Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations

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Abstract

Given an unstable hybrid neutral stochastic differential equation (NSDE), can we design a delay feedback control to make the controlled hybrid NSDE become stable? It has been proved that this is possible under the linear growth condition. However, there is no answer to the question if the drift and diffusion coefficients of the given unstable NSDE satisfy highly nonlinear growth condition. The aim of this paper is to design delay feedback controls in order to stabilise a class of highly nonlinear hybrid NSDEs whose coefficients satisfy the polynomial growth condition.

Keywords: Neutral stochastic differential equation, delay feedback control, highly nonlinear, asymptotic stability, Markovian switching.

1. Introduction

Many stochastic dynamical systems do not only depend on present and past states but also involve derivatives with delays. Neutral stochastic differential equations are often used to model such systems. On other hand, many systems in the real word may experience abrupt changes in their structures and parameters due to sudden changes of system factors. NSDEs with Markovian switching (also known as hybrid NSDEs) form an important class of hybrid dynamical systems. They have been successfully applied in practice, such as in traffic control, switching power converters, neural networks, and so on (see, e.g., [1, 9, 20]). The hybrid NSDEs can be described by

$$d[x(t) - D(x(t-\tau),r(t),t)] = f(x(t),x(t-\tau),r(t),t)dt + g(x(t),x(t-\tau),r(t),t)dB(t),$$

where $x(t) \in R^n$ is the state, τ stands for time delay, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1,2,\cdots,N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$.

One of the important issue in the study of hybrid NSDEs is the analysis of stability (see, e.g., [7, 8, 10, 18, 21, 23, 24]). In the case when a given hybrid NSDE is unstable, it is classical to find a feedback control $u(x(t),r(t),t)$, based on the current state $x(t)$, for the controlled system

$$d[x(t)-D(x(t-\tau),r(t),t)] = [f(x(t),x(t-\tau),r(t),t) + u(x(t),r(t),t)]dt + g(x(t),x(t-\tau),r(t),t)dB(t)$$

to become stable. However, taking into account a time lag δ ($\delta > 0$) between the time when the observation of the state is made and the time when the feedback control reaches the system, it is more realistic that the control depends on a past state $x(t-\delta)$ (see, e.g., [2, 3, 17, 25]). Accordingly, we assume that the control is the form $u(x(t-\delta),r(t),t)$. In this paper, we assume that $\delta \le \tau$. Hence, the stabilisation problem becomes to design a delay feedback control $u(x(t-\delta),r(t),t)$ for the controlled system

$$d[x(t)-D(x(t-\tau),r(t),t)] = [f(x(t),x(t-\tau),r(t),t) + u(x(t-\delta),r(t),t)]dt + g(x(t),x(t-\tau),r(t),t)dB(t)$$

to be stable. Mao et al. [15] were the first to study stabilisation problem by the delay feedback control for hybrid SDEs and there have been some further developments since then (see, e.g., [14, 26]). Chen et al. investigated the stabilisation problem for hybrid NSDEs by the delay feedback control(see, e.g., [2, 3]). The common restrict condition imposed in these existing papers is that both drift coefficient and diffusion one need to satisfy the linear growth condition. In fact, this restrict condition excludes many

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NSDE models in the real world, for example, the following scalar hybrid NSDE

$$d[x(t) - D(x(t-\tau), r(t), t)] = f(x(t), x(t-\tau), r(t), t)dt + g(x(t), x(t-\tau), r(t), t)dB(t)$$

where

$$\begin{aligned} f(x, y, 1, t) &= 0.5x + y^3 - 6x^3, \\ f(x, y, 2, t) &= x + y^3 - 4x^3, \\ g(x, y, 1, t) &= g(x, y, 2, t) = 0.5y^2. \end{aligned} \quad (1)$$

It is therefore necessary and important to establish a new theory which shows how to design delay feedback controls in order to stabilise highly nonlinear hybrid NSDEs. Recently, a number of new results of feedback control are obtained for highly nonlinear SDEs. For example, Lu et al. [12] explored stabilisation of highly nonlinear hybrid SDEs by delay feedback control, while Fei et al.[4] discussed stabilisation of highly nonlinear hybrid SDEs based on discrete-time observation feedback control. However, the unstable systems they considered are highly nonlinear SDEs without delay. Only very recently Li and Mao [11] established a new theory on the stabilisation by delay feedback control for highly nonlinear SDDEs. Unfortunately, there is so far no answer to the question if the given system is highly nonlinear hybrid delay NSDEs. In this paper, we will explore the stabilisation of highly nonlinear hybrid NSDEs by delay feedback control. The key challenge of this paper lies in the difficulties arisen from the highly nonlinear drift and diffusion coefficients.

2. Notation and Assumption

Throughout this paper, unless otherwise specified, we use the following notation. If $x \in R^n$, then $|x|$ is its Euclidean norm. For $\tau > 0$, denote by $C([-\tau, 0]; R^n)$ the family of continuous functions φ from $[-\tau, 0] \to R^n$ with the norm $\|\varphi\| = \sup_{-\tau \le u \le 0} |\varphi(u)|$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

In order to make our stability analysis more understandable, we will only consider the simple case in this paper where the neutral term D is independent of the mode and time. That is we consider the underlying unstable system which is described by following highly nonlinear hybrid NSDE

$$d[x(t) - D(x(t-\tau))] = f(x(t), x(t-\tau), r(t), t)dt + g(x(t), x(t-\tau), r(t), t)dB(t) \quad (2)$$

on $t \ge 0$ with initial data

$$\{x(t) : -\tau \le t \le 0\} = \varphi \in C([-\tau, 0]; R^n) \text{ and } r(0) = r_0 \in S, \quad (3)$$

where

$$f : R^n \times R^n \times S \times R_+ \to R^n, \quad g : R^n \times R^n \times S \times R_+ \to R^{n\times m}, \quad D : R^n \to R^n$$

are Borel measurable functions. For the sake of simplicity, we denote $\tilde{x}(t) = x(t) - D(x(t-\tau))$. Hence, we are required to design a feedback control $u(x(t-\delta), r(t), t)$ in the drift part so that the controlled system

$$d\tilde{x}(t) = [f(x(t), x(t-\tau), r(t), t) + u(x(t-\delta), r(t), t)]dt + g(x(t), x(t-\tau), r(t), t)dB(t) \quad (4)$$

becomes stable.

