = (\bar{d}_{\iota j}^R)_{n\times n}$, $\bar{D}^I = (\bar{d}_{\iota j}^I)_{n\times n}$, $\tilde{D}^R = \text{diag}(\sum_{j=1}^n(\tilde{d}_{1j}^R)^2, \sum_{j=1}^n(\tilde{d}_{2j}^R)^2, \cdots, \sum_{j=1}^n(\tilde{d}_{nj}^R)^2)$, $\tilde{D}^I = \text{diag}(\sum_{j=1}^n(\tilde{d}_{1j}^I)^2, \sum_{j=1}^n(\tilde{d}_{2j}^I)^2, \cdots, \sum_{j=1}^n(\tilde{d}_{nj}^I)^2)$.

In this section, we consider the following CCVMNNs consisting of N CVMNNs (1):

$$\dot{Z}_s(t) = -AZ_s(t) + B(Z_s(t))h(\overline{Z_s(t)}) + D(Z_s(t))f(Z_s(t)) + u_s(t) + g\sum_{\kappa=1}^{N} G_{s\kappa}MZ_\kappa(t) + x_s(t),\ s = 1,2,\cdots,N, \tag{2}$$

where $Z_s(t) = (Z_{s1}(t), Z_{s2}(t), \cdots, Z_{sn}(t)) \in \mathbb{C}^n$ represents the complex-valued state variable of the s-th node. $0 < A = \text{diag}(a_1, a_2, \cdots, a_n) \in \mathbb{R}^{n\times n}$. $\overline{Z_s(t)} = (Z_{s1}(t-\tau_1(t)), Z_{s2}(t-\tau_2(t)), \cdots, Z_{sn}(t-\tau_n(t)))^T \in \mathbb{C}^n$. $h(\overline{Z_s(t)}) = (h_1(Z_{s1}(t-\tau_1(t))), h_2(Z_{s2}(t-\tau_2(t))), \cdots, h_n(Z_{sn}(t-\tau_n(t))))^T \in \mathbb{C}^n$. $f(Z_s(t)) = (f_1(Z_{s1}(t)), f_2(Z_{s2}(t)), \cdots, f_n(Z_{sn}(t)))^T \in \mathbb{C}^n$; $M \in \mathbb{R}^{n\times n}$ symbols the inner coupling matrix. $B(Z_s(t)) = (b_{\iota j}(Z_{s\iota}(t)))_{n\times n} \in \mathbb{C}^{n\times n}$, $D(Z_s(t)) = (d_{\iota j}(Z_{s\iota}(t)))_{n\times n} \in \mathbb{C}^{n\times n}$, where $\iota, j = 1,2,\cdots,n$. $u_s(t) = u_s^R(t) + iu_s^I(t) = (u_{s1}(t), u_{s2}(t), \cdots, u_{sn}(t))^T \in \mathbb{C}^n$ is the controller to be designed for obtaining a certain control objective. $x_s(t) = x_s^R(t) + ix_s^I(t) = (x_{s1}(t), x_{s2}(t), \cdots, x_{sn}(t))^T \in \mathbb{C}^n$ denotes the external input of the network. $g > 0$ is the overall coupling strength. $G = (G_{s\kappa})_{N\times N}$ stands for coupling weight between nodes, where $G_{s\kappa} = G_{\kappa s} > 0$ if and only if there exists a connection between node s and node κ; if not, $G_{s\kappa} = G_{\kappa s} = 0 (s \neq \kappa)$; and

$$G_{ss} = -\sum_{\substack{\kappa=1\\ \kappa\neq s}}^{N} G_{s\kappa},\ s = 1,2,\cdots,N.$$

Then, the network (2) can be separated into real and imaginary parts as follows:

$$\begin{aligned}
\dot{Z}_s^R(t) =& -AZ_s^R(t) + B^R(Z_s^R(t))h^R(\overline{Z_s^R(t)}) + u_s^R(t) + x_s^R(t)\\
&-B^I(Z_s^I(t))h^I(\overline{Z_s^I(t)}) + D^R(Z_s^R(t))f^R(Z_s^R(t))\\
&- D^I(Z_s^I(t))f^I(Z_s^I(t)) + g\sum_{\kappa=1}^{N} G_{s\kappa}MZ_\kappa^R(t),
\end{aligned} \tag{3}$$

$$\begin{aligned}
\dot{Z}_s^I(t) =& -AZ_s^I(t) + B^R(Z_s^R(t))h^I(\overline{Z_s^I(t)}) + u_s^I(t) + x_s^I(t)\\
&+B^I(Z_s^I(t))h^R(\overline{Z_s^R(t)}) + D^R(Z_s^R(t))f^I(Z_s^I(t))\\
&+D^I(Z_s^I(t))f^R(Z_s^R(t)) + g\sum_{\kappa=1}^{N} G_{s\kappa}MZ_\kappa^I(t),
\end{aligned} \tag{4}$$

where $h^R(\overline{Z_s^R(t)}) = (h_1^R(Z_{s1}^R(t-\tau_1(t))), h_2^R(Z_{s2}^R(t-\tau_2(t))), \cdots, h_n^R(Z_{sn}^R(t-\tau_n(t))))^T$, $h^I(\overline{Z_s^I(t)}) = (h_1^I(Z_{s1}^I(t-\tau_1(t))), h_2^I(Z_{s2}^I(t-

$\tau_2(t))),\cdots,h_n^I(Z_{sn}^I(t-\tau_n(t))))^T$, $f^R(Z_s^R(t)) = (f_1^R(Z_{s1}^R(t)),f_2^R(Z_{s2}^R(t)),\cdots,f_n^R(Z_{sn}^R(t)))^T$, $f^I(Z_s^I(t)) = (f_1^I(Z_{s1}^I(t)),f_2^I(Z_{s2}^I(t)),\cdots,f_n^I(Z_{sn}^I(t)))^T$, $Z_s^R(t) = (Z_{s1}^R(t),Z_{s2}^R(t),\cdots,Z_{sn}^R(t))^T$, $B^R(\cdot) = (b_{\iota j}^R(\cdot))_{n\times n}$, $Z_s^I(t)=(Z_{s1}^I(t),Z_{s2}^I(t),\cdots,Z_{sn}^I(t))^T$, $D^R(\cdot) = (d_{\iota j}^R(\cdot))_{n\times n}$, $u_s^R(t)=(u_{s1}^R(t),u_{s2}^R(t),\cdots,u_{sn}^R(t))^T$, $B^I(\cdot) = (b_{\iota j}^I(\cdot))_{n\times n}$, $u_s^I(t)=(u_{s1}^I(t),u_{s2}^I(t),\cdots,u_{sn}^I(t))^T$, $D^I(\cdot) = (d_{\iota j}^I(\cdot))_{n\times n}$, $x_s^R(t)=(x_{s1}^R(t),x_{s2}^R(t),\cdots,x_{sn}^R(t))^T$, $x_s^I(t) = (x_{s1}^I(t),x_{s2}^I(t),\cdots,x_{sn}^I(t))^T$.

