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Discrete fragmentation systems in weighted ℓ^1 spaces

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Abstract. We investigate an infinite, linear system of ordinary differential equations that models the evolution of fragmenting clusters. We assume that each cluster is composed of identical units (monomers), and we allow mass to be lost, gained or conserved during each fragmentation event. By formulating the initial-value problem for the system as an abstract Cauchy problem (ACP), posed in an appropriate weighted ℓ^1 space, and then applying perturbation results from the theory of operator semigroups, we prove the existence and uniqueness of physically relevant, classical solutions for a wide class of initial cluster distributions. Additionally, we establish that it is always possible to identify a weighted ℓ^1 space on which the fragmentation semigroup is analytic, which immediately implies that the corresponding ACP is well posed for any initial distribution belonging to this particular space. We also investigate the asymptotic behaviour of solutions and show that, under appropriate restrictions on the fragmentation coefficients, solutions display the expected long-term behaviour of converging to a purely monomeric steady state. Moreover, when the fragmentation semigroup is analytic, solutions are shown to decay to this steady state at an explicitly defined exponential rate.

1. Introduction

There are many diverse situations arising in nature and industrial processes where clusters of particles can merge together (coagulate) to produce larger clusters and can break apart (fragment) to produce smaller clusters. Particular examples can be found in polymer science [1,24,25], in the formation of aerosols [16] and in the powder production industry [21,23]. It is often appropriate when modelling such processes to regard cluster size as a discrete variable, with a cluster of size n, an n-mer, composed of n identical units (monomers). By scaling the mass, we can assume that each monomer has unit mass and so an n-mer has mass n. The aim is to use the mathematical model to obtain information on how clusters of different sizes evolve. In this paper, we restrict our attention to the case when no coagulation occurs, and consequently the evolution of clusters can be described by a linear, infinite system of ordinary differential equations. With the number density of clusters of size n (i.e. mass n) at time t denoted by $u_n(t)$, this fragmentation system is given by

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$$\begin{aligned} u_n'(t) &= -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t), \qquad t > 0; \\ u_n(0) &= \mathring{u}_n, \qquad n = 1, 2, \ldots, \end{aligned} \tag{1.1}$$

where a_n is the rate at which clusters of size n are lost, $b_{n,j}$ is the rate at which clusters of size n are produced when a larger cluster of size j fragments and $\mathring{u}_n$ is the initial density of clusters of size n at time $t = 0$. Equation (1.1) was first introduced in [25] to deal with the case of binary fragmentation, where it is assumed that each fragmentation event results in the creation of exactly two daughter clusters. As in [7,10,18,19], we consider the more general case, where each fragmentation event can result in the creation of two or more clusters. Since (1.1) is an infinite system, it is convenient to express solutions as time-dependent sequences of the form $u(t) := (u_n(t))_{n=1}^{\infty}$.

Throughout this paper, we need various assumptions on the fragmentation coefficients a_n and $b_{n,j}$. We list these assumptions here and will refer to them in the sequel when required.

Assumption 1.1. (i) For all $n \in \mathbb{N}$,

$$a_n \geq 0. \tag{1.2}$$

(ii) For all $n, j \in \mathbb{N}$,

$$b_{n,j} \geq 0 \qquad \text{and} \qquad b_{n,j} = 0 \quad \text{when } n \geq j. \tag{1.3}$$

The total mass of daughter clusters resulting from the fragmentation of a j-mer is given by $\sum_{n=1}^{j-1} n b_{n,j}$. In most papers that have dealt with discrete fragmentation systems, it is assumed that

$$\sum_{n=1}^{j-1} n b_{n,j} \leq j \qquad \text{for all } j = 2, 3, \ldots, \tag{1.4}$$

i.e. there is no increase in mass at fragmentation events. If there is strict inequality in (1.4), then mass is lost by some other mechanism. However, for most of our results we do not assume that (1.4) holds; this means that mass could even be gained at fragmentation events. We can specify the local mass loss or mass gain with real parameters λ_j, $j = 2, 3, \ldots$, such that

$$\sum_{n=1}^{j-1} n b_{n,j} = (1 - \lambda_j) j, \qquad j = 2, 3, \ldots. \tag{1.5}$$

In terms of the densities $u_n(t)$, the total mass of all clusters in the system at time t is given by the first moment, $M_1(u(t))$, of $u(t)$, where

$$M_1\big(u(t)\big) := \sum_{n=1}^{\infty} n u_n(t). \tag{1.6}$$

A formal calculation establishes that if u is a solution of (1.1), then

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1\big(u(t)\big) = -a_1u_1(t) - \sum_{j=2}^{\infty} j\lambda_j a_j u_j(t). \tag{1.7}$$

The expression in (1.7) gives the rate at which mass may be lost from the system or gained and also shows that, at least formally, the total mass is conserved when $a_1 = 0$ and $\lambda_j = 0$ for all $j = 2, 3, \ldots$, i.e. when

$$a_1 = 0 \quad \text{and} \quad \sum_{n=1}^{j-1} nb_{n,j} = j \quad \text{for all } j = 2, 3, \ldots. \tag{1.8}$$

Note that monomers cannot fragment to produce smaller clusters, and hence the case when $a_1 > 0$ is interpreted as a situation in which monomers are removed from the system.

In this paper, the approach we use to investigate (1.1) relies on the theory of semigroups of bounded linear operators and entails formulating (1.1) as an abstract Cauchy problem (ACP) in an appropriate Banach space. The existence and uniqueness of solutions to the ACP are established via the application of perturbation results for operator semigroups. Of particular relevance is the Kato–Voigt perturbation theorem for substochastic semigroups [5,22] that was first applied to (1.1) in [18], and subsequently in similar semigroup-based investigations into (1.1), such as [8,19]. We use a refined version of this theorem proved by Thieme and Voigt in [20].

In previous studies, including [18,19], the ACP associated with the fragmentation system has been formulated in the space

$$X_{[1]} := \left\{ f = (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} n|f_n| < \infty \right\}. \tag{1.9}$$

Equipped with the norm

$$\|f\|_{[1]} = \sum_{n=1}^{\infty} n|f_n|, \qquad f \in X_{[1]}, \tag{1.10}$$

$X_{[1]}$ is a Banach space, and

$$\|f\|_{[1]} = M_1(f) \tag{1.11}$$

if $f \in X_{[1]}$ is such that $f_n \geq 0$, $n \in \mathbb{N}$. This means that whenever $u : [0, \infty) \to X_{[1]}$ is a non-negative solution of the fragmentation system, the norm, $\|u(t)\|_{[1]}$, gives the total mass at time t. Other Banach spaces, with norms related to higher-order moments, have also played a prominent role [8,11], with $X_{[1]}$ being replaced by $X_{[p]}$, $p > 1$, where

$$X_{[p]} := \left\{ f = (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \|f\|_{[p]} := \sum_{n=1}^{\infty} n^p|f_n| < \infty \right\}. \tag{1.12}$$

Rather than restricting our investigations to spaces of the type $X_{[p]}$, we choose to work within the framework of more general weighted ℓ^1 spaces. As we shall demonstrate, this additional flexibility will enable us to establish desirable semigroup properties and results that may not always be possible in an $X_{[p]}$ setting. Therefore, we let $w = (w_n)_{n=1}^{\infty}$ be such that $w_n > 0$ for all $n \in \mathbb{N}$, and define

$$\ell_w^1 = \left\{ f = (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} w_n |f_n| < \infty \right\}. \tag{1.13}$$

Equipped with the norm

$$\|f\|_w = \sum_{n=1}^{\infty} w_n |f_n|, \qquad f \in \ell_w^1, \tag{1.14}$$

ℓ_w^1 is a Banach space, which we refer to as the weighted ℓ^1 space with weight w.

Motivated by the terms in (1.1), we introduce the formal expressions

$$\mathcal{A} : (f_n)_{n=1}^{\infty} \mapsto (-a_n f_n)_{n=1}^{\infty} \qquad \text{and} \qquad \mathcal{B} : (f_n)_{n=1}^{\infty} \mapsto \left(\sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \right)_{n=1}^{\infty}.$$

Operator realisations, $A^{(w)}$ and $B^{(w)}$, of $\mathcal{A}$ and $\mathcal{B}$, respectively, are defined in ℓ_w^1 by

$$A^{(w)} f = \mathcal{A} f, \qquad \mathcal{D}(A^{(w)}) = \left\{ f \in \ell_w^1 : \mathcal{A} f \in \ell_w^1 \right\} \tag{1.15}$$

and

$$B^{(w)} f = \mathcal{B} f, \qquad \mathcal{D}(B^{(w)}) = \left\{ f \in \ell_w^1 : \mathcal{B} f \in \ell_w^1 \right\}. \tag{1.16}$$

Here, and in the sequel, $\mathcal{D}(T)$ denotes the domain of the designated operator T. Similarly, we shall represent the resolvent, $(\lambda I - T)^{-1}$, of T by $R(\lambda, T)$.

An ACP version of (1.1), posed in the space ℓ_w^1, can be formulated as

$$u'(t) = A^{(w)} u(t) + B^{(w)} u(t), \quad t > 0; \qquad u(0) = \mathring{u}. \tag{1.17}$$

Note that this reformulation of (1.1) imposes additional constraints on both the initial data and the sought solutions since we now require $\mathring{u} \in \ell_w^1$ and also that the solution $u(t) \in \mathcal{D}(A^{(w)}) \cap \mathcal{D}(B^{(w)})$ for all $t > 0$. Moreover, as the derivative on the left-hand side of (1.17) is defined in terms of $\|\cdot\|_w$, it is customary to look for a solution $u \in C^1((0,\infty), \ell_w^1) \cap C([0,\infty), \ell_w^1)$. Such a solution is referred to as a classical solution of (1.17) and has the property that $\|u(t) - \mathring{u}\|_w \to 0$ as $t \to 0^+$.

It turns out that often, instead of using the operator $A^{(w)} + B^{(w)}$ on the right-hand side of (1.17), one has to use its closure, which leads to the ACP

$$u'(t) = \overline{(A^{(w)} + B^{(w)})} u(t), \quad t > 0; \qquad u(0) = \mathring{u}. \tag{1.18}$$

Yet another option for an operator on the right-hand side is the maximal operator, $G_{\mathsf{max}}^{(w)}$, which is defined by

$$G_{\mathsf{max}}^{(w)} f = \mathcal{A} f + \mathcal{B} f, \qquad \mathcal{D}\big(G_{\mathsf{max}}^{(w)}\big) = \left\{ f \in \ell_w^1 : \mathcal{A} f + \mathcal{B} f \in \ell_w^1 \right\}. \tag{1.19}$$

However, the domain of this operator is too large in general to ensure uniqueness of solutions; see Example 4.3 and also [6] where a continuous fragmentation equation is studied.

There are a number of benefits to be gained by working in more general weighted ℓ^1 spaces, least of which is the derivation of existence and uniqueness results for (1.1) in ℓ_w^1 that reduce to those established in earlier $X_{[p]}$-based investigations by choosing $w_n = n^p$. For example, in Theorem 3.4 we prove that $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup. While this result has already been shown for the specific case $w_n = n^p$ for $p \geq 1$, see [8,18], Theorem 3.4 is formulated for more general weights and is proved by means of an alternative and novel argument that is based on theory presented in [20]. Our approach also leads to an additional invariance result, which can be used to establish the existence of solutions to the fragmentation system (1.17) for a certain specified class of initial conditions.

A further major advantage of working in the more general setting of ℓ_w^1 is that it yields results on the analyticity of the related fragmentation semigroups, which do not necessarily hold in the restricted case of $w_n = n^p$, $p \geq 1$. In particular, in Theorem 5.5 we prove that, for *any* fragmentation coefficients, we can *always* find a weight w such that $A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup on ℓ_w^1. In connection with this, it should be noted that there are no known general results that guarantee the analyticity of the fragmentation semigroup on the space $X_{[1]}$. Indeed, this provided the motivation for previous investigations into fragmentation ACPs posed in higher moment spaces, which led to a sufficient condition being found in [8] for $A^{(w)} + B^{(w)}$ to generate an analytic semigroup on $X_{[p]}$ for some $p > 1$. However, simple examples are also given in [8] of fragmentation coefficients where the semigroup is not analytic in $X_{[p]}$ for any $p \geq 1$; see Example 5.6.

The importance of establishing the analyticity of the semigroup associated with the fragmentation system is that analytic semigroups have extremely useful properties. For example, if $A^{(w)} + B^{(w)}$ generates an analytic semigroup on ℓ_w^1, then it follows immediately that the ACP (1.17) has a unique classical solution for any $\mathring{u} \in \ell_w^1$. In addition, when coagulation is introduced into the system, the analyticity of the semigroup generated by $A^{(w)} + B^{(w)}$ can be used to weaken the assumptions that are required on the cluster coagulation rates to obtain the existence and uniqueness of solutions to the corresponding coagulation–fragmentation system of equations. Such coagulation–fragmentation systems will be considered in a subsequent publication.

Once the well-posedness of the fragmentation ACP has been satisfactorily dealt with, the next question to be addressed is that of the long-term behaviour of solutions. Results on the asymptotic behaviour of solutions to (1.17) are given in [7,12,15] for the specific case where the weight is $w_n = n^p$ for $p \geq 1$, $n \in \mathbb{N}$. In particular, for mass-conserving fragmentation processes, where (1.8) holds, it is shown that the solution of (1.17) converges to a state where there are only monomers present if and only if $a_n > 0$ for all $n \geq 2$. In Sect. 6, we continue to work with more general weights and, in the mass loss case, show that the solution of (1.17) decays to the zero state over time if and only if $a_n > 0$ for all $n \in \mathbb{N}$. This mass loss result can then be used to deduce that the solution, in the mass-conserving case, converges to the monomer state if and only if $a_n > 0$ for all $n \geq 2$, this result now holding in the general weighted space ℓ_w^1.

