

Truncated Euler-Maruyama method for classical and time-changed non-autonomous stochastic differential equations

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Abstract

The truncated Euler-Maruyama (EM) method is proposed to approximate a class of non-autonomous stochastic differential equations (SDEs) with the Hölder continuity in the temporal variable and the super-linear growth in the state variable. The strong convergence with the convergence rate is proved. Moreover, the strong convergence of the truncated EM method for a class of highly non-linear time-changed SDEs is studied.

Keywords: Truncated Euler-Maruyama method, non-autonomous stochastic differential equations, strong convergence, super-linear coefficients, time-changed stochastic differential equations.

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1. Introduction

2 Stochastic differential equations (SDEs) have broad applications in many
3 areas such as finance, physics, chemistry and biology [1, 29]. However, most
4 SDEs do not have the explicit expressions of the true solutions. Therefore,
5 the numerical methods and the rigorous numerical analyses of those methods
6 become extremely important [17, 27].

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In this paper, we investigate the numerical approximation to the solutions of a class of non-autonomous stochastic differential equations of the Itô type

$$\begin{cases} dx(t) = \mu(t, x(t)) dt + \sum_{r=1}^m \sigma^r(t, x(t)) dW^r(t), & t \in [t_0, T], \\ x(t_0) = x_0, \end{cases}$$

7 where the coefficients obey the Hölder continuity in the temporal variable and
8 the super-linear growth condition in the state variable. The detailed mathemat-
9 ical descriptions can be found in Section 2.

10 For non-autonomous SDEs with the Hölder continuous temporal variable in
11 the coefficients, the randomized techniques are used to construct the Euler type
12 method [30] and the Milstein type method [19]. However, most papers that in-
13 vestigate non-autonomous SDEs only consider the global Lipschitz condition for
14 the state variable. Thus, one aim of this paper is to study the non-autonomous
15 SDEs whose coefficients may grow super-linearly in the state variable, known
16 as highly non-linear SDEs.

17 The classic Euler-Maruyama (EM) method has been proved divergent for
18 highly non-linear SDEs [11]. While bearing in mind the idea that explicit meth-
19 ods have their advantages in simple algorithm structure and relatively lower
20 computational cost in the simulations of a large number of sample paths [9],
21 the tamed Euler method [12] and the truncated EM method [24] are developed
22 to approximate the solutions of highly non-linear SDEs. Some other interest-
23 ing works on explicit methods for highly non-linear SDEs are, for example,
24 [3, 7, 8, 10, 13, 20, 31, 33, 35, 36, 38] and the references therein. However, those
25 explicit methods proposed to tackle the super-linearity in the state variable do
26 not take the non-autonomous SDEs into consideration.

27 When both the Hölder continuity in the temporal variable and the super-
28 linearity in the state variable appear together in one SDE, few works have been
29 done on the numerical approximation to its solution. To fill up this gap, we
30 investigate the truncated EM method for this type of SDEs in this paper.

31 The time-changed SDEs, where the time variable t is replaced by some
32 stochastic process $E(t)$ (see Section 4 for the details), have attracted lots of

33 attentions in recent years [5, 22, 25, 28, 32, 34, 37]. Due to the change of the
 34 time, the solution to the time-changed SDE is understood as a subdiffusion
 35 process, which could be used to describe diffusion phenomena that move slower
 36 than the Brownian motion [2, 26]. Numerical approximations to such type of
 37 SDEs are also important, as the explicit forms of the true solutions are rarely
 38 obtained. Only recently, authors in [16] studied the classical EM method for a
 39 class of time-changed SDEs, both of whose drift and diffusion coefficients satisfy
 40 the global Lipschitz condition. To our best knowledge, [16] is the first paper
 41 to investigate the numerical approximation to time-changed SDEs by directly
 42 discretising the equations. More recently, the semi-implicit EM method was
 43 proposed in [4] to approximate some time-changed SDEs with the global Lips-
 44 chitz condition on the drift coefficient being replaced by the one-sided Lipschitz
 45 condition. Both of those two works used the duality principle proposed in [18],
 46 which, briefly speaking, relates the time-changed SDEs to certain kind of SDEs
 47 (see Section 4 for more details). In [14], the authors investigated the classical
 48 EM method for a larger class of time-changed SDEs without the application of
 49 the duality principle, though the drift and diffusion coefficients still satisfy the
 50 global Lipschitz condition. All the three works [4, 14, 16] investigated either the
 51 L^1 or L^2 convergence.

In this paper, the truncated EM method is used to approximate a class of
 time-changed SDEs of the form

$$dy(t) = \mu(E(t), y(t))dE(t) + \sigma(E(t), y(t))dW(E(t)).$$

52 To our best knowledge, this is the first work devoted to numerical approxima-
 53 tions to time-changed SDEs, whose drift and diffusion coefficients are allowed
 54 to grow super-linearly. Moreover, we consider the $L^{\bar{q}}$ convergence for any $\bar{q} \geq 2$.

55 The main contributions of our work are as follows.

- 56 • The truncated EM method, which is an explicit method, is proved to be
 57 convergent to SDEs with the Hölder continuity in the temporal variable
 58 and the super-linearity in the state variable.

- 59 • The convergence rate of $\min\{\alpha, \gamma, \frac{1}{2} - \varepsilon\}$ is given, where α and γ are the
60 Hölder continuity indexes in the drift and diffusion coefficients, and $\varepsilon > 0$
61 could be arbitrarily small.
- 62 • The strong convergence of the truncated EM method for a class of time-
63 changed SDEs, whose coefficients can grow super-linearly, is proved.

64 The paper is constructed as follows. Section 2 briefly introduces the truncated
65 EM method and some useful lemmas. The strong convergence with the rate
66 for classical SDEs is presented and proved in Section 3. The truncated EM
67 method for time-changed SDEs is discussed in 4. Numerical examples are given
68 in Section 5 to demonstrate the theoretical results.

69 2. Mathematical preliminaries

70 This section is divided into three parts. In Section 2.1, the notations and
71 assumptions are introduced. To keep the paper self-contained, the truncated
72 EM method is briefed in Section 2.2. Some useful lemmas are presented in
73 Section 2.3.

74 2.1. Notations and assumptions

75 Throughout this paper, unless otherwise specified, we let $(\Omega_W, \mathcal{F}^W, \mathbb{P}_W)$ be
76 a complete probability space with a filtration $\{\mathcal{F}_t^W\}_{t \in [0, T]}$ satisfying the usual
77 conditions (that is, it is right continuous and increasing while \mathcal{F}_0^W contains all
78 \mathbb{P}_W -null sets), and let \mathbb{E}_W denote the probability expectation with respect to
79 \mathbb{P}_W . If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. Let x^T denote the transposition
80 of x . Moreover, for two real numbers a and b , we use $a \vee b = \max(a, b)$ and
81 $a \wedge b = \min(a, b)$.