Assumption 1. *For any $\alpha_1,\alpha_2\in\mathbb{R}$, the real part $f_s^R(\cdot)$ and the imaginary part $f_s^I(\cdot)$ of function $f_s(\cdot)$ and the real part $h_s^R(\cdot)$ and the imaginary part $h_s^I(\cdot)$ of function $h_s(\cdot)$ satisfy*

$$\begin{aligned}
&|f_s^R(\cdot)|\leqslant F_s^R,\ |f_s^I(\cdot)|\leqslant F_s^I,\\
&|h_s^R(\cdot)|\leqslant H_s^R,\ |h_s^I(\cdot)|\leqslant H_s^I,\\
&|f_s^R(\alpha_1)-f_s^R(\alpha_2)|\leqslant l_s^R|\alpha_1-\alpha_2|,\\
&|f_s^I(\alpha_1)-f_s^I(\alpha_2)|\leqslant l_s^I|\alpha_1-\alpha_2|,\\
&|h_s^R(\alpha_1)-h_s^R(\alpha_2)|\leqslant \eta_s^R|\alpha_1-\alpha_2|,\\
&|h_s^I(\alpha_1)-h_s^I(\alpha_2)|\leqslant \eta_s^I|\alpha_1-\alpha_2|,
\end{aligned}$$

where F_s^R, F_s^I, H_s^R, H_s^I, l_s^R, l_s^I, η_s^R, η_s^I are positive constants.

Suppose $Z_0(t)=(Z_{01}(t),Z_{02}(t),\cdots,Z_{0n}(t))^T\in\mathbb{C}^n$ is an arbitrary solution of the network (2), then

$$\dot{Z}_0(t)=-AZ_0(t)+B(Z_0(t))h(Z_0(t))+D(Z_0(t))f(Z_0(t)),\quad (5)$$

where $Z_0(t)=Z_0^R(t)+iZ_0^I(t)$. Then, Eq. (5) can be separated into real and imaginary parts as follows:

$$\begin{aligned}
\dot{Z}_0^R(t) =&-AZ_0^R(t)+B^R(Z_0^R(t))h^R(Z_0^R(t))-B^I(Z_0^I(t))h^I(Z_0^I(t))\\
&+D^R(Z_0^R(t))f^R(Z_0^R(t))-D^I(Z_0^I(t))f^I(Z_0^I(t)),\\
\dot{Z}_0^I(t) =&-AZ_0^I(t)+B^R(Z_0^R(t))h^I(Z_0^I(t))+B^I(Z_0^I(t))h^R(Z_0^R(t))\\
&+D^R(Z_0^R(t))f^I(Z_0^I(t))+D^I(Z_0^I(t))f^R(Z_0^R(t)).
\end{aligned}$$

Let $e_s(t)=Z_s(t)-Z_0(t)$, then

$$\begin{aligned}
\dot{e}_s(t)=&-Ae_s(t)+B(Z_s(t))h(\overline{Z_s(t)})-B(Z_0(t))h(Z_0(t))\\
&+D(Z_s(t))f(Z_s(t))+u_s(t)-D(Z_0(t))f(Z_0(t))\\
&+g\sum_{\kappa=1}^{N}G_{s\kappa}Me_\kappa(t)+x_s(t),\ s=1,2,\cdots,N \quad (6)
\end{aligned}$$

where $e_s(t)=(e_{s1}(t),e_{s2}(t),\cdots,e_{sn}(t))^T$.

By separating (6) into real and imaginary parts, one has

$$\begin{aligned}
\dot{e}_s^R(t)=&-Ae_s^R(t)+D^R(Z_s^R(t))P^R(e_s^R(t))+x_s^R(t)+u_s^R(t)\\
&-D^I(Z_s^I(t))P^I(e_s^I(t))+B^R(Z_s^R(t))Q^R(\overline{e_s^R(t)})\\
&-B^I(Z_s^I(t))Q^I(\overline{e_s^I(t)})+g\sum_{\kappa=1}^{N}G_{s\kappa}Me_\kappa^R(t)\\
&+(D^R(Z_s^R(t))-D^R(Z_0^R(t)))f^R(Z_0^R(t))\\
&+(B^R(Z_s^R(t))-B^R(Z_0^R(t)))h^R(Z_0^R(t))\\
&-(D^I(Z_s^I(t))-D^I(Z_0^I(t)))f^I(Z_0^I(t))\\
&-(B^I(Z_s^I(t))-B^I(Z_0^I(t)))h^I(Z_0^I(t)),
\end{aligned}$$

$$\begin{aligned}
\dot{e}_s^I(t)=&-Ae_s^I(t)+D^R(Z_s^R(t))P^I(e_s^I(t))+x_s^I(t)+u_s^I(t)\\
&+D^I(Z_s^I(t))P^R(e_s^R(t))+B^R(Y_s^R(t))Q^I(\overline{e_s^I(t)})\\
&+B^I(Z_s^I(t))Q^R(\overline{e_s^R(t)})+g\sum_{\kappa=1}^{N}G_{s\kappa}Me_\kappa^I(t)\\
&+(D^R(Z_s^R(t))-D^R(Z_0^R(t)))f^I(Z_0^I(t))\\
&+(D^I(Z_s^I(t))-D^I(Z_0^I(t)))f^R(Z_0^R(t))\\
&+(B^R(Z_s^R(t))-B^R(Z_0^R(t)))h^I(Z_0^I(t))\\
&+(B^I(Z_s^I(t))-B^I(Z_0^I(\varrho,t)))h^R(Z_0^R(t)),
\end{aligned}$$

where $e_s^R(t)=(e_{s1}^R(t),e_{s2}^R(t),\cdots,e_{sn}^R(t))^T$, $e_s^I(t)=(e_{s1}^I(t),e_{s2}^I(t),\cdots,e_{sn}^I(t))^T$, $\overline{e_s^R(t)}=(e_{s1}^R(t-\tau_1(t)),e_{s2}^R(t-\tau_2(t)),\cdots,e_{sn}^R(t-\tau_n(t)))^T$, $\overline{e_s^I(t)}=(e_{s1}^I(t-\tau_1(t)),e_{s2}^I(t-\tau_2(t)),\cdots,e_{sn}^I(t-\tau_n(t)))^T$, $P^R(e_s^R(t))=f^R(Z_s^R(t))-f^R(Z_0^R(t))$, $P^I(e_s^I(t))=f^I(Z_s^I(t))-f^I(Z_0^I(t))$, $Q^R(\overline{e_s^R(t)})=h^R(\overline{Z_s^R(t)})-h^R(Z_0^R(t))$ and $Q^I(\overline{e_s^I(t)})=h^I(\overline{Z_s^I(t)})-h^I(Z_0^I(t))$.