The well-known conditions imposed for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition [13, 16, 22]. In this paper, we need the local Lipschitz condition . However, we impose the polynomial growth condition instead of the linear growth condition [5, 6]. Let us state these conditions as an assumption for the use of this paper.

Assumption 2.1. *Assume that for any $h > 0$, there exists a positive constant K_h such that*

$$|f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)| \vee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)| \le K_h(|x - \bar{x}| + |y - \bar{y}|) \quad (5)$$

for all $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le h$ and all $(i, t) \in S \times R_+$. Assume also that there exist three constants $K > 0$, $q_1 \ge 1$ and $q_2 \ge 1$ such that

$$\begin{aligned} |f(x, y, i, t)| &\le K(1 + |x|^{q_1} + |y|^{q_1}), \\ |g(x, y, i, t)| &\le K(1 + |x|^{q_2} + |y|^{q_2}) \end{aligned} \quad (6)$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$*. Assume moreover that there is a constant* $\kappa \in (0, 1)$ *such that*

$$|D(u) - D(v)| \le \kappa |u - v| \qquad (7)$$

for all $u, v \in R$*, and* $D(0) = 0$*.*

If $q_1 = q_2 = 1$, then condition (6) is the familiar linear growth condition. However, we emphasise once again that we are here interested in highly nonlinear NSDEs which have either $q_1 > 1$ or $q_2 > 1$. We will refer to condition (6) as the polynomial growth condition.

Let $C^{2,1}(R^n \times S \times R_+; R_+)$ denote the family of non-negative functions $U(x, i, t)$ on $R^n \times S \times R_+$, which are continuously twice differentiable in x and once in t. Let

$$U_t(x, i, t) = \frac{\partial U(x, i, t)}{\partial t},$$
$$U_x(x, i, t) = \Big(\frac{\partial U(x, i, t)}{\partial x_1}, \cdots, \frac{\partial U(x, i, t)}{\partial x_d}\Big)$$
$$U_{xx}(x, i, t) = \Big(\frac{\partial^2 U(x, i, t)}{\partial x_k \partial x_l}\Big)_{d\times d}.$$

Define an operator $\mathbb{L}\bar{U} : R^n \times R^n \times S \times R_+ \to R$ by

$$\begin{aligned}&\mathbb{L}\bar{U}(x, y, i, t) = \bar{U}_t(x - D(y), i, t)\\&+ \bar{U}_x(x - D(y), i, t) f(x, y, i, t)\\&+ \frac{1}{2}\text{trace}[g^T(x, y, i, t)\bar{U}_{xx}(x - D(y), i, t)g(x, y, i, t)]\\&+ \sum_{j\in S} \gamma_{ij} \bar{U}(x - D(y), j, t).\end{aligned}$$

To avoid possible explosion, we need to impose an additional as an assumption.

Assumption 2.2. *Assume that there exists a pair of functions* $\bar{U} \in C^{2,1}(R^n \times S \times R_+; R_+)$ *and* $\Lambda \in C(R^n \times [-\tau, \infty); R_+)$*, as well as positive numbers* c_1, c_2, c_3, c_4 *and* $q \ge 2(q_1 \vee q_2)$*, such that*

$$c_3 + c_4 < c_2, \quad |x|^q \le \bar{U}(x, i, t) \le \Lambda(x, t),$$

$\forall (x, i, t) \in R^n \times S \times R_+$ *and*

$$\begin{aligned}&\mathbb{L}\bar{U}(x, y, i, t) + \bar{U}_x(x - D(y), i, t)u(z, i, t)\\&\quad \le c_1 - c_2\Lambda(x, t) + c_3\Lambda(y, t - \tau) + c_4\Lambda(z, t - \delta),\end{aligned}$$

$\forall (x, y, i, t) \in R^n \times R^n \times S \times R_+$.

We now cite a result from [23] as a lemma for the use of this paper.

Lemma 2.3. *Under Assumptions 2.1 and 2.2, the NSDE (4) with the initial data (3) has the unique global solution* $x(t)$ *on* $t \ge -\tau$ *and the solution has the property that*

$$\sup_{-\tau \le t < \infty} E|x(t)|^q < \infty.$$

3. Asymptotic Stabilisation

In this section, we will use the method of Lyapunov functionals to investigate the asymptotic stabilisation. We define two segments $\bar{x}_t := \{x(t + s) : -2\tau \le s \le 0\}$ and $\bar{r}_t := \{r(t + s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\bar{x}_t$ and $\bar{r}_t$ to be well defined for $0 \le t < 2\tau$, we set $x(s) = \varphi(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. The Lyapunov functional used in this paper is defined by

$$V(\bar{x}_t, \bar{r}_t, t) = U(\tilde{x}(t), r(t), t) + \rho \int_{-\delta}^{0} \int_{t+s}^{t} H(v)dvds$$

for $t \ge 0$, where $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ such that