82 For $d, m \in \mathbb{N}$, let $W : [t_0, T] \times \Omega_W \rightarrow \mathbb{R}^m$ be a standard $\{\mathcal{F}_t^W\}_{t \in [t_0, T]}$ -Wiener
83 process. Moreover, let $x : [t_0, T] \times \Omega_W \rightarrow \mathbb{R}^d$ be an $\{\mathcal{F}_t^W\}_{t \in [t_0, T]}$ -adapted
84 stochastic process that is a solution to Itô type stochastic differential equation

$$\begin{cases} dx(t) = \mu(t, x(t)) dt + \sum_{r=1}^m \sigma^r(t, x(t)) dW^r(t), & t \in [t_0, T], \\ x(t_0) = x_0, \end{cases} \quad (1)$$

85 where $\mathbb{E}_W |x_0|^p < \infty$ for any $p > 0$, the drift coefficient function $\mu : [t_0, T] \times$
 86 $\mathbb{R}^d \rightarrow \mathbb{R}^d$ and the diffusion coefficient function $\sigma^r : [t_0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ for
 87 $r \in \{1, 2, \dots, m\}$.

88 We impose the following assumptions on the drift and diffusion coefficients.

Assumption 2.1. *Assume that there exist positive constants β and M such that*

$$|\mu(t, x) - \mu(t, y)| \vee |\sigma^r(t, x) - \sigma^r(t, y)| \leq M(1 + |x|^\beta + |y|^\beta)|x - y|,$$

89 for all $t \in [t_0, T]$, any $x, y \in \mathbb{R}^d$ and any $r \in \{1, 2, \dots, m\}$.

90 It can be observed from Assumption 2.1 that all $t \in [t_0, T]$, $r \in \{1, 2, \dots, m\}$
 91 and $x \in \mathbb{R}^d$

$$|\mu(t, x)| \vee |\sigma^r(t, x)| \leq K(1 + |x|^{\beta+1}), \quad (2)$$

92 where K depends on M and $\sup_{t_0 \leq t \leq T} (|\mu(t, 0)| + \max_{1 \leq r \leq m} |\sigma^r(t, 0)|)$.

Assumption 2.2. *Assume that there exists a pair of constants $q > 2$ and $L_1 > 0$ such that*

$$(x - y)^T (\mu(t, x) - \mu(t, y)) + \frac{q-1}{2} \sum_{r=1}^m |\sigma^r(t, x) - \sigma^r(t, y)|^2 \leq L_1 |x - y|^2,$$

93 for all $t \in [t_0, T]$ and any $x, y \in \mathbb{R}^d$.

Assumption 2.3. *Assume that there exists a pair of constants $p > 2$ and $L_2 > 0$ such that*

$$x^T \mu(t, x) + \frac{p-1}{2} \sum_{r=1}^m |\sigma^r(t, x)|^2 \leq L_2(1 + |x|^2), \quad (3)$$

94 for all $t \in [t_0, T]$ and any $x \in \mathbb{R}^d$.

95 **Remark 2.4.** It is clear that Assumption 2.3 may be derived from Assumption
 96 2.2 but with more complicated coefficient in front of $|\sigma^r(t, x)|^2$. To keep the
 97 notation simple, we state Assumption 2.3 as a new assumption.

98 **Assumption 2.5.** Assume that there exist constants $\gamma \in (0, 1]$, $\alpha \in (0, 1]$,
 99 $K_1 > 0$ and $K_2 > 0$ such that

$$\begin{aligned} |\mu(t_1, x) - \mu(t_2, x)| &\leq K_1(1 + |x|^{\beta+1})|t_1 - t_2|^\gamma, \\ |\sigma^r(t_1, x) - \sigma^r(t_2, x)| &\leq K_2(1 + |x|^{\beta+1})|t_1 - t_2|^\alpha, \end{aligned}$$

100 for all $t \in [t_0, T]$, any $x \in \mathbb{R}^d$ and any $r \in \{1, 2, \dots, m\}$, where the β is the
 101 same as that in Assumption 2.1.

102 2.2. The truncated EM method for non-autonomous SDEs

103 This part is to recall the truncated EM numerical scheme. To define the
 104 truncated EM numerical solutions with time t , we choose a strictly increasing
 105 continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f(u) \rightarrow \infty$ as $u \rightarrow \infty$ and

$$\sup_{t_0 \leq t \leq T} \sup_{|x| \leq u} (|\mu(t, x)| \vee |\sigma(t, x)|) \leq f(u), \quad \forall u \geq 1.$$

Denote by f^{-1} the inverse function of f . It is clear that f^{-1} is a strictly increasing continuous function from $[f(0), \infty)$ to \mathbb{R}_+ . We also choose a constant $\hat{h} \geq 1 \wedge |f(1)|$ and a strictly decreasing function $\kappa : (0, 1] \rightarrow [f(1), \infty)$ such that

$$\lim_{\Delta \rightarrow 0} \kappa(\Delta) = \infty, \quad \Delta^{\frac{1}{4}} \kappa(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1].$$

For a given step size $\Delta \in (0, 1]$ let us define the truncated mapping $\pi_\Delta : \mathbb{R}^d \rightarrow \{x \in \mathbb{R}^d : |x| \leq f^{-1}(\kappa(\Delta))\}$ by

$$\pi_\Delta(x) = \left(|x| \wedge f^{-1}(\kappa(\Delta)) \right) \frac{x}{|x|},$$

106 where we set $x/|x| = 0$ when $x = 0$.

Define the truncated functions by

$$\mu_\Delta(t, x) = \mu(t, \pi_\Delta(x)), \quad \sigma_\Delta(t, x) = \sigma(t, \pi_\Delta(x)), \quad (4)$$

for $x \in \mathbb{R}^d$. It is easy to see that for any $t \in [t_0, T]$ and all $x \in \mathbb{R}^d$

$$|\mu_\Delta(t, x)| \vee |\sigma_\Delta(t, x)| \leq f(f^{-1}(\kappa(\Delta))) = \kappa(\Delta).$$

The discrete-time truncated EM numerical solutions $x_\Delta(t_k)$, to approximate $x(t_k)$ for $t_k = k\Delta + t_0$, are formed by setting $x_\Delta(t_0) = x_0$ and computing

$$x_\Delta(t_{k+1}) = x_\Delta(t_k) + \mu_\Delta(t_k, x_\Delta(t_k))\Delta + \sum_{r=1}^m \sigma_\Delta^r(t_k, x_\Delta(t_k))\Delta W_k^r,$$

107 for $k = 0, 1, \dots, N_\Delta$, where N_Δ is the integer part of T/Δ and we will set
 108 $t_{N_\Delta+1} = T$ while $\Delta W_k^r = W^r(t_{k+1}) - W^r(t_k)$.

To form the continuous versions of truncated EM numerical schemes, we define

$$\tau(t) = \sum_{k=0}^{N_\Delta} t_k I_{[t_k, t_{k+1})}(t), \quad t \in [t_0, T].$$

There are two versions of the continuous-time truncated EM solutions. The first one is defined by

$$\bar{x}_\Delta(t) = \sum_{k=0}^{N_\Delta} x_\Delta(t_k) I_{[t_k, t_{k+1})}(t),$$

which is a simple step process. The other one is defined by

$$x_\Delta(t) = x_0 + \int_{t_0}^t \mu_\Delta(\tau(s), \bar{x}_\Delta(s)) ds + \sum_{r=1}^m \int_{t_0}^t \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s)) dW^r(s),$$

109 which is continuous in $t \in [t_0, T]$.