Definition III.1. *For any $t_2,t_1\in\mathbb{R}^+$ and $t_2\geqslant t_1$, if there exists a constant $\rho>0$ satisfy*

$$\begin{aligned}
&\int_{t_1}^{t_2}[(y^R(t))^Tx^R(t)+(y^I(t))^Tx^I(t)]dt\geqslant\\
&V(t_2)-V(t_1)-\rho\int_{t_1}^{t_2}[(x^R(t))^Tx^R(t)+(x^I(t))^Tx^I(t)]dt,
\end{aligned}$$

where $V(t):\mathbb{R}^+\to\mathbb{R}^+$ is the storage function, then the network (6) is called passive.

Definition III.2. *The network (2) is synchronized if*

$$\lim_{t\to\infty}\|Z_s(t)-Z_0(t)\|=0,\quad s=1,2,\cdots,N,$$

under the condition $x_s(t)=0$, $s=1,2,\cdots,N$.

B. Passivity control

The following state feedback controller is designed for the network (2):

$$\begin{cases}
u_s^R(t)=-\Upsilon^Re_s^R(t)-\mathrm{sign}(e_s^R(t))(\bar{D}^R\bar{F}^R+\bar{D}^I\bar{F}^I\\
\qquad\qquad+\bar{B}^R\bar{H}^R+\bar{B}^I\bar{H}^I),\\
u_s^I(t)=-\Upsilon^Ie_s^I(t)-\mathrm{sign}(e_s^I(t))(\bar{D}^R\bar{F}^I+\bar{D}^I\bar{F}^R\\
\qquad\qquad+\bar{B}^R\bar{H}^I+\bar{B}^I\bar{H}^R),
\end{cases}\quad (7)$$

where $s=1,2,\cdots,N$, $\Upsilon^R=\mathrm{diag}(\upsilon_1^R,\upsilon_2^R,\cdots,\upsilon_n^R)\in\mathbb{R}^{n\times n}$ and $\Upsilon^I=\mathrm{diag}(\upsilon_1^I,\upsilon_2^I,\cdots,\upsilon_n^I)\in\mathbb{R}^{n\times n}$ are the positive definite controller gain matrices. $\mathbb{R}\ni\upsilon_\iota^R>0$ and $\mathbb{R}\ni\upsilon_\iota^I>0$. $\bar{F}^R=(F_1^R,F_2^R,\cdots,F_n^R)^T$, $\bar{F}^I=(F_1^I,F_2^I,\cdots,F_n^I)^T$, $\bar{H}^R=(H_1^R,H_2^R,\cdots,H_n^R)^T$, and $\bar{H}^I=(H_1^I,H_2^I,\cdots,H_n^I)^T$. $\mathrm{sign}(e_s^R(t))=\mathrm{diag}(\mathrm{sign}(e_{s1}^R(t)),\mathrm{sign}(e_{s2}^R(t)),\cdots,\mathrm{sign}(e_{sn}^R(t)))$ and $\mathrm{sign}(e_s^I(t))=\mathrm{diag}(\mathrm{sign}(e_{s1}^I(t)),\mathrm{sign}(e_{s2}^I(t)),\cdots,\mathrm{sign}(e_{sn}^I(t)))$.

The output vector $y_s(t)\in\mathbb{C}^n$ of the system (6) is described as follows:

$$y_s(t)=W_1e_s(t)+W_2x_s(t),$$

where $W_1\in\mathbb{R}^{n\times n}$ and $W_2\in\mathbb{R}^{n\times n}$.

For convenience, we denote

$$\begin{aligned}
e^R(t) &= ((e_1^R(t))^T, (e_2^R(t))^T, \cdots, (e_N^R(t))^T)^T,\\
e^I(t) &= ((e_1^I(t))^T, (e_2^I(t))^T, \cdots, (e_N^I(t))^T)^T,\\
\overline{e^R(t)} &= ((\overline{e_1^R(t)})^T, (\overline{e_2^R(t)})^T, \cdots, (\overline{e_N^R(t)})^T)^T,\\
\overline{e^I(t)} &= ((\overline{e_1^I(t)})^T, (\overline{e_2^I(t)})^T, \cdots, (\overline{e_N^I(t)})^T)^T,\\
L^R &= \mathrm{diag}((l_1^R)^2, (l_2^R)^2, \cdots, (l_n^R)^2),\\
L^I &= \mathrm{diag}((l_1^I)^2, (l_2^I)^2, \cdots, (l_n^I)^2),\\
\zeta^R &= \mathrm{diag}((\eta_1^R)^2, (\eta_2^R)^2, \cdots, (\eta_n^R)^2),\\
\zeta^I &= \mathrm{diag}((\eta_1^I)^2, (\eta_2^I)^2, \cdots, (\eta_n^I)^2),\\
x(t) &= (x_1^H(t), x_2^H(t), \cdots, x_N^H(t))^H,\\
y(t) &= (y_1^H(t), y_2^H(t), \cdots, y_N^H(t))^H,\\
\Gamma &= \mathrm{diag}(\frac{1}{1-\gamma_1}, \frac{1}{1-\gamma_2}, \cdots, \frac{1}{1-\gamma_n}).
\end{aligned}$$

Theorem III.1. *If there exists a constant $\rho > 0$ such that*

$$\begin{pmatrix} \Psi_1^R & \Xi^R \\ (\Xi^R)^T & \Psi_2^R \end{pmatrix} \leqslant 0 \ and \ \begin{pmatrix} \Psi_1^I & \Xi^I \\ (\Xi^I)^T & \Psi_2^I \end{pmatrix} \leqslant 0, \tag{8}$$

where $\Psi_1^R = I_N \otimes (-2A + \tilde{D}^R + 2L^R + \tilde{D}^I + \tilde{B}^R + \tilde{B}^I - \Upsilon^R + 2\zeta^R\Gamma) + gG \otimes (M + M^T)$, $\Xi^R = \Xi^I = I_N \otimes (I_n - \frac{1}{2}W_1^T)$, $\Psi_1^I = I_N \otimes (-2A + 2L^I + \tilde{D}^R + \tilde{D}^I + \tilde{B}^I + \tilde{B}^R - \Upsilon^I + 2\zeta^I\Gamma) + gG \otimes (M + M^T)$, $\Psi_2^R = \Psi_2^I = I_N \otimes (-\frac{1}{2}(W_2^T + W_2) - \rho I_n)$, then the network (6) *is said to be passive under the controller* (7).

