

# Do we have enough data? Robust reliability via uncertainty quantification

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## Abstract

A generalised probabilistic framework is proposed for reliability assessment and uncertainty quantification under a lack of data. The developed computational tool allows the effect of epistemic uncertainty to be quantified and has been applied to assess the reliability of an electronic circuit and a power transmission network. The strength and weakness of the proposed approach are illustrated by comparison to traditional probabilistic approaches. In the presence of both aleatory and epistemic uncertainty, classic probabilistic approaches may lead to misleading conclusions and a false sense of confidence which may not fully represent the quality of the available information. In contrast, generalised probabilistic approaches are versatile and powerful when linked to a computational tool that permits their applicability to realistic engineering problems.

*Keywords:* Uncertainty quantification, Information quality, Probability boxes, Dempster-Shafer, Computational tool, Reliability

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## 1. Introduction

Nowadays it is generally well accepted that estimating the effect of uncertainty is a necessity, e.g. due to variation in parameters, operational conditions and in the modelling and simulations [1, 2]. In practical applications, situations are common where the analyst has to deal with poor quality data, few available specimens or inconsistent information. A typical example is a situation where very expensive samples have to be

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7 collected, such as field proprieties of a deep reservoir [3] or performance of satellites  
8 [4]. In these cases, the amount of data will be scarce due to economic and time con-  
9 straints and in several cases, expert elicitation (i.e. the best estimate of an expert) may  
10 be the only viable way of carrying on with the analysis [5].

11 As a consequence, strong assumptions may be needed to apply classical probabilistic  
12 methods given poor information quality, which can lead to erroneous reliability esti-  
13 mations and a false sense of confidence [6]. Generalised approaches, which fit in the  
14 framework of imprecise probability [6], are powerful methodologies for dealing with  
15 imprecise information and lack of data. These methodologies can be coupled to tradi-  
16 tional probabilistic approaches in order to give a different prospective on the results,  
17 whilst avoiding the inclusion of unjustified assumptions and enhancing the overall ro-  
18 bustness of the analysis. Generalised methods are rarely used in practice and this is  
19 probably due to lack of proper guidance, simulation tools, as well as some misconcep-  
20 tion in the interpretation of the results. Further comparison of different methodologies,  
21 both in theoretical aspects and in their applicability to real case studies, are required.

22  
23 An original throughout analysis of the applicability of different methodologies to  
24 deal with different level of imprecision is presented. In addition, this paper presents  
25 a novel computational framework for generalised probabilistic analysis that can be  
26 adopted to deal with low quality data, few available samples and inconsistent informa-  
27 tion. Efficient and generally applicable computational strategies have been developed  
28 and implemented into OpenCossan [7]. The proposed framework is applied to assess  
29 the reliability of an electric series RLC circuit (a problem proposed by the NAFEMS  
30 Stochastics Working Group [8]) and of a power transmission network, both affected by  
31 a lack of data.

32 Generally speaking, different system performance indicators may be affected very dif-  
33 ferently by the same (lack of) data. The extent of a lack of information is not a-priori  
34 quantifiable and depends on the context of the analysis. The proposed approach is used  
35 to assess the information quality by comparison to classical probabilistic results and  
36 with respect to system reliability estimates. One of the main contributions of this work  
37 is a detailed comparison between classical and generalised probabilistic approaches  
38 from a straightforward applicative point of view and under different levels of impreci-  
39 sion. This serves as guidance for engineering practitioners to solve problems affected  
40 by a lack of data.

41

42 The rest of the paper is structured as follows: Section 2, presents the mathematical  
43 framework. In Section 3, a synthetic overview of the numerical framework and the  
44 proposed approach is proposed. The NAFEMS reliability problem is described and  
45 solved in Section 4. A lack of data problem for power network reliability estimation  
46 is solved in Section 5. A discussion on the limitations of the different approaches is  
47 presented in Section 6 and Section 7 closes the paper.

## 48 **2. Mathematical Framework**

49 Uncertainty is generally classified into two categories, aleatory and epistemic un-  
50 certainty. Aleatory uncertainty (Type I or irreducible uncertainty), represents stochastic  
51 behaviours and randomness of events and variables. Hence, due to its intrinsic random  
52 nature it is normally regarded as irreducible. Some examples of aleatory uncertainty  
53 are future weather conditions, stock market prices or chaotic phenomenon. Epistemic  
54 uncertainty (Type II or reducible uncertainty), is commonly associated with lack of  
55 knowledge about phenomena, imprecision in measurements and poorly designed mod-  
56 els. It is considered to be reducible since further data can decrease the level of un-  
57 certainty, but this might not always be practical or feasible. In recent decades, efforts  
58 were focused on the explicit treatment of imprecise knowledge, non-consistent infor-  
59 mation and both epistemic and aleatory uncertainty. The methodologies are discussed  
60 in literature by different mathematical concepts: Evidence theory [9], interval prob-  
61 abilities [10], Fuzzy-based approaches [11], info-gap approaches [12] and Bayesian  
62 frameworks [13] are some of the most intensively applied concepts.

63 In this paper, Dempster-Shafer structures and probability boxes are used to model quan-  
64 tities affected by epistemic uncertainty, by aleatory uncertainty, or by a combination of  
65 the two. In addition, the Kolmogorov-Smirnov test [14] and Kernel Density Estimator  
66 [15] have been used to characterise the parameter uncertainty in case of small sample  
67 sizes.

### 68 *2.1. Dempster-Shafer Structures and Probability Boxes*

69 The Dempster-Shafer (DS) theory is a well-suited framework to represent both  
70 aleatory and epistemic uncertainty. The difference between the axioms of classical  
71 probability theory and the DS theory is that the latter slacken the strict assumption of a

72 single probability measure for an event. It can be seen as a generalisation of Bayesian  
 73 probability [16]. Mathematically, a Dempster-Shafer structure on the real line  $\mathbb{R}$  can  
 74 be identified with a basic probability assignment, that is a map as follows:

$$m : 2^{\mathbb{R}} \rightarrow [0, 1] \quad (1)$$

75 where the probability mass is  $m([\underline{x}_i, \bar{x}_i]) = p_i$  for each focal element  $[\underline{x}_i, \bar{x}_i] \subseteq \mathbb{R}$  with  
 76  $i = 1, \dots, n$ .  $m(S)$  is equal 0 for the empty set  $S = \emptyset$  and for  $S \neq [\underline{x}_i, \bar{x}_i]$ , such that  $p_i >$   
 77  $0 \forall i$  and  $\sum_i p_i = 1$ . The upper bound on probability is referred as plausibility and the  
 78 lower bound as belief, the cumulative plausibility function  $Pl(x)$  and cumulative be-  
 79 lief function  $Bel(x)$  can be computed as  $Pl(x) = \sum_{\underline{x}_i \leq x} m_i$  and  $Bel(x) = \sum_{\bar{x}_i \leq x} m_i$ . The  
 80 continuous equivalents of DS structures are the so-called probability boxes or P-boxes.  
 81 Mathematically, a P-box is a pair of lower and upper cumulative distribution functions  
 82  $[\underline{F}_X, \bar{F}_X]$  from the possibility space  $\Theta$  to  $[0,1]$  such that  $\underline{F}_X(x) \leq \bar{F}_X(x) \forall x \in \Omega$   
 83 and  $\Omega$  is a classical probability space. The upper and lower bounds for the CDFs are  
 84  $\bar{F}_X = \bar{P}(X \leq x)$  and  $\underline{F}_X = \underline{P}(X \leq x)$ , respectively. Note that the probability distri-  
 85 bution family associated with the random variable  $x$  can be either specified or not speci-  
 86 fied. The former are generally named distributional P-boxes, or parametric P-boxes, the  
 87 latter are named distribution-free P-boxes, or non-parametric P-boxes [13]. The wider  
 88 the distance between the upper and the lower bound is, the higher the incertitude asso-  
 89 ciated to the random variable. P-boxes and DS structures offer a straightforward way  
 90 to deal with multiple and overlapping intervals, inconsistent sources of information  
 91 and small sample sizes. The drawback is that the computational cost of propagating  
 92 P-boxes and DS structures through the system is generally quite high, especially for a  
 93 large number of intervals (i.e. focal elements) and time-consuming models. Neverthe-  
 94 less, the quantification approaches are generally not-intrusive and hence applicable to  
 95 any model.