Regarding the rate at which solutions approach the steady state, the case where mass is conserved and $w_n = n^p$ for $p > 1$ is examined in [12, Section 4], and it is shown that solutions decay to the monomer state at an exponential rate, which, however, is not quantified. In Sect. 6, we obtain results regarding the exponential rate of decay of solutions, both for the mass-conserving and for the mass loss cases, by working in a space ℓ_w^1 in which $A^{(w)} + B^{(w)}$ generates an analytic semigroup. The approach we use enables us to quantify the exponential decay rate.

In [19], the theory of Sobolev towers is used to investigate a specific example of (1.1) that has been proposed as a model of random bond annihilation. Of particular note is the fact that the resulting analysis provides a rigorous explanation of an apparent non-uniqueness of solutions that emanate from a zero initial condition. We shall establish that an approach involving Sobolev towers can also be used to obtain results on (1.1) for general fragmentation coefficients. By writing (1.1) as an ACP in ℓ_w^1, where w is such that $A^{(w)} + B^{(w)}$ generates an analytic, substochastic C_0-semigroup on ℓ_w^1, we are able to construct a Sobolev tower and then use this to prove the existence of unique, non-negative solutions of (1.17) for a wider class of non-negative initial conditions than those in ℓ_w^1; see Theorem 7.2.

The paper is structured as follows. In Sect. 2, we provide some prerequisite results and definitions. Following this, we begin our examination of (1.1) in Sect. 3, obtaining, in particular, the aforementioned Theorem 3.4, which is then used to draw conclusions on the existence and uniqueness of solutions to (1.17) and (1.18), both in the space $X_{[1]}$ and in more general ℓ_w^1 spaces. We consider the pointwise system (1.1) in Sect. 4 and show that for any $\mathring{u} \in \ell_w^1$, a solution of (1.1) can be expressed in terms of the semigroup generated by $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$. We then use this result to show that $G^{(w)}$ is a restriction of the maximal operator $G_{\mathsf{max}}^{(w)}$. This is important in investigations into the full coagulation–fragmentation system as it allows the fragmentation terms to be completely described by the operator $G^{(w)}$. Results on the analyticity of the fragmentation semigroup are presented in Sect. 5 and then applied both in Sect. 6, where the asymptotic behaviour of solutions is investigated, and in Sect. 7, where the theory of Sobolev towers is applied to establish the well-posedness of (1.17) for more general initial conditions.

2. Preliminaries

We begin by recalling some terminology. The following notions are well known and can be found in various sources, including [9,13]. Let X be a real vector lattice with norm $\|\cdot\|$. The positive cone, X_+, of X is the set of non-negative elements in X and, similarly, for a subspace D of X, we denote the set of non-negative elements in D

by D_+. If X is a vector lattice, then for each $f \in X$ the vectors $f_\pm := \sup\{\pm f, 0\}$ are well defined and satisfy $f_+, f_- \in X_+$ and $f = f_+ - f_-$. A vector lattice, equipped with a lattice norm $\|\cdot\|$, is said to be a *Banach lattice* if X is complete under $\|\cdot\|$. Moreover, if the lattice norm satisfies

$$\|f + g\| = \|f\| + \|g\|$$

for all $f, g \in X_+$, then X is an *AL-space*. It can be shown that, when X is an AL-space, there exists a unique, bounded linear functional, ϕ, that extends $\|\cdot\|$ from X_+ to X; see [9, Theorems 2.64 and 2.65].

We now turn our attention to C_0-semigroups which are crucial to our investigation into the pure fragmentation system. The notions and results given here can be found in [17]. First, we note that if $(S(t))_{t\geq 0}$ is a C_0-semigroup on a Banach space X, then there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and the growth bound, ω_0, of $(S(t))_{t\geq 0}$ is defined by

$$\omega_0 := \inf\left\{\omega \in \mathbb{R} : \text{there exists } M_\omega \geq 1 \text{ such that } \|S(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0\right\}.$$

Analytic semigroups, see [17, Definition II.4.5], are of particular importance in Sect. 5. Semigroups of this type have a number of useful properties that make them desirable to work with. For example, if G is the generator of an analytic semigroup, $(S(t))_{t\geq 0}$, on a Banach space X, then $S(t)f \in \mathcal{D}(G^n)$ for all $t > 0$, $n \in \mathbb{N}$ and $f \in X$, and $S(\cdot)$ is infinitely differentiable.

When dealing with many physical problems, such as the fragmentation system, meaningful solutions must be non-negative, and this requirement has to be taken into account in any semigroup-based investigation. In connection with this, we say that a C_0-semigroup $(S(t))_{t\geq 0}$ on an ordered Banach space X, such as a Banach lattice, is *positive* if $S(t)f \geq 0$ for all $f \in X_+$; it is called *substochastic* (resp. *stochastic*) if, additionally, $\|S(t)f\| \leq \|f\|$ (resp. $\|S(t)f\| = \|f\|$) for all $f \in X_+$. It follows that if G generates a substochastic semigroup $(S(t))_{t\geq 0}$, then the associated ACP

$$u'(t) = Gu(t), \quad t > 0; \qquad u(0) = \mathring{u},$$

has a unique, non-negative classical solution, given by $u(t) = S(t)\mathring{u}$, for any $\mathring{u} \in D(G)_+$.

A result on substochastic semigroups and their generators that we shall exploit is due to Thieme and Voigt [20, Theorem 2.7]. This result gives sufficient conditions under which the closure of the sum of two operators, such as $A^{(w)} + B^{(w)}$ in (1.17), generates a substochastic semigroup. The existence of an invariant subspace under the resulting semigroup is also established. As we demonstrate in Proposition 2.4, it is possible to adapt the Thieme–Voigt result to produce a modified version that is ideally suited for applying to the fragmentation system. We first provide some prerequisite results that are used in the proof of this proposition.

Lemma 2.1. *Let A be a closable operator in a Banach space X. If $G = \overline{A}$ is the generator of a C_0-semigroup on X, then no other extension of A is the generator of a C_0-semigroup on X.*

Proof. Suppose that $G = \overline{A}$ and $H \supseteq A$ are generators of C_0-semigroups with growth bounds ω_1 and ω_2, respectively, and assume that $H \neq G$. Clearly, $H \supseteq G$ since H is closed. Let $\lambda > \max\{\omega_1, \omega_2\}$. Then, $\lambda \in \rho(G) \cap \rho(H)$ and hence $\lambda I - G : \mathcal{D}(G) \to X$ and $\lambda I - H : \mathcal{D}(H) \to X$ are both bijective. This is a contradiction since $\lambda I - H$ is a proper extension of $\lambda I - G$. □

The following lemma, which is a special case of [9, Remark 6.6], will also be used. For the convenience of the reader, we present a short proof.

Lemma 2.2. *Let G be the generator of a positive C_0-semigroup on a Banach lattice X. Then, for every $f \in \mathcal{D}(G)$, there exist $g, h \in \mathcal{D}(G)_+$ such that $f = g - h$.*

Proof. Let $f \in \mathcal{D}(G)$. Further, let ω_0 be the growth bound of the semigroup generated by G, fix $\lambda > \omega_0$ and set $f_0 := (\lambda I - G)f$. Since X is a Banach lattice, we have $f_0 = f_+ - f_-$ with $f_+, f_- \in X_+$. Now, let $g := R(\lambda, G)f_+$ and $h := R(\lambda, G)f_-$. The fact that G generates a positive semigroup implies that $R(\lambda, G)$ is a positive operator, and therefore $g, h \in \mathcal{D}(G)_+$. Moreover,

$$f = R(\lambda, G)f_0 = R(\lambda, G)(f_+ - f_-) = R(\lambda, G)f_+ - R(\lambda, G)f_- = g - h,$$

which proves the result. □

When the fragmentation coefficients satisfy Assumption 1.1 and (1.8), then, as mentioned in the previous section, a formal calculation shows that the total mass is conserved. Consequently, if u is a non-negative solution of the fragmentation system, and it is known that $u(t) \in X_{[1]}$ for $t \geq 0$, then we would expect u to satisfy

$$\|u(t)\|_{[1]} = \sum_{n=1}^{\infty} n u_n(t) = \sum_{n=1}^{\infty} n \mathring{u} = \|\mathring{u}\|_{[1]} \qquad \text{for all } t \geq 0.$$

Clearly, this mass conservation property will hold whenever the solution can be written in terms of a stochastic semigroup on $X_{[1]}$. To this end, the following proposition will prove useful.

Proposition 2.3. *Let $(S(t))_{t\geq 0}$ be a positive C_0-semigroup on an AL-space, X, with generator G, and let ϕ be the unique bounded linear extension of the norm $\|\cdot\|$ from X_+ to X.*

(i) *The semigroup $(S(t))_{t\geq 0}$ is stochastic if and only if*

$$\phi\big(S(t)f\big) = \phi(f) \qquad \textit{for all } f \in X. \tag{2.1}$$

(ii) *If $\phi(Gf) = 0$ for all $f \in \mathcal{D}(G)_+$, then (2.1) holds and hence the semigroup $(S(t))_{t\geq 0}$ is stochastic.*

(iii) *Let G_0 be an operator such that $G = \overline{G_0}$. If $\phi(G_0 f) = 0$ for all $f \in \mathcal{D}(G_0)_+$ and each $f \in \mathcal{D}(G_0)$ can be written as $f = g - h$, where $g, h \in \mathcal{D}(G_0)_+$, then (2.1) holds and hence $(S(t))_{t\geq 0}$ is stochastic.*

Proof. (i) Assume that $(S(t))_{t\geq 0}$ is stochastic and let $f \in X$ and $t \geq 0$. Then, $f = f_+ - f_-$, where $f_+, f_- \in X_+$, and therefore

$$\begin{aligned}\phi\big(S(t)f\big) &= \phi\big(S(t)f_+\big) - \phi\big(S(t)f_-\big) = \|S(t)f_+\| - \|S(t)f_-\| \\ &= \|f_+\| - \|f_-\| = \phi(f_+) - \phi(f_-) = \phi(f).\end{aligned}$$

Conversely, when (2.1) holds, we have $\|S(t)f\| = \phi(S(t)f) = \phi(f) = \|f\|$ for $f \in X_+$ and $t \geq 0$.

(ii) Let $f \in \mathcal{D}(G)$. From Lemma 2.2, there exist $g, h \in \mathcal{D}(G)_+$ such that $f = g - h$. Then,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\phi(S(t)f)\big) &= \phi\left(\frac{\mathrm{d}}{\mathrm{d}t}\big(S(t)f\big)\right) = \phi\big(GS(t)f\big) \\ &= \phi\big(GS(t)g\big) - \phi\big(GS(t)h\big) = 0\end{aligned}$$

since $S(t)g, S(t)h \in \mathcal{D}(G)_+$. Thus, $\phi(S(t)f) = \phi(f)$ for all $f \in \mathcal{D}(G)$, and hence also for all $f \in X$, since $\mathcal{D}(G)$ is dense in X.

(iii) Let $f \in \mathcal{D}(G_0)$. Then, $f = g - h$ for some $g, h \in \mathcal{D}(G_0)_+$ by assumption, and

$$\phi(G_0 f) = \phi\big(G_0(g - h)\big) = \phi(G_0 g) - \phi(G_0 h) = 0.$$

Thus, $\phi(G_0 f) = 0$ for all $f \in \mathcal{D}(G_0)$. Now, let $f \in \mathcal{D}(G)$. Then, there exist $f^{(n)} \in \mathcal{D}(G_0)$, $n \in \mathbb{N}$, such that $f^{(n)} \to f$ and $G_0 f^{(n)} \to Gf$ as $n \to \infty$. Therefore,

$$\phi(Gf) = \phi\Big(\lim_{n\to\infty} G_0 f^{(n)}\Big) = \lim_{n\to\infty} \phi(G_0 f^{(n)}) = 0,$$

and the result follows from part (ii).

□

We now use [20, Theorem 2.7] to obtain the following proposition, which will later be applied to the fragmentation problem.

Proposition 2.4. *Let $(X, \|\cdot\|)$ and $(Z, \|\cdot\|_Z)$ be AL-spaces, such that*

(i) *Z is dense in X,*

(ii) *$(Z, \|\cdot\|_Z)$ is continuously embedded in $(X, \|\cdot\|)$.*

Also, let ϕ and ϕ_Z be the linear extensions of $\|\cdot\|$ from X_+ to X and of $\|\cdot\|_Z$ from Z_+ to Z, respectively. Let $A : \mathcal{D}(A) \to X$, $B : \mathcal{D}(B) \to X$ be operators in X such that $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. Assume that the following conditions are satisfied.

(a) *$-A$ is positive;*

(b) *A generates a positive C_0-semigroup, $(T(t))_{t\geq 0}$, on X;*

(c) *the semigroup* $(T(t))_{t\geq 0}$ *leaves* Z *invariant and its restriction to* Z *is a (necessarily positive)* C_0*-semigroup on* $(Z, \|\cdot\|_Z)$*, with generator* $\widetilde{A}$ *given by*

$$\widetilde{A}f = Af \qquad \text{for all } f \in \mathcal{D}(\widetilde{A}) = \left\{f \in \mathcal{D}(A) \cap Z : Af \in Z\right\};$$

(d) $B|_{\mathcal{D}(A)}$ *is a positive linear operator;*
(e) $\phi((A+B)f) \leq 0$ *for all* $f \in \mathcal{D}(A)_+$*;*
(f) $(A+B)f \in Z$ *and* $\phi_Z((A+B)f) \leq 0$ *for all* $f \in \mathcal{D}(\widetilde{A})_+$*;*
(g) $\|Af\| \leq \|f\|_Z$ *for all* $f \in \mathcal{D}(\widetilde{A})_+$.