Proof. We construct a Lyapunov functional as follows:

$$\begin{aligned}
V(t) =& \sum_{s=1}^{N} (e_s^R(t))^T e_s^R(t) + 2\sum_{s=1}^{N}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \frac{(\eta_j^R e_{sj}^R(\delta))^2}{1-\gamma_j} d\delta\\
&+2\sum_{s=1}^{N}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \frac{(\eta_j^I e_{sj}^I(\delta))^2}{1-\gamma_j} d\delta + \sum_{s=1}^{N}(e_s^I(t))^T e_s^I(t).
\end{aligned}$$

Then,

$$\begin{aligned}
\dot{V}(t) \leqslant& 2\sum_{s=1}^{N}(e_s^R(t))^T(-Ae_s^R(t) + D^R(Z_s^R(t))P^R(e_s^R(t)) + x_s^R(t)\\
&- D^I(Z_s^I(t))P^I(e_s^I(t)) + B^R(Z_s^R(t))Q^R(\overline{e_s^R(t)}) - \Upsilon^R e_s^R(t)\\
&- B^I(Z_s^I(t))Q^I(\overline{e_s^I(t)}) + g\sum_{\kappa=1}^{N} G_{s\kappa} M e_\kappa^R(t) + (D^R(Z_s^R(t))\\
&- D^R(Z_0^R(t)))f^R(Z_0^R(t)) - \mathrm{sign}(e_s^R(t))(\bar{D}^R\bar{F}^R + \bar{D}^I\bar{F}^I\\
&+ \bar{B}^R\bar{H}^R + \bar{B}^I\bar{H}^I) - (D^I(Z_s^I(t)) - D^I(Z_0^I(t)))f^I(Z_0^I(t))\\
&+ (B^R(Z_s^R(t)) - B^R(Z_0^R(t)))h^R(Z_0^R(t)) - (B^I(Z_s^I(t))\\
&- B^I(Z_0^I(t)))h^I(Z_0^I(t))) + 2\sum_{s=1}^{N}(e_s^I(t))^T(-Ae_s^I(t) + x_s^I(t)\\
&+ D^R(Z_s^R(t))P^I(e_s^I(t)) + D^I(Z_s^I(t))P^R(e_s^R(t)) - \Upsilon^I e_s^I(t)\\
&+ B^R(Z_s^R(t))Q^I(\overline{e_s^I(t)}) + B^I(Z_s^I(t))Q^R(\overline{e_s^R(t)}) + (D^R(Z_s^R(t))\\
&- D^R(Z_0^R(t)))f^I(Z_0^I(t)) + g\sum_{\kappa=1}^{N} G_{s\kappa} M e_\kappa^I(t) + (D^I(Z_s^I(t))\\
&- D^I(Z_0^I(t)))f^R(Z_0^R(t)) + (B^R(Z_s^R(t)) - B^R(Z_0^R(t)))\\
&\times h^I(Z_0^I(t)) - \mathrm{sign}(e_s^I(t))(\bar{D}^R\bar{F}^I + \bar{D}^I\bar{F}^R + \bar{B}^R\bar{H}^I + \bar{B}^I\bar{H}^R)\\
&+ (B^I(Z_s^I(t)) - B^I(Z_0^I(t)))h^R(Z_0^R(t))) + 2(e^R(t))^T(I_N\\
&\otimes(\zeta^R\Gamma))e^R(t) - 2(\overline{e^R(t)})^T(I_N \otimes \zeta^R)\overline{e^R(t)} + 2(e^I(t))^T\\
&\times(I_N \otimes (\zeta^I\Gamma))e^I(t) - 2(\overline{e^I(t)})^T(I_N \otimes \zeta^I)\overline{e^I(t)}.
\end{aligned} \tag{9}$$

From Assumption 1, one has

$$\begin{aligned}
&2\sum_{s=1}^{N}(e_s^R(t))^T D^R(Z_s^R(t))P^R(e_s^R(t))\\
\leqslant& 2\sum_{s=1}^{N}\sum_{\iota=1}^{n}\sum_{j=1}^{n}|e_{s\iota}^R(t)|\tilde{d}_{\iota j}^R|f_j^R(Z_{sj}^R(t)) - f_j^R(Z_{0j}^R(t))|\\
\leqslant& \sum_{s=1}^{N}\sum_{\iota=1}^{n}\sum_{j=1}^{n}(e_{s\iota}^R(t))^2(\tilde{d}_{\iota j}^R)^2 + \sum_{s=1}^{N}\sum_{j=1}^{n}(l_j^R)^2(e_{sj}^R(t))^2\\
=& (e^R(t))^T(I_N \otimes \tilde{D}^R)e^R(t) + (e^R(t))^T(I_N \otimes L^R)e^R(t).
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
&-2\sum_{s=1}^{N}(e_s^R(t))^T D^I(Z_s^I(t))P^I(e_s^I(t))\\
\leqslant& (e^R(t))^T(I_N \otimes \tilde{D}^I)e^R(t) + (e^I(t))^T(I_N \otimes L^I)e^I(t),
\end{aligned} \tag{11}$$

$$\begin{aligned}
&2\sum_{s=1}^{N}(e_s^I(t))^T D^R(Z_s^R(t))P^I(e_s^I(t))\\
\leqslant& (e^I(t))^T(I_N \otimes \tilde{D}^R)e^I(t) + (e^I(t))^T(I_N \otimes L^I)e^I(t),
\end{aligned} \tag{12}$$

$$\begin{aligned}
&2\sum_{s=1}^{N}(e_s^I(t))^T D^I(Z_s^I(t))P^R(e_s^R(t))\\
\leqslant& (e^I(t))^T(I_N \otimes \tilde{D}^I)e^I(t) + (e^R(t))^T(I_N \otimes L^R)e^R(t).
\end{aligned} \tag{13}$$

Moreover,

$$\begin{aligned}
&2\sum_{s=1}^{N}(e_s^R(t))^T B^R(Z_s^R(t))Q^R(\overline{e_s^R(t)})\\
\leqslant& (e^R(t))^T(I_N \otimes \tilde{B}^R)e^R(t) + (\overline{e^R(t)})^T(I_N \otimes \zeta^R)\overline{e^R(t)}.
\end{aligned} \tag{14}$$

Similarly,

$$\begin{aligned}
&-2\sum_{s=1}^{N}(e_s^R(t))^T B^I(Z_s^I(t))Q^I(\overline{e_s^I(t)})\\
\leqslant& (e^R(t))^T(I_N \otimes \tilde{B}^I)e^R(t) + (\overline{e^I(t)})^T(I_N \otimes \zeta^I)\overline{e^I(t)},
\end{aligned} \tag{15}$$

$$\begin{aligned}
&2\sum_{s=1}^{N}(e_s^I(t))^T B^R(Z_s^R(t))Q^I(\overline{e_s^I(t)})\\
\leqslant& (e^I(t))^T(I_N \otimes \tilde{B}^R)e^I(t) + (\overline{e^I(t)})^T(I_N \otimes \zeta^I)\overline{e^I(t)},
\end{aligned} \tag{16}$$

$$\begin{aligned}
&2\sum_{s=1}^{N}(e_s^I(t))^T B^I(Z_s^I(t))Q^R(\overline{e_s^R(t)})\\
\leqslant& (e^I(t))^T(I_N \otimes \tilde{B}^I)e^I(t) + (\overline{e^R(t)})^T(I_N \otimes \zeta^R)\overline{e^R(t)}.
\end{aligned} \tag{17}$$