$$\lim_{|x|\to\infty} \big[\inf_{(i,t)\in S\times R_+} U(x, i, t)\big] = \infty,$$

$$\begin{aligned}H(t) =&\delta|f(x(t), x(t - \tau), r(t), t) + u(x(t - \delta), r(t), t)|^2\\&+ |g(x(t), x(t - \tau), r(t), t)|^2\end{aligned}$$

and ρ is a positive number to be determined later. Here we set

$$\begin{aligned}&f(x, y, i, s) = f(x, y, i, 0), \quad u(z, i, s) = u(z, i, 0),\\&g(x, y, i, s) = g(x, y, i, 0)\end{aligned}$$

for $(x, y, i, s) \in R^n \times R^n \times S \times [-2\tau, 0)$. Applying the generalized Itô formula to $U(\tilde{x}(t), r(t), t)$, we get

$$\begin{aligned}&dU(\tilde{x}(t), r(t), t) = \Big(U_t(\tilde{x}(t), r(t), t) + U_x(\tilde{x}(t), r(t), t)\\&\times [f(x(t), x(t - \tau), r(t), t) + u(x(t - \delta), r(t), t)]\\&+ \frac{1}{2}\text{trace}[g^T(x(t), x(t - \tau), r(t), t)\\&\times U_{xx}(\tilde{x}(t), r(t), t)g(x(t), x(t - \tau), r(t), t)]\\&+ \sum_{j=1}^{N} \gamma_{r(t),j} U(\tilde{x}(t), r(t), t)\Big)dt + dM(t),\end{aligned}$$

for $t \ge 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ (see, e.g., [19, Theorem 1.45 on page 48]). Rearranging terms gives

$$\begin{aligned}&dU(\tilde{x}(t), r(t), t) = \Big(U_t(\tilde{x}(t), r(t), t)\\&+ U_x(\tilde{x}(t), r(t), t)[u(x(t - \delta), r(t), t) - u(x(t), r(t), t)]\\&+ \mathcal{L}U(x(t), x(t - \delta), r(t), t)\Big)dt + dM(t),\end{aligned}$$

where the function $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ is defined by

$$\begin{aligned}\mathcal{L}U(x,y,i,t) &= U_t(x-D(y),i,t)\\ &+ U_x(x-D(y),i,t)[f(x,y,i,t)+u(x,i,t)]\\ &+ \frac{1}{2}\text{trace}[g^T(x,y,i,t)U_{xx}(x-D(y),i,t)g(x,y,i,t)]\\ &+ \sum_{j=1}^{N}\gamma_{ij}U(x-D(y),j,t). \end{aligned} \tag{8}$$

Lemma 3.1. *With the notation above, $V(\bar{x}_t,\bar{r}_t,t)$ is an Itô process on $t\ge 0$ with its Itô differential*

$$dV(\bar{x}_t,\bar{r}_t,t) = LV(\bar{x}_t,\bar{r}_t,t)dt + dM(t),$$

where $M(t)$ is a continuous local martingale with $M(0)=0$ and

$$\begin{aligned}LV(\bar{x}_t,\bar{r}_t,t) &= \mathcal{L}U(x(t),y(t),r(t),t)\\ &+ U_x(\tilde{x}(t),r(t),t)[u(x(t-\delta),r(t),t)-u(x(t),r(t),t)]\\ &+ \rho\delta H(t) - \rho\int_{t-\delta}^{t}H(v)dv.\end{aligned}$$

To study the asymptotic stability of the controlled NSDE (4), we need to impose a couple of new assumptions.

Assumption 3.2. *Assume that there are functions $U \in C^{2,1}(R^n\times S\times R_+;R_+)$, $G\in C(R^n;R_+)$ and positive numbers α,λ and $\lambda_i(i=0,1,2,3)$ such that*

$$\alpha<1, \quad \lambda_0<\lambda$$

and

$$\begin{aligned}&\mathcal{L}U(x,y,i,t)+\lambda_1|U_x(\tilde{x},i,t)|^2\\ &+\lambda_2|f(x,y,i,t)|^2+\lambda_3|g(x,y,i,t)|^2\\ &\le -\lambda|x|^2+\lambda_0|y|^2-G(x)+\alpha G(y),\end{aligned}\tag{9}$$

for all $(x,y,i,t)\in R^n\times R^n\times S\times R_+$.

Assumption 3.3. *Assume that there exists a positive number λ_4 such that*

$$|u(x,i,t)-u(y,i,t)|\le\lambda_4|x-y| \tag{10}$$

for all $(x,y,i,t)\in R^n\times R^n\times S\times R_+$. Moreover, for the stability purpose, assume that $u(0,i,t)=0$.

This assumption implies

$$|u(x,i,t)|\le\lambda_4|x|, \quad \forall(x,i,t)\in R^n\times S\times R_+. \tag{11}$$

Theorem 3.4. *Let Assumptions 2.1, 3.2 and 3.3 hold. Assume also that*

$$\delta \le \frac{2(1-\kappa)^2\lambda_1\lambda_3}{\lambda_4^2}\wedge\frac{(1-\kappa)\sqrt{\lambda_1\lambda_2}}{\lambda_4}$$
$$\text{and} \quad \delta<\frac{(1-\kappa)\sqrt{\lambda_1(\lambda-\lambda_0)}}{\lambda_4^2}. \tag{12}$$