Then, there exists a unique substochastic C_0*-semigroup on* X *which is generated by an extension,* G*, of* $A+B$*. The operator* G *is the closure of* $A+B$*. Moreover, the semigroup* $(S(t))_{t\geq 0}$ *generated by* G *leaves* Z *invariant. If* $\phi((A+B)f) = 0$ *for all* $f \in \mathcal{D}(A)_+$*, then* $(S(t))_{t\geq 0}$ *is stochastic.*

Proof. We first show that the conditions of [20, Theorem 2.7] hold. From (ii) and the fact that $(Z, \|\cdot\|_Z)$ is an AL-space, it is clear that [20, Assumption 2.5] is satisfied. Also, from (f) and (g) we obtain that

$$\phi_Z\big((A+B)f\big) \leq 0 \leq \|f\|_Z - \|Af\|$$

for all $f \in \mathcal{D}(\widetilde{A})_+$. Moreover, (f) and the definition of $\widetilde{A}$ imply that $Bf \in Z$ for all $f \in \mathcal{D}(\widetilde{A})_+$. Consequently, if we now take $f \in D(\widetilde{A})$ and use Lemma 2.2 to express Bf as $Bg - Bh$, where $g, h \in \mathcal{D}(\widetilde{A})_+$, then it follows easily that $B(\mathcal{D}(\widetilde{A})) \subseteq Z$. Thus, all the assumptions of [20, Theorem 2.7] are satisfied and therefore $G = \overline{A+B}$ is the generator of a substochastic semigroup $(S(t))_{t\geq 0}$, which leaves Z invariant. That no other extension of $A+B$ can generate a C_0-semigroup on X is an immediate consequence of Lemma 2.1. Finally, since A generates a substochastic C_0-semigroup, it follows from Lemma 2.2 that we can write any $f \in \mathcal{D}(A) = \mathcal{D}(A+B)$ as $f = g - h$, where $g, h \in \mathcal{D}(A)_+$. An application of Proposition 2.3(iii) then yields the stochasticity result. □

3. The fragmentation semigroup

In this section, we begin our analysis of the fragmentation system (1.1) by investigating the associated ACP (1.17), which we recall takes the form

$$u'(t) = A^{(w)}u(t) + B^{(w)}u(t), \quad t > 0; \qquad u(0) = \mathring{u},$$

where $A^{(w)}$ and $B^{(w)}$ are defined in ℓ_w^1 by (1.15) and (1.16), respectively. A direct application of Proposition 2.4 will establish that, under appropriate conditions on the weight w, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ generates a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. As no other extension of $A^{(w)} + B^{(w)}$ generates a C_0-semigroup on ℓ_w^1, we shall refer to $(S^{(w)}(t))_{t\geq 0}$ as *the* fragmentation semigroup on ℓ_w^1. In the process of proving the existence of the fragmentation semigroup, we shall also obtain explicit subspaces of ℓ_w^1 which are invariant under $(S^{(w)}(t))_{t\geq 0}$.

First, we note that ℓ_w^1 is an AL-space, with positive cone

$$\left(\ell_w^1\right)_+ = \left\{f = (f_n)_{n=1}^\infty \in \ell_w^1 : f_n \geq 0 \text{ for all } n \in \mathbb{N}\right\},$$

whenever $w = (w_n)_{n=1}^\infty$ is a positive sequence. Moreover, in this case the unique bounded linear functional, ϕ_w, that extends $\|\cdot\|_w$ from $(\ell_w^1)_+$ to ℓ_w^1 is given by

$$\phi_w(f) = \sum_{n=1}^\infty w_n f_n \qquad \text{for all } f \in \ell_w^1. \tag{3.1}$$

We recall also that if we take $w_n = n$ for all $n \in \mathbb{N}$, then $\ell_w^1 = X_{[1]}$ and $\|\cdot\|_w = \|\cdot\|_{[1]}$. For this specific case, we shall represent ϕ_w, $A^{(w)}$ and $B^{(w)}$ by M_1, A_1 and B_1, respectively, and consequently the ACP (1.17) on $X_{[1]}$ will be written as

$$u'(t) = A_1 u(t) + B_1 u(t), \quad t > 0; \qquad u(0) = \mathring{u}. \tag{3.2}$$

From physical considerations, it is clear that the initial condition, $\mathring{u}$, in the ACP (1.17) must necessarily be non-negative, and similarly, if $u : [0, \infty) \to \ell_w^1$ is the corresponding solution, then we require $u(t)$ to be non-negative for all $t \geq 0$. Moreover, if we assume (1.4) to hold, or, equivalently, (1.5) with $\lambda_j \in [0, 1]$, we expect from (1.7) that mass is either lost or conserved during fragmentation. From (1.6) and the definition of the norm on $X_{[1]}$, this is equivalent to

$$\|u(t)\|_{[1]} \leq \|\mathring{u}\|_{[1]} \qquad \text{for all } t \geq 0, \tag{3.3}$$

with equality being required in the mass-conserving case, provided that w is such that $\ell_w^1 \subseteq X_{[1]}$.

For convenience, we include the following elementary result which states that the operator $A^{(w)}$ generates a substochastic semigroup on ℓ_w^1 for any non-negative weight w.

Lemma 3.1. *Let ℓ_w^1 and $\|\cdot\|_w$ be defined by* (1.13) *and* (1.14)*, respectively, and let* (1.2) *hold. Then, the operator $A^{(w)}$, defined by* (1.15)*, is the generator of a substochastic C_0-semigroup, $(T^{(w)}(t))_{t\geq 0}$, on ℓ_w^1, which is given, for $t \geq 0$, by the infinite diagonal matrix* diag$(v_1(t), v_2(t), \ldots)$*, where $v_n(t) = e^{-a_n t}$ for all $n \in \mathbb{N}$.*

For the remainder of this section, the weight, w, will be required to satisfy the following assumption.

Assumption 3.2. (i) $w_n \geq n$ for all $n \in \mathbb{N}$.

(ii) There exists $\kappa \in (0, 1]$ such that

$$\sum_{n=1}^{j-1} w_n b_{n,j} \leq \kappa w_j \qquad \text{for all } j = 2, 3, \ldots. \tag{3.4}$$

Remark 3.3. Let w be such that $(w_n/n)_{n=1}^{\infty}$ is increasing and let (1.4) hold. Then,

$$\sum_{n=1}^{j-1} w_n b_{n,j} = \sum_{n=1}^{j-1} \frac{w_n}{n} n b_{n,j} \leq \frac{w_j}{j} \sum_{n=1}^{j-1} n b_{n,j} \leq \frac{w_j}{j} j = w_j.$$

Hence, (3.4) is satisfied with $\kappa = 1$. In particular, if (1.4) holds, then Assumption 3.2 is automatically satisfied by any weight of the form $w_n = n^p$, $p \geq 1$.

It is an immediate consequence of Assumption 3.2 that, for any $f \in \mathcal{D}(A^{(w)})_+$, we have

$$\begin{aligned} \phi_w\big(B^{(w)} f\big) &= \sum_{n=1}^{\infty} w_n \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j = \sum_{j=2}^{\infty} \left(\sum_{n=1}^{j-1} w_n b_{n,j} \right) a_j f_j \\ &\leq \kappa \sum_{j=1}^{\infty} w_j a_j f_j = -\kappa \phi_w\big(A^{(w)} f\big). \end{aligned} \tag{3.5}$$

Consequently, for all $f \in \mathcal{D}(A^{(w)})$,

$$\begin{aligned} \|B^{(w)} f\|_w &= \sum_{n=1}^{\infty} w_n \left| \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \right| \leq \phi_w\big(B^{(w)} |f|\big) \\ &\leq -\kappa \phi_w\big(A^{(w)} |f|\big) = \kappa \|A^{(w)} f\|_w, \end{aligned} \tag{3.6}$$

from which it follows that

$$\mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(B^{(w)}) \quad \text{and} \quad \mathcal{D}\big(A^{(w)} + B^{(w)}\big) = \mathcal{D}(A^{(w)}) \cap \mathcal{D}(B^{(w)}) = \mathcal{D}(A^{(w)}). \tag{3.7}$$

We now apply Proposition 2.4 to the operators $A^{(w)}$ and $B^{(w)}$. This involves the construction of a suitable subspace of ℓ_w^1, and to this end we require a sequence $(c_n)_{n=1}^{\infty}$ that satisfies

$$c_n \leq c_{n+1} \qquad \text{and} \qquad a_n \leq c_n \qquad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

Note that such a sequence can always be found. For example, we can take

$$c_n = \max\{a_1, \ldots, a_n\} \qquad \text{for } n = 1, 2, \ldots. \tag{3.9}$$

Let $C^{(w)}$ be the corresponding multiplication operator, defined by

$$[C^{(w)} f]_n = -c_n f_n, \quad n \in \mathbb{N}, \qquad \mathcal{D}(C^{(w)}) = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n c_n |f_n| < \infty \right\}, \tag{3.10}$$

and equip $\mathcal{D}(C^{(w)})$ with the graph norm

$$\|f\|_{C^{(w)}} = \|f\|_w + \|C^{(w)} f\|_w = \sum_{n=1}^{\infty} (w_n + w_n c_n) |f_n|, \qquad f \in \mathcal{D}(C^{(w)}). \tag{3.11}$$

Clearly, $(\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}}) = (\ell^1_{\widetilde{w}}, \|\cdot\|_{\widetilde{w}})$ with weight $\widetilde{w} = (\widetilde{w}_n)_{n=1}^\infty$ where

$$\widetilde{w}_n = w_n + w_n c_n, \qquad n \in \mathbb{N}, \tag{3.12}$$

and hence $(\ell^1_{\widetilde{w}}, \|\cdot\|_{\widetilde{w}})$ is an AL-space, and the unique linear extension of $\|\cdot\|_{\widetilde{w}}$ from $(\ell^1_{\widetilde{w}})_+$ to $\ell^1_{\widetilde{w}}$ is given by $\phi_{\widetilde{w}}(f) = \sum_{n=1}^\infty \widetilde{w}_n f_n$ for $f \in \ell^1_{\widetilde{w}}$.

We note that the choice (3.9) for $(c_n)_{n=1}^\infty$ is 'maximal' in the sense that if $(\hat{c}_n)_{n=1}^\infty$ is any other monotone increasing sequence that dominates $(a_n)_{n=1}^\infty$ and $\widehat{C}$ is defined analogously to (3.10), then $\mathcal{D}(\widehat{C}^{(w)}) \subseteq \mathcal{D}(C^{(w)})$.

Theorem 3.4. *Let Assumptions 1.1 and 3.2 hold. Then, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ^1_w. Moreover, the semigroup $(S^{(w)}(t))_{t\geq 0}$ leaves $\mathcal{D}(C^{(w)}) = \ell^1_{\widetilde{w}}$ invariant, where $\mathcal{D}(C^{(w)})$ and $\widetilde{w}$ are defined in (3.10) and (3.12), respectively, and $(c_n)_{n=1}^\infty$ satisfies (3.8). If, in addition, (1.8) holds and $w_n = n$ for all $n \in \mathbb{N}$, then the semigroup, $(S_1(t))_{t\geq 0}$, generated by $G_1 = \overline{A_1 + B_1}$ is stochastic on $X_{[1]}$.*

Proof. We show that the conditions (i), (ii) and (a)–(g) of Proposition 2.4 are all satisfied when $A = A^{(w)}$, $B = B^{(w)}$ and the AL-spaces $(X, \|\cdot\|)$ and $(Z, \|\cdot\|_Z)$ are, respectively, ℓ^1_w and $(\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}}) = (\ell^1_{\widetilde{w}}, \|\cdot\|_{\widetilde{w}})$.

Clearly, $\ell^1_{\widetilde{w}}$ is dense in ℓ^1_w and continuously embedded since $w_n \leq \widetilde{w}_n$, $n \in \mathbb{N}$. It follows that (i) and (ii) both hold.

Condition (a) is obviously satisfied by $A^{(w)}$, and, for (b), we apply Lemma 3.1 to establish that $A^{(w)}$ generates a substochastic C_0-semigroup, $(T^{(w)}(t))_{t\geq 0}$, on ℓ^1_w. It is easy to see that the semigroup $(T^{(w)}(t))_{t\geq 0}$ leaves $\ell^1_{\widetilde{w}}$ invariant and the generator of the restriction to $\ell^1_{\widetilde{w}}$ is $A^{(\widetilde{w})}$, the part of $A^{(w)}$ in $\ell^1_{\widetilde{w}}$; this shows (c).

It is also clear that $B^{(w)}$ is positive. From (3.5), we obtain that, for $f \in \mathcal{D}(A^{(w)})_+$,

$$\begin{aligned} \phi_w\big((A^{(w)} + B^{(w)})f\big) &= \phi_w(A^{(w)}f) + \phi_w(B^{(w)}f) \\ &\leq \phi_w(A^{(w)}f) - \kappa\phi_w(A^{(w)}f) \leq 0. \end{aligned} \tag{3.13}$$

Hence, (d) and (e) hold.

Since $w_n \geq n$, by Assumption 3.2 (i), we have $\widetilde{w}_n = w_n + w_n c_n \geq n$, $n \in \mathbb{N}$. Moreover, the monotonicity of $(c_n)_{n=1}^\infty$ and Assumption 3.2 (ii) imply that

$$\sum_{n=1}^{j-1} \widetilde{w}_n b_{n,j} = \sum_{n=1}^{j-1} (1 + c_n) w_n b_{n,j} \leq (1 + c_j) \sum_{n=1}^{j-1} w_n b_{n,j} \leq \kappa(1 + c_j) w_j = \kappa \widetilde{w}_j$$

for all $j \in \mathbb{N}$. This means that Assumption 3.2 also holds for the weight $\widetilde{w}$. Therefore, we obtain from (3.7) and (3.13) that $\mathcal{D}(A^{(\widetilde{w})}) \subseteq \mathcal{D}(B^{(\widetilde{w})})$ and $\phi_{\widetilde{w}}((A^{(\widetilde{w})} + B^{(\widetilde{w})})f) \leq 0$ for $f \in \mathcal{D}(A^{(\widetilde{w})})_+$, and so (f) is also satisfied. That (g) holds follows from

$$\|A^{(w)}f\|_w = \sum_{n=1}^{\infty} w_n a_n |f_n| \leq \sum_{n=1}^{\infty} w_n c_n |f_n| \leq \sum_{n=1}^{\infty} \widetilde{w}_n |f_n| = \|f\|_{\widetilde{w}}$$

for $f \in \mathcal{D}(\tilde{A}^{(w)})_+$.