In addition,

$$2g\sum_{s=1}^{N}\sum_{\kappa=1}^{N}(e_s^R(t))^T G_{s\kappa} M e_\kappa^R(t)$$

$$=g(e^R(t))^T(G\otimes(M+M^T))e^R(t), \tag{18}$$

$$2g\sum_{s=1}^{N}\sum_{\kappa=1}^{N}(e_s^I(t))^T G_{s\kappa}Me_\kappa^I(t)$$
$$=g(e^I(t))^T(G\otimes(M+M^T))e^I(t). \tag{19}$$

What's more,

$$2\sum_{s=1}^{N}(e_s^R(t))^T(D^R(Z_s^R(t))-D^R(Z_0^R(t)))f^R(Z_0^R(t))$$
$$\leqslant 2\sum_{s=1}^{N}\sum_{\iota=1}^{n}\sum_{j=1}^{n}|e_{s\iota}^R(t)||\hat{d}_{\iota j}^R-\check{d}_{\iota j}^R|F_j^R$$
$$=2\sum_{s=1}^{N}|(e_s^R(t))^T|\bar{D}^R\bar{F}^R. \tag{20}$$

Similarly,

$$2\sum_{s=1}^{N}(e_s^R(t))^T(D^I(Z_0^I(t))-D^I(Z_s^I(t)))f^I(Z_0^I(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^R(t))^T|\bar{D}^I\bar{F}^I, \tag{21}$$

$$2\sum_{s=1}^{N}(e_s^I(t))^T(D^R(Z_s^R(t))-D^R(Z_0^R(t)))f^I(Z_0^I(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^I(t))^T|\bar{D}^R\bar{F}^I, \tag{22}$$

$$2\sum_{s=1}^{N}(e_s^I(t))^T(D^I(Z_s^I(t))-D^I(Z_0^I(t)))f^R(Z_0^R(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^I(t))^T|\bar{D}^I\bar{F}^R, \tag{23}$$

$$2\sum_{s=1}^{N}(e_s^R(t))^T(B^R(Z_s^R(t))-B^R(Z_0^R(t)))h^R(Z_0^R(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^R(t))^T|\bar{B}^R\bar{H}^R, \tag{24}$$

$$2\sum_{s=1}^{N}(e_s^R(t))^T(B^I(Z_0^I(t))-B^I(Z_s^I(t)))h^I(Z_0^I(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^R(t))^T|\bar{B}^I\bar{H}^I, \tag{25}$$

$$2\sum_{s=1}^{N}(e_s^I(t))^T(B^R(Z_s^R(t))-B^R(Z_0^R(t)))h^I(Z_0^I(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^I(t))^T|\bar{B}^R\bar{H}^I, \tag{26}$$

$$2\sum_{s=1}^{N}(e_s^I(t))^T(B^I(Z_s^I(t))-B^I(Z_0^I(t)))h^R(Z_0^R(t))$$
$$\leqslant 2\sum_{s=1}^{N}|(e_s^I(t))^T|\bar{B}^I\bar{H}^R. \tag{27}$$

Eqs. (9)-(27) yield

$$\begin{aligned}\dot{V}(t)\leqslant&(e^R(t))^T\big(I_N\otimes(-2A+\tilde{D}^R+2L^R+\tilde{D}^I+\tilde{B}^R+\tilde{B}^I-\Upsilon^R\\&+2\zeta^R\Gamma)+gG\otimes(M+M^T)\big)e^R(t)+(e^I(t))^T\big(I_N\otimes(\\&-2A+2L^I+\tilde{D}^R+\tilde{D}^I+\tilde{B}^I+\tilde{B}^R-\Upsilon^I+2\zeta^I\Gamma)\\&+gG\otimes(M+M^T)\big)e^I(t)+2(e^R(t))^Tx^R(t)\\&+2(e^I(t))^Tx^I(t).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\dot{V}(t)-[(y^R(t))^Tx^R(t)+(y^I(t))^Tx^I(t)]\\&-\rho[(x^R(t))^Tx^R(t)+(x^I(t))^Tx^I(t)]\\\leqslant&(\varphi^R(t))^T\begin{pmatrix}\Psi_1^R & \Xi^R\\(\Xi^R)^T & \Psi_2^R\end{pmatrix}\varphi^R(t)\\&+(\varphi^I(t))^T\begin{pmatrix}\Psi_1^I & \Xi^I\\(\Xi^I)^T & \Psi_2^I\end{pmatrix}\varphi^I(t),\end{aligned}$$

where $\varphi^R(t)=((e^R(t))^T,(x^R(t))^T)^T$ and $\varphi^I(t)=((e^I(t))^T,(x^I(t))^T)^T$. From (8), it is easy to obtain

$$\begin{aligned}\dot{V}(t)\leqslant&(y^R(t))^Tx^R(t)+(y^I(t))^Tx^I(t)\\&+\rho[(x^R(t))^Tx^R(t)+(x^I(t))^Tx^I(t)].\end{aligned} \tag{28}$$

By integrating (28) about t over the time period from t_1 to t_2, one has

$$\begin{aligned}V(t_2)-V(t_1)\leqslant&\int_{t_1}^{t_2}[(y^R(t))^Tx^R(t)+(y^I(t))^Tx^I(t)]dt\\&+\rho\int_{t_1}^{t_2}[(x^R(t))^Tx^R(t)+(x^I(t))^Tx^I(t)]dt,\end{aligned}$$

where $t_2\geqslant t_1$. Namely,

$$\begin{aligned}&\int_{t_1}^{t_2}[(y^R(t))^Tx^R(t)+(y^I(t))^Tx^I(t)]dt\geqslant V(t_2)-V(t_1)\\&-\rho\int_{t_1}^{t_2}[(x^R(t))^Tx^R(t)+(x^I(t))^Tx^I(t)]dt\end{aligned}$$

for any $t_2,t_1\in\mathbb{R}^+$ and $t_2\geqslant t_1$.

According to Definition III.1, we can obtain that the network (6) is passive under the controller (7). □

C. Synchronization control

Theorem III.2. *The network* (2) *is synchronized under the controller* (7) *if*

$$\Psi_1^R<0 \text{ and } \Psi_1^I<0, \tag{29}$$

where $\Psi_1^R=I_N\otimes(-2A+\tilde{D}^R+2L^R+\tilde{D}^I+\tilde{B}^R+\tilde{B}^I-\Upsilon^R+2\zeta^R\Gamma)+gG\otimes(M+M^T)$, $\Psi_1^I=I_N\otimes(-2A+2L^I+\tilde{D}^R+\tilde{D}^I+\tilde{B}^I+\tilde{B}^R-\Upsilon^I+2\zeta^I\Gamma)+gG\otimes(M+M^T)$.