110 2.3. Some useful lemmas

111 In this subsection, some lemmas that will be essential for the proof of the
 112 main result in Section 3 are presented. The proofs of these lemmas are either
 113 straightforward or can be found in references. Therefore, to focus our attention
 114 on the proof of the main result, those lemmas are stated without proofs.

Lemma 2.6. *Let Assumptions 2.1 and 2.3 hold. The SDE (1) has a unique global solution $x(t)$. Moreover,*

$$\sup_{t_0 \leq t \leq T} \mathbb{E}_W |x(t)|^p < \infty.$$

115 The proof of the above lemma can be found in , for example, Theorem 4.1
 116 in Page 59 of [23].

Lemma 2.7. For any $\Delta \in (0, 1]$ and any $\bar{p} > 0$, we have

$$\mathbb{E}_W |x_\Delta(t) - \bar{x}_\Delta(t)|^{\bar{p}} \leq C_{\bar{p}} \Delta^{\frac{\bar{p}}{2}} (\kappa(\Delta))^{\bar{p}}, \quad \forall t \in [t_0, T],$$

where $C_{\bar{p}}$ is a positive constant dependent only on \bar{p} . Consequently

$$\lim_{\Delta \rightarrow 0} \mathbb{E}_W |x_\Delta(t) - \bar{x}_\Delta(t)|^{\bar{p}} = 0, \quad \forall t \in [t_0, T].$$

Lemma 2.8. Let Assumptions 2.1 and 2.3 hold. Then

$$\sup_{0 < \Delta \leq 1} \sup_{t_0 \leq t \leq T} \mathbb{E}_W |x_\Delta(t)|^p \leq C,$$

117 where C is a positive constant independent of Δ .

118 From now on, the constants C, C_1, C_2, C_3, C_{31} and C_{32} stand for generic
119 positive constants that are independent of Δ and their values may change be-
120 tween occurrences.

121 The proofs of Lemmas 2.7 and 2.8 follow straightforwardly from the proofs
122 of Lemmas 3.1 and 3.2 in [24], by substituting $\mu_\Delta(t, \bar{x}_\Delta(s))$ and $\sigma_\Delta(t, \bar{x}_\Delta(s))$
123 for $\mu_\Delta(\bar{x}_\Delta(s))$ and $\sigma_\Delta(\bar{x}_\Delta(s))$, respectively.

Remark 2.9. From Lemma 2.8, it is easily obtained that

$$\sup_{0 < \Delta < 1} \sup_{t_0 \leq t \leq T} \mathbb{E}_W |\bar{x}_\Delta(t)|^p \leq C.$$

124 3. Main results on classical SDEs

125 In this section, the strong convergence of the truncated EM method is proved
126 and the convergence rate is given. The main theorem of this paper is as follows.

Theorem 3.1. Let Assumptions 2.1, 2.2 and 2.5 hold. In addition, assume that
(3) in Assumption 2.3 is true for any $p > 2$. Then for any $\bar{q} \geq 2$, $\Delta \in (0, 1]$
and any $\varepsilon \in (0, 1/4)$,

$$\sup_{t_0 \leq t \leq T} \mathbb{E}_W |x(t) - x_\Delta(t)|^{\bar{q}} \leq C \Delta^{\min(\gamma, \alpha, \frac{1}{2} - \varepsilon)\bar{q}},$$

and

$$\sup_{t_0 \leq t \leq T} \mathbb{E}_W |x(t) - \bar{x}_\Delta(t)|^{\bar{q}} \leq C \Delta^{\min(\gamma, \alpha, \frac{1}{2} - \varepsilon)\bar{q}}.$$

127 **Remark 3.2.** To obtain the results hold for any $\bar{q} \geq 2$ and arbitrarily $\varepsilon \in$
128 $(0, 1/4)$, Assumption 2.3 is strengthened by requiring (3) to hold for any $p > 2$
129 instead of some $p > 2$ in Theorem 3.1. In this circumstance, the L_2 in (3) is no
130 longer dependent on p .

131 To prove Theorem 3.1, we show Theorem 3.3 firstly, in which the format of
132 the convergence rate is a bit complicated. The proof of Theorem 3.1 is postponed
133 after the proof of the following theorem.

134 It should be noted that in Theorem 3.3 the Assumption 2.3 is not required
135 to be strengthened compared with Theorem 3.1.

Theorem 3.3. *Let Assumptions 2.1, 2.2, 2.3 and 2.5 hold with $p > (1 + \beta)q$.
Then, for any $\bar{q} \in [2, q]$ and $\Delta \in (0, 1]$*

$$\mathbb{E}_W |x(t) - x_\Delta(t)|^{\bar{q}} \leq C \left((f^{-1}(\kappa(\Delta)))^{[(1+\beta)\bar{q}-p]} + \Delta^{\bar{q}/2} (\kappa(\Delta))^{\bar{q}} + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right), \quad (5)$$

and

$$\mathbb{E}_W |x(t) - \bar{x}_\Delta(t)|^{\bar{q}} \leq C \left((f^{-1}(\kappa(\Delta)))^{[(1+\beta)\bar{q}-p]} + \Delta^{\bar{q}/2} (\kappa(\Delta))^{\bar{q}} + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right). \quad (6)$$

136 *Proof.* Fix $\bar{q} = [2, q]$ and $\Delta \in (0, 1]$ arbitrarily. Let $e_\Delta(t) = x(t) - x_\Delta(t)$ for
137 $t \in [t_0, T]$. By the Itô formula, we have for any $t_0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E}_W |e_\Delta(t)|^{\bar{q}} &\leq \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(s) [\mu(s, x(s)) - \mu_\Delta(\tau(s), \bar{x}_\Delta(s))] \right. \\ &\quad \left. + \frac{\bar{q}-1}{2} \sum_{r=1}^m |\sigma^r(s, x(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2 \right) ds. \end{aligned} \quad (7)$$

138 By the Young inequality $2ab \leq \varepsilon a^2 + b^2/\varepsilon$ for any $a, b \geq 0$ and ε arbitrary,

139 choosing $\varepsilon = (q - \bar{q})/(\bar{q} - 1)$ leads to

$$\begin{aligned}
& \frac{\bar{q} - 1}{2} \sum_{r=1}^m |\sigma^r(s, x(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2 \\
\leq & \frac{\bar{q} - 1}{2} \sum_{r=1}^m \left(\left(1 + \frac{q - \bar{q}}{\bar{q} - 1}\right) |\sigma^r(s, x(s)) - \sigma^r(s, x_\Delta(s))|^2 \right. \\
& \left. + \left(1 + \frac{\bar{q} - 1}{q - \bar{q}}\right) |\sigma^r(s, x_\Delta(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2 \right) \\
= & \frac{q - 1}{2} \sum_{r=1}^m |\sigma^r(s, x(s)) - \sigma^r(s, x_\Delta(s))|^2 \\
& + \frac{(\bar{q} - 1)(q - 1)}{2(q - \bar{q})} \sum_{r=1}^m |\sigma^r(s, x_\Delta(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2.
\end{aligned}$$