Then for any given initial data (3), the solution of the NSDE (4) has the properties that

$$\int_0^\infty E[|x(t)|^2+G(x(t))]dt<\infty, \tag{13}$$

$$\sup_{0\le t<\infty} EU(x(t)-D(x(t-\tau)),r(t),t)<\infty. \tag{14}$$

Proof: Fix the initial data $\varphi\in C([-\tau,0];R^n)$ and $r_0\in S$ arbitrarily. Let $k_0>0$ be a sufficiently large integer such that $\|\varphi\| := \sup_{-\tau\le s\le 0}\varphi(s)<k_0$. For each integer $k>k_0$, define the stopping time

$$\sigma_k=\inf\{t\ge 0: |x(t)|\ge k\},$$

where throughout this paper we set $\inf\emptyset=\infty$ (as usual $\emptyset$ denotes the empty set). By Lemma 2.3 and [23], we can see that σ_k is increasing as $k\to\infty$ and $\lim_{k\to\infty}\sigma_k=\infty$ a.s. By the generalized Itô formula we obtain from Lemma 3.1 that

$$\begin{aligned}&EV(\bar{x}_{t\wedge\sigma_k},\bar{r}_{t\wedge\sigma_k},t\wedge\sigma_k)\\ &=V(\bar{x}_0,\bar{r}_0,0)+E\int_0^{t\wedge\sigma_k}LV(\bar{x}_s,\bar{r}_s,s)ds\end{aligned}\tag{15}$$

for any $t\ge 0$ and $k\ge k_0$. Let $\rho=\frac{\lambda_4^2}{2\lambda_1(1-\kappa)^2}$. By Assumption 3.3, it is easy to see that

$$\begin{aligned}&U_x(\tilde{x}(t),r(t),t)[u(x(t-\delta),r(t),t)-u(x(t),r(t),t)]\\ &\le\lambda_1|U_x(\tilde{x}(t),r(t),t)|^2+\frac{\lambda_4^2}{4\lambda_1}|x(t)-x(t-\delta)|^2.\end{aligned}$$

By condition (12), we also have

$$2\rho\delta^2\le\lambda_2 \quad \text{and} \quad \rho\delta\le\lambda_3.$$

It then follows from Lemma 3.1 that

$$\begin{aligned}LV(\bar{x}_s,\bar{r}_s,s)&\le\mathcal{L}U(x(s),y(s),r(s),s)\\ &+\lambda_1|U_x(\tilde{x}(s),r(s),s)|^2+\lambda_2|f(x(s),x(s-\tau),r(s),s)|^2\\ &+\lambda_3|g(x(s),x(s-\tau),r(s),s)|^2\\ &+\frac{\lambda_4^2}{4\lambda_1}|x(s)-x(s-\delta)|^2+2\rho\delta^2\lambda_4^2|x(s-\delta)|^2\\ &-\frac{\lambda_4^2}{2\lambda_1(1-\kappa)^2}\int_{s-\delta}^{s}H(v)dv.\end{aligned}$$

By Assumption 3.2, we then have

$$
\begin{aligned}
LV(\bar{x}_s,\bar{r}_s,s) &\le -\lambda|x|^2+\lambda_0|y|^2-G(x)+\alpha G(y)\\
&+\frac{\lambda_4^2}{4\lambda_1}|x(s)-x(s-\delta)|^2+2\rho\delta^2\lambda_4^2|x(s-\delta)|^2\\
&-\frac{\lambda_4^2}{2\lambda_1(1-\kappa)^2}\int_{s-\delta}^{s}H(v)dv.
\end{aligned}
$$

Substituting this into (15) implies

$$
\begin{aligned}
EV(\bar{x}_{t\wedge\sigma_k},&\bar{r}_{t\wedge\sigma_k},t\wedge\sigma_k)\\
&\le V(\bar{x}_0,\bar{r}_0,0)+\Pi_1+\Pi_2+\Pi_3-\Pi_4, \qquad (16)
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_1&=E\int_0^{t\wedge\sigma_k}[-\lambda|x(s)|^2+\lambda_0|x(s-\tau)|^2\\
&\quad+2\rho\delta^2\lambda_4^2|x(s-\delta)|^2]ds,\\
\Pi_2&=E\int_0^{t\wedge\sigma_k}[-G(x(s))+\alpha G(x(s-\tau))]ds,\\
\Pi_3&=\frac{\lambda_4^2}{4\lambda_1}E\int_0^{t\wedge\sigma_k}|x(s)-x(s-\delta)|^2ds,\\
\Pi_4&=\frac{\lambda_4^2}{2\lambda_1(1-\kappa)^2}E\int_0^{t\wedge\sigma_k}\int_{s-\delta}^{s}H(v)dvds.
\end{aligned}
$$

Noting that

$$
\int_0^{t\wedge\sigma_k}|x(s-\tau)|^2ds\le\int_{-\tau}^{t\wedge\sigma_k}|x(v)|^2dv,
$$

$$
\int_0^{t\wedge\sigma_k}|x(s-\delta)|^2ds\le\int_{-\delta}^{t\wedge\sigma_k}|x(v)|^2dv\le\int_{-\tau}^{t\wedge\sigma_k}|x(v)|^2dv
$$

and

$$
\int_0^{t\wedge\sigma_k}G(x(s-\tau))ds\le\int_{-\tau}^{t\wedge\sigma_k}G(x(v))dv,
$$

we have

$$
\begin{aligned}
\Pi_1&\le(\lambda_0+2\rho\delta^2\lambda_4^2)\int_{-\tau}^{0}|x(s)|^2ds\\
&\quad-(\lambda-\lambda_0-2\rho\delta^2\lambda_4^2)E\int_0^{t\wedge\sigma_k}|x(s)|^2ds,\\
\Pi_2&\le\alpha\int_{-\tau}^{0}G(x(s))ds-(1-\alpha)\int_0^{t\wedge\sigma_k}G(x(s))ds.
\end{aligned}
$$