Thus, the conditions of Proposition 2.4 are all satisfied and therefore $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1, which also leaves $\mathcal{D}(C^{(w)}) = \ell_{\widetilde{w}}^1$ invariant.

Finally, assume that (1.8) is satisfied and $w_n = n$ for all $n \in \mathbb{N}$. Then, equality holds in (3.4) with $\kappa = 1$ and hence also in (3.5), and so, from Proposition 2.4, the semigroup generated in this case is stochastic. □

Remark 3.5. Consider the case where $w_n = n$ for all $n \in \mathbb{N}$, so that $\ell_w^1 = X_{[1]}$, and let Assumption 1.1 and (1.4) hold. Then, by Remark 3.3, (3.4) is also satisfied, and therefore, from Theorem 3.4, the operator $G_1 = \overline{A_1 + B_1}$ is the generator of a substochastic C_0-semigroup, $(S_1(t))_{t\geq 0}$, on $X_{[1]}$. It follows that the ACP

$$u'(t) = G_1 u(t), \quad t > 0; \qquad u(0) = \mathring{u}, \tag{3.14}$$

with $\mathring{u} \in \mathcal{D}(G_1)$, has a unique classical solution, given by $u(t) = S_1(t)\mathring{u}$ for all $t \geq 0$. Moreover, if $\mathring{u} \geq 0$, then this solution is non-negative. Now suppose that $\mathring{u} \in \mathcal{D}(G_1)_+$ and, in addition, assume that (1.8) holds. Then, the semigroup $(S_1(t))_{t\geq 0}$ is stochastic on $X_{[1]}$ and so, from (1.11),

$$M_1\big(u(t)\big) = \|u(t)\|_{[1]} = \|S_1(t)\mathring{u}\|_{[1]} = \|\mathring{u}\|_{[1]} = M_1(\mathring{u}) \qquad \text{for all } t \geq 0,$$

showing that $u(t)$ is a mass-conserving solution.

With the help of Remark 3.5, we obtain the following corollary.

Corollary 3.6. *Let Assumptions 1.1 and 3.2 hold and let $\mathring{u} \in \mathcal{D}(G^{(w)})$, where $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ as in Theorem 3.4. Then, the ACP*

$$u'(t) = G^{(w)} u(t), \quad t > 0; \qquad u(0) = \mathring{u} \tag{3.15}$$

has a unique classical solution, given by $u(t) = S^{(w)}(t)\mathring{u}$. This solution is non-negative if $\mathring{u} \in \mathcal{D}(G^{(w)})_+$. Moreover, if (1.8) holds and $\mathring{u} \in \mathcal{D}(G^{(w)})_+$, then this solution is mass conserving.

Proof. It follows immediately from Theorem 3.4 that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (3.15) for all $\mathring{u} \in \mathcal{D}(G^{(w)})$. Moreover, since $(S^{(w)}(t))_{t\geq 0}$ is substochastic, this solution is non-negative if $\mathring{u} \in \mathcal{D}(G^{(w)})_+$.

Now, assume that (1.8) holds and $\mathring{u} \in \mathcal{D}(G^{(w)})_+$. Then, $(S_1(t))_{t\geq 0}$ is a stochastic C_0-semigroup on $X_{[1]}$. Additionally, since $w_n \geq n$ for all $n \in \mathbb{N}$, ℓ_w^1 is continuously embedded in $X_{[1]}$ and so, as $u(t)$ is differentiable in ℓ_w^1, $u(t)$ is also differentiable in $X_{[1]}$ and the derivatives must coincide. Moreover, since $G^{(w)}$ is the part of G_1 in ℓ_w^1, we have $u(t) \in \mathcal{D}(G_1)$. Therefore, $u(t) = S^{(w)}(t)\mathring{u}$ is also a solution of (3.14), and, by uniqueness of solutions, it follows that $S^{(w)}(t)\mathring{u} = S_1(t)\mathring{u}$ for $t \geq 0$. Remark 3.5 then establishes that $u(t) = S^{(w)}(t)\mathring{u}$ is a mass-conserving solution. □

Note that even if $\mathring{u} \in \mathcal{D}(A^{(w)})$, the solution, $u(t)$, of (3.15) need not belong to $\mathcal{D}(A^{(w)})$ for any $t > 0$. Hence, the existence of a solution of (1.17) is not guaranteed in general; one only has uniqueness of solutions. However, the next theorem shows that under the stronger assumption $\mathring{u} \in \mathcal{D}(C^{(w)})$ on the initial condition, the ACP (1.17) is well posed.

Theorem 3.7. *Let Assumptions 1.1 and 3.2 hold. For $\mathring{u} \in \mathcal{D}(C^{(w)})$, the ACP (1.17) has a unique classical solution given by $u(t) = S^{(w)}(t)\mathring{u}$, $t \geq 0$. If $\mathring{u} \in \mathcal{D}(C^{(w)})_+$, then this solution is non-negative. Moreover, if (1.8) holds and $\mathring{u} \in \mathcal{D}(C^{(w)})_+$, then the solution is mass conserving.*

Proof. We know that $G^{(w)}$ and $A^{(w)} + B^{(w)}$ coincide on $\mathcal{D}(A^{(w)})$ and also that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique solution of (3.15) for $\mathring{u} \in \mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(G^{(w)})$. Since $(S^{(w)}(t))_{t\geq 0}$ leaves $\mathcal{D}(C^{(w)})$ invariant, it follows that $S^{(w)}(t)\mathring{u} \in \mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$. The result then follows from Corollary 3.6. □

The next proposition shows that if the sequence $(a_n)_{n=1}^{\infty}$ has a certain additional property, then a unique solution of (1.17) exists for $\mathring{u} \in \mathcal{D}(A^{(w)})$.

Proposition 3.8. *Let $(a_n)_{n=1}^{\infty}$ be an unbounded sequence such that (1.2) holds. Further, define the sequence $(c_n)_{n=1}^{\infty}$ by (3.9) and let $w = (w_n)_{n=1}^{\infty}$ be such that $w_n > 0$ for all $n \in \mathbb{N}$. Then, $\mathcal{D}(C^{(w)}) = \mathcal{D}(A^{(w)})$ if and only if*

$$\liminf_{n\to\infty} \frac{a_n}{c_n} > 0. \tag{3.16}$$

Proof. Note first that the unboundedness of $(a_n)_{n=1}^{\infty}$ implies that $c_n \to \infty$ as $n \to \infty$. Since $c_n \geq a_n$ for all $n \in \mathbb{N}$, we have $\mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$. If (3.16) holds, then there exist $\gamma > 0$, $N \in \mathbb{N}$ such that $a_n \geq \gamma c_n$ for all $n \geq N$. Let $f \in \mathcal{D}(A^{(w)})$. Then,

$$\begin{aligned} \|C^{(w)} f\|_w = \sum_{n=1}^{\infty} w_n c_n |f_n| &\leq \sum_{n=1}^{N-1} w_n c_n |f_n| + \frac{1}{\gamma} \sum_{n=N}^{\infty} w_n a_n |f_n| \\ &\leq \sum_{n=1}^{N-1} w_n c_n |f_n| + \frac{1}{\gamma} \|A^{(w)} f\|_w < \infty, \end{aligned}$$

and so $\mathcal{D}(A^{(w)}) = \mathcal{D}(C^{(w)})$.

Now, suppose that $\liminf_{n\to\infty}(a_n/c_n) = 0$. Then, there exists a subsequence, $(a_{n_k}/c_{n_k})_{k=1}^{\infty}$, such that

$$c_{n_k} \neq 0, \quad \frac{a_{n_k}}{c_{n_k}} \leq \frac{1}{k} \quad \text{and} \quad \frac{1}{c_{n_k}} \leq \frac{1}{k} \qquad \text{for all } k \in \mathbb{N}.$$

Let f be such that

$$f_j = \begin{cases} 1/(c_{n_k} w_{n_k} k) & \text{when } j = n_k, \\ 0 & \text{otherwise.} \end{cases} \tag{3.17}$$

Then,

$$\sum_{n=1}^{\infty} a_n w_n |f_n| = \sum_{k=1}^{\infty} a_{n_k} w_{n_k} \frac{1}{c_{n_k} w_{n_k} k} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$
$$\sum_{n=1}^{\infty} w_n |f_n| = \sum_{k=1}^{\infty} w_{n_k} \frac{1}{c_{n_k} w_{n_k} k} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$
$$\sum_{n=1}^{\infty} c_n w_n |f_n| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

It follows that $f \in \mathcal{D}(A^{(w)}) \backslash \mathcal{D}(C^{(w)})$, showing that $\mathcal{D}(C^{(w)})$ is a proper subset of $\mathcal{D}(A^{(w)})$. □

Remark 3.9. If $(a_n)_{n=1}^{\infty}$ is unbounded and eventually monotone increasing, then $(c_n)_{n=1}^{\infty}$, given by (3.9), satisfies (3.16). Note that, in $X_{[1]}$, the invariance of $\mathcal{D}(A^{(w)})$ under the fragmentation semigroup has already been established in [18, Theorem 3.2] for the case when $(a_n)_{n=1}^{\infty}$ is monotone increasing.

We end this section by obtaining an infinite matrix representation of the fragmentation semigroup $(S^{(w)}(t))_{t\geq 0}$ on ℓ_w^1, which is used in Sect. 6. Let Assumptions 1.1 and 3.2 be satisfied so that $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. For $n \in \mathbb{N}$, let $e_n \in \ell_w^1$ be given by

$$(e_n)_k = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise,} \end{cases} \tag{3.18}$$

and let $(s_{m,n}(t))_{m,n\in\mathbb{N}}$ be the infinite matrix defined by

$$s_{m,n}(t) = (S^{(w)}(t)e_n)_m \qquad \text{for all } m, n \in \mathbb{N}.$$

Note that, since $(S^{(w)}(t))_{t\geq 0}$ is positive, $s_{m,n}(t) \geq 0$ for all $m, n \in \mathbb{N}$. Now, each $f \in \ell_w^1$ can be expressed as $f = \sum_{n=1}^{\infty} f_n e_n$, where the infinite series is convergent in ℓ_w^1. Hence,

$$\left(S^{(w)}(t)f\right)_m = \left(\sum_{n=1}^{\infty} f_n S^{(w)}(t)e_n\right)_m = \sum_{n=1}^{\infty} f_n s_{m,n}(t) \qquad \text{for all } m \in \mathbb{N},$$

and therefore $(S^{(w)}(t))_{t\geq 0}$ can be represented by the matrix $(s_{m,n}(t))_{m,n\in\mathbb{N}}$. To determine $s_{m,n}(t)$ more explicitly, fix $n \in \mathbb{N}$ and let $(u_1(t), \ldots, u_n(t))$ be the unique solution of the n-dimensional system

$$u_m'(t) = -a_m u_m(t) + \sum_{j=m+1}^{n} a_j b_{m,j} u_j(t), \quad t > 0; \qquad m = 1, 2, \ldots, n; \tag{3.19}$$

$$u_n(0) = 1; \qquad u_m(0) = 0 \quad \text{for } m < n. \tag{3.20}$$

It is straightforward to check that $u(t) = (u_1(t), \dots, u_n(t), 0, 0, \dots)$ solves (1.1) with $\mathring{u} = e_n$. Since $u(t) \in \mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(G^{(w)})$, the function u coincides with the unique solution of (3.15), and hence $u(t) = S^{(w)}(t)e_n$, which yields

$$s_{m,n}(t) = \begin{cases} u_m(t), & m = 1, 2, \dots, n, \\ 0, & m > n. \end{cases} \tag{3.21}$$

For $m = n$, the differential equation in (3.19) reduces to $u_n'(t) = -a_n u_n(t)$, which implies that $s_{n,n}(t) = u_n(t) = e^{-a_n t}$. Since n was arbitrary, it follows that, for all $t \geq 0$,

$$S^{(w)}(t) = \left[\begin{array}{c|ccc} e^{-a_1 t} & s_{1,2}(t) & s_{1,3}(t) & \cdots \\ \hline 0 & e^{-a_2 t} & s_{2,3}(t) & \cdots \\ 0 & 0 & e^{-a_3 t} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right] = \begin{bmatrix} e^{-a_1 t} & S_{(12)}^{(w)}(t) \\ \mathbf{0} & S_{(22)}^{(w)}(t) \end{bmatrix}, \tag{3.22}$$

where $\mathbf{0}$ is an infinite column vector consisting entirely of zeros, $S_{(12)}^{(w)}(t)$ is a non-negative infinite row vector and $S_{(22)}^{(w)}(t)$ is an infinite-dimensional, non-negative, upper triangular matrix. We note that, in the particular case when $\ell_w^1 = X_{[1]}$ and mass is conserved, Banasiak obtains the infinite matrix representation (3.22) for the semigroup $(S_1(t))_{t \geq 0}$ in [7, Equation (10) and Lemma 1]. In [7], an explicit expression is also found for $s_{m,n}(t)$, $m < n$, but we omit this here since it is not required for the results that follow. As observed in [7, pp. 363], it follows from (3.22) that, for all $N \in \mathbb{N}$, we have $S(t)f \in \operatorname{span}\{e_1, e_2, \dots, e_N\}$ for all $f \in \operatorname{span}\{e_1, e_2, \dots, e_N\}$. Also, note that the functions $s_{m,n}$ are independent of the weight w, which implies that, whenever $\widehat{w}$ is another weight satisfying Assumption 3.2, $S^{(w)}(t)$ and $S^{(\widehat{w})}(t)$ coincide on $\ell_w^1 \cap \ell_{\widehat{w}}^1$.