140 We can get from (7) that

$$\begin{aligned}
& \mathbb{E}_W |e_\Delta(t)|^{\bar{q}} \\
\leq & \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(s) [\mu(s, x(s)) - \mu(s, x_\Delta(s))] \right. \\
& \left. + \frac{q - 1}{2} \sum_{r=1}^m [\sigma^r(s, x(s)) - \sigma^r(s, x_\Delta(s))]^2 \right) ds \\
& + \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} e_\Delta^T(t) [\mu(s, x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))] ds \\
& + \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} e_\Delta^T(t) [\mu(\tau(s), x_\Delta(s)) - \mu_\Delta(\tau(s), \bar{x}_\Delta(s))] ds \\
& + \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \frac{(\bar{q} - 1)(q - 1)}{(q - \bar{q})} \sum_{r=1}^m |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 ds \\
& + \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \frac{(\bar{q} - 1)(q - 1)}{(q - \bar{q})} \sum_{r=1}^m |\sigma^r(\tau(s), x_\Delta(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2 ds.
\end{aligned}$$

This implies

$$\mathbb{E}_W |e_\Delta(t)|^{\bar{q}} \leq I_1 + I_2 + I_3,$$

141 where

$$\begin{aligned}
I_1 &= \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(s) [\mu(s, x(s)) - \mu(s, x_\Delta(s))] \right. \\
& \left. + \frac{q - 1}{2} \sum_{r=1}^m [\sigma^r(s, x(s)) - \sigma^r(s, x_\Delta(s))]^2 \right) ds,
\end{aligned}$$

$$\begin{aligned}
I_2 &= \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(s) [\mu(s, x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))] \right. \\
&\quad \left. + \frac{(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 \right) ds,
\end{aligned}$$

142 and

$$\begin{aligned}
I_3 &= \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(s) [\mu(\tau(s), x_\Delta(s)) - \mu_\Delta(\tau(s), \bar{x}_\Delta(s))] \right. \\
&\quad \left. + \frac{(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m |\sigma^r(\tau(s), x_\Delta(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2 \right) ds.
\end{aligned}$$

By Assumption 2.2, we have

$$I_1 \leq C_1 \mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds, \quad (8)$$

143 where $C_1 = L_1 \bar{q}$. Using the Young inequality and Assumption 2.5, we can derive

$$\begin{aligned}
I_2 &\leq \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(\frac{1}{2} |e_\Delta(s)|^2 + \frac{1}{2} |\mu(s, x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))|^2 \right. \\
&\quad \left. + \frac{(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^2 \right) ds \\
&\leq C_2 \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \mathbb{E}_W \int_{t_0}^t |\mu(s, x_\Delta(s)) - \mu(\tau(s), x_\Delta(s))|^{\bar{q}} ds \right. \\
&\quad \left. + \frac{2(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m \mathbb{E}_W \int_{t_0}^t |\sigma^r(s, x_\Delta(s)) - \sigma^r(\tau(s), x_\Delta(s))|^{\bar{q}} ds \right) \\
&\leq C_2 \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \mathbb{E}_W \int_{t_0}^t K_1^{\bar{q}} (1 + |x_\Delta(s)|^{(\beta+1)\bar{q}}) \Delta^{\gamma\bar{q}} ds \right. \\
&\quad \left. + \mathbb{E}_W \int_{t_0}^t K_2^{\bar{q}} (1 + |x_\Delta(s)|^{(\beta+1)\bar{q}}) \Delta^{\alpha\bar{q}} ds \right).
\end{aligned}$$

Then by Lemma 2.8, we obtain

$$I_2 \leq C_2 \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right). \quad (9)$$

144 Rearranging I_3 gives

$$\begin{aligned}
I_3 &\leq \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(t) [\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), \bar{x}_\Delta(s))] \right. \\
&\quad \left. + \frac{2(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), \bar{x}_\Delta(s))|^2 \right) ds \\
&\quad + \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(e_\Delta^T(t) [\mu(\tau(s), \bar{x}_\Delta(s)) - \mu_\Delta(\tau(s), \bar{x}_\Delta(s))] \right. \\
&\quad \left. + \frac{2(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m |\sigma^r(\tau(s), \bar{x}_\Delta(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^2 \right) ds \\
&:= I_{31} + I_{32}. \tag{10}
\end{aligned}$$

145 By using the Young inequality and Assumption 2.1 we can show that

$$\begin{aligned}
I_{31} &\leq \mathbb{E}_W \int_{t_0}^t \bar{q} |e_\Delta(s)|^{\bar{q}-2} \left(\frac{1}{2} |e_\Delta^T(t)|^2 + \frac{1}{2} |\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), \bar{x}_\Delta(s))|^2 \right. \\
&\quad \left. + \frac{2(\bar{q}-1)(q-1)}{(q-\bar{q})} \sum_{r=1}^m |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), \bar{x}_\Delta(s))|^2 \right) ds \\
&\leq C_{31} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \mathbb{E}_W \int_{t_0}^t |\mu(\tau(s), x_\Delta(s)) - \mu(\tau(s), \bar{x}_\Delta(s))|^{\bar{q}} \right. \\
&\quad \left. + \sum_{r=1}^m |\sigma^r(\tau(s), x_\Delta(s)) - \sigma^r(\tau(s), \bar{x}_\Delta(s))|^{\bar{q}} ds \right) \\
&\leq C_{31} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds \right. \\
&\quad \left. + M \mathbb{E}_W \int_{t_0}^t (1 + |x_\Delta(s)|^{\beta\bar{q}} + |\bar{x}_\Delta(s)|^{\beta\bar{q}}) |x_\Delta(s) - \bar{x}_\Delta(s)|^{\bar{q}} ds \right).
\end{aligned}$$

146 Then, by the Hölder inequality, Lemma 2.7 and Lemma 2.8, we arrive at

$$\begin{aligned}
I_{31} &\leq C_{31} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \int_{t_0}^t (\mathbb{E}_W |x_\Delta(s) - \bar{x}_\Delta(s)|^p)^{\frac{\bar{q}}{p}} ds \right) \\
&\leq C_{31} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \Delta^{\frac{\bar{q}}{2}} (\kappa(\Delta))^{\bar{q}} ds \right). \tag{11}
\end{aligned}$$

147 Similarly, we can show that

$$\begin{aligned}
I_{32} &\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \mathbb{E}_W \int_{t_0}^t |\mu(\tau(s), \bar{x}_\Delta(s)) - \mu_\Delta(\tau(s), \bar{x}_\Delta(s))|^{\bar{q}} \right. \\
&\quad \left. + \sum_{r=1}^m |\sigma^r(\tau(s), \bar{x}_\Delta(s)) - \sigma_\Delta^r(\tau(s), \bar{x}_\Delta(s))|^{\bar{q}} ds \right).
\end{aligned}$$