### 96 3. Generalised Probabilistic Reliability Analysis and Numerical Implementation

97 In modern engineering systems and critical infrastructures uncertainty quantifica-  
 98 tion must be performed to assure an adequate level of safety and reliability. A broadly  
 99 applied numerical approach, often used to deal with uncertainty propagation, is the  
 100 Monte Carlo (MC) method. Typically the Monte Carlo algorithm allows uncertainty to

101 be propagated from inputs characterised by well-defined probability distribution func-  
 102 tions (PDF) [17]. It is flexible, unbiased and one of the most well-established method-  
 103 ologies to propagate uncertainty, but its classical implementation does not differenti-  
 104 ate between aleatory and epistemic uncertainty. This is a disadvantage from several  
 105 points of view. First, it makes the analyst unable to grasp how much of the uncer-  
 106 tainty is due to inherent variability and to what extent the uncertainty is due to poor  
 107 data quality (therefore suitable to be reduced in principle). Secondly, it relies upon a  
 108 good characterization of the variables to be sampled, which usually requires a consider-  
 109 able body of empirical information in order to properly define probability distributions.  
 110 To overcome such limitations, more sophisticated MC algorithms can be used within  
 111 generalised probabilistic frameworks to propagate both types of uncertainties without  
 112 mixing them. For instance, the so-called double loop Monte Carlo algorithm [18] can  
 113 be used. In this work, using classical probabilistic approaches the uncertain factors are  
 114 described by probability distribution functions (PDFs) and a traditional MC approach  
 115 is employed to propagate uncertainty. When generalised probabilistic approaches are  
 116 adopted, parameters are characterised by P-boxes or DS structures and the uncertainty  
 117 is propagated using the proposed double loop MC or Dempster-Shafer structures prop-  
 118 agation algorithms (adopted as presented in Figure 1).

119 The double loop MC is presented by Figure 1-(a). A first loop (outer loop) samples  
 120 from the epistemic uncertainty space  $\Theta$ . Each realisation corresponds to a classical  
 121 probabilistic model for which only aleatory uncertainties must be considered. Then, a  
 122 traditional MC simulation can be used (inner loop) to propagate aleatory uncertainty.  
 123 The quantity  $N_e$  is the number of realisations in the epistemic space and  $N_a$  is the  
 124 number of samples from the aleatory space.  $\theta_j$  is the set of uncertain parameters of  
 125 the epistemic space realizations  $j$ , sampled from a known set of intervals  $[\underline{\theta}, \bar{\theta}]$ . The  
 126 quantity  $x_{k,i}$  is the sample  $i$  of the random variable  $k$  obtained from the inverse trans-  
 127 form of the associated CDF  $F_{X_k|\theta_j}(x)$ , which depends on the epistemic realization  $\theta_j$ .  
 128 The cumulative distribution  $F_{Y|\theta_j}(y)$  of the reliability performance  $y$  can be used to  
 129 compute  $P_{f,j}$ , which is the system failure probability given the epistemic realization  $j$ .  
 130 The probability results of the inner loop are not to be averaged over the outer loop but  
 131 only collected. Then the minimum and maximum can be selected to obtain bounds on  
 132 the quantity of interest [19].

133

134 The Dempster-Shafer structures propagation procedure in Figure 1-(b), works as

135 follows:

- 136 1. First,  $n$  “Parameter cells” are constructed by Cartesian product of the focal ele-  
137 ments. Hence, each parameter cell  $\omega$  is an hypercube  $\omega : \{\underline{\mathbf{x}}_\omega \leq \mathbf{x} \leq \bar{\mathbf{x}}_\omega \forall \mathbf{x}\}$ .
- 138 2. The minimum and maximum values of the system performance  $y$  are calculated  
139 based on optimization technique and constrained by the  $\omega$  bounds.
- 140 3. The  $n$  resulting min-max intervals (i.e. propagation of the focal elements) are  
141 used to construct Dempster-Shafer structures.
- 142 4. Finally, Dempster-Shafer structures are converted to **distribution-free** P-boxes  
143 and the system reliability bounds  $[\underline{P}_f, \bar{P}_f]$  obtained.

144 The computational cost of the procedure is proportional to the number of input inter-  
145 vals to be propagated and the time needed to simulate the system. Applicability for  
146 complex systems with highly non-regular behaviour, which are hence computationally  
147 expensive, can require a meta-modelling approach to speed-up the propagation proce-  
148 dure (e.g. Polynomial Chaos, Artificial Neural Networks).

149 OpenCossan [7] is a collection of methods and tools under continuous development  
150 at the Institute for Risk and Uncertainty, University of Liverpool, coded exploiting  
151 the object-oriented Matlab programming environment. It allows specialised solution  
152 sequences to be defined including a wide variety of reliability methods. Novel optimi-  
153 sation algorithms, reliability methods, and uncertainty quantification and propagation  
154 techniques can be easily integrated into the main software body. For these reasons, the  
155 developed methods (i.e. Algorithms in Figure 1) have been integrated into OpenCos-  
156 san and adopted for the solution of two reliability assessments, see Sections 4.2 and  
157 5. As a result of such development, OpenCossan can be used to perform uncertainty  
158 quantification adopting classical and generalised probabilistic methods.

## 159 **4. Case Study I: The NAFEMS Challenge Problem**

### 160 *4.1. Problem Definition*

161 The challenge problem, prepared by the NAFEMS Stochastics Working Group [8],  
162 consists of four uncertainty quantification and information qualification tasks moti-  
163 vated by the need to promote best practices to deal with uncertainty to industry. The  
164 analysts are asked to evaluate the reliability of an electronic resistive, inductive, capac-  
165 itive (RLC) series circuit. Four different cases (A, B, C and D) have been proposed

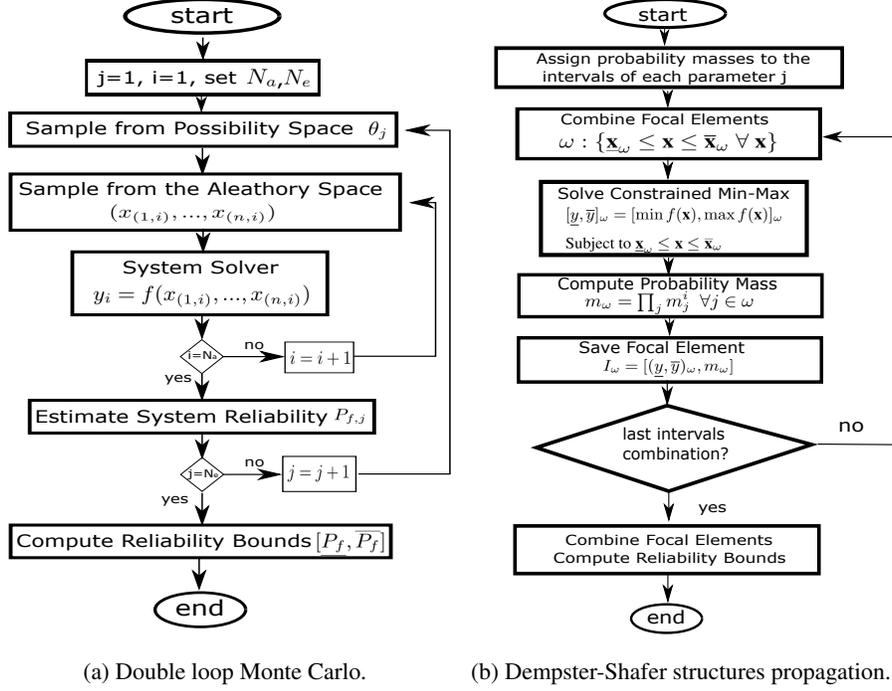


Figure 1: Flow charts for the double loop Monte Carlo (a) and the DS structures propagation (b).