Proof. We construct the same Lyapunov functional as in Theorem III.1 in this subsection. Then, one obtains

$$\begin{aligned}\dot{V}(t)\leqslant&(e^R(t))^T\big(I_N\otimes(-2A+\tilde{D}^R+2L^R+\tilde{D}^I+\tilde{B}^R+\tilde{B}^I\\&-\Upsilon^R+2\zeta^R\Gamma)+gG\otimes(M+M^T)\big)e^R(t)\\&+(e^I(t))^T\big(I_N\otimes(-2A+2L^I+\tilde{D}^R+\tilde{D}^I+\tilde{B}^I+\tilde{B}^R\end{aligned}$$

$$
\begin{aligned}
&- \Upsilon^I + 2\zeta^I\Gamma) + gG \otimes (M + M^T)\big)e^I(t) \\
\leqslant &\alpha\|e(t)\|^2, \qquad (30)
\end{aligned}
$$

where $\alpha = \max\{\lambda_M(\Psi_1^R), \lambda_M(\Psi_1^I)\}$.

According to (30) and the definition of $V(t)$, we can obtain $V(t)$ is non-increasing and bounded. Hence, $\lim_{t\to+\infty}V(t)$ exists and satisfies $\lim_{t\to+\infty}V(t) \geqslant 0$. In addition, from (30), we can get

$$
\|e(t)\|^2 \leqslant \frac{\dot{V}(t)}{\alpha}. \qquad (31)
$$

From (31), it is easy to derive that $\lim_{t\to+\infty}\int_0^t \|e(\delta)\|^2 d\delta$ exists and is a nonnegative real number. Moreover,

$$
\begin{aligned}
0 \leqslant &\lim_{t\to+\infty} \sum_{s=1}^{N}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \frac{2(\eta_j^R e_{sj}^R(\delta))^2}{1-\gamma_j} d\delta \\
\leqslant &\lim_{t\to+\infty} \int_{t-\tau}^{t} (e^R(\delta))^T (I_N \otimes (2\zeta^R\Gamma)) e^R(\delta) d\delta \\
\leqslant &\lambda_M(I_N \otimes (2\zeta^R\Gamma)) \lim_{t\to+\infty} \int_{t-\tau}^{t} \|e^R(\delta)\|^2 d\delta \\
= &0. \qquad (32)
\end{aligned}
$$

Similarly,

$$
0 \leqslant \lim_{t\to+\infty} \sum_{s=1}^{N}\sum_{j=1}^{n}\int_{t-\tau_j(t)}^{t} \frac{2(\eta_j^I e_{sj}^I(\delta))^2}{1-\gamma_j} d\delta = 0. \qquad (33)
$$

From (32) and (33), we can easily know that $\lim\limits_{t\to+\infty} \sum_{s=1}^{N}[(e_s^R(t))^T e_s^R(t) + (e_s^I(t))^T e_s^I(t)]$ exists and is a nonnegative real number. Suppose that

$$
\lim_{t\to+\infty} \sum_{s=1}^{N}[(e_s^R(t))^T e_s^R(t) + (e_s^I(t))^T e_s^I(t)] = \beta > 0.
$$

Then, there exists a real number $\epsilon > 0$ satisfying

$$
\sum_{s=1}^{N}[(e_s^R(t))^T e_s^R(t) + (e_s^I(t))^T e_s^I(t)] > \frac{\beta}{2} \text{ for } t \geqslant \epsilon.
$$

Then, one has

$$
\|e(t)\|^2 > \frac{\beta}{2}, \quad t \geqslant \epsilon. \qquad (34)
$$

Combined (30) and (34), one has

$$
\dot{V}(t) < \frac{\alpha\beta}{2}, \quad t \geqslant \epsilon. \qquad (35)
$$

By (35), we can acquire

$$
-V(\epsilon) \leqslant V(+\infty) - V(\epsilon) = \int_{\epsilon}^{+\infty} \dot{V}(t)dt < \int_{\epsilon}^{+\infty} \frac{\alpha\beta}{2} dt = -\infty,
$$

which is unreasonable. Therefore,

$$
\lim_{t\to+\infty} \sum_{s=1}^{N}[(e_s^R(t))^T e_s^R(t) + (e_s^I(t))^T e_s^I(t)] = 0.
$$

Then, we can obtain

$$
\lim_{t\to+\infty} \|e(t)\| = 0.
$$

Consequently, the network (2) achieves synchronization. □

IV. Numerical Examples

Example IV.1. Consider the following CCVMNN:

$$
\begin{aligned}
\dot{Z}_s(t) = &- AZ_s(t) + B(Z_s(t))h(\overline{Z_s(t)}) + u_s(t) + x_s(t) \\
&+ D(Z_s(t))f(Z_s(t)) + g\sum_{\kappa=1}^{N} G_{s\kappa} M Z_\kappa(t), \qquad (36)
\end{aligned}
$$

where $s = 1, 2, \cdots, 6$, $f_i^R(\omega) = f_i^I(\omega) = h_i^R(\omega) = h_i^I(\omega) = \frac{|\omega+1|-|\omega-1|}{4}$ $(i = 1, 2, 3)$, $A = \mathrm{diag}(1.3, 0.8, 1.2)$, $M = \mathrm{diag}(0.5, 0.4, 0.6)$, $g = 0.3$, $\tau_j(t) = 1 - \frac{1}{2+j}e^{-t}$, $\tau = 1$, $\gamma_j = \frac{1}{2+j}$, $j = 1, 2, 3$, and the matrices $B(Z_s(t))$, $D(Z_s(t))$, $G = (G_{s\kappa})_{6\times 6}$ are selected as follows:

$$
b_{11}^R(z_{s1}^R(t)) = \begin{cases} 0.36, & |z_{s1}^R(t)| \leqslant 1.5, \\ -0.28, & |z_{s1}^R(t)| > 1.5, \end{cases}
$$

$$
b_{12}^R(z_{s1}^R(t)) = \begin{cases} -0.25, & |z_{s1}^R(t)| \leqslant 1.5, \\ -0.42, & |z_{s1}^R(t)| > 1.5, \end{cases}
$$

$$
b_{13}^R(z_{s1}^R(t)) = \begin{cases} -0.22, & |z_{s1}^R(t)| \leqslant 1.5, \\ 0.33, & |z_{s1}^R(t)| > 1.5, \end{cases}
$$

$$
b_{21}^R(z_{s2}^R(t)) = \begin{cases} 0.33, & |z_{s2}^R(t)| \leqslant 1.5, \\ -0.27, & |z_{s2}^R(t)| > 1.5, \end{cases}
$$