Substituting this into (16) yields

$$
\begin{aligned}
&EV(\bar{x}_{t\wedge\sigma_k},\bar{r}_{t\wedge\sigma_k},t\wedge\sigma_k)\\
&\le C_1-(\lambda-\lambda_0-2\rho\delta^2\lambda_4^2)E\int_0^{t\wedge\sigma_k}|x(s)|^2ds\\
&-(1-\alpha)E\int_0^{t\wedge\sigma_k}G(x(s))ds+\Pi_3-\Pi_4, \qquad (17)
\end{aligned}
$$

where C_1 is a constant defined by

$$
\begin{aligned}
C_1=V(\bar{x}_0,\bar{r}_0,0)&+(\lambda_0+2\rho\delta^2\lambda_4^2)\int_{-\tau}^{0}|x(s)|^2ds\\
&+\alpha\int_{-\tau}^{0}G(x(s))ds.
\end{aligned}
$$

Applying the classical Fatou lemma and letting $k\to\infty$ in (17), we obtain

$$
\begin{aligned}
EV(\bar{x}_t,\bar{r}_t,t)&\le C_1-(\lambda-\lambda_0-2\rho\delta^2\lambda_4^2)E\int_0^t|x(s)|^2ds\\
&-(1-\alpha)E\int_0^t G(x(s))ds+\bar{\Pi}_3-\bar{\Pi}_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{\Pi}_3&=\frac{\lambda_4^2}{4\lambda_1}E\int_0^t|x(s)-x(s-\delta)|^2ds,\\
\bar{\Pi}_4&=\frac{\lambda_4^2}{2\lambda_1(1-\kappa)^2}E\int_0^t\int_{s-\delta}^{s}H(v)dvds.
\end{aligned}
$$

By the well-known Fubini theorem, we have

$$
\bar{\Pi}_3=\frac{\lambda_4^2}{4\lambda_1}\int_0^t E|x(s)-x(s-\delta)|^2ds.
$$

For $t\in[0,\delta]$, we have

$$
\begin{aligned}
\bar{\Pi}_3&\le\frac{\lambda_4^2}{2\lambda_1}\int_0^\delta(E|x(s)|^2+E|x(s-\delta)|^2)ds\\
&\le\frac{\delta\lambda_4^2}{\lambda_1}\Big(\sup_{-\delta\le v\le\delta}E|x(v)|^2\Big)=:C_2,
\end{aligned}
$$

while for $t>\delta$, we have

$$
\bar{\Pi}_3\le C_2+\frac{\lambda_4^2}{4\lambda_1}\int_\delta^t E|x(s)-x(s-\delta)|^2ds.
$$

Noting that

$$
\begin{aligned}
&|x(s)-x(s-\delta)|\le|\tilde{x}(s)-\tilde{x}(s-\delta)|\\
&+|D(x(s-\tau))-D(x(s-\delta-\tau))|\\
&\le\kappa|x(s-\tau)-x(s-\delta-\tau)|\\
&+|\int_{s-\delta}^{s}[f(x(v),x(v-\tau),r(v),v)\\
&+u(x(v),x(v-\delta),r(v),v)]dv\\
&+\int_{s-\delta}^{s}g(x(v),x(v-\tau),r(v),v)dB(v)|.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}
&E|x(s)-x(s-\delta)|^2 \\
&\le (1+\theta)\kappa^2 E|x(s-\tau)-x(s-\delta-\tau)|^2 \\
&+(1+\frac{1}{\theta})E|\int_{s-\delta}^{s}[f(x(v),x(v-\tau),r(v),v) \\
&+u(x(v),x(v-\delta),r(v),v)]dv \\
&+\int_{s-\delta}^{s} g(x(v),x(v-\tau),r(v),v)dB(v)|^2 \\
&\le (1+\theta)\kappa^2 E|x(s-\tau)-x(s-\delta-\tau)|^2 \\
&+2(1+\frac{1}{\theta})E\int_{s-\delta}^{s} H(v)dv.
\end{aligned}$$

Setting $\theta=\frac{1}{\kappa}-1$, then we have

$$\begin{aligned}
&\int_{\delta}^{t} E|x(s)-x(s-\delta)|^2 ds \\
&\le \kappa\int_{\delta}^{t} E|x(s-\tau)-x(s-\delta-\tau)|^2 ds \\
&+\frac{2}{1-\kappa}E\int_{\delta}^{t}\int_{s-\delta}^{s} H(v)dvds \\
&\le \kappa\int_{-\tau+\delta}^{t} E|x(s)-x(s-\delta)|^2 ds \\
&+\frac{2}{1-\kappa}E\int_{\delta}^{t}\int_{s-\delta}^{s} H(v)dvds.
\end{aligned}$$

Noting that $0<\kappa<1$, it follows that

$$\begin{aligned}
&\int_{\delta}^{t} E|x(s)-x(s-\delta)|^2 ds \\
&\le \frac{\kappa}{1-\kappa}\int_{-\tau+\delta}^{\delta} E|x(s)-x(s-\delta)|^2 ds \\
&+\frac{2}{(1-\kappa)^2}E\int_{\delta}^{t}\int_{s-\delta}^{s} H(v)dvds.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_{-\tau+\delta}^{\delta} E|x(s)-x(s-\delta)|^2 ds \\
&\le 2E\int_{-\tau+\delta}^{\delta} |x(s)|^2 ds+|x(s-\delta)|^2 ds \\
&\le 4E\int_{-\tau}^{\tau} |x(s)|^2 ds \le 8\tau \sup_{-\tau\le v\le\tau} E|x(v)|^2
\end{aligned}$$