4. The pointwise fragmentation problem and the fragmentation generator

We established in Theorem 3.7 that if Assumptions 1.1 and 3.2 are satisfied, then $u(t) = S^{(w)}(t)\mathring{u}$ is the unique, non-negative classical solution of the fragmentation ACP (1.17) for all $\mathring{u} \in \mathcal{D}(C^{(w)})_+$. Moreover, when (1.8) holds, then this solution is mass conserving. Clearly, $u(t) = S^{(w)}(t)\mathring{u}$ will also satisfy the fragmentation system (1.1) in a pointwise manner when $\mathring{u} \in \mathcal{D}(C^{(w)})_+$. However, at this stage we do not know in what sense, if any, the semigroup $(S^{(w)}(t))_{t\geq 0}$ provides a non-negative solution for a general $\mathring{u} \in (\ell_w^1)_+$. In this section, we show that a non-negative solution of the pointwise system (1.1) can be determined for any given initial condition in $(\ell_w^1)_+$ by using the semigroup $(S^{(w)}(t))_{t\geq 0}$.

As before, we require Assumptions 1.1 and 3.2 to hold, and we define a sequence $(c_n)_{n=1}^{\infty}$ by (3.9), with the associated multiplication operator $C^{(w)}$ given by (3.10).

Then, $a_n \leq c_n$ for all $n \in \mathbb{N}$ and it follows that $\mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$. From Proposition 3.4, $\mathcal{D}(C^{(w)})$ is invariant under the substochastic semigroup $(S^{(w)}(t))_{t\geq 0}$ generated by $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$. Consequently, $u(t) = S^{(w)}(t)\mathring{u}$ is the unique, non-negative classical solution of (1.17) for each $\mathring{u} \in \mathcal{D}(C^{(w)})_+$, and therefore

$$u_n(t) - \mathring{u}_n = -a_n \int_0^t u_n(s)\,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(s)\,\mathrm{d}s, \tag{4.1}$$

for $n = 1, 2, \ldots$. We use this integrated version of the pointwise fragmentation system (1.1) to prove the following result.

Theorem 4.1. *Let Assumptions 1.1 and 3.2 hold, and let $\mathring{u} \in \ell_w^1$. Then, $u(t) = S^{(w)}(t)\mathring{u}$ satisfies the system (1.1) for almost all $t \geq 0$. Moreover, if $\mathring{u} \geq 0$, then $u(t) \geq 0$ for $t \geq 0$.*

Proof. Let $\mathring{u} \in (\ell_w^1)_+$ and, for $N \in \mathbb{N}$, define the operator $P_N : \ell_w^1 \to \ell_w^1$ by

$$P_N f := \sum_{n=1}^{N} f_n e_n = (f_1, f_2, \ldots, f_N, 0, \ldots), \qquad f \in \ell_w^1.$$

Then, $P_N\mathring{u} \in \mathcal{D}(C^{(w)})_+$ for all $N \in \mathbb{N}$, and so, on setting $u^{(N)}(t) = S^{(w)}(t)P_N\mathring{u}$, we have

$$u_n^{(N)}(t) = P_N\mathring{u}_n - a_n \int_0^t u_n^{(N)}(s)\,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(N)}(s)\,\mathrm{d}s, \tag{4.2}$$

for $n = 1, 2, \ldots, N$. Clearly, $P_N\mathring{u} \to \mathring{u}$ in ℓ_w^1 as $N \to \infty$, and so, by the continuity of $S^{(w)}(t)$, it follows that $u_n^{(N)}(t) \to u_n(t)$ as $N \to \infty$ for all $n \in \mathbb{N}$ and $t \geq 0$. Moreover, if $N_2 \geq N_1$ then $u^{(N_2)}(t) - u^{(N_1)}(t) \geq 0$ for all $t \geq 0$, since $(S^{(w)}(t))_{t\geq 0}$ is linear and positive. Similarly, $u(t) - u^{(N)}(t) \geq 0$ for all $N \in \mathbb{N}$ and $t \geq 0$. Hence, $(u^{(N)}(t))_{N=1}^{\infty}$ is monotone increasing and bounded above by $u(t)$, and therefore, for each fixed $n \in \mathbb{N}$, $(u_n^{(N)}(t))_{N=1}^{\infty}$ is monotone increasing and bounded above by $u_n(t)$. On allowing $N \to \infty$ in (4.2), and using the monotone convergence theorem, we obtain

$$u_n(t) = \mathring{u}_n - a_n \int_0^t u_n(s)\,\mathrm{d}s + \lim_{N\to\infty} \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(N)}(s)\,\mathrm{d}s.$$

From this, we deduce that

$$\lim_{N\to\infty} \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(N)}(s)\,\mathrm{d}s$$

exists, and a further application of the monotone convergence theorem shows that

$$\lim_{N\to\infty} \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(N)}(s)\,\mathrm{d}s = \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(s)\,\mathrm{d}s.$$

Thus, for all $\mathring{u} \in (\ell^1_w)_+$,

$$(S^{(w)}(t)\mathring{u})_n = \mathring{u}_n + \int_0^t \left(-a_n (S^{(w)}(s)\mathring{u})_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} (S^{(w)}(s)\mathring{u})_j \right) \mathrm{d}s. \quad (4.3)$$

It follows that $(S^{(w)}(t)\mathring{u})_n$ is absolutely continuous with respect to t for each $n = 1, 2, \ldots$ and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(S^{(w)}(t)\mathring{u}\big)_n = -a_n\big(S^{(w)}(t)\mathring{u}\big)_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} (S^{(w)}(t)\mathring{u})_j, \qquad n \in \mathbb{N}, \quad (4.4)$$

for all $\mathring{u} \in (\ell^1_w)_+$ and almost every $t \geq 0$.

When $\mathring{u}$ is a general, and therefore not necessarily non-negative, sequence in ℓ^1_w, we can express $\mathring{u} = \mathring{u}_+ - \mathring{u}_- \in \ell^1_w$. It then follows immediately from the first part of the proof that $u(t) = S^{(w)}(t)\mathring{u}$ also satisfies (1.1) for almost all $t \geq 0$.

The last statement of the theorem follows immediately from the positivity of the semigroup $(S^{(w)}(t))_{t\geq 0}$. □

Note that, in general, solutions of (1.1) are not unique; see the discussion in Example 4.3.

We now turn our attention to obtaining a simple representation of the generator $G^{(w)}$. Although we know that $G^{(w)}$ coincides with $A^{(w)} + B^{(w)}$ on $\mathcal{D}(A^{(w)})$, and also that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (3.15) for $\mathring{u} \in \mathcal{D}(G^{(w)})$, we have yet to ascertain an explicit expression that describes the action of $G^{(w)}$ on $\mathcal{D}(G^{(w)})$. This matter is resolved by the following theorem, which shows that $G^{(w)}$ is a restriction of the maximal operator, $G^{(w)}_{\mathsf{max}}$, defined in (1.19). In the specific case of $X_{[p]}$, the result has been obtained from [9, Theorem 6.20], which uses extension techniques first introduced by Arlotti in [4] and which is applied in [8, Theorem 2.1]. We present an alternative proof, which avoids the use of such extensions.

Theorem 4.2. *Let Assumptions 1.1 and 3.2 hold. Then, for all $g \in \mathcal{D}(G^{(w)})$, we have*

$$\big[G^{(w)}g\big]_n = -a_n g_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} g_j, \qquad n \in \mathbb{N}. \quad (4.5)$$

Proof. It follows from Lemma 2.2 and its proof that, for every $g \in \mathcal{D}(G^{(w)})$, there exist $g_1, g_2 \in \mathcal{D}(G^{(w)})_+$ such that $g = g_1 - g_2$ and $f_j := (I - G^{(w)})g_j \in (\ell^1_w)_+$ for $j = 1, 2$. This and the linearity of $G^{(w)}$ allow us to assume that $g \in \mathcal{D}(G^{(w)})_+$ such that $f := (I - G^{(w)})g \in (\ell^1_w)_+$. Defining $u(t) = S^{(w)}(t)f$, we have from (4.3) that

$$\begin{aligned}\big[R(1, G^{(w)})f\big]_n &= \int_0^\infty e^{-t}[S^{(w)}(t)f]_n \,\mathrm{d}t = \int_0^\infty e^{-t} u_n(t)\,\mathrm{d}t \\ &= f_n - \int_0^\infty \int_0^t e^{-t} a_n u_n(s)\,\mathrm{d}s\,\mathrm{d}t + \int_0^\infty \int_0^t \sum_{j=n+1}^{\infty} e^{-t} a_j b_{n,j} u_j(s)\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

By Tonelli's theorem, we have

$$\int_0^\infty \int_0^t e^{-t} a_n u_n(s)\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty \int_s^\infty e^{-t} a_n u_n(s)\,\mathrm{d}t\,\mathrm{d}s$$
$$= a_n \int_0^\infty e^{-s} u_n(s)\,\mathrm{d}s = a_n\big[R(1, G^{(w)}) f\big]_n.$$

Using Tonelli's theorem and the monotone convergence theorem, we obtain

$$\int_0^\infty e^{-t} \int_0^t \sum_{j=n+1}^\infty a_j b_{n,j} u_j(s)\,\mathrm{d}s\,\mathrm{d}t = \sum_{j=n+1}^\infty a_j b_{n,j}\big[R(1, G^{(w)}) f\big]_j.$$

Thus,

$$g_n = \big[R\big(1, G^{(w)}\big) f\big]_n = f_n - a_n\big[R\big(1, G^{(w)}\big) f\big]_n + \sum_{j=n+1}^\infty a_j b_{n,j}\big[R\big(1, G^{(w)}\big) f\big]_j$$
$$= \big[\big(I - G^{(w)}\big) g\big]_n - a_n\big[R\big(1, G^{(w)}\big) f\big]_n + \sum_{j=n+1}^\infty a_j b_{n,j}\big[R\big(1, G^{(w)}\big) f\big]_j$$
$$= g_n - \big[G^{(w)} g\big]_n - a_n g_n + \sum_{j=n+1}^\infty a_j b_{n,j} g_j,$$

and (4.5) follows. □

We note that the formula (4.5) is independent of the weight $w = (w_n)_{n=1}^\infty$. Being able to express the action of $G^{(w)}$ in this way is important when investigating the full coagulation–fragmentation system, as it enables the fragmentation terms to be described by means of an explicit formula for the operator $G^{(w)}$. We shall return to this in a subsequent paper.

Example 4.3. Let us consider the system

$$\begin{aligned} u_n'(t) &= -(n-1)u_n(t) + 2\sum_{j=n+1}^\infty u_j(t), \qquad t > 0; \\ u_n(0) &= \mathring{u}_n, \qquad n = 1, 2, \ldots, \end{aligned} \tag{4.6}$$

which coincides with (1.1) if one sets

$$a_n = n - 1, \quad b_{n,j} = \frac{2}{j-1}, \qquad n, j \in \mathbb{N}, \quad j > n. \tag{4.7}$$

The system (4.6) models random scission; see, e.g. [25, equation (49)] and [14, equation (10)]. It is easily seen that (1.8) is satisfied, and hence, mass is conserved. The example (4.6) is closely related to the example that is studied in [19, §3] and which models random bond annihilation. More precisely, if we denote the operators for the example from [19] by $\widetilde{A}^{(w)}$, $\widetilde{B}^{(w)}$, $\widetilde{G}^{(w)}$ etc., then

$$A^{(w)} = \widetilde{A}^{(w)} + I, \quad B^{(w)} = \widetilde{B}^{(w)}, \quad G^{(w)} = \widetilde{G}^{(w)} + I,$$

and hence $S^{(w)}(t) = e^t \widetilde{S}^{(w)}(t)$, $t \geq 0$. For the particular case when $w_n = n$, $n \in \mathbb{N}$, we have similar relations for the operators A_1, $\widetilde{A}_1$ etc. It follows from [19, Lemma 3.6] that every $\lambda > 0$ is an eigenvalue of the maximal operator $G_{1,\mathsf{max}}$ [i.e. the operator $G^{(w)}_{\mathsf{max}}$ defined in (1.19) for $w_n = n$] with eigenvector $g^{(\lambda)} = (g_n^{(\lambda)})_{n\in\mathbb{N}}$ where

$$g_n^{(\lambda)} = \frac{1}{(\lambda + n - 1)(\lambda + n)(\lambda + n + 1)}, \qquad n \in \mathbb{N}. \tag{4.8}$$

The existence of positive eigenvalues of $G_{1,\mathsf{max}}$ implies that $G_{1,\mathsf{max}}$ is a proper extension of G_1. Note that the domain of $\widetilde{G}_1$ is determined explicitly in [19, Theorem 3.7], from which we obtain that

$$\mathcal{D}(G_1) = \left\{ f = (f_k)_{k\in\mathbb{N}} \in \mathcal{D}(G_{1,\mathsf{max}}) : \lim_{n\to\infty} \left(n^2 \sum_{k=n+1}^{\infty} f_k \right) = 0 \right\}. \tag{4.9}$$

Using the eigenvectors $g^{(\lambda)}$ from (4.8), we can define the function

$$u^{(\lambda)}(t) := e^{\lambda t} g^{(\lambda)}, \qquad t \geq 0,$$

which is a solution of the ACP

$$u'(t) = G_{1,\mathsf{max}} u(t), \quad t > 0; \qquad u(0) = \mathring{u} \tag{4.10}$$

with $\mathring{u} = g^{(\lambda)}$. On the other hand, since the semigroup $(S_1(t))_{t\geq 0}$ is analytic by [19, Theorem 3.4], the function $u(t) = S_1(t) g^{(\lambda)}$, $t \geq 0$, is also a solution of (4.10) and is distinct from $u^{(\lambda)}$. This shows that, in general, one does not have uniqueness of solutions of the ACP, (4.10), corresponding to the maximal operator, $G_{1,\mathsf{max}}$, and hence, also solutions of (1.1) are not unique.