$$
b_{22}^R(z_{s2}^R(t)) = \begin{cases} 0.26, & |z_{s2}^R(t)| \leqslant 1.5, \\ 0.25, & |z_{s2}^R(t)| > 1.5, \end{cases}
$$

$$
b_{23}^R(z_{s2}^R(t)) = \begin{cases} 0.18, & |z_{s2}^R(t)| \leqslant 1.5, \\ -0.14, & |z_{s2}^R(t)| > 1.5, \end{cases}
$$

$$
b_{31}^R(z_{s3}^R(t)) = \begin{cases} 0.25, & |z_{s3}^R(t)| \leqslant 1.5, \\ -0.45, & |z_{s3}^R(t)| > 1.5, \end{cases}
$$

$$
b_{32}^R(z_{s3}^R(t)) = \begin{cases} -0.34, & |z_{s3}^R(t)| \leqslant 1.5, \\ 0.25, & |z_{s3}^R(t)| > 1.5, \end{cases}
$$

$$
b_{33}^R(z_{s3}^R(t)) = \begin{cases} 0.17, & |z_{s3}^R(t)| \leqslant 1.5, \\ 0.21, & |z_{s3}^R(t)| > 1.5, \end{cases}
$$

$$
b_{11}^I(z_{s1}^I(t)) = \begin{cases} -0.35, & |z_{s1}^I(t)| \leqslant 1.5, \\ 0.27, & |z_{s1}^I(t)| > 1.5, \end{cases}
$$

$$
b_{12}^I(z_{s1}^I(t)) = \begin{cases} 0.26, & |z_{s1}^I(t)| \leqslant 1.5, \\ 0.16, & |z_{s1}^I(t)| > 1.5, \end{cases}
$$

$$
b_{13}^I(z_{s1}^I(t)) = \begin{cases} -0.22, & |z_{s1}^I(t)| \leqslant 1.5, \\ 0.13, & |z_{s1}^I(t)| > 1.5, \end{cases}
$$

$$
b_{21}^I(z_{s2}^I(t)) = \begin{cases} -0.32, & |z_{s2}^I(t)| \leqslant 1.5, \\ 0.26, & |z_{s2}^I(t)| > 1.5, \end{cases}
$$

$$
b_{22}^I(z_{s2}^I(t)) = \begin{cases} 0.34, & |z_{s2}^I(t)| \leqslant 1.5, \\ 0.12, & |z_{s2}^I(t)| > 1.5, \end{cases}
$$

$$
b_{23}^I(z_{s2}^I(t)) = \begin{cases} 0.12, & |z_{s2}^I(t)| \leqslant 1.5, \\ 0.11, & |z_{s2}^I(t)| > 1.5, \end{cases}
$$

$$
b_{31}^I(z_{s3}^I(t)) = \begin{cases} 0.25, & |z_{s3}^I(t)| \leqslant 1.5, \\ -0.33, & |z_{s3}^I(t)| > 1.5, \end{cases}
$$

$$b_{32}^{I}(z_{s3}^{I}(t)) = \begin{cases} 0.14, & |z_{s3}^{I}(t)| \leqslant 1.5, \\ -0.25, & |z_{s3}^{I}(t)| > 1.5, \end{cases}$$

$$b_{33}^{I}(z_{s3}^{I}(t)) = \begin{cases} 0.31, & |z_{s3}^{I}(t)| \leqslant 1.5, \\ 0.25, & |z_{s3}^{I}(t)| > 1.5, \end{cases}$$

$$d_{11}^{R}(z_{s1}^{R}(t)) = \begin{cases} -0.36, & |z_{s1}^{R}(t)| \leqslant 1.5, \\ -0.15, & |z_{s1}^{R}(t)| > 1.5, \end{cases}$$

$$d_{12}^{R}(z_{s1}^{R}(t)) = \begin{cases} -0.26, & |z_{s1}^{R}(t)| \leqslant 1.5, \\ -0.25, & |z_{s1}^{R}(t)| > 1.5, \end{cases}$$

$$d_{13}^{R}(z_{s1}^{R}(t)) = \begin{cases} 0.24, & |z_{s1}^{R}(t)| \leqslant 1.5, \\ -0.13, & |z_{s1}^{R}(t)| > 1.5, \end{cases}$$

$$d_{21}^{R}(z_{s2}^{R}(t)) = \begin{cases} 0.13, & |z_{s2}^{R}(t)| \leqslant 1.5, \\ 0.25, & |z_{s2}^{R}(t)| > 1.5, \end{cases}$$

$$d_{22}^{R}(z_{s2}^{R}(t)) = \begin{cases} -0.25, & |z_{s2}^{R}(t)| \leqslant 1.5, \\ -0.17, & |z_{s2}^{R}(t)| > 1.5, \end{cases}$$

$$d_{23}^{R}(z_{s2}^{R}(t)) = \begin{cases} -0.12, & |z_{s2}^{R}(t)| \leqslant 1.5, \\ 0.23, & |z_{s2}^{R}(t)| > 1.5, \end{cases}$$

$$d_{31}^{R}(z_{s3}^{R}(t)) = \begin{cases} -0.25, & |z_{s3}^{R}(t)| \leqslant 1.5, \\ 0.27, & |z_{s3}^{R}(t)| > 1.5, \end{cases}$$

$$d_{32}^{R}(z_{s3}^{R}(t)) = \begin{cases} 0.23, & |z_{s3}^{R}(t)| \leqslant 1.5, \\ 0.11, & |z_{s3}^{R}(t)| > 1.5, \end{cases}$$

$$d_{33}^{R}(z_{s3}^{R}(t)) = \begin{cases} -0.13, & |z_{s3}^{R}(t)| \leqslant 1.5, \\ -0.14, & |z_{s3}^{R}(t)| > 1.5, \end{cases}$$

$$d_{11}^{I}(z_{s1}^{I}(t)) = \begin{cases} -0.16, & |z_{s1}^{I}(t)| \leqslant 1.5, \\ -0.26, & |z_{s1}^{I}(t)| > 1.5, \end{cases}$$

$$d_{12}^{I}(z_{s1}^{I}(t)) = \begin{cases} 0.26, & |z_{s1}^{I}(t)| \leqslant 1.5, \\ 0.15, & |z_{s1}^{I}(t)| > 1.5, \end{cases}$$

$$d_{13}^{I}(z_{s1}^{I}(t)) = \begin{cases} 0.21, & |z_{s1}^{I}(t)| \leqslant 1.5, \\ -0.13, & |z_{s1}^{I}(t)| > 1.5, \end{cases}$$

$$d_{21}^{I}(z_{s2}^{I}(t)) = \begin{cases} 0.22, & |z_{s2}^{I}(t)| \leqslant 1.5, \\ 0.28, & |z_{s2}^{I}(t)| > 1.5, \end{cases}$$

$$d_{22}^{I}(z_{s2}^{I}(t)) = \begin{cases} -0.24, & |z_{s2}^{I}(t)| \leqslant 1.5, \\ 0.16, & |z_{s2}^{I}(t)| > 1.5, \end{cases}$$