166 in [8], each one having incomplete, scarce or imprecise information about the system  
 167 parameters, as shown in Table 1. In CASE-A single intervals, i.e. one upper bound  
 168 and one lower bound for parameter R, L and C are given. In CASE-B, each parame-  
 169 ter can lay within multiple intervals, i.e. three upper and lower bounds. In CASE-C,  
 170 ten sampled points for each parameter are provided. Finally, for CASE-D, imprecise  
 171 bounds and nominal values is the only available information. The last case is similar  
 172 to CASE-A, but one bound is not precisely defined. The equations governing the RLC  
 173 circuit, although very simple, are provided by the challengers and reported here for  
 174 completeness. The transfer function of the system is defined as:

$$\frac{V_c(t)}{V} = \frac{\omega^2}{S^2 + \frac{R}{L}S + \omega^2} \quad (2)$$

175 Depending on the values of R, L and C, the system may be classified as under-damped  
 176 ( $Z < 1$ ), critically damped ( $Z = 1$ ) or over-damped ( $Z > 1$ ) and having different solu-  
 177 tions as detailed below.

Table 1: The available information for CASE-A, CASE-B, CASE-C, and CASE-D (data taken from[8]).

CASE	R [ $\Omega$ ]	L [mH]	C [ $\mu$ F]
A: Interval	[40,1000]	[1,10]	[1,10]
B: source 1	[40,1000]	[1,10]	[1,10]
B: source 2	[600,1200]	[10,100]	[1,10]
B: source 3	[10,1500]	[4,8]	[0.5,4]
C: Samples	861, 87, 430, 798, 219, 152, 64, 361, 224, 61	4.1, 8.8, 4.0, 7.6, 0.7, 3.9, 7.1, 5.9, 8.2, 5.1	9.0, 5.2, 3.8, 4.9, 2.9, 8.3, 7.7, 5.8, 10, 0.7
D: Interval	[40, $R_{U1}$ ]	[1, $L_{U1}$ ]	[ $C_{L1}$ ,10]
D: Other info	$R_{U1} > 650$	$L_{U1} > 6$	$C_{L1} < 7$
D: Nominal Val.	650	6	7

178

$$V_c(t) = \begin{cases} V + (A_1 \cos(\omega t) + A_2 \sin(\omega t)) \exp^{-\alpha t} & \text{if } Z < 1 \\ V + (A_1 + A_2 t) \exp^{-\alpha t} & \text{if } Z = 1 \\ V + (A_1 \exp^{S_1 t} + A_2 \exp^{S_2 t}) & \text{se } Z > 1 \end{cases} \quad (3)$$

179 Where  $\alpha = \frac{R}{2L}$ ,  $\omega = \frac{1}{\sqrt{LC}}$ , the damping factor is  $Z = \frac{\alpha}{\omega}$  and roots obtained as  $S_{1,2} =$   
180  $-\alpha \pm \sqrt{\alpha^2 - \omega^2}$ . Coefficients  $A_1$  and  $A_2$  are determined by assuming the initial volt-  
181 age and voltage derivative equal zero and a unitary step voltage function is considered.  
182 In [this case study](#), the main goals consist in qualifying the value of information and  
183 evaluating the reliability of the system with respect to three requirements:

$$V_c(t = 10ms) > 0.9 V \quad , \quad t_r = t(V_c = 0.9V) \leq 8 ms \quad , \quad Z \leq 1 \quad (4)$$

184 where  $t_r$  is the voltage rise time, i.e. the time required to increase  $V_c$  from 0 to 90% of  
185 the input voltage, and it has to be less than or equal 8 ms. The first two requirements  
186 are on the voltage at the capacitance  $V_c$ , the third requirement is on the damping factor,  
187 which assures that under-damped system responses are discharged ( $Z \leq 1$ ). Specific-  
188 ally,  $V_c(10ms)$ ,  $V_c(8ms)$  and  $Z$  are regarded as performance variable for the system,  
189 and if these conditions are not satisfied the system is considered to have failed. Prob-  
190 abilistic and generalised probabilistic approaches are adopted to tackle the four cases  
191 and uncertainty characterization and propagation are presented for each case. Depend-  
192 ing on the approach selected, CDFs or P-boxes are obtained for the three performance  
193 variables (see Eq.(4)). If  $V_c(10ms)$ ,  $V_c(8ms)$  and  $Z$  result in crisp CDFs, the probability

194 of failure is computed by estimating the CDF values at 0.9 Volts for the requirements  
195 on  $V_c$  and voltage rise time  $t_r$  as well as the CDF value at  $Z=1$  for the requirement  
196 on the damping factor. Similarly, if bounds on the CDFs are obtained (i.e. P-boxes),  
197 then bounds on probability of not meeting the requirements are computed as explained  
198 in Sections 2, which are  $[P_{V_{c10}}, \bar{P}_{V_{c10}}]$ ,  $[P_{t_r}, \bar{P}_{t_r}]$ , and  $[P_Z, \bar{P}_Z]$ , respectively. This  
199 case study was previously tackled by different groups and the author using different  
200 approaches. For further reading the reader is reminded to Refs. [8]-[20]. This work  
201 presents additional analyses of the NAFEMS challenge problem by adopting novel al-  
202 gorithms in a unified computational framework.

#### 203 4.2. CASE-A and CASE-B

204 In CASE-A, a single interval was provided for the parameters while multiple inter-  
205 vals were available in CASE-B (see, Table 1). CASE-B degenerates to CASE-A if the  
206 probability mass equal one is assigned to the first source of information. This because  
207 in CASE-B intervals values for source 1 corresponds to the interval values in CASE-A.  
208 Due to the considerations made, the two cases are presented and solved together.

#### 209 Probabilistic Approach

210 In the CASE-A the intervals were propagated using a single loop Monte Carlo by  
211 assuming a uniform distribution within the bounds on R, L and C, which is an assump-  
212 tion made with respect to the principle of maximum entropy. The reliability is assessed  
213 by evaluating if the system requirements are met as shown in Eq. (4). For the solu-  
214 tions of CASE-A, failure probabilities have been estimated using  $10^7$  samples and are  
215  $P_{V_{c10}}=0.243$ ,  $P_{t_r}=0.345$  and  $P_Z=0.031$ . The probability of failure for requirement  
216 one is lower than the probability of failure for requirement two.

217 For the solution of the CASE-B, each interval is considered individually. Hence, three  
218 different uniform distributions for each R, L, and C parameter are used, one for each  
219 source of information. The reliability analyses have been performed to compute 3  
220 probabilities of failure and results are shown in Table 2. The Source 3 has the lowest  
221 estimated probability of failure while the Source 1 shows an intermediate failure prob-  
222 ability. On the right-hand side of Figure 2 the resulting CDFs for the three sources of  
223 information and three requirements are displayed.