Hence

$$\begin{aligned}
\bar{\Pi}_3 &\le C_2+\frac{2\kappa\tau\lambda_4^2}{(1-\kappa)\lambda_1}\sup_{-\tau\le v\le\tau} E|x(v)|^2+\bar{\Pi}_4 \\
&= C_3+\bar{\Pi}_4,
\end{aligned}$$

where $C_3=C_2+\frac{2\kappa\tau\lambda_4^2}{(1-\kappa)\lambda_1}\sup_{-\tau\le v\le\tau} E|x(v)|^2$. That means

$$\begin{aligned}
&(\lambda-\lambda_0-2\rho\delta^2\lambda_4^2)E\int_0^t |x(s)|^2 ds \\
&+(1-\alpha)E\int_0^t G(x(s))ds \le C_1+C_3.
\end{aligned}$$

By condition (12), we have

$$\lambda-\lambda_0-2\rho\delta^2\lambda_4^2=\lambda-\lambda_0-\frac{\delta^2\lambda_4^4}{\lambda_1(1-\kappa)^2}>0.$$

Letting $t\to\infty$, the assertion (13) can be obtained.

Similarly, we can see from (16) that

$$\begin{aligned}
&EU\Big(x(t\wedge\sigma_k)-D(x(t\wedge\sigma_k-\tau)),r(t\wedge\sigma_k),t\wedge\sigma_k\Big) \\
&\le C_1-(\lambda-\lambda_0-2\rho\delta^2\lambda_4^2)E\int_0^{t\wedge\sigma_k} |x(s)|^2 ds \\
&-(1-\alpha)\int_0^{t\wedge\sigma_k} G(x(s))ds+\Pi_3-\Pi_4, \qquad (18)
\end{aligned}$$

Letting $k\to\infty$, we get

$$EU(x(t)-D(x(t-\tau)),r(t),t)\le C_1+C_3<\infty,$$

which shows the assertion (14). Thus the proof is complete. □

The following corollary will give a criterion on H_∞-stability.

Corollary 3.5. *Let the conditions of Theorem 3.4 hold. If there exists a pair of positive constants c and $\bar{p}>2$ such that*

$$c|x|^{\bar{p}}\le G(x), \quad \forall (x,t)\in R^n\times R_+,$$

then the solution of the controlled system (4) has the property that for any $p\in[2,\bar{p}]$ and any given initial data (3)

$$\int_0^\infty E|x(t)|^p dt<\infty. \qquad (19)$$

That is, the NSDE (4) is H_∞-stable in L^p for any $p\in[2,\bar{p}]$.

We can see this corollary follows from (13) obviously. However, it does not follow from (19) that $\lim_{t\to\infty} E|x(t)|^p=0$.

Theorem 3.6. *Let the conditions of Corollary 3.5 hold. If, moreover,*

$$p\ge 2 \quad \textit{and} \quad (p+q_1-1)\vee(p+2q_2-2)\le q,$$

then the solution of the NSDE (4) satisfies

$$\lim_{t\to\infty} E|x(t)|^p = 0$$

for any initial data (3). That is, the NSDE (4) is asymptotically stable in L^p.

Proof: Fix the initial data (3) arbitrarily. For any $0 \le t_1 < t_2 < \infty$, by the Itô formula, we obatain

$$\begin{aligned} &E|\tilde{x}(t_2)|^p - E|\tilde{x}(t_1)|^p \\ &\quad = E\int_{t_1}^{t_2} \Big(p|\tilde{x}(t)|^{p-2}\tilde{x}^T(t) \\ &\qquad\qquad \times [f(x(t),x(t-\tau),r(t),t) + u(x(t-\delta),r(t),t)] \\ &\quad + \frac{p}{2}|\tilde{x}(t)|^{p-2}|g(x(t),x(t-\tau),r(t),t)|^2 \\ &\quad + \frac{p(p-2)}{2}|\tilde{x}(t)|^{p-4}|\tilde{x}^T(t)g(x(t),x(t-\tau),r(t),t)|^2\Big)dt. \end{aligned}$$

This implies

$$\begin{aligned} &\big|E|\tilde{x}(t_2)|^p - E|\tilde{x}(t_1)|^p\big| \le E\int_{t_1}^{t_2} \Big(p|\tilde{x}(t)|^{p-1} \\ &\times |f(x(t),x(t-\tau),r(t),t) + u(x(t-\delta),r(t),t)| \\ &+ \frac{p(p-1)}{2}|\tilde{x}(t)|^{p-2}|g(x(t),x(t-\tau),r(t),t)|^2\Big)dt \\ &\le E\int_{t_1}^{t_2} \Big(p|\tilde{x}(t)|^{p-1} \\ &\times \big[K(1+|x(t)|^{q_1} + |x(t-\tau)|^{q_1}) + \lambda_4|x(t-\delta)|\big] \\ &+ \frac{3p(p-1)K^2}{2}|\tilde{x}(t)|^{p-2}\big[1+|x(t)|^{2q_2} + |x(t-\tau)|^{2q_2}\big]\Big)dt \\ &\le C_4(t_2-t_1), \end{aligned}$$

where C_4 is a constant independent t_1, t_2, Thus we have $E|\tilde{x}(t)|^p$ is uniformly continuous in t on R_+. By (19), we can have

$$\begin{aligned} &\int_0^\infty E|\tilde{x}(t)|^p dt \le \int_0^\infty 2^{p-1}E\Big(|x(t)|^p + \kappa^p|x(t-\tau)|^p\Big)dt \\ &\le 2^{p-1}(1+\kappa^p)\int_0^\infty E|x(t)|^p dt + 2^{p-1}\kappa^p\|\varphi\| < \infty, \end{aligned}$$

so we obtain $\lim_{t\to\infty} E|\tilde{x}(t)|^p = 0$. By the inequality (7), we can get

$$\begin{aligned} E|x(t)|^p &\le E[|\tilde{x}(t)| + |D(x(t-\tau))|]^p \\ &\le E[(1+\varrho)^{p-1}(|\tilde{x}(t)|^p + \varrho^{1-p}\kappa^p|x(t-\tau)|^p)]. \end{aligned}$$