More generally, a specific characterisation of $\mathcal{D}(G^{(w)})$ is given by Banasiak and Arlotti [9, Theorem 6.20], but this does not lead to an explicit description, such as that obtained in Example 4.3.

5. Analyticity of the fragmentation semigroup

In Sect. 3, we established that Assumptions 1.1 and 3.2 are sufficient conditions for $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ to be the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. This enabled us to obtain results on the existence and uniqueness of solutions to (1.17). We now investigate the analyticity of $(S^{(w)}(t))_{t\geq 0}$ and prove that, given *any* fragmentation coefficients, it is always possible to construct a weight, w, such that $A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup on ℓ^1_w. This particular result, which is one of the main motivations for carrying out an analysis of the fragmentation system in general weighted ℓ^1 spaces, requires a stronger assumption on the weight w. Note that when dealing with analytic semigroups, we use complex versions of the spaces ℓ^1_w.

Assumption 5.1. (i) $w_n \geq n$ for all $n \in \mathbb{N}$.
(ii) There exists $\kappa \in (0, 1)$ such that

$$\sum_{n=1}^{j-1} w_n b_{n,j} \leq \kappa w_j \qquad \text{for all } j = 2, 3, \ldots . \tag{5.1}$$

Note that Assumption 5.1 is obtained from Assumption 3.2 by simply replacing $\kappa \in (0, 1]$ with $\kappa \in (0, 1)$. By removing the possibility of $\kappa = 1$, we can obtain the following improved version of Theorem 3.4.

Theorem 5.2. *Let Assumptions 1.1 and 5.1 hold. Then, the operator $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ^1_w.*

Proof. Let $(T^{(w)}(t))_{t\geq 0}$ be as in Lemma 3.1. For $\alpha > 0$ and $f \in \mathcal{D}(A^{(w)})_+$, we obtain from (3.6) that

$$\begin{aligned}
&\int_0^\alpha \left\| B^{(w)} T^{(w)}(t) f \right\| \mathrm{d}t \\
&\quad \leq \kappa \int_0^\alpha \left\| A^{(w)} T^{(w)}(t) f \right\|_w \mathrm{d}t \\
&\quad = \kappa \int_0^\alpha \phi_w\big(-A^{(w)} T^{(w)}(t) f\big) \,\mathrm{d}t = \kappa \phi_w\left(-\int_0^\alpha A^{(w)} T^{(w)}(t) f \,\mathrm{d}t\right) \\
&\quad = \kappa \phi_w\left(-\int_0^\alpha \frac{\mathrm{d}}{\mathrm{d}t}\big(T^{(w)}(t) f\big) \,\mathrm{d}t\right) = \kappa \phi_w\big(f - T^{(w)}(\alpha) f\big) \\
&\quad = \kappa \|f\|_w - \kappa \|T^{(w)}(\alpha) f\|_w \leq \kappa \|f\|_w.
\end{aligned}$$

Since $\kappa < 1$, it follows from [20, Theorem A.2] that $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of a positive C_0-semigroup. The proof of [20, Theorem A.2] establishes that this semigroup is substochastic since $\kappa < 1$. Moreover, by Lemma 3.1, $A^{(w)}$ is also the generator of a substochastic C_0-semigroup, $(T^{(w)}(t))_{t\geq 0}$, on ℓ^1_w, and a routine calculation shows that

$$\left\| R(\lambda, A^{(w)}) f \right\|_w = \sum_{n=1}^{\infty} w_n \frac{1}{|\lambda + a_n|} |f_n| \leq \frac{1}{|\operatorname{Im} \lambda|} \|f\|_w, \lambda \in \mathbb{C}\backslash\mathbb{R} \text{ with } \operatorname{Re} \lambda > 0,$$

for all $f \in \ell^1_w$. Therefore, by [17, Theorem II.4.6], $(T^{(w)}(t))_{t\geq 0}$ is an analytic semigroup. Also, the positivity of $(S^{(w)}(t))_{t\geq 0}$ implies that $A^{(w)}+B^{(w)}$ is resolvent positive. Hence, by [2, Theorem 1.1], $(S^{(w)}(t))_{t\geq 0}$ is analytic. □

Remark 5.3. (i) Although Assumption 5.1 is never satisfied when (1.8) holds and $w_n = n$ for all $n \in \mathbb{N}$, this does not rule out the possibility of an analytic fragmentation semigroup on $X_{[1]}$ existing. Indeed, the semigroup $(S_1(t))_{t\geq 0}$ in Example 4.3 is analytic, which follows from [19, Theorem 3.4] as mentioned above.

(ii) If there exists $\lambda_0 > 0$ such that (1.5) holds with $\lambda_j \geq \lambda_0$ for all $j \geq 2$ (which corresponds to a 'uniform' mass loss case), then Assumption 5.1 immediately holds with $w_n = n$ for all $n \in \mathbb{N}$, and $\kappa = 1 - \lambda_0$.

The following lemma gives sufficient conditions under which Assumption 5.1 holds.

Lemma 5.4. *Let w be such that*

$$1 \leq \frac{w_n}{n} \leq \delta \frac{w_{n+1}}{n+1} \qquad \textit{for all } n \in \mathbb{N}, \tag{5.2}$$

where $\delta \in (0,1)$. Moreover, let (1.4) hold. Then, Assumption 5.1 is satisfied with $\kappa = \delta$.

Proof. Since

$$\frac{w_n}{n} \leq \delta^{j-n} \frac{w_j}{j} \leq \delta \frac{w_j}{j} \qquad \text{for all } n = 1, \ldots, j-1,$$

it follows that

$$\sum_{n=1}^{j-1} w_n b_{n,j} = \sum_{n=1}^{j-1} \frac{w_n}{n} n b_{n,j} \leq \delta \frac{w_j}{j} \sum_{n=1}^{j-1} n b_{n,j} \leq \delta w_j$$

for $j = 2, 3, \ldots$, where (1.4) is used to obtain the last inequality. Since $\delta \in (0,1)$, the result follows immediately. □

This leads to the main result of this section.

Theorem 5.5. *For any given fragmentation coefficients for which Assumption 1.1 holds, we can always find a weight, $w = (w_n)_{n=1}^{\infty}$, such that $A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup on ℓ_w^1. If, in addition, (1.4) holds, we can choose $w_n = r^n$ with arbitrary $r > 2$ and $\kappa = 2/r$ so that (5.1) holds.*

Proof. For the first statement, note that we can choose $w_n \geq n$ iteratively so that (5.1) is satisfied. The claim then follows from Theorem 5.2.

Now, assume that (1.4) holds. Let $r > 2$, $w_n = r^n$ for $n \in \mathbb{N}$, and $\delta = 2/r$, which satisfies $\delta < 1$. Then, $w_n \geq n$ and

$$\delta \frac{w_{n+1}}{n+1} = \frac{2}{r} \cdot \frac{r^{n+1}}{n+1} = \frac{2r^n}{n+1} \geq \frac{2r^n}{n+n} = \frac{r^n}{n} = \frac{w_n}{n},$$

which shows that (5.2) is satisfied. Hence, Lemma 5.4 implies that Assumption 5.1 is fulfilled. □

As mentioned earlier, analytic semigroups have a number of desirable properties, and Theorem 5.5 will play an important role when we investigate the full coagulation–fragmentation system in a subsequent paper. In particular, Theorem 5.5 will enable us to relax the usual assumptions that are imposed on the coagulation rates in order to obtain the existence and uniqueness of solutions to the full coagulation–fragmentation system.

It should be noted that a condition that is equivalent to Assumption 5.1 has previously been used as a condition for analyticity in the mass-conserving case by Banasiak; see [8, Theorem 2.1]. However, the choice of weights in [8] is restricted to $w_n = n^p$, $p > 1$, and Assumption 5.1 need not be satisfied for these weights for any $p > 1$ as the following example shows.

Example 5.6. Consider the mass-conserving case where a cluster of mass n breaks into two clusters, with respective masses 1 and $n-1$. The corresponding fragmentation coefficients take the form

$$b_{1,2} = 2; \quad b_{1,j} = b_{j-1,j} = 1,\ j \geq 3; \quad b_{n,j} = 0,\ 2 \leq n \leq j-2. \tag{5.3}$$

For the choice

$$a_0 = 0; \quad a_n = n, \quad n \geq 2; \qquad w_n = n^p, \quad n \in \mathbb{N}; \qquad p \geq 1,$$

it is proved in [7, Theorem 3] (for $p = 1$) and [8, Theorem A.3] (for $p > 1$) that the semigroup generated by $G^{(w)}$ is not analytic. On the other hand, Theorem 5.5 guarantees the existence of exponentially growing weights w_n such that $G^{(w)} = A^{(w)} + B^{(w)}$ generates an analytic semigroup. It is easy to show that for this particular example one can also choose powers of 2, namely $w_1 = 1$ and $w_n = 2^n$ for $n \geq 2$, in which case $\kappa = 5/8$.

6. Asymptotic behaviour of solutions

There have been several earlier investigations into the long-term behaviour of solutions to the mass-conserving fragmentation system (1.1), when (1.8) holds. In particular, the case of mass-conserving binary fragmentation is dealt with in [15], where it is shown that, under suitable assumptions, the unique solution emanating from $\mathring{u}$ must converge in the space $X_{[1]}$ to the expected steady-state solution $M_1(\mathring{u})e_1$, where $M_1(\mathring{u})$ and e_1 are given by (1.6) and (3.18), respectively. This was followed by [7], and [12] where, once again, the expected long-term steady-state behaviour is established, but now for the mass-conserving multiple fragmentation system. More specifically, in [7], a semigroup-based approach is used to prove that, for any $\mathring{u} \in X_{[1]}$,

$$\lim_{t\to\infty} \|S_1(t)\mathring{u} - M(\mathring{u})e_1\|_{[1]} = 0 \qquad \text{if and only if} \qquad a_n > 0 \ \text{ for all } n = 2, 3, \ldots.$$

That the corresponding result is also valid in the higher moment spaces $X_{[p]}$, $p > 1$, is established in [12], and, under additional assumptions on the fragmentation coefficients, it is shown in [12, Theorem 4.3] that there exist constants $L > 0$ and $\alpha > 0$ such that the fragmentation semigroup $(S_p(t))_{t\geq 0}$ on $X_{[p]}$, $p > 1$, satisfies

$$\|S_p(t)\mathring{u} - M_1(\mathring{u})e_1\|_{[p]} \leq Le^{-\alpha t}\|\mathring{u}\|_{[p]}, \tag{6.1}$$

for all $\mathring{u} \in X_{[p]}$. It follows from [3] that the fragmentation semigroup $(S_p(t))_{t\geq 0}$ has the asynchronous exponential growth (AEG) property (with $\lambda^* = 0$ in [3, equation (3)], i.e. with trivial growth). The assumptions required in [12] to prove that (6.1) holds in some $X_{[p]}$ space are somewhat technical and not straightforward to check. Moreover, no information on the size of the constant α, and hence the exponential rate of decay to the steady state is provided. Our aim in this section is to address these issues. Working within the framework of more general weighted ℓ^1 spaces, we study the long-term dynamics of solutions in both the mass-conserving and mass loss cases. When mass is conserved, we establish simpler conditions under which the fragmentation semigroup $(S^{(w)}(t))_{t\geq 0}$ satisfies an inequality of the form (6.1) on some space ℓ^1_w, and also quantify α.

We begin by considering the general fragmentation system (1.1), where the coefficients a_n and $b_{n,j}$ satisfy Assumption 1.1, and recall that $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ^1_w whenever Assumption 3.2 holds. Furthermore, $(S^{(w)}(t))_{t\geq 0}$ is analytic, with generator $A^{(w)} + B^{(w)}$ when the more restrictive Assumption 5.1 is satisfied.

Theorem 6.1. *Let Assumptions 1.1 and 3.2 hold.*

(i) *Then,*

$$\lim_{t\to\infty} \|S^{(w)}(t)\mathring{u}\|_w = 0 \tag{6.2}$$

for all $\mathring{u} \in \ell^1_w$ if and only if $a_n > 0$ for all $n \in \mathbb{N}$.

(ii) *If, additionally, we choose w such that Assumption 5.1 is satisfied, and set $a_0 := \inf_{n\in\mathbb{N}} a_n$, then*

$$\|S^{(w)}(t)\| \leq e^{-(1-\kappa)a_0 t}, \tag{6.3}$$

and hence, if $a_0 > 0$ and $\alpha \in [0, (1-\kappa)a_0)$, we have

$$\lim_{t\to\infty} e^{\alpha t}\|S^{(w)}(t)\mathring{u}\|_w = 0 \quad \textit{for every } \mathring{u} \in \ell^1_w. \tag{6.4}$$

If $\alpha > a_0$, then (6.4) does not hold. In particular, if $a_0 = 0$, then (6.4) does not hold for any $\alpha > 0$.