Table 2: The results for CASE-B obtained by Monte Carlo method and  $10^7$  samples.

CASE-B	Source 1	Source 2	Source 3
$P_{V_{c10}}$	0.243	0.549	0.052
$P_{tr}$	0.340	0.660	0.129
$P_Z$	0.031	$1.25 \cdot 10^{-5}$	0.069

224 *Generalised Probabilistic Approach*

Possible values of the parameters (interval) can be represented by means of the generalised probabilistic approach without defining a probability distribution. Parameter uncertainty has been characterised using Dempster-Shafer structures. For CASE-A three Dempster-Shafer structures composed by a single focal element have been defined as  $\{\underline{R}_1, \overline{R}_1\}, \{\underline{L}_1, \overline{L}_1\}$  and  $\{\underline{C}_1, \overline{C}_1\}$ , where the probability mass  $m_1$  is equal one. For CASE-B, each DS structure is defined as:

$$\{([\underline{X}_1, \overline{X}_1], m_1), ([\underline{X}_2, \overline{X}_2], m_2), ([\underline{X}_3, \overline{X}_3], m_3)\}$$

225 where  $[\underline{X}_i, \overline{X}_i]$  represents the  $i^{th}$  interval source for one of the parameters (R, L or  
 226 C) and  $m_i$  is the associated probability mass. The CASE-B degenerate to the CASE-  
 227 A if the probability mass  $m_2$  and  $m_3$  are set equal to zero. It was not possible here  
 228 to establish if some sources of information are better, thus, pieces of information de-  
 229 rived from different sources are assumed as equally likely, i.e.  $m_1 = m_2 = m_3 = 1/3$ .  
 230 Twenty-seven parameter cells are constructed by the permutation of the intervals. Then,  
 231 minimizations and maximisations of  $V_c(8ms)$ ,  $V_c(10ms)$  and  $Z$  were performed to iden-  
 232 tify the bounds of the system performance. The output Dempster-Shafer structures are  
 233 used to create probability boxes for the system performances  $V_c(8ms)$ ,  $V_c(10ms)$  and  
 234  $Z$  and the corresponding failure probabilities obtained.

235 Applying the procedure to the CASE-A, the resulting P-boxes give no valuable infor-  
 236 mation on the failure probability for the three performance requirements. The proba-  
 237 bility of failure is in fact just bounded in the interval  $[0,1]$  for all the requirements. The  
 238 CASE-B includes all the information available for the CASE-A plus two additional  
 239 sources of information. The additional intervals contribute to reducing the uncertainty  
 240 on the system performance as shown on the right-hand side of Figure 2. Resulting  
 241 bounds are also presented in Table 3 and it can be noticed that the outputs have high  
 242 associated uncertainty, but less than that in the CASE-A.  $P_{tr}$  lays within the interval

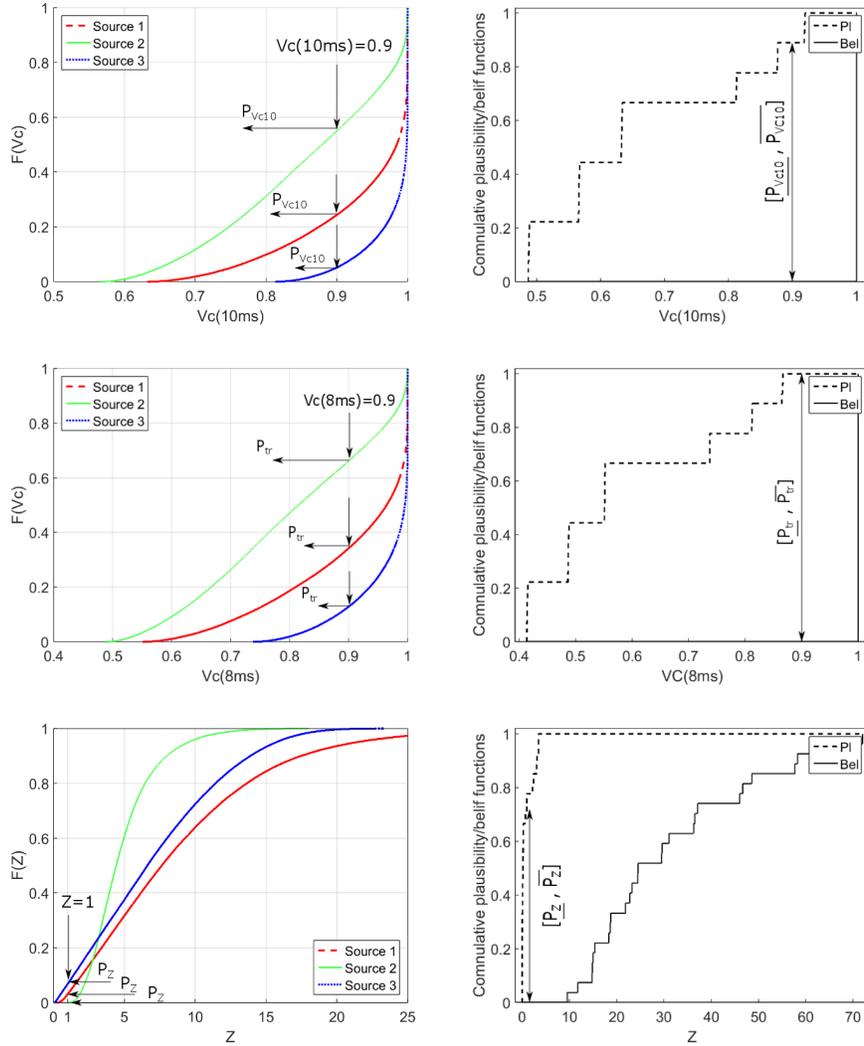


Figure 2: Comparison of the  $V_c(8ms)$ ,  $V_c(10ms)$  and  $Z$  results for CASE-B, respectively. Resulting CDFs obtained using the probabilistic approach (on the left) and P-boxes obtained from the generalised approach (on the right).

243  $[0,0.9]$ ,  $P_{V_{c10}}$  within  $[0,1]$  and  $P_Z$  lays within the interval  $[0,0.7]$ . Hence, failure prob-  
 244 ability for requirement two does not show any reduction in the uncertainty.

245 The failure probability computed by adopting classical approaches always lays within  
 246 the bounds obtained using the Dempster-Shafer methodology, as shown in Figure 2.

247 The maximum failure probability for the  $Z$  requirement is 0.069 (source 3), while the

Table 3: The results of CASE-B obtained adopting generalised probabilistic approach.

CASE-B	Source 1	Source 2	Source 3	All Sources
$P_{V_{c10}}$	[0,1]	[0,1]	[0,1]	[0,0.9]
$P_{tr}$	[0,1]	[0,1]	[0,1]	[0,1]
$P_Z$	[0,1]	[0,1]	[0,1]	[0,0.7]

248 generalised approach bounds the results between 0 and 0.7. This reliability overesti-  
 249 mation was due to the assumption made on the parent distribution needed to apply the  
 250 classical methodology. In fact, by selecting a PDF we explicitly assume a well-defined  
 251 stochastic behaviour for the parameters. As a matter of fact, no information was given  
 252 to assume a random behaviour at all, and the imprecise information could be due to  
 253 different experts advising for different scale ranges to be analysed.