148 Recalling the definition of truncated EM method (4) and Assumption 2.1

149 gives

$$\begin{aligned}
I_{32} &\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \mathbb{E}_W \int_{t_0}^t |\mu(\tau(s), \bar{x}_\Delta(s)) - \mu(\tau(s), \pi_\Delta(\bar{x}_\Delta(s)))|^{\bar{q}} \right. \\
&\quad \left. + \sum_{r=1}^m |\sigma^r(\tau(s), \bar{x}_\Delta(s)) - \sigma_\Delta^r(\tau(s), \pi_\Delta(\bar{x}_\Delta(s)))|^{\bar{q}} ds \right) \\
&\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds \right. \\
&\quad \left. + M \mathbb{E}_W \int_{t_0}^t (1 + |\bar{x}_\Delta(s)|^{\beta\bar{q}} + |\pi_\Delta(\bar{x}_\Delta(s))|^{\beta\bar{q}}) |\bar{x}_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))|^{\bar{q}} ds \right).
\end{aligned}$$

150 By the Hölder inequality, we obtain

$$\begin{aligned}
I_{32} &\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \int_{t_0}^t [\mathbb{E}_W (1 + |\bar{x}_\Delta(s)|^p + |\pi_\Delta(\bar{x}_\Delta(s))|^p)]^{\frac{\beta\bar{q}}{p}} \right. \\
&\quad \left. \times (\mathbb{E}_W |\bar{x}_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))|^{\frac{p\bar{q}}{p-\beta\bar{q}}})^{\frac{p-\beta\bar{q}}{p}} ds \right) \\
&\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \int_{t_0}^t (\mathbb{E}_W [I_{\{|\bar{x}_\Delta(s)| > f^{-1}(\kappa(\Delta))\}} |x_\Delta(s)|^{\frac{p\bar{q}}{p-\beta\bar{q}}}]^{\frac{p-\beta\bar{q}}{p}} ds \right) \\
&\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds \right. \\
&\quad \left. + \int_{t_0}^t ([\mathbb{P}\{|\bar{x}_\Delta(s)| > f^{-1}(\kappa(\Delta))\}]^{\frac{p-\beta\bar{q}-\bar{q}}{p-\beta\bar{q}}} [\mathbb{E}_W |\bar{x}_\Delta(s)|^p]^{\frac{\bar{q}}{p-\beta\bar{q}}})^{\frac{p-\beta\bar{q}}{p}} ds \right) \\
&\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \int_{t_0}^T \left(\frac{\mathbb{E}_W |\bar{x}_\Delta(s)|^p}{(f^{-1}(\kappa(\Delta)))^p} \right)^{\frac{p-\beta\bar{q}-\bar{q}}{p}} ds \right) \\
&\leq C_{32} \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + (f^{-1}(\kappa(\Delta)))^{(\beta+1)\bar{q}-p} \right). \tag{12}
\end{aligned}$$

Substituting (11) and (12) into (10), we arrive at

$$I_3 \leq C_3 \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + \Delta^{\frac{\bar{q}}{2}} (\kappa(\Delta))^{\bar{q}} + (f^{-1}(\kappa(\Delta)))^{(\beta+1)\bar{q}-p} \right). \tag{13}$$

Then (8), (9) and (13) together imply that

$$\mathbb{E}_W |e_\Delta(t)|^{\bar{q}} \leq C \left(\mathbb{E}_W \int_{t_0}^t |e_\Delta(s)|^{\bar{q}} ds + (f^{-1}(\kappa(\Delta)))^{(\beta+1)\bar{q}-p} + \Delta^{\frac{\bar{q}}{2}} (\kappa(\Delta))^{\bar{q}} + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right).$$

An application of the Gronwall inequality yields that

$$\mathbb{E}_W |e_\Delta(t)|^{\bar{q}} \leq C \left((f^{-1}(\kappa(\Delta)))^{(\beta+1)\bar{q}-p} + \Delta^{\frac{\bar{q}}{2}} (\kappa(\Delta))^{\bar{q}} + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right),$$

151 which is the required assertion (5). The other assertion (6) follows from (5) and
 152 Lemma 2.7. Therefore, the proof is completed. \square

153 Now, we are ready to give the proof of Theorem 3.1.

154 **Proof of Theorem 3.1**

Recalling (2), we then define

$$f(u) = Ku^{\beta+2}, \quad u \geq 1,$$

which implies that

$$f^{-1}(u) = \left(\frac{u}{K}\right)^{\frac{1}{\beta+2}}.$$

Let

$$\kappa(\Delta) = \Delta^{-\varepsilon} \text{ for some } \varepsilon \in (0, \frac{1}{4}) \text{ and } \hat{h} \geq 1.$$

Following Theorem 3.3, we obtain

$$\mathbb{E}_W |x(t) - x_\Delta(t)|^{\bar{q}} \leq C \left(\Delta^{\frac{\varepsilon(p - \beta\bar{q} - \bar{q})}{\beta+2}} + \Delta^{\frac{\bar{q}(1-2\varepsilon)}{2}} + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right), \quad (14)$$

and

$$\mathbb{E}_W |x(t) - \bar{x}_\Delta(t)|^{\bar{q}} \leq C \left(\Delta^{\frac{\varepsilon(p - \beta\bar{q} - \bar{q})}{\beta+2}} + \Delta^{\frac{\bar{q}(1-2\varepsilon)}{2}} + \Delta^{\gamma\bar{q}} + \Delta^{\alpha\bar{q}} \right). \quad (15)$$

Choosing p sufficiently large for

$$\frac{\varepsilon(p - \beta\bar{q} - \bar{q})}{\beta + 2} > \min(\gamma, \alpha, \frac{1}{2} - \varepsilon)\bar{q},$$

155 we can draw the assertions from (14) and (15) immediately. \square

156 **4. Main results on time-changed SDEs**

157 This section is divided into two parts. In Section 4.1, mathematical pre-
 158 liminaries about time-changed SDEs are presented together with some useful
 159 lemmas. The result on the strong convergence of the truncated EM method is
 160 presented in Section 4.2.

161 *4.1. Mathematical preliminaries for time-changed SDEs*

Let $D(t)$ be a right continuous with the existence of the left limit increasing Lévy process defined on a complete probability space $(\Omega_D, \mathcal{F}^D, \mathbb{P}_D)$ with a filtration $\{\mathcal{F}_t^D\}_{t \geq 0}$ satisfying the usual conditions. Let \mathbb{E}_D denote the expectation under the probability measure \mathbb{P}_D . $D(t)$ is called subordinator starting from 0 if the Laplace transform is given by

$$\mathbb{E}_D e^{-\lambda D(t)} = e^{-t\phi(\lambda)},$$

where the Laplace exponent is

$$\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda x}) \nu(dx),$$

162 with $\int_0^\infty (x \wedge 1) \nu(dx) < \infty$. We focus on the case when the Lévy measure ν is
 163 infinity, i.e. $\nu(0, \infty) = \infty$, which implies that $D(t)$ has strictly increasing paths
 164 with infinitely many jumps and excludes the compound Poisson subordinator.

Let $E(t)$ be the inverse of $D(t)$, i.e.

$$E(t) := \inf\{u > 0; D(u) > t\}, \quad t \geq 0.$$

165 We call $E(t)$ an inverse subordinator.

Assume that $W(t)$ and $D(t)$ are independent. Define the product probability space by

$$(\Omega, \mathcal{F}, \mathbb{P}) := (\Omega_W \times \Omega_D, \mathcal{F}^W \otimes \mathcal{F}^D, \mathbb{P}_W \otimes \mathbb{P}_D).$$

166 Let \mathbb{E} denote the expectation under the probability measure \mathbb{P} . It is clear that
 167 $\mathbb{E}(\cdot) = \mathbb{E}_D \mathbb{E}_W(\cdot) = \mathbb{E}_W \mathbb{E}_D(\cdot)$.