Proof. (i) First assume that $a_n > 0$ for all $n \in \mathbb{N}$. Let $\mathring{u} \in \ell^1_w$, and, as in Sect. 4, let $P_N\mathring{u} = (\mathring{u}_1, \mathring{u}_2, \ldots, \mathring{u}_N, 0, \ldots)$, $N \in \mathbb{N}$. For each fixed $n \in \mathbb{N}$, we know from (3.22) that $(S^{(w)}(t)e_n)_m = s_{m,n}(t) = 0$ for $m > n$. Furthermore, $(s_{1,n}, s_{2,n}, \ldots, s_{n,n})$, with the identification (3.21), is the unique solution of the n-dimensional system (3.19). Our assumption on the coefficients a_n means that all eigenvalues of the matrix associated with (3.19) are negative. It follows that $s_{m,n}(t) \to 0$ as $t \to \infty$ for $m = 1, \dots, n$, and therefore

$$\lim_{t\to\infty} \|S^{(w)}(t)e_n\|_w = \lim_{t\to\infty} \sum_{m=1}^{n} w_m s_{m,n}(t) = 0,$$

for all $n \in \mathbb{N}$. This in turn implies that

$$\|S^{(w)}(t)P_N\mathring{u}\|_w \leq \sum_{n=1}^{N} |\mathring{u}_n| \|S^{(w)}(t)e_n\|_w \to 0 \qquad \text{as } t \to \infty,$$

for each $N \in \mathbb{N}$. Given any $\varepsilon > 0$, we can always find $N \in \mathbb{N}$ and $t_0 > 0$ such that

$$\|\mathring{u} - P_N\mathring{u}\|_w < \frac{\varepsilon}{2} \qquad \text{and} \qquad \|S^{(w)}(t)P_N\mathring{u}\|_w < \frac{\varepsilon}{2} \quad \text{for all } t \geq t_0.$$

Then,

$$\begin{aligned}\|S^{(w)}(t)\mathring{u}\|_w &\leq \left\|S^{(w)}(t)\big(\mathring{u} - P_N\mathring{u}\big)\right\|_w + \|S^{(w)}(t)P_N\mathring{u}\|_w \\ &\leq \|\mathring{u} - P_N\mathring{u}\|_w + \|S^{(w)}(t)P_N\mathring{u}\|_w < \varepsilon \qquad \text{for all } t \geq t_0,\end{aligned}$$

which establishes (6.2).

On the other hand, suppose that $a_N = 0$ for some $N \in \mathbb{N}$. Then, we have that the unique solution of (1.17), with $\mathring{u} = e_N$, is $u(t) = S^{(w)}(t)e_N = (s_{m,N}(t))_{m=1}^{\infty}$. Since $s_{N,N}(t) = e^{-a_N t} = 1$, it is clear that $u(t) \not\to 0$ as $t \to \infty$.

(ii) Now, let Assumption 5.1 hold and let $\mathring{u} \in (\ell_w^1)_+$. From Theorem 5.2, $A^{(w)} + B^{(w)}$ generates an analytic, substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1, and $u(t) = S^{(w)}(t)\mathring{u}$ is the unique, non-negative classical solution of (1.17). Let $t > 0$. Using (3.5), we obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi_w\big(u(t)\big) &= \phi_w\big(u'(t)\big) = \phi_w\big(A^{(w)}u(t)\big) + \phi_w\big(B^{(w)}u(t)\big) \\ &\leq \phi_w\big(A^{(w)}u(t)\big) - \kappa\phi_w\big(A^{(w)}u(t)\big) \\ &= -(1-\kappa)\sum_{n=1}^{\infty} w_n a_n u_n(t) \\ &\leq -(1-\kappa)a_0\phi_w\big(u(t)\big).\end{aligned}$$

Therefore,

$$\phi_w\big(u(t)\big) \leq \phi_w(\mathring{u})e^{-(1-\kappa)a_0 t} \quad \text{and hence} \quad \|S^{(w)}(t)\mathring{u}\|_w \leq e^{-(1-\kappa)a_0 t}\|\mathring{u}\|_w,$$

and (6.3) then follows from the positivity of $(S^{(w)}(t))_{t\geq 0}$ and [9, Proposition 2.67]. If $a_0 > 0$ and $\alpha \in [0, (1-\kappa)a_0)$, then (6.4) holds.

On the other hand, if we choose $\alpha > a_0$, then there exists $N \in \mathbb{N}$ such that $a_N < \alpha$, in which case $(S^{(w)}(t)e_N)_N = e^{-a_N t} > e^{-\alpha t}$ for $t > 0$, and so

$$e^{\alpha t}\|S^{(w)}(t)e_N\|_w \geq e^{\alpha t} w_N \big(S^{(w)}(t)e_N\big)_N > e^{\alpha t} w_N e^{-\alpha t} = w_N.$$

Hence, (6.4) cannot hold for any $\alpha > a_0$. □

Remark 6.2. When the assumptions of Theorem 6.1 are satisfied and $a_n > 0$ for all $n \in \mathbb{N}$, then (6.2) shows that the only equilibrium solution of (1.17) is $u(t) \equiv 0$, and this equilibrium is a global attractor for the system. On the other hand, if $a_n = 0$ for at least one $n \in \mathbb{N}$, then $u(t) \equiv 0$ is not a global attractor.

We now examine the mass-conserving case and assume that (1.8) holds. Note that, in this mass-conserving case, the fragmentation semigroup $(S_1(t))_{t\geq 0}$ is stochastic on the space $X_{[1]}$. Our aim is to establish an ℓ^1_w version of the results obtained in [7,12,15]. To this end, we recall the matrix representation of $S^{(w)}(t)$ given by (3.22) and also define a sequence space $Y^{(w)}$, and its norm $\|\cdot\|_{Y^{(w)}}$, by

$$Y^{(w)} = \{\tilde{f} = (f_n)_{n=2}^{\infty} : f = (f_n)_{n=1}^{\infty} \in \ell^1_w\} \quad \text{and} \quad \|f\|_{Y^{(w)}} = \sum_{n=2}^{\infty} w_n |f_n|,$$

respectively. Clearly, $Y^{(w)}$ is a weighted ℓ^1 space and can be identified with $\ell^1_{\widehat{w}}$, where $\widehat{w}_n = w_{n+1}$ for $n \in \mathbb{N}$. Moreover, we define the embedding operator $J : Y^{(w)} \to \ell^1_w$ by

$$Jf = (0, f_2, f_3, \ldots) \quad \text{for all } f \in \ell^1_w.$$

Lemma 6.3. *Let $\alpha \geq 0$ and $f \in \ell^1_w$ be fixed, and define $\tilde{f} := (f_n)_{n=2}^{\infty}$. If Assumptions 1.1, 3.2 and (1.8) hold, then*

$$\|S^{(w)}_{(22)}(t)\tilde{f}\|_{Y^{(w)}} \leq \|S^{(w)}(t)f - M_1(f)e_1\|_w \leq (w_1 + 1)\|S^{(w)}_{(22)}(t)\tilde{f}\|_{Y^{(w)}} \tag{6.5}$$

for all $t \geq 0$.

Proof. It follows from (3.22) that

$$S^{(w)}(t)f = \big(f_1 + S^{(w)}_{(12)}(t)\tilde{f}\big)e_1 + JS^{(w)}_{(22)}(t)\tilde{f}. \tag{6.6}$$

From this, we deduce that

$$\|S^{(w)}(t)f - M_1(f)e_1\|_w = w_1\Big|f_1 + S^{(w)}_{(12)}(t)\tilde{f} - M_1(f)\Big| + \|S^{(w)}_{(22)}(t)\tilde{f}\|_{Y^{(w)}} \tag{6.7}$$

and so

$$\|S^{(w)}(t)f - M_1(f)e_1\|_w \geq \|S^{(w)}_{(22)}(t)\tilde{f}\|_{Y^{(w)}},$$

which is the first inequality in (6.5).

On the other hand, from Proposition 2.3 (i) and the stochasticity of $(S_1(t))_{t\geq 0}$ on $X_{[1]}$, we know that $M_1(S^{(w)}(t)f) = M_1(S_1(t)f) = M_1(f)$. Using (6.6), we obtain

$$\begin{aligned}\left|f_1 + S^{(w)}_{(12)}(t)\tilde{f} - M_1(f)\right| &= \left|M_1(f) - M_1\Big(\big(f_1 + S^{(w)}_{(12)}(t)\tilde{f}\big)e_1\Big)\right| \\ &= \left|M_1\big(S^{(w)}(t)f\big) - M_1\Big(\big(f_1 + S^{(w)}_{(12)}(t)\tilde{f}\big)e_1\Big)\right| \\ &\le M_1\Big(\left|S^{(w)}(t)f - \big(f_1 + S^{(w)}_{(12)}(t)\tilde{f}\big)e_1\right|\Big) \\ &\le \phi_w\Big(\left|S^{(w)}(t)f - \big(f_1 + S^{(w)}_{(12)}(t)\tilde{f}\big)e_1\right|\Big) \\ &= \phi_w\Big(\left|(J S^{(w)}_{(22)}(t)\tilde{f}\right|\Big) \\ &= \|S^{(w)}_{(22)}(t)\tilde{f}\|_{Y^{(w)}}.\end{aligned}$$

The second inequality in (6.5) then follows from (6.7). □

We are now in a position to prove the main theorem of this section. The first part confirms that $S^{(w)}(t)\mathring{u} \to M_1(\mathring{u})e_1$ in ℓ^1_w as $t \to \infty$, for all $\mathring{u} \in \ell^1_w$, provided that Assumption 3.2 holds and the fragmentation rates, a_n, are positive for all $n \ge 2$. In the second part, which deals with quantifying the rate of convergence to equilibrium, the fragmentation coefficients are assumed additionally to be bounded below by a positive constant, and Assumption 3.2 is strengthened to Assumption 5.1. In this case, the decay to zero of $\|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_w$ is shown to occur at an exponential rate, defined explicitly in terms of the rate coefficients and the constant $\kappa \in (0, 1)$ in Assumption 5.1.

Theorem 6.4. *Let Assumptions 1.1 and 3.2, and (1.8) hold and let M_1 be as in (1.6).*

(i) *We have*

$$\lim_{t\to\infty} \|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_w = 0 \tag{6.8}$$

for all $\mathring{u} \in \ell^1_w$ if and only if $a_n > 0$ for all $n \ge 2$.

(ii) *Choose w such that Assumption 5.1 holds and let $\widehat{a}_0 := \inf_{n\in\mathbb{N}:n\ge 2} a_n$. Then, for all $\mathring{u} \in \ell^1_w$,*

$$\|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_w \le (w_1 + 1)e^{-(1-\kappa)\widehat{a}_0 t}\|\mathring{u}\|_w, \tag{6.9}$$

and so

$$\lim_{t\to\infty} e^{\alpha t}\|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_w = 0, \tag{6.10}$$

whenever $\widehat{a}_0 > 0$ and $\alpha \in [0, (1-\kappa)\widehat{a}_0)$.

Equation (6.10) does not hold for any $\alpha > \widehat{a}_0$. In particular, if $\widehat{a}_0 = 0$, then (6.10) does not hold for any $\alpha > 0$.

Proof. Removing the equation for u_1 from (1.1) leads to a reduced fragmentation system that can be formulated as an ACP in $Y^{(w)} = \ell^1_{\widehat{w}}$, where, as before, $\widehat{w}_n = w_{n+1}$ for all $n \in \mathbb{N}$. The fragmentation coefficients, $(\widehat{a}_n)_{n=1}^\infty$ and $(\widehat{b}_{n,j})_{n,j\in\mathbb{N}:n<j}$, associated with the reduced system are given by $\widehat{a}_n = a_{n+1}$ and $\widehat{b}_{n,j} = b_{n+1,j+1}$. Clearly, $\widehat{a}_n \ge 0$

and $\widehat{b}_{n,j} \geq 0$ for all $n, j \in \mathbb{N}$ and $\widehat{b}_{n,j} = 0$ if $n \geq j$, and $\widehat{w}_n = w_{n+1} \geq n+1 > n$ for all $n \in \mathbb{N}$. Moreover, for $j = 2, 3, \ldots$,

$$\sum_{n=1}^{j-1} \widehat{w}_n \widehat{b}_{n,j} = \sum_{n=1}^{j-1} w_{n+1} b_{n+1,j+1} = \sum_{k=2}^{j} w_k b_{k,j+1} \leq \sum_{k=1}^{j} w_k b_{k,j+1}$$
$$\leq \kappa w_{j+1} = \kappa \widehat{w}_j.$$

Hence, Assumptions 1.1 and 3.2 are satisfied by $\widehat{w}$, $\widehat{a}_n$ and $\widehat{b}_{n,j}$, and it follows from Theorem 3.4 and (3.22) that associated with the reduced system is a substochastic C_0-semigroup on $Y^{(w)}$, which can be represented by the infinite matrix

$$\begin{bmatrix} e^{-\widehat{a}_1 t} & \widehat{s}_{1,2}(t) & \widehat{s}_{1,3}(t) & \cdots \\ 0 & e^{-\widehat{a}_2 t} & \widehat{s}_{2,3}(t) & \cdots \\ 0 & 0 & e^{-\widehat{a}_3 t} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} e^{-a_2 t} & \widehat{s}_{1,2}(t) & \widehat{s}_{1,3}(t) & \cdots \\ 0 & e^{-a_3 t} & \widehat{s}_{2,3}(t) & \cdots \\ 0 & 0 & e^{-a_4 t} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{6.11}$$

where, for all $n \in \mathbb{N}$, $m = 1, \ldots, n-1$, $t \geq 0$, $\widehat{s}_{m,n}(t)$ is the unique solution of

$$\widehat{s}'_{m,n}(t) = -\widehat{a}_m \widehat{s}_{m,n}(t) + \sum_{j=m+1}^{n} \widehat{a}_j \widehat{b}_{m,j} \widehat{s}_{j,n}(t)$$
$$= -a_{m+1} \widehat{s}_{m,n}(t) + \sum_{k=m+2}^{n+1} a_k b_{m+1,k} \widehat{s}_{k-1,n}(t).$$

An inspection of (3.19), together with (3.21), shows that $\widehat{s}_{m,n}(t) = s_{m+1,n+1}(t)$ for all $n \in \mathbb{N}$, $m = 1, \ldots, n-1$, $t \geq 0$, and therefore the substochastic semigroup on $Y^{(w)}$ is given by $(S^{(w)}_{(22)}(t))_{t \geq 0}$, where $(S^{(w)}_{(22)}(t))_{t \geq 0}$ is the infinite matrix that features in (3.22).