254 The computational time for CASE-B using classical Monte Carlo simulation was about  
 255 6.7 seconds. The generalised solution to CASE-A and CASE-B was relatively compu-  
 256 tationally inexpensive, taking about 0.07-0.08 seconds for the solution of each min-max  
 257 problem. Thus, the DS structures propagation for the 3 reliability requirements took  
 258 just 5-6 seconds for CASE-B on a 4 cores machine with 8.00 Gb ram and a 2.00 GHz  
 259 Intel<sup>®</sup> Core<sup>™</sup> i5-4590T processor.

### 260 4.3. CASE-C

#### 261 *Probabilistic Approach*

262 For the solution of CASE-C, two methodologies were adopted. Firstly a uniform  
 263 distribution approach and secondly a Kernel Density estimation (KDE) approach [15].  
 264 The uniform distribution approach allows the values of the parameters to change within  
 265 the sampled range (but not outside). The bounds are assumed equal to the minimum  
 266 and maximum values of the samples. Then,  $10^5$  MC run have been performed ob-  
 267 taining estimated probabilities of failure of  $P_{V_{c10}}=0.183$ ,  $P_{tr}=0.273$ ,  $P_Z=0.016$ , re-  
 268 spectively. The Kernel Density Estimator is a well-known approach that allows a  
 269 probability distribution to be constructed based on sample data without assuming its  
 270 distribution form. Different Kernels can be used and the Gaussian Kernel is a popu-  
 271 lar choice which has been adopted in this work because it allows the incorporation of  
 272 measurement error. The optimal bandwidth value was obtained using Silverman's rule  
 273 of thumb [15]. By adopting KDE the estimated failure probabilities are  $P_{V_{c10}}=0.232$ ,

274  $P_{tr}=0.292$ ,  $P_Z=0.121$ , respectively. These values are slightly larger compared to the  
275 one obtained with the uniform distribution approach. Higher values of the probability  
276 of failure are due to the tails of the Kernel fitted probability distribution (displayed in  
277 Figure 3) which allows the value of the parameter to change outside the range of the  
278 samples. Plots on the left-hand side in Figure 4 show the output CDFs when adopting  
279 uniform distributions and KDE to model parameter uncertainty. The CDF of  $Z$  has  
280 been zoomed around the value  $Z=1$  for graphical reasons. The failure probabilities  
281 calculated using sampled values of R, L and C are also lower if compared to the ones  
282 obtained in CASE-A and CASE-B. This is due to the smaller upper bound on R in  
283 CASE-C (861 Ohm).

#### 284 *Generalised Probabilistic Approach*

285 CASE-C is solved by applying the Kolmogorov-Smirnov (KS) test to characterise  
286 the uncertainty of the input parameters as shown in [14], and obtaining bounds on the  
287 empirical cumulative probability distribution function. Maximum and minimum values  
288 of the parameters are assumed and the CDF upper and lower bounds are truncated ac-  
289 cordingly. [Due to the underlying physics governing the system](#), all the parameters must  
290 be positive and this condition allows the lower bounds to be set. Truncating the tails  
291 of the distributions, especially in reliability analysis, can lead to erroneous results and  
292 safety overconfidence. Thus a relatively high upper bound for the CDF truncation was  
293 selected, which was assumed equal to the sample mean plus three times the sample's  
294 standard deviation. In Figure 3 the upper and lower bounds (dashed and solid lines)  
295 are shown for the empirical CDF (square marker blue line) and the Kernel density es-  
296 timator (blue dot-dashed line). Three different confidence levels for the KS test are  
297 used for each parameter. The bounds on the left-hand side plots refer to a confidence  
298 level  $\alpha=0.05$  and they are compared to the plots on the right-hand side which show the  
299 obtained bounds for  $\alpha=0.01$  (dashed and solid star marker lines),  $\alpha=0.1$  (dashed and  
300 solid blue lines) and  $\alpha=0.2$  (dashed and solid circle marker lines).

301 The obtained P-boxes are propagated through the system. On the right plots of Fig-  
302 ure 4, the voltage at the  $10^{th}$  ms,  $8^{th}$  ms and damping factor P-boxes are presented,  
303 red blue and black colour lines with different markers refer to confidence level  $\alpha=0.01$ ,  
304  $\alpha=0.1$  and  $\alpha=0.2$  respectively. The P-box of the damping factor has been zoomed  
305 around the value  $Z=1$  to improve the readability of the plot. The bounds on the prob-  
306 abilities of failure are presented in the Table 4. It can be observed that the intervals

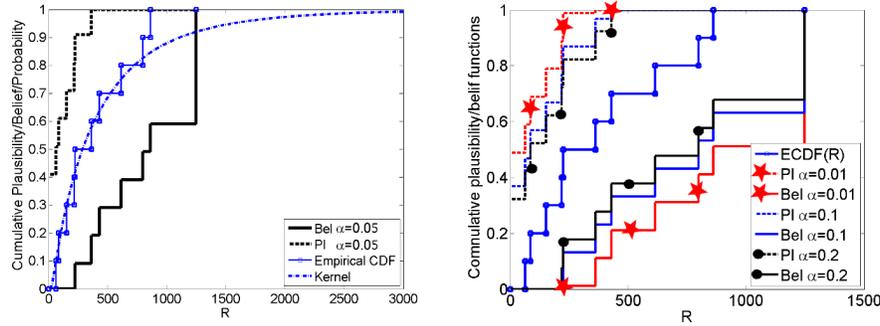


Figure 3: The Kernel fitting (on the right panel) and the P-box bounds (on the right panel) of the resistance R for the CASE-C.

Table 4: The results for CASE-C, the probability bounds for the three requirements and the three confidence levels.

CASE-C	$\alpha=0.01$	$\alpha=0.1$	$\alpha=0.2$
$P_{Vc10}$	[0,0.87]	[0,0.7]	[0,0.63]
$P_{tr}$	[0,0.92]	[0,0.77]	[0,0.7]
$P_Z$	[0,0.83]	[0,0.7]	[0,0.64]

307 on the failure probability are quite wide, as already observed for CASE-A and CASE-  
 308 B. Nevertheless, the failure probability bounds appear to be narrower if compared to  
 309 CASE-A and CASE-B. This shows that the information provided for CASE-C is of  
 310 higher quality, which allows less imprecise reliability estimates to be obtained. The  
 311 results show that the uncertainty in the system reliability was underestimated by us-  
 312 ing the Monte Carlo method because precise probability distribution functions were  
 313 assumed despite the small sample size. The failure probabilities estimated by adopting  
 314 the classical approach lay within the probability interval obtained by adopting gener-  
 315 alised approaches.

316 Using the same machine adopted for solving the previous cases, the classical proba-  
 317 bilistic solution of CASE-C required about 0.07 seconds for the fitting and propagation  
 318 of the Kernel probability densities and additional 0.05 seconds for the propagation  
 319 of uniform probability densities. Conversely to the generalised solution to CASE-A  
 320 and CASE-B, the computational time needed for the propagation of the focal elements  
 321 is generally higher when compared to its classical probabilistic counterpart. The DS