In this section, we consider the following time-changed SDE

$$dy(t) = \mu(E(t), y(t))dE(t) + \sigma(E(t), y(t))dW(E(t)), \quad t \in [0, T], \quad (16)$$

168 with the initial value $y(0) = y_0$. Here, for the simplicity of the notation, we
 169 only consider the scale Wiener process W (i.e. $m = 1$ in Sections 2 and 3).

According to the duality principle, the time-changed SDE (16) and the classical SDE of Itô type

$$dx(t) = \mu(t, x(t))dt + \sigma(t, x(t))dW(t) \quad (17)$$

170 have a deep connection. The next theorem states such a relation more precisely,
 171 which is borrowed from Theorem 4.2 in [18].

172 **Theorem 4.1.** *Suppose Assumptions 2.1, 2.2, 2.3 and 2.5 hold. If $x(t)$ is the*
 173 *unique solution to the SDE (17), then the time-changed process $x(E(t))$, which*
 174 *is an $\mathcal{F}_{E(t)}^W$ -semimartingale, is the unique solution to the time-changed SDE*
 175 *(16). On the other hand, if $y(t)$ is the unique solution to the time-changed SDE*
 176 *(16), then the process $y(D(t))$, which is an \mathcal{F}_t^W -semimartingale, is the unique*
 177 *solution to the SDE (17).*

178 The plan to numerically approximate the time-changed SDE (16) in this
 179 section is as follows. Firstly, we discretize the inverse subordinator $E(t)$ to get
 180 $E_\Delta(t)$. Then the combination, $x_\Delta(E_\Delta(t))$, of the truncated EM solution to the
 181 SDE (17), $x_\Delta(t)$, and the discretized inverse subordinator, $E_\Delta(t)$, is used to
 182 approximate the solution to the time-changed SDE (16).

To approximate the $E(t)$ in a given time interval $[0, T]$, we follow the idea
 in [6]. Firstly, we simulate the path of $D(t)$ by $D_\Delta(t_i) = D_\Delta(t_{i-1}) + \xi_i$ with
 $D(t_0) = 0$ and $t_i = t_0 + i\Delta$, where ξ_i is independently identically sequence with
 $\xi_i = D(t_1) - D(t_0)$ in distribution. The process is stopped when

$$T \in [D_\Delta(t_n), D_\Delta(t_{n+1})),$$

for some n . Then the approximate $E_\Delta(t)$ to $E(t)$ is generated by

$$E_\Delta(t) = (\min\{n; D_\Delta(t_n) > t\} - 1)\Delta, \quad (18)$$

for $t \in [0, T]$. It is easy to see

$$E_\Delta(t) = i\Delta, \quad \text{when } t \in [D_\Delta(t_i), D_\Delta(t_{i+1})).$$

183 It is not hard to observe that $E_\Delta(t)$ leads to the uniform partition in the time-
 184 changed case as well. The next lemma provides the approximation error of
 185 $E_\Delta(t)$ to $E(t)$, whose proof can be found in Theorem 2 in [21].

Lemma 4.2. *Let $E(t)$ be the inverse of a subordinator $D(t)$ with infinite Lévy*
measure. Then for any $t \in [0, T]$

$$E(t) - \Delta \leq E_\Delta(t) \leq E(t) \quad a.s.$$

186 The following lemma states that any inverse subordinator $E(t)$ with infinite
 187 Lévy measure is known to have the exponential moment, see for example Lemma
 188 2.7 in [4].

Lemma 4.3. *Let $E(t)$ be the inverse of a subordinator $D(t)$ with Laplace exponent ϕ and infinite Lévy measure, then for any $C \in \mathbb{R}$ and $t \geq 0$,*

$$\mathbb{E}_D \left(e^{CE(t)} \right) < \infty.$$

189 We also need the continuity of the solution to (17) presented in the next
 190 lemma. The proof is not hard to obtain by using the standard approach (see
 191 for example Theorem 4.3 in Page 61 of [23]).

Lemma 4.4. *Suppose that Assumptions 2.1 and 2.3 hold. Then for any $q < p/(\beta + 1)$ and $|t - s| < 1$, the solution to (17) satisfies*

$$\mathbb{E}_W |x(t) - x(s)|^q \leq C_4 |t - s|^{q/2} e^{C_4 t},$$

192 where C_4 is a constant independent of t and s .

193 4.2. Strong convergence of the truncated EM method for time-changed SDEs

194 Before the main result is presented, we make some remarks on the constant,
 195 C , in Theorem 3.1. Since the main purpose of Theorem 3.1 is to show the
 196 convergence rate, we do not give the explicit form of the constant C . But it is
 197 not hard by going through the proof to see that $C(t) := C$ contains the time
 198 variable t only in the form of $\exp(\text{some constant} \times t)$. This means that if we
 199 replace t by $E(t)$, we have $\mathbb{E}_D(C(E(t))) < \infty$ by Lemma 4.3.

Theorem 4.5. *Let Assumptions 2.1, 2.2 and 2.5 hold. In addition, assume that (3) in Assumption 2.3 is true for any $p > 2$. Then the combination of the truncated EM solution and the discretized inverse subordinator, i.e. $x_\Delta(E_\Delta(t))$, converges strongly to the solution of (16)*

$$\mathbb{E} |y(t) - x_\Delta(E_\Delta(t))|^{\bar{q}} \leq C_{tc} \Delta^{\min(\gamma, \alpha, \frac{1}{2} - \varepsilon)\bar{q}},$$

200 for any $\bar{q} \geq 2$, $\Delta \in (0, 1]$, $\varepsilon \in (0, 1/4)$ and $t \in [0, T]$, where C_{tc} is constant
 201 independent from Δ .

Proof. By Theorem 4.1 and the elementary inequality, we have

$$\begin{aligned} & \mathbb{E} |y(t) - x_\Delta(E_\Delta(t))|^{\bar{q}} \\ &= \mathbb{E} |x(E(t)) - x_\Delta(E_\Delta(t))|^{\bar{q}} \\ &\leq 2^{\bar{q}-1} \left(\mathbb{E} |x(E(t)) - x(E_\Delta(t))|^{\bar{q}} + \mathbb{E} |x(E_\Delta(t)) - x_\Delta(E_\Delta(t))|^{\bar{q}} \right). \end{aligned}$$

By Lemmas 4.2, 4.3 and 4.4, we can see

$$\mathbb{E} |x(E(t)) - x(E_\Delta(t))|^{\bar{q}} \leq C_4 \Delta^{\bar{q}/2} \mathbb{E}_D \left(e^{C_4 E(t)} \right) \leq C_5 \Delta^{\bar{q}/2}, \quad (19)$$

where C_5 is a constant independent from Δ . By Lemma 4.2 and Theorem 3.1, we obtain

$$\mathbb{E} |x(E_\Delta(t)) - x_\Delta(E_\Delta(t))|^{\bar{q}} \leq \mathbb{E}_D(C) \Delta^{\min(\gamma, \alpha, \frac{1}{2} - \varepsilon)\bar{q}} \leq C_6 \Delta^{\min(\gamma, \alpha, \frac{1}{2} - \varepsilon)\bar{q}}, \quad (20)$$

202 where C_6 is a constant independent from Δ . Combining (19) and (20), we have
 203 the required assertion. \square

204 5. Numerical Simulations

205 This section is divided into two parts. The numerical simulations on SDEs
 206 are presented in Section 5.1 and time-changed SDEs are displayed in Section
 207 5.2.