(i) Let $\widehat{\mathring{u}} = (\mathring{u}_2, \mathring{u}_3, \ldots)$ for each $\mathring{u} \in \ell^1_w$. From Theorem 6.1, we deduce that

$$\lim_{t \to \infty} \left\| S^{(w)}_{(22)}(t) \widehat{\mathring{u}} \right\|_{Y^{(w)}} = \lim_{t \to \infty} \left\| S^{(w)}_{(22)}(t) \widehat{\mathring{u}} \right\|_{\widehat{w}} = 0,$$

if and only if $a_n > 0$ for all $n \geq 2$, and the result is then an immediate consequence of Lemma 6.3.

(ii) The calculations above show that, when Assumption 3.2 holds for w and the coefficients $(b_{n,j})$, it is also satisfied by $\widehat{w}$ and $(\widehat{b}_{n,j})$ with exactly the same value of κ. Therefore, from Theorem 6.1,

$$\| S^{(w)}_{(22)}(t) \| \leq e^{-(1-\kappa)\widehat{a}_0 t},$$

and (6.9) follows immediately from Lemma 6.3. Moreover, if $\widehat{a}_0 > 0$ and $\alpha \in [0, (1-\kappa)\widehat{a}_0)$, then we obtain (6.10).

If $\alpha > \widehat{a}_0$, then, from Theorem 6.1, the result

$$\lim_{t \to \infty} e^{\alpha t} \left\| S^{(w)}_{(22)}(t) \widehat{\mathring{u}} \right\|_{Y^{(w)}} = \lim_{t \to \infty} e^{\alpha t} \left\| S^{(w)}_{(22)}(t) \widehat{\mathring{u}} \right\|_{\widehat{w}} = 0,$$

does not hold for all $\mathring{u} \in \ell^1_w$. Hence, from Lemma 6.3, (6.10) does not hold if $\alpha > \widehat{a}_0$. □

Remark 6.5. When the assumptions of Theorem 6.4 are satisfied, then it follows from (3.22) that $\overline{u}_M = Me_1$ is an equilibrium solution of the mass-conserving fragmentation system for all $M \in \mathbb{R}$. In addition, the basin of attraction for $\overline{u}_M$ is given by $\{\mathring{u} \in \ell_w^1 : M_1(\mathring{u}) = M\}$ provided that the assumptions of Theorem 6.4 hold and $a_n > 0$ for all $n \geq 2$. On the other hand, if $a_N = 0$ for some $N \geq 2$, then Me_N is also an equilibrium solution for every $M \in \mathbb{R}$.

7. Sobolev towers

In this section, we use a Sobolev tower construction to obtain existence and uniqueness results relating to the pure fragmentation system for a larger class of initial conditions. Sobolev towers appear to have been first applied to the discrete fragmentation system (1.1) in [19], where the authors examine a specific example and use Sobolev towers to explain an apparent non-uniqueness of solutions. As we demonstrate below, the theory of Sobolev towers is applicable to more general fragmentation systems and, in the following, the only restrictions that are imposed are that the fragmentation coefficients satisfy Assumption 1.1, and also that a weight, $w = (w_n)_{n=1}^{\infty}$, has been chosen so that Assumption 5.1 holds. These restrictions imply that $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. Let ω_0 be the growth bound of $(S^{(w)}(t))_{t\geq 0}$. Choosing $\mu > \omega_0$, we rescale $(S^{(w)}(t))_{t\geq 0}$ to obtain an analytic semigroup, $(\mathscr{S}^{(w)}(t))_{t\geq 0} = (e^{-\mu t}S^{(w)}(t))_{t\geq 0}$, with a strictly negative growth bound. The generator of $(\mathscr{S}^{(w)}(t))_{t\geq 0}$ is $\mathcal{G}^{(w)} = G^{(w)} - \mu I$. We set $X_0^{(w)} = \ell_w^1$, $\|\cdot\|_0 := \|\cdot\|_w$, $\mathscr{S}_0^{(w)}(t) = \mathscr{S}^{(w)}(t)$, $S_0^{(w)}(t) = S^{(w)}(t)$, and $\mathcal{G}_0^{(w)} = \mathcal{G}^{(w)}$.

As described in [17, §II.5(a)], $(\mathscr{S}^{(w)}(t))_{t\geq 0}$ can be used to construct a Sobolev tower, $(X_n^{(w)})_{n\in\mathbb{N}}$, via

$$X_n^{(w)} := \left(\mathcal{D}\big((\mathcal{G}^{(w)})^n\big), \|\cdot\|_n\right); \qquad \|f\|_n = \left\|(\mathcal{G}^{(w)})^n f\right\|_w, \quad f \in \mathcal{D}\big((\mathcal{G}^{(w)})^n\big), \quad n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, $X_n^{(w)}$ is referred to as the Sobolev space of order n associated with the semigroup $(\mathscr{S}^{(w)}(t))_{t\geq 0}$. We also define the operator $\mathcal{G}_n^{(w)} : X_n^{(w)} \supseteq \mathcal{D}(\mathcal{G}_n^{(w)}) \to X_n^{(w)}$ to be the restriction of $\mathcal{G}^{(w)}$ to

$$\mathcal{D}(\mathcal{G}_n^{(w)}) = \left\{f \in X_n^{(w)} : \mathcal{G}^{(w)} f \in X_n^{(w)}\right\} = \mathcal{D}\big((\mathcal{G}^{(w)})^{n+1}\big) = X_{n+1}^{(w)},$$

for each $n \in \mathbb{N}$.

Sobolev spaces of negative order, $-n$, $n \in \mathbb{N}$, are defined recursively by

$$X_{-n}^{(w)} = \left(X_{-n+1}^{(w)}, \|\cdot\|_{-n}\right)^{\sim}; \qquad \|f\|_{-n} = \left\|(\mathcal{G}_{-n+1}^{(w)})^{-1} f\right\|_{-n+1}, \quad f \in X_{-n+1}^{(w)}, \tag{7.1}$$

where $(X, \|\cdot\|)^{\sim}$ denotes the completion of the normed vector space $(X, \|\cdot\|)$. Operators $\mathcal{G}_{-n}^{(w)}$ can then be obtained in a similar recursive manner for each $n \in \mathbb{N}$, with $\mathcal{G}_{-n}^{(w)}$ defined as the unique extension of $\mathcal{G}_{-n+1}^{(w)}$ from $\mathcal{D}(\mathcal{G}_{-n+1}^{(w)}) = X_{-n+2}^{(w)}$ to $\mathcal{D}(\mathcal{G}_{-n}^{(w)}) = X_{-n+1}^{(w)}$; see [17, §II.5(a)].

From [17, §II.5(a)], it follows that $\mathcal{G}_n^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup, $(\mathscr{S}_n^{(w)}(t))_{t\geq 0}$, on $X_n^{(w)}$ for all $n \in \mathbb{Z}$, where $\mathscr{S}_{-n}^{(w)}(t)$ is the unique, continuous extension of $\mathscr{S}^{(w)}(t)$ from $X_0^{(w)}$ to $X_{-n}^{(w)}$ for each $t \geq 0$ and $n \in \mathbb{N}$. Since $\mathscr{S}^{(w)}(t) = e^{-\mu t} S^{(w)}(t)$, we also obtain the analytic, substochastic C_0-semigroup, $(S_{-n}^{(w)}(t))_{t\geq 0}$, defined on $X_{-n}^{(w)}$ by $S_{-n}^{(w)}(t) = e^{\mu t}\mathscr{S}_{-n}^{(w)}(t)$. More generally, it is known that $\mathscr{S}_n^{(w)}(t)$ is the unique, continuous extension of $\mathscr{S}_m^{(w)}(t)$ from $X_m^{(w)}$ to $X_n^{(w)}$ when $m, n \in \mathbb{Z}$ with $m \geq n$. The analyticity of $(\mathscr{S}_n^{(w)}(t))_{t\geq 0}$ on $X_n^{(w)}$, also enables us to prove the following key result.

Lemma 7.1. *Let $\mathring{u} \in X_n^{(w)}$ for some fixed $n \in \mathbb{Z}$. Then, $\mathscr{S}_n^{(w)}(t)\mathring{u} \in X_m^{(w)}$ for all $m \geq n$ and $t > 0$.*

Proof. It is obvious that $\mathscr{S}_n^{(w)}(t)\mathring{u} \in X_n^{(w)}$ for all $t \geq 0$ and $\mathring{u} \in X_n^{(w)}$. Also, if $\mathscr{S}_n^{(w)}(t)\mathring{u} \in X_m^{(w)}$ for some $m \geq n$ and all $t > 0$, then, on choosing $t_0 \in (0, t)$, we have

$$\mathscr{S}_n^{(w)}(t)\mathring{u} = \mathscr{S}_m^{(w)}(t - t_0)\mathscr{S}_n^{(w)}(t_0)\mathring{u} \in \mathcal{D}(\mathcal{G}_m^{(w)}) = X_{m+1}^{(w)},$$

where we have used the fact that $\mathscr{S}_n^{(w)}(t)$ and $\mathscr{S}_m^{(w)}(t)$ coincide on $X_m^{(w)}$ together with the analyticity of $\mathscr{S}_m^{(w)}(t)$. The result then follows by induction. □

We can now prove the following result regarding the solvability of (1.17).

Theorem 7.2. *Let Assumptions 1.1 and 5.1 hold. Further, let $n \in \mathbb{N}$. Then, the ACP (1.17) has a unique, non-negative solution $u \in C^1((0,\infty), \ell_w^1) \cap C([0,\infty), X_{-n}^{(w)})$ for all $\mathring{u} \in (X_{-n}^{(w)})_+$. This solution is given by $u(t) = S_{-n}^{(w)}(t)\mathring{u},\ t \geq 0$.*

Proof. Let $\mathring{u} \in (X_{-n}^{(w)})_+$ and let $u(t) = S_{-n}^{(w)}(t)\mathring{u} = e^{\mu t} v(t)$, $t \geq 0$, where $v(t) = \mathscr{S}_{-n}^{(w)}(t)\mathring{u}$. Then, $v \in C^1((0,\infty), X_{-n}^{(w)}) \cap C([0,\infty), X_{-n}^{(w)})$ is the unique classical solution of

$$v'(t) = \mathcal{G}_{-n}^{(w)} v(t), \quad t > 0; \qquad v(0) = \mathring{u}. \tag{7.2}$$

Also, from Lemma 7.1, $\mathscr{S}_{-n}^{(w)}(t)\mathring{u} \in X_1^{(w)} = \mathcal{D}(\mathcal{G}^{(w)})$ for all $t > 0$. Since $(\mathscr{S}_{-n}^{(w)}(t))_{t\geq 0}$ coincides with $(\mathscr{S}^{(w)}(t))_{t\geq 0}$ on $\mathcal{D}(\mathcal{G}^{(w)})$, it follows that

$$\mathscr{S}_{-n}^{(w)}(t)\mathring{u} = \mathscr{S}^{(w)}(t - t_0)\mathscr{S}_{-n}^{(w)}(t_0)\mathring{u}, \qquad \text{where } t_0 \in (0, t).$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathscr{S}_{-n}^{(w)}(t)\mathring{u}\big) = \mathcal{G}^{(w)}\mathscr{S}^{(w)}(t - t_0)\mathscr{S}_{-n}^{(w)}(t_0)\mathring{u} = \mathcal{G}^{(w)}\mathscr{S}_{-n}^{(w)}(t)\mathring{u}, \qquad t > 0,$$

where the derivative is with respect to the norm on $X_0^{(w)} = \ell_w^1$. This establishes that $u \in C^1((0,\infty), \ell_w^1) \cap C([0,\infty), X_{-n}^{(w)})$ and also that u satisfies (1.17). The non-negativity of u follows from the substochasticity of the semigroups.

For uniqueness, we observe first that the construction of the Sobolev tower ensures that $X_0^{(w)}$ is continuously embedded in $X_{-n}^{(w)}$. Moreover, $\mathcal{G}^{(w)}$ is the restriction of $\mathcal{G}_{-n}^{(w)}$

to $X_1^{(w)} = \mathcal{D}(\mathcal{G}^{(w)})$. Consequently, if $u_1, u_2 \in C^1((0,\infty), \ell_w^1) \cap C([0,\infty), X_{-n}^{(w)})$ both satisfy (1.17), and we set $v_i(t) = e^{-\mu t} u_i(t)$, $i = 1, 2$, then the difference $v_1 - v_2$ is the unique classical solution of (7.2) with $\mathring{u} = 0$, and so $v_1 = v_2$, from which it follows that $u_1 = u_2$. □

Finally, we make the following remark on the solvability of (3.2).

Remark 7.3. For fixed $n \in \mathbb{N}$, the previous theorem establishes that the ACP (1.17) has a unique, non-negative solution $u \in C^1((0,\infty), \ell_w^1) \cap C([0,\infty), X_{-n}^{(w)})$, given by $u(t) = S_{-n}^{(w)}(t)\mathring{u}$, for all $\mathring{u} \in (X_{-n}^{(w)})_+$, provided that Assumptions 1.1 and 5.1 are satisfied. Recalling that we also assume that $w_n \geq n$ for all $n \in \mathbb{N}$, we have that ℓ_w^1 is continuously embedded in $X_{[1]}$, and from this we deduce that if $u(t)$ is differentiable with respect to the norm on ℓ_w^1 then it is also differentiable with respect to the norm on $X_{[1]}$, and the derivatives coincide. Since $A_1 + B_1$ is an extension of $G^{(w)} = A^{(w)} + B^{(w)}$, we conclude that $u(t) = S_{-n}^{(w)}(t)\mathring{u}$ also satisfies (3.2).

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