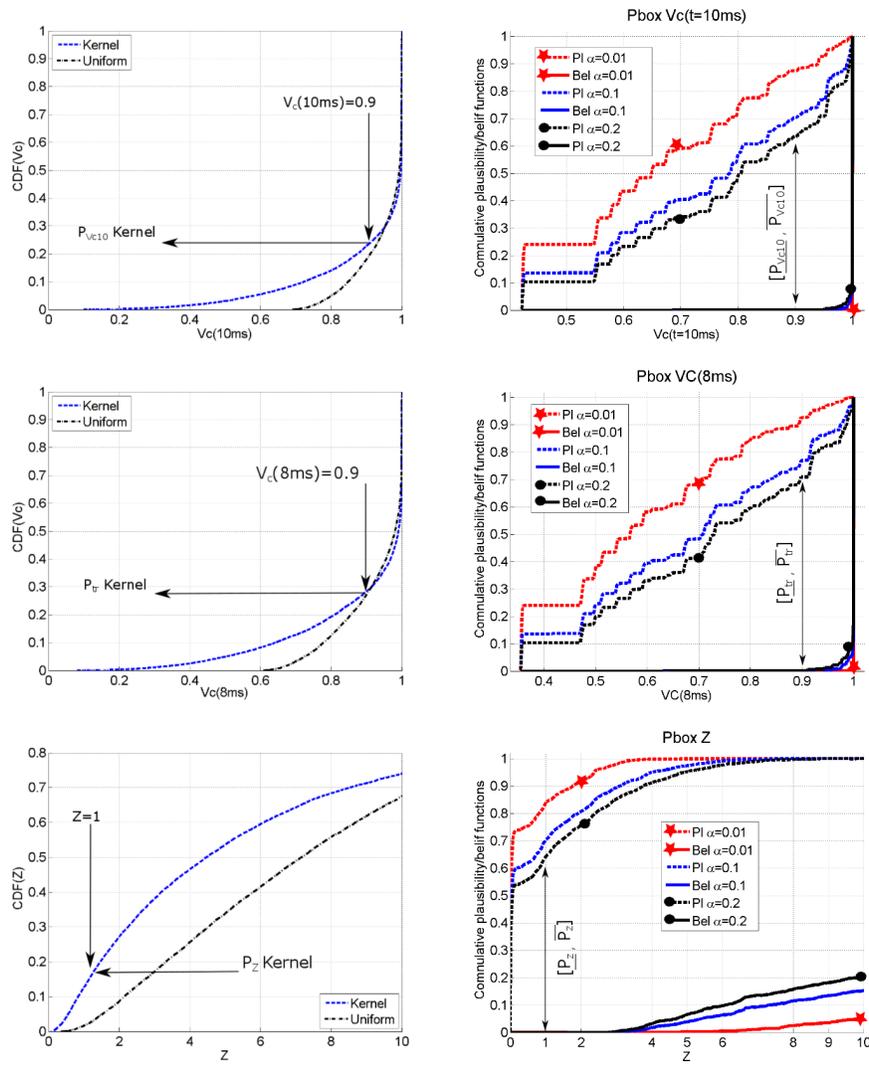


Figure 4: CDFs (on the left panel) and P-boxes (on the right panel) of  $V_c(10ms)$ ,  $V_c(8ms)$  and  $Z$  for the CASE-C

322 structures propagation took about 461 seconds for each confidence level  $\alpha$  (i.e. about  
 323 23 minutes for the 3 confidence levels). The higher computational cost is attributable  
 324 to the larger number of min-max optimisations performed (i.e. 2197 combinations of  
 325 focal elements).

326 *4.4. CASE-D*

327 Similarly to CASE-A, the bounds of the parameters are provided. However, just  
328 one bound is precisely defined for each parameter. The upper bounds of R and L and  
329 the lower bounds of C are imprecisely defined as shown in the last row of Table 1. In  
330 addition, the nominal values for the parameter are provided. The problem has been  
331 tackled by defining upper bounds of R and L, which were redefined as  $T$  times their  
332 nominal value while the lower bound of C was redefined as its nominal value divided  
333 by  $T$ , where  $T = 10$ . Thus, the maximum truncation bounds are  $\bar{R}_n = 6500 \Omega$ ,  $\bar{L}_n = 60$   
334 mH and  $\underline{C}_n = 0.7 \mu\text{F}$ . The quantity  $T$  is defined as ‘truncation level’ and  $n = 10$  linearly  
335 spaced intermediate bounds are also considered.

336 *Probabilistic Approach*

337 Uniform PDFs are assumed within the defined intervals and all combinations of  
338 upper and lower bounds are propagated by the Monte Carlo method. Having reduced  
339 the semi-definite intervals to a set of defined intervals, it is now possible to estimate  
340 the reliability of the systems by adopting the same approach as CASE-B. For the first  
341 two requirements, the probability of failure increases from 0.1 up to 0.9. The MC  
342 method is not efficient in providing solutions for the lower bounds of the intervals. In  
343 fact, the probability of having  $Z < 1$  goes from a maximum of 0.2 to a minimum of  
344 approximately 0.0005 (requiring at least  $10^5$  samples for a rough estimation).

345 *Generalised Probabilistic Approach*

346 The parameters’ uncertainty has been characterised using a set of  $n$  multiple inter-  
347 vals translated into DS structures. A probability mass function equal to  $1/n$  has been  
348 assigned to each interval (for normalization reasons) defining Dempster-Shafter struc-  
349 tures for the parameters, for instance the structure of R is  $\{([\underline{R}, \bar{R}_1], \frac{1}{n}), \dots, ([\underline{R}, \bar{R}_n], \frac{1}{n})\}$ .  
350 The three probabilities of failure lay within the interval  $[0, 1]$ . In particular, the im-  
351 precision associated with the last requirement indicates a severe misjudgement of the  
352 real uncertainty when the only classical probabilistic solution is considered (obtain-  
353 ing a maximum  $P_Z = 0.2$ ). In order to investigate the effect of the assumptions on  
354 the results, a sensitivity analysis of the values of  $\bar{R}_n$ ,  $\bar{L}_n$  and  $\underline{C}_n$  is performed. The  
355 sensitivity approach adopted is similar to the one-at-a-time method presented in [21].  
356 The selected base-case has truncation level  $T = 10$  and truncation bounds  $\bar{R}_n = 6500 \Omega$ ,  
357  $\bar{L}_n = 60$  mH and  $\underline{C}_n = 0.7 \mu\text{F}$ . A total of 27 sensitivity cases are defined by selecting 9

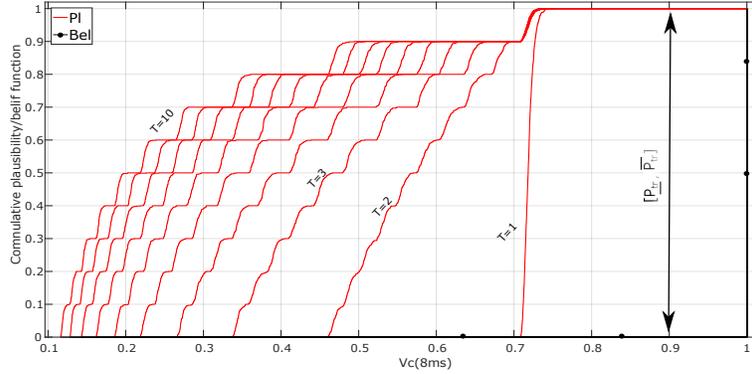


Figure 5: Variation in the probability bounds due different values of  $\bar{R}_n$ . The truncation values are  $\bar{R}_n=650 \cdot T \cdot \Omega$  with  $T=1, \dots, 10$  and  $\bar{L}_n=60$  mH and  $\underline{C}_n=0.7 \mu\text{F}$ .

358 truncation level to  $T=9, 8, 7, \dots, 1$  for each one of the parameters taken one-at-a-time. Then  
 359 uncertainty propagation is carried out for the sensitivity cases and results compared to  
 360 the bounds of the base case. The comparison shows that the shape of the P-boxes is  
 361 affected most by  $\bar{R}_n$ . On the other hand, it does not have relevant effects on the bounds  
 362 of the failure probability. Figure 5 displays the sensitivity analysis performed by vary-  
 363 ing  $\bar{R}_n$ .