208 5.1. Simulations for SDEs

209 Two examples with the different theoretical convergence rates are presented
 210 in this part. Computer simulations are conducted to verify the theoretical re-
 211 sults.

Example 5.1. Consider a scalar stochastic differential equation

$$\begin{cases} dx(t) = \left([t(1-t)]^{\frac{1}{4}} x^2(t) - 2x^5(t) \right) dt + \left([t(1-t)]^{\frac{1}{4}} x^2(t) \right) dW(t), \\ x(t_0) = 2, \end{cases} \quad (21)$$

212 with $t_0 = 0$ and $T = 1$.

213 For any $q > 2$, we can see

$$\begin{aligned}
& (x-y)^T(\mu(t,x) - \mu(t,y)) + \frac{q-1}{2}|\sigma^r(t,x) - \sigma^r(t,y)|^2 \\
\leq & (x-y)^2 \left([t(1-t)]^{\frac{1}{4}}(x+y) - 2(x^4 + x^3y + x^2y^2 + xy^3 + y^4) \right. \\
& \left. + \frac{q-1}{2}[t(1-t)]^{\frac{1}{2}}(x+y)^2 \right).
\end{aligned}$$

But

$$-2(x^3y + xy^3) = -2xy(x^2 + y^2) \leq (x^2 + y^2)^2 = x^4 + y^4 + 2x^2y^2.$$

214 Therefore, for any $t \in [0, 1]$

$$\begin{aligned}
& (x-y)^T(\mu(t,x) - \mu(t,y)) + \frac{q-1}{2}|\sigma^r(t,x) - \sigma^r(t,y)|^2 \\
\leq & (x-y)^2 \left([t(1-t)]^{\frac{1}{4}}(x+y) - x^4 - y^4 + (q-1)[t(1-t)]^{\frac{1}{2}}(x^2 + y^2) \right) \\
\leq & C(x-y)^2,
\end{aligned}$$

215 where the last inequality is due to the fact that polynomials with the negative
216 coefficient for the highest order term can always be bounded from above. This
217 indicates that Assumption 2.2 holds.

218 In the similar manner, for any $p > 2$ and any $t \in [0, 1]$, we have

$$\begin{aligned}
x^T \mu(t,x) + \frac{p-1}{2}|\sigma(t,x)|^2 &= (t(1-t))^{\frac{1}{4}}x^3 - 2x^6 + \frac{p-1}{2}(t(1-t))^{\frac{1}{2}}x^4 \\
&\leq C(1 + |x|^2),
\end{aligned}$$

219 which means that Assumption 2.3 is satisfied.

220 Using the mean value theorem for the temporal variable, Assumptions 2.1 and
221 2.5 are satisfied with $\alpha = \gamma = 1/4$ and $\beta = 4$. According to Theorem 3.3, we
222 know that

$$\mathbb{E}_W |x(t) - x_\Delta(t)|^{\bar{q}} \leq C \left((f^{-1}(\kappa(\Delta)))^{(5\bar{q}-p)} + \Delta^{\bar{q}/2}(\kappa(\Delta))^{\bar{q}} + \Delta^{\bar{q}/4} \right),$$

and

$$\mathbb{E}_W |x(t) - \bar{x}_\Delta(t)|^{\bar{q}} \leq C \left((f^{-1}(\kappa(\Delta)))^{(5\bar{q}-p)} + \Delta^{\bar{q}/2}(\kappa(\Delta))^{\bar{q}} + \Delta^{\bar{q}/4} \right).$$

223 Due to that

$$\sup_{t_0 \leq t \leq T} \sup_{|x| \leq u} (|\mu(t, x) \vee |\sigma(t, x)|) \leq 3u^5, \quad \forall u \geq 1,$$

we choose $f(u) = 3u^5$ and $\kappa(\Delta) = \Delta^{-\varepsilon}$, for any $\varepsilon \in (0, 1/4)$. As a result, $f^{-1}(u) = (u/3)^{1/5}$ and $f^{-1}(\kappa(\Delta)) = (\Delta^{-\varepsilon}/3)^{1/5}$. Choosing p sufficiently large, we can get from Theorem 3.1 that

$$\sup_{0 \leq t \leq 1} \mathbb{E}_W |x(t) - x_\Delta(t)|^{\bar{q}} \leq C\Delta^{\bar{q}/4},$$

and

$$\sup_{0 \leq t \leq 1} \mathbb{E}_W |x(t) - \bar{x}_\Delta(t)|^{\bar{q}} \leq C\Delta^{\bar{q}/4},$$

224 which imply that the convergence rate of truncated EM method for the SDE
 225 (21) is $1/4$.

226 Let us compute the approximation of the mean square error. We run $M =$
 227 1000 independent trajectories for every different step sizes, 10^{-1} , 10^{-2} , 10^{-3} ,
 228 10^{-8} . Because it is hard to find the true solution for the SDE, the numerical
 229 solution with the step size 10^{-8} is regarded as the exact solution.

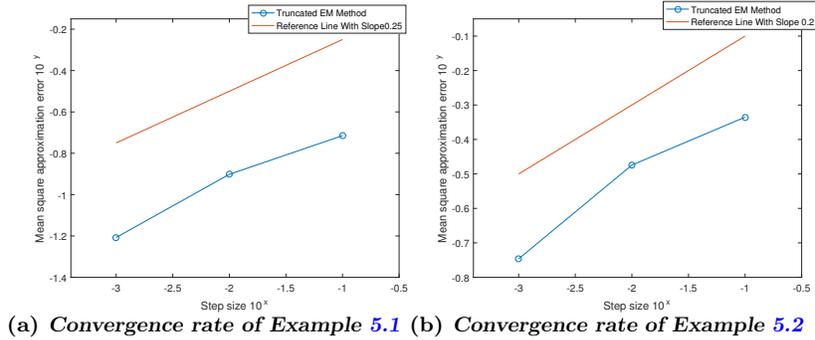


Figure 1: The L^1 errors between the exact solution and the numerical solutions for step sizes $\Delta = 10^{-1}$, 10^{-2} , 10^{-3} .

230 By the linear regression, also shown in the Figure 1(a), the slope of the
 231 errors against the step sizes is approximately 0.24629, which is quite close to
 232 the theoretical result.

Example 5.2. Consider the scalar stochastic differential equation

$$\begin{cases} dx(t) = \left([(t-1)(2-t)]^{\frac{1}{5}} x^2(t) - 2x^5(t) \right) dt + \left([(t-1)(2-t)]^{\frac{2}{5}} x^2(t) \right) dW(t), \\ x(t_0) = 2, \end{cases} \quad (22)$$

233 where $t_0 = 1$ and $T = 2$. In the similar way as Example 5.1, we can verify that
 234 Assumptions 2.2 and 2.3 hold.