Setting $\varrho = \kappa/(1-\kappa)$, we obatin

$$E|x(t)|^p \le (\frac{1}{1-\kappa})^{p-1}E|\tilde{x}(t)|^p + \kappa E|x(t-\tau)|^p, \qquad (20)$$

letting $t \to \infty$, we have

$$\lim_{t\to\infty}\sup E|x(t)|^p \le \kappa \lim_{t\to\infty}\sup E|x(t)|^p.$$

This, together with the Lemma 2.3, yields

$$\lim_{t\to\infty} E|x(t)|^p = 0.$$

Thus the proof is complete. □

4. An Example

In this section, we will use an example to illustrate our results.

Example 4.1. Consider the following scalar hybrid NSDE

$$\begin{aligned} d[x(t) - D(x(t-\tau))] &= f(x(t),x(t-\tau),r(t),t)dt \\ &+ g(x(t),x(t-\tau),r(t),t)dB(t), \end{aligned} \qquad (21)$$

where $r(t)$ is a Markov chain on the state space $S = \{1,2\}$ with its generator

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix},$$

the coefficients f and g are given by (1) and $D(y) = 0.1y$. Setting $\tau = 2$, letting the initial data $x(u) = 2 + \cos(u)$ for $u \in [-2,0], r(0) = 2$. The sample paths of the Markov chain and the solution of the NSDE (21) are plotted in Figure 1, which indicates that the NSDE is unstable. We

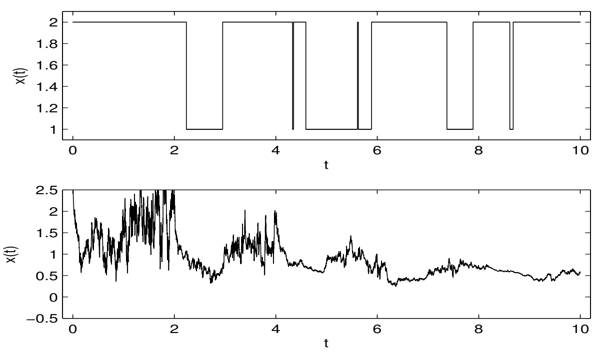

Figure 1: The computer simulation of the sample paths of the Markov chain and the NSDE (21) with $\tau = 2$ using the Euler-Maruyama method.

will use the control function $u : R \times S \times R_+ \to R$ defined by

$$u(x,1,t) = -x, \; u(x,2,t) = -2x. \qquad (22)$$

Thus, the controlled hybrid NSDE has the form

$$
\begin{aligned}
&d[x(t) - D(x(t-\tau))] \\
&\quad = [f(x(t), x(t-\tau), r(t), t) + u(x(t-\delta), r(t), t)]dt \\
&\qquad + g(x(t), x(t-\tau), r(t), t)dB(t), \qquad (23)
\end{aligned}
$$

In order to get the bound on δ, we need to verify the assumptions imposed in Section 2. It is easy to see that Assumption 2.1 is satisfied with $q_1 = 3$ and $q_2 = 2$.

$$
\begin{aligned}
&\mathbb{L}\bar{U}(x, y, i, t) + \bar{U}_x(x - D(y), i, t)u(z, i, t) \\
&= 6|x - D(y)|^5 f(x, y, i, t) + \frac{15}{4}|x - D(y)|^4|y|^4 \\
&+ 6|x - D(y)|^5 u(z, i, t).
\end{aligned}
$$

By Young inequality and (20), we have

$$
\begin{aligned}
&\mathbb{L}\bar{U}(x, y, i, t) + \bar{U}_x(x - D(y), i, t)u(z, i, t) \\
&\le \begin{cases} -16.961x^8 + 5.762y^8 \\ +16.785x^6 + y^6 + 1.1z^6, & \text{if } i = 1, \\ -8.229x^8 + 4.971y^8 \\ +24.41x^6 + 1.5y^6 + 2.2z^6, & \text{if } i = 2. \end{cases} \\
&\le -8.229x^8 + 5.762y^8 + 24.41x^6 + 1.5y^6 + 2.2z^6 \\
&\le c_1 - 8(x^8 + 2.2x^6) + 5.8(y^8 + 2.2y^6) + (z^8 + 2.2z^6),
\end{aligned}
$$

where $c_1 = \sup_{x\in R}(33x^6 - 0.229x^8) < \infty$. Therefore, Assumption 2.2 is fulfilled with $\Lambda(x, t) = x^8 + 2.2x^6, c_2 = 8, c_3 = 5.8, c_4 = 1$ and $q = 6$.

To apply our theorems established in the previous section, we need to verify assumptions imposed there. Let us do so one by one. To verify Assumption 3.2, we define

$$
U(x, i, t) = \begin{cases} 2x^2 + x^4, & \text{if } i = 1, \\ x^2 + x^4, & \text{if } i = 2, \end{cases}
$$

for $(x, i, t) \in R \times S \times R_+$.