364 The computational time required to solve CASE-D is about 200 seconds by using the  
 365 DS structure propagation algorithm whilst the classical approach required 1400 sec-  
 366 onds for the solution (selecting  $10^5$  samples for the Mone Carlo and propagating all  
 367 the combinations of upper and lower bounds).

## 368 5. Case Study II: Analysis of a Power Transmission Network

369 The case study selected for the analysis is a 6-bus and 11-lines power transmission  
 370 network [22]. Figure 6 displays the network topology, nodes indices and load names.  
 371 The nodes 1-3 represent the generator buses while the nodes 4-6 are the demand buses.  
 372 To simplify the reliability assessment, loads correlation is neglected and grid stress is  
 373 increased. The reference loads  $L_{d4}$ ,  $L_{d5}$  and  $L_{d6}$  and the decreased maximum power  
 374 capacity of the generators are reported in Figure 6.

375 It is assumed that a lack of data is affecting the failure rate of the transmission lines.  
 376 This is a common situation for highly reliable components for which at best only a few  
 377 failures have been observed. A common practice used to estimate the failure rate of

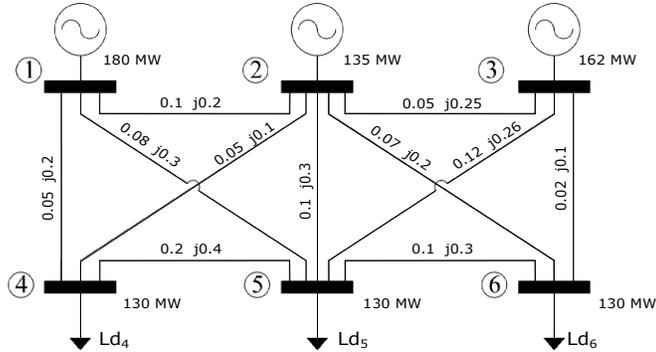


Figure 6: The 6-bus power network system.

378 transmission lines is to merge the few available failure samples between similar lines.  
 379 This procedure is named “data pooling” and assumes that the behaviour of similar  
 380 components can be described by the same probabilistic model. This is often a ratio-  
 381 nal assumption. However, when (similar) components are subjected to different work  
 382 loads (e.g. close/far from their thermal limits), different conditions (e.g. in a harsh/mild  
 383 environment) or with different maintenance policies such assumptions are rarely true.  
 384 Different endogenous and operational-environmental factors will most likely influence  
 385 the ageing of the components and produce a very different failure behaviour even for  
 386 identical lines. For more details on the problem, the reader is referred to [23].  
 387 The transmission links in the system are assumed to be LGJ-300 and for this specific  
 388 line, an estimation of the failure rates ( $\lambda_l$ ) for each link  $l$  is presented in [24]. The  
 389 available data consists of 40 failure times collected over 10 years for a first line and 5  
 390 years of failure times for a second. Over the first 5 years, the estimated  $\lambda_l$  is 0.00027  
 391 [failure/h] while for the last 3 years the failure rate increases to 0.00042 [failure/h]  
 392 (possibly due to a poorly described ageing effect). Similarly to CASE-A in the first  
 393 case study (Section 4), an interval data source is considered for each line failure rate  $\lambda_l$   
 394 with  $l = 1, \dots, 11$ . The failure rate is imprecisely defined during the ageing of the line  
 395 (e.g. between 5 years to 8 years from installation) and this might affect the estimation  
 396 of the power network reliability.  
 397 The Energy-not-Supplied (ENS) is a well-known reliability indicator for power grids  
 398 and is employed here to assess the network failure probability. The power network is

399 simulated for a given period of time (e.g. 1 day) and random components' failures are  
 400 sampled from probability distributions used to model the components' failure times.  
 401 The probability of failure for a line is assumed to follow a Poisson distribution and ob-  
 402 tained similarly to [2]. During the simulation, the network power flow equations are  
 403 solved and in the case of occurred failures or unsatisfied constraints (e.g. thermal or  
 404 generators capacity limits), part of the power load can be curtailed. The power grid will  
 405 fail to meet the performance requirement if the energy not provided to the customers  
 406 is larger than a predefined threshold level (i.e.  $ENS > ENS_{tsh}$ ). The  $ENS_{tsh}$  has  
 407 been set equal to 0.05 % of the total load demand. Further details on the reliability  
 408 model are available in Ref. [25]. First, a classical probabilistic approach is used to as-  
 409 sess the power grid reliability. The probabilistic model for the grid has to be precisely  
 410 defined. Hence, a point value for the failure rate of each ageing line has been se-  
 411 lected and set equal to the mean failure rate (0.000345 [outage/years]). A plain Monte  
 412 Carlo is employed to propagate  $10^4$  independent realisations of the power grid history.  
 413 In each MC run, failures can randomly occur according to the line failure probabil-  
 414 ity and the  $ENS$  is computed for the sampled network state. The resulting CDF of  
 415 the Energy-Not-Supplied ( $F_{ENS}$ ) is displayed by the blue circle markers line in Fig-  
 416 ure 7. It can be used to obtain the probability of failure for the network as follows:  
 417  $P(ENS > ENS_{tsh}) = 1 - F_{ENS}(ENS_{tsh})$ .

The imprecise information available for the failure rate has been propagated using a

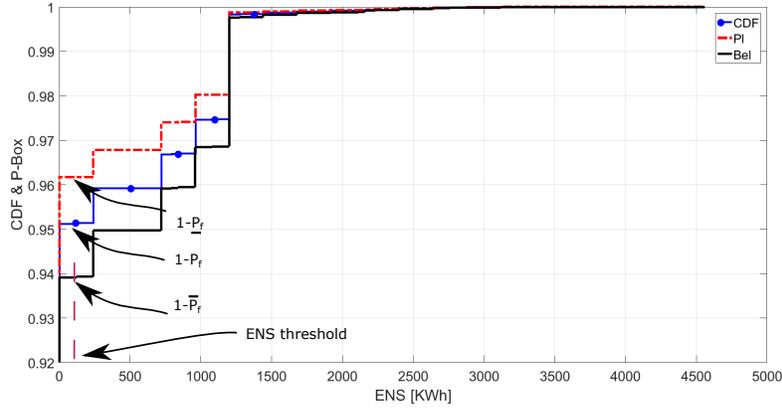


Figure 7: The CDF, Cumulative belief and Cumulative plausibility functions for the  $ENS$  in [MWh]. The plot is zoomed in to better display the reliability results and  $ENS_{tsh}$ .

418

Table 5: The probability bounds resulting from the generalised approach accounting for 4 levels of imprecision for the nodal load demand,  $L_{di}$ .

Imprecision on $L_{di}$	5%	10%	15%	20%
$\overline{P}_f$	0.0874	0.0964	0.0964	1
$\underline{P}_f$	0.0389	0.0387	0.0384	0.032

419 double loop Monte Carlo approach as presented in Section 3. In the outer loop, 50 val-  
 420 ues of the failure rates are sampled from the interval [0.00027,0.00042] failure/h and  
 421 forwarded to the inner loop. In the inner loop, analogously to the classical probabilis-  
 422 tic analysis, a Monte Carlo simulation is used to obtain independent histories for the  
 423 power network, sampling failed components and obtaining the  $ENS$ . The results are  
 424 cumulative belief (black solid line) and plausibility (dot-dashed red line), displayed in  
 425 Figure 7. The threshold  $ENS_{tsh}$  is also presented with a dashed line. The resulting  
 426 reliability interval is  $[3.89, 6.09] \cdot 10^{-2}$  which includes the single-valued reliability es-  
 427 timator obtained by the classical probabilistic approach,  $4.99 \cdot 10^{-2}$ .