235 Moreover, the mean value theorem is used to verify that Assumptions 2.1
 236 and 2.5 are satisfied with $\alpha = 2/5$, $\gamma = 1/5$ and $\beta = 4$.

We can get from Theorem 3.1 that

$$\sup_{1 \leq t \leq 2} \mathbb{E}_W |x(t) - x_{\Delta}(t)|^{\bar{q}} \leq C \Delta^{\bar{q}/5},$$

and

$$\sup_{1 \leq t \leq 2} \mathbb{E}_W |x(t) - \bar{x}_{\Delta}(t)|^{\bar{q}} \leq C \Delta^{\bar{q}/5},$$

237 which implies that the convergence rate of truncated EM method for the SDE
 238 (22) is $1/5$. Simulation is conducted using the same strategy as that in Example
 239 5.1. Using the linear regression, also seen in the figure 1(b), the slope of the
 240 errors against the step sizes is approximately 0.20550, which coincides with the
 241 theoretical result.

242 5.2. Simulations for time-changed SDEs

Example 5.3. A two-dimensional time-changed SDE

$$\begin{cases} dy_1(t) = -2y_1^4(t) dE(t) + y_2^2(t) dW(E(t)), \\ dy_2(t) = -2y_2^4(t) dE(t) + y_1^2(t) dW(E(t)), \end{cases} \quad (23)$$

243 is considered with the initial data $y_1(0) = 1$ and $y_2(0) = 2$.

244 For a given step size h , one path of the numerical solution to (23) is simulated
 245 in the following way.

Step 1. The truncated EM method with the step size Δ is used to simulate the numerical solution, $x_\Delta(t_k)$, for $k = 1, 2, 3, \dots$, to the duel SDE

$$\begin{cases} dx_1(t) = -2x_1^4(t) dt + x_2^2(t) dW(t), \\ dx_2(t) = -2x_2^4(t) dt + x_1^2(t) dW(t). \end{cases}$$

246 **Step 2.** One path of the subordinator $D(t)$ is simulated with the same step
247 size Δ . (see for example Chapter 2.2 in [15]).

248 **Step 3.** The $E_h(t)$ is found by using (18).

249 **Step 4.** The combination, $x_\Delta(E_h(t))$, is used to approximate the solution to
250 (23).

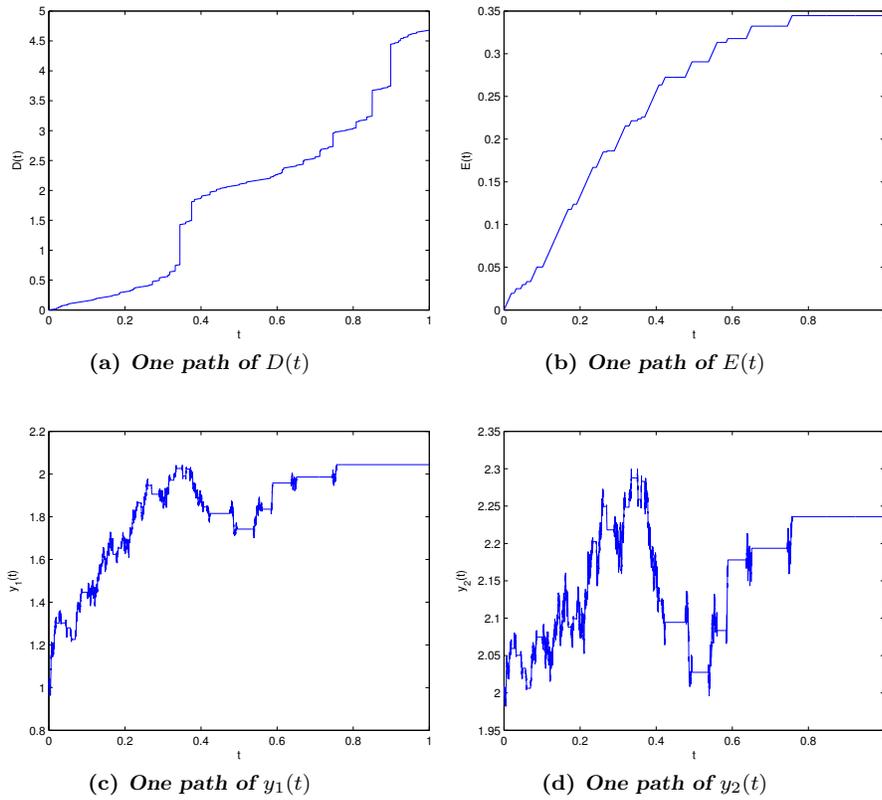


Figure 2: Numerical simulations of $D(t)$, $E(t)$, $y_1(t)$ and $y_2(t)$

251 For $t \in [0, 1]$ and $\Delta = 10^{-4}$, Figure 2(a) shows one path of $D(t)$ and Figure

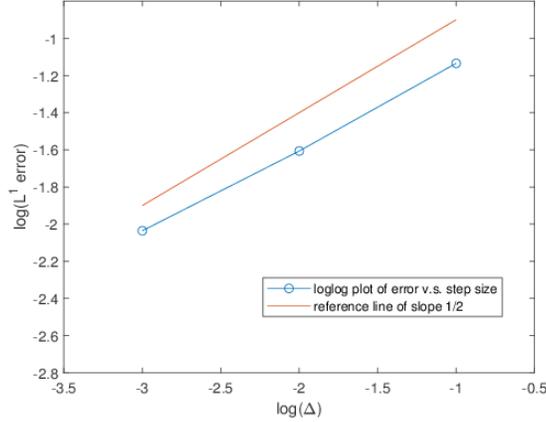


Figure 3: Blue line: Loglog plot of the strong L^1 error against the step size. Red Line: The reference line with the slope of $1/2$.

252 2(b) displays one path of $E(t)$. Paths of $y_1(t)$ and $y_2(t)$ are plotted in Figures
 253 2(c) and 2(d), respectively.

254 Now we demonstrate the strong convergence rate. Since the explicit solution
 255 is hard to obtain, we treat the numerical solution with $\Delta = 10^{-8}$ as the true
 256 solution. Two hundred samples are used to compute the strong convergence
 257 with the step sizes 10^{-1} , 10^{-2} and 10^{-3} . We pick up $\epsilon = 0.01$, by Theorem 4.5
 258 a strong convergence rate that is closed to 0.5 is expected. The loglog plot of the
 259 L^1 errors against the step sizes is shown in Figure 3. By the linear regression,
 260 the slope is 0.4506 that is not far away from the theoretical result.

261 6. Conclusion

262 In this paper, we apply the truncated EM method for a class of non-autonomous
 263 classical SDEs with the Hölder continuity in the temporal variable and the
 264 super-linear growth in the state variable. The strong convergence with the rate
 265 is proved.

266 In addition, the results on the classical SDEs are used to prove that the
 267 truncated EM method can also work well for a class of highly non-linear time-

268 changed SDEs. Such a result provides a trusted numerical method for a much
269 larger class of time-changed SDEs than those in existing works.

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