By the well-known Young inequality, we can show that

$$
\begin{aligned}
&\mathcal{L}U(x, y, i, t) \\
&\le \begin{cases} -2.75x^2 - 12.149x^4 - 15.418x^6 \\ +0.23y^2 + 2.002y^4 + 4.577y^6 & \text{if } i = 1, \\ -2.7x^2 - 16.899x^4 - 9.422x^6 \\ +0.31y^2 + 3.803y^4 + 4.173y^6 & \text{if } i = 2. \end{cases}
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&|U_x(x - D(y), i, t)|^2 \\
&\le \begin{cases} 17.6x^2 + 1.76y^2 + 43.9x^4 \\ +3.2y^4 + 27.1x^6 + 1.6y^6, & \text{if } i = 1, \\ 4.4x^2 + 0.44y^2 + 21.95x^4 \\ +1.6y^4 + 27.1x^6 + 1.6y^6, & \text{if } i = 2; \end{cases}
\end{aligned}
$$

$$
|f(x, y, i, t)|^2 \le \begin{cases} x^2 - 11.5x^4 + 1.5y^4 + 42x^6 + 7y^6, \\ \text{if } i = 1, \\ x^2 - 7.5x^4 + 1.5y^4 + 20x^6 + 5y^6, \\ \text{if } i = 2; \end{cases}
$$

$$
|g(x, y, 1, t)|^2 = |g(x, y, 2, t|^2 = 0.25y^4.
$$

Setting $\lambda_1 = 0.05,\ \lambda_2 = 0.1,\ \lambda_3 = 4$, we obtain that

$$
\begin{aligned}
&\mathcal{L}U(x, y, i, t) + \lambda_1|U_x(x - D(y), i, t)|^2 \\
&+ \lambda_2|f(x, y, i, t)|^2 + \lambda_3|g(x, y, i, t)|^2 \\
&\le \begin{cases} -1.77x^2 + 0.318y^2 - 11.099x^4 \\ +3.312y^4 - 10.863x^6 + 5.357y^6, & \text{if } i = 1, \\ -2.38x^2 + 0.332y^2 - 15.051x^4 \\ +5.033y^4 - 6.067x^6 + 4.753y^6, & \text{if } i = 2 \end{cases} \\
&\le -1.77x^2 + 0.332y^2 - 6(x^4 + x^6) + 5.4(y^4 + y^6).
\end{aligned}
$$

Thus, Assumption 3.2 is satisfied with $G(x) = 6(x^4 + x^6), \lambda = 1.77, \lambda_0 = 0.332$ and $\alpha = 0.9$. Moreover, we can see that definition (10) is satisfied with $\lambda_4 = 2$ and $\kappa = 0.1$. Thus, condition (12) becomes $\delta \le 0.0318$. By Theorem 3.4, we can therefore conclude that the solution of the NSDE (21) has the properties that

$$
\int_0^\infty (x^2(t) + x^4 + x^6(t))dt < \infty \ a.s.
$$

$$
\text{and} \int_0^\infty E(x^2(t) + x^4(t) + x^6(t))dt < \infty.
$$

Moreover, as $|x(t)|^p \le x^2(t) + x^4(t) + x^6(t)$ for any $p \in [2, 6]$, we have $\int_0^\infty E|x(t)|^p dt < \infty$.

Recalling $q_1 = 3,\ q_2 = 2$ and $q = 6$, we see that for $p = 4$, all the conditions of Theorem 3.6 are satisfied and hence we have $\lim_{t\to\infty} E|x(t)|^4 = 0$.

We perform a computer simulation with the time-delay $\tau = 2$ and feedback control time-delay $\delta = 0.03$ for all $t \geq 0$ and the initial data $x(u) = 2 + \cos(u)$ for $u \in [-2, 0]$ and $r(0) = 2$. The sample paths of the Markov chain and the solution of the NSDE (23) are plotted in Figure 2. The simulation supports our theoretical results.

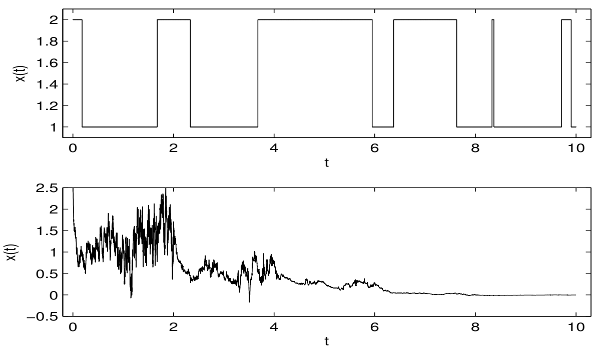

Figure 2: The computer simulation of the sample paths of the Markov chain and the NSDE (23) with $\tau = 2, \delta = 0.03$ using the Euler-Maruyama method.

5. Conclusion

In this paper we have discussed the stabilisation of highly nonlinear hybrid NSDEs by delay feedback controls. We should point out, up to now, there's no result on the stabilisation for NSDEs without the linear growth condition. There is hence a need to develop a new theory on the stabilisation by delay feedback controls for the highly nonlinear NSDE models. In this paper we have successfully used the method of Lyapunov functionals to study this stabilisation problem by delay feedback controls. We have showed that a class of highly nonlinear unstable hybrid NSDEs whose coefficients satisfy the polynomial growth condition can be stabilised by delay feedback controls. An example and computer simulations have been used to illustrate our theory.

Acknowledgements

The authors would like to thank the Natural Science Foundation of China (71571001, 61703003), the Natural Science Foundation of University of Anhui (KJ2016A064, KJ2017A105, KJ2019A0141), the Promoting Plan of Higher Education of Anhui (TSKJ2016B11), the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), and the EPSRC (EP/K503174/1) for their financial support.

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