428 The analysis has been extended by accounting for imprecision in the power loads  $L_{d4}$ ,  
 429  $L_{d5}$  and  $L_{d6}$ . In Ref. [26], power demand is affected by imprecision and modelled  
 430 using two interval cases. Similarly, 4 imprecision levels on the power demanded (from  
 431 5% to 20% of the design load) are considered here, due for instance to measurement  
 432 errors or forecast incertitude. Table 5 summarises the result for increasing imprecision  
 433 on the load value and Figure 8 displays the output cumulative  $Pl$  and  $Bel$ . The reli-  
 434 ability bound gets wider the larger the imprecision surrounding the system loads is. It  
 435 is worth noticing that when the load interval is increased from 15 to 20 % the upper  
 436 failure probability increases drastically, from  $9.64 \cdot 10^{-2}$  to 1 (dashed marked lines in  
 437 Figure 8). This because within the parameter cell  $\omega : \{\underline{L}_{di} \leq L_{di} \leq \overline{L}_{di} \forall i = 4, 5, 6\}$   
 438 exists at least one combinations of loads ( $L_{d4}, L_{d5}, L_{d6}$ ) for which the power flow can  
 439 not satisfy the given constraints (i.e. power balance, thermal limit and generators ca-  
 440 pacity constraints). As consequence, the power flow solver curtails a significant amount  
 441 of load even for undamaged grid conditions and for each realisation within the inner  
 442 loop the  $ENS$  exceeds  $ENS_{tsh}$ .

443 In this final application, the developed framework has been tested using a more com-  
 444 plex engineering application. Comparing the results obtained using the classical and  
 445 generalised probabilistic approaches helped to understand the quality of the informa-

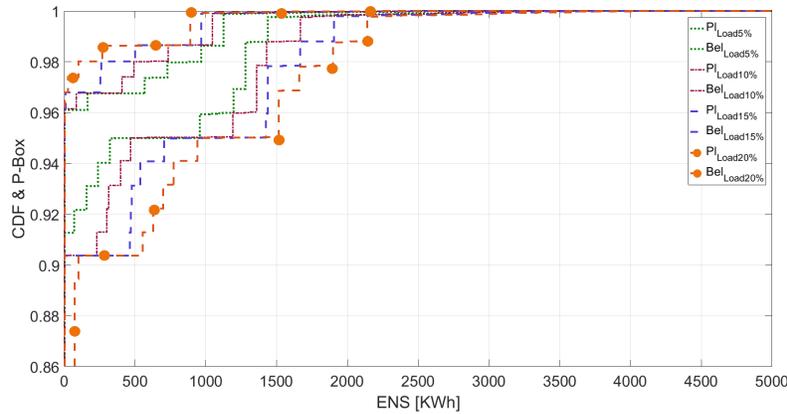


Figure 8: The Cumulative belief and plausibility functions for different levels of imprecision on the loads. The plot is zoomed in to improve the graphical output.

446 tion on  $\lambda_l$  and loads and their impact on the network reliability. In the first case, the  
 447 information quality was good and the imprecise data resulted in a moderate (but defini-  
 448 nitely observable) effect on the network reliability. In the second analysis, an increasing  
 449 level of imprecision affecting the power demand is considered. The results showed that  
 450 more imprecision in the input load increases the imprecision in the reliability estimate.  
 451 Moreover, the generalised approach pointed out that increasing the imprecision in the  
 452 load up to 20%, drastically stretched the reliability bounds (about  $[0,1]$ ). This is in-  
 453 deed an indicator of a severe lack of the available information quality, which has been  
 454 successfully pointed out by the generalised approach. The computational time for the  
 455 solution was about 98 seconds using classical approaches (MC with  $10^4$  samples) and  
 456 about 4900 seconds for the generalised approach (50 outer loop samples and  $10^4$  inner  
 457 loop samples).

## 458 6. Limitation Faced and Discussions

459 Classical probabilistic approaches require the estimation of (or assuming) PDFs to  
 460 describe parameters. Uncertainty and uniform distributions and Kernel density estima-  
 461 tors have been used to characterise parameter uncertainty. In both cases, it has been  
 462 explicitly assumed that the analysed parameters have some sort of stochastic nature,  
 463 which in reality might not be true. One of the strongest limitations of classical prob-  
 464 abilistic approaches is the need to represent the epistemic uncertainty as aleatory and

465 then mix these two types of uncertainty. The analysed NAFEMS reliability problem has  
466 confirmed that artificial model assumptions might lead to an underestimation of the un-  
467 certainty. Hence the reliability analysis might not represent precisely the real quality of  
468 the available data. For extreme cases, a severe lack of data can lead to non-informative  
469 bounds  $[0,1]$ . The large epistemic uncertainty about the system parameters may sug-  
470 gest considering an investment in collecting more empirical data rather than refining  
471 the model for the reliability assessment. The overall outcomes of the study highlighted  
472 some of the positive and negative aspects of employ a generalised approach with re-  
473 spect to classical uncertainty quantification methodologies.

474 The reliability assessments were affected by severe uncertainty when, if tackled using  
475 classical probabilistic approaches, the analyst is forced to make unjustified assump-  
476 tions leading to a strong underestimation the true output uncertainty. A case affected  
477 by a severe lack of data was the NAFEMS reliability problem for which the epistemic  
478 component appeared to be a dominant part of the outcomes' uncertainty. On the other  
479 hand, a reliability problem affected by a mild lack of data would have had results less  
480 sensitive to the epistemic uncertainty. This might be well-represented by the power  
481 grid reliability problem for which the failure rate imprecision influenced moderately  
482 (but visibly) the precision of reliability estimate. Similar results have been obtained for  
483 imprecision on the load demand up to 15 %. On the other hand, higher imprecision on  
484 the load (20%) drastically widened the reliability bounds. This has been pointed out  
485 thanks to the proposed comparison framework for classical and generalised probabilis-  
486 tic approaches.

## 487 **7. Conclusions**

488 In order to define a precise and 'exact' probabilistic model, a very high amount of  
489 data (possibly infinite) would be necessary. Unfortunately, a lack of information always  
490 affects engineering analysis and its extent cannot be quantified a priori. In general, the  
491 quality of the available information is context and scope-dependent, e.g. different sys-  
492 tems performance indicators may react very differently to the same lack of data. The  
493 proposed framework provides a simple but effective way to assess a data deficiency  
494 by comparing the system reliability bounds (obtained through generalised probabilistic  
495 approaches) against single-valued probability indicators (obtained adopting classical  
496 probabilistic methods). If the lack of knowledge is mild, the system reliability will  
497 result in relatively narrow bounds which include the point reliability estimator. In this

498 case, classical approaches will be well-suited to tackle the problem. Otherwise, the  
499 lack of data will be severe and reliability bounds wide or, for extreme cases, even  
500 non-informative ( $[0,1]$ ). Combination of pure probabilistic approaches (e.g. Monte  
501 Carlo Simulation) and generalised uncertainty quantification approaches (e.g. based on  
502 Dempster-Shafer structures and probability boxes), implemented in a common compu-  
503 tational framework, are unavoidable tools for the industry which may rely on multiple  
504 accurate information qualification approaches. This will aid understanding if the data  
505 is of high quality or poor quality, with the aim of designing safer and more reliable  
506 systems and components. The NAFEMS uncertainty quantification challenge prob-  
507 lem and a power system reliability assessment have been selected as representative test  
508 cases and have been solved using the proposed computational tool. Essential informa-  
509 tion has been provided and the quality of the available data assessed.

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