$$d_{23}^{I}(z_{s2}^{I}(t)) = \begin{cases} 0.22, & |z_{s2}^{I}(t)| \leqslant 1.5, \\ -0.11, & |z_{s2}^{I}(t)| > 1.5, \end{cases}$$

$$d_{31}^{I}(z_{s3}^{I}(t)) = \begin{cases} -0.11, & |z_{s3}^{I}(t)| \leqslant 1.5, \\ -0.16, & |z_{s3}^{I}(t)| > 1.5, \end{cases}$$

$$d_{32}^{I}(z_{s3}^{I}(t)) = \begin{cases} -0.24, & |z_{s3}^{I}(t)| \leqslant 1.5, \\ 0.11, & |z_{s3}^{I}(t)| > 1.5, \end{cases}$$

$$d_{33}^{I}(z_{s3}^{I}(t)) = \begin{cases} -0.34, & |z_{s3}^{I}(t)| \leqslant 1.5, \\ 0.25, & |z_{s3}^{I}(t)| > 1.5, \end{cases}$$

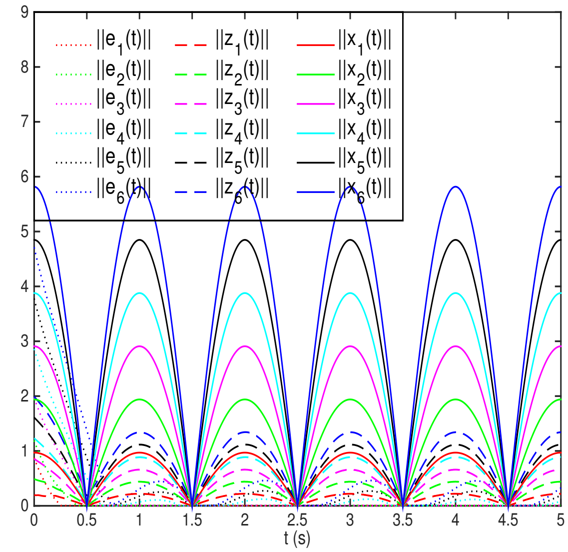

Fig. 1. The norms of $e_s(t)$, $z_s(t)$, $x_s(t)$, $s = 1, 2, \cdots, 6$.

$$H = \begin{pmatrix} -0.6 & 0.2 & 0.1 & 0 & 0.2 & 0.1 \\ 0.2 & -0.7 & 0.1 & 0 & 0.3 & 0.1 \\ 0.1 & 0.1 & -0.9 & 0.1 & 0.5 & 0.1 \\ 0 & 0 & 0.1 & -0.4 & 0.2 & 0.1 \\ 0.2 & 0.3 & 0.5 & 0.2 & -1.4 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.2 & -0.6 \end{pmatrix}.$$

Obviously, $f_i^R(\cdot)$, $f_i^I(\cdot)$, $h_i^R(\cdot)$, and $h_i^I(\cdot)(i = 1, 2, 3)$ satisfy Assumption 1 with $F_i^R = F_i^I = H_s^R = H_s^I = 0.5$ and $l_s^R = l_s^I = \eta_s^R = \eta_s^I = 0.5$. The input $x_{s1}(t) = 0.6s\cos(t) + i0.3s\cos(t)$, $x_{s2}(t) = 0.2s\cos(t) + i0.2s\cos(t)$, $x_{s3}(t) = 0.4s\cos(t) + i0.5s\cos(t)$. The parameters in the controller (7) are chosen as follows: $\Upsilon^R = \mathrm{diag}(0.9, 1.2, 0.6)$, $\Upsilon^I = \mathrm{diag}(0.8, 1.0, 0.9)$. Take W_1 and W_2 as follows:

$$W_1 = \begin{pmatrix} 0 & 0.4 & 0 \\ 0.2 & -0.5 & 0.2 \\ 0.3 & 0 & -0.2 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -0.1 & 0.2 & 0.3 \\ 0 & -0.4 & 0.1 \\ 0.2 & 0.5 & -0.3 \end{pmatrix}.$$

By using the MATLAB, it is easy to obtain that $\rho = 3.7220$ which satisfies the condition (8). On the basis of Theorem III.1, the system (36) is passive under controller (7). Fig. 1 shows the evolutions of error, output and input of six nodes when the system (36) is passive. Similarly, through a simple operation based on the above parameters by utilizing the MATLAB, we can obtain

$$\begin{aligned} \lambda(\Psi_1^R) = \{ & -1.6951, -1.5403, -1.4313, -1.4060, -1.3738, \\ & -1.2863, -1.2558, -1.2236, -1.1933, -1.1625, \\ & -1.1547, -1.0496, -1.0448, -1.0245, -0.9988, \\ & -0.9288, -0.9011, -0.8297\}, \\ \lambda(\Psi_1^I) = \{ & -1.8403, -1.5951, -1.5236, -1.4933, -1.4547, \\ & -1.3496, -1.3313, -1.3060, -1.2738, -1.2011, \\ & -1.1863, -1.0625, -1.0558, -0.8448, -0.8245, \end{aligned}$$

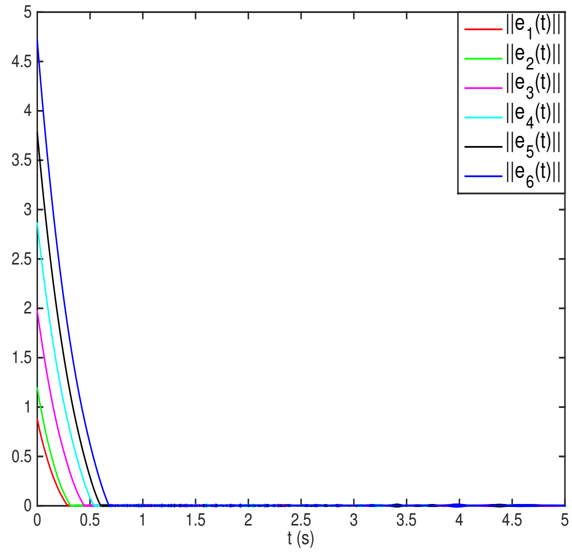

Fig. 2. Time evolutions of $e_s(t),\ s = 1, 2, \cdots, 6$.

$$- 0.7988, -0.7288, -0.6297\},$$

which satisfy the condition (29). According to Theorem III.2, the system (36) achieves synchronization. Fig. 2 depicts the simulation result of synchronization.

V. Conclusion

This study has concerned with a type of CCVMNNs. By using certain inequalities, Lyapunov functionals as well as the design of suitable controller, a novel criterion for ensuring passivity of the considered network has been derived. Similarly, we have also carried out some discussion on the synchronization of CCVMNNs. A simulation example has been performed to confirm the correctness of our results at the end.

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