IET Research Journals

Submission Template for IET Research Journal Papers

ISSN 1751-8644
doi: 0000000000
www.ietdl.org

Exponential Stabilisation of Continuous-time Periodic Stochastic Systems by Feedback Control Based on Periodic Discrete-time Observations

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Abstract: Since Mao in 2013 discretised the system observations for stabilisation problem of hybrid SDEs (stochastic differential equations with Markovian switching) by feedback control, the study of this topic using a constant observation frequency has been further developed. However, the time-varying observation frequencies have not been considered yet. Particularly, an observational more efficient way is to consider the time-varying property of the system and observe a periodic SDE system at the periodic time-varying frequencies. This paper investigates how to stabilise a periodic hybrid SDE by a periodic feedback control, based on periodic discrete-time observations. This paper provides sufficient conditions under which the controlled system can achieve pth moment exponential stability for $p > 1$ and almost sure exponential stability. The Lyapunov method and inequalities are main tools of our derivation and analysis. The existence of observation interval sequence is verified and one way of its calculation is provided. Finally, an example is given for illustration. Our new techniques not only reduce the observational cost by reducing observation frequency dramatically, but also offer the flexibility on system observation settings. This paper allows readers to set observation frequencies for some time intervals according to their needs to some extent.

Keywords: Stochastic differential equations, exponential stabilisation, Markovian switching, Periodic stochastic systems, Feedback control, discrete-time observations.

1 Introduction

In the past decades, stochastic differential equations have been playing a critical role in many areas including engineering, finance, population ecology, etc., and catching increasing attentions from scientists and engineers. For example, due to its ability to capture the influence of noise, SDE has been used as an important tool in explorations of autonomous vehicles in recent years (see e.g. [1]-[3]). In particular, hybrid SDEs have been widely used for modelling systems that may undergo abrupt changes in structures and parameters, which can be caused by environmental disturbances or accidents. An intriguing topic for SDEs is automatic control. Different stabilities for various systems including uncertain, jump and singular systems etc. using different control schemes including feed forward, feedback and sliding mode control, etc. have been studied (e.g. [4]-[17]).

Consider a hybrid SDE system

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{1.1}$$

on $t \geq 0$, where $x(t) \in \mathbb{R}^n$ is the system state, $B(t)$ is a Brownian motion, $r(t)$ is a Markov chain (please see Section 2 for formal definitions) which represents the system mode. If system (1.1) is not stable and need to be stabilized by a feedback control, a traditional controller based on continuous-time observations are not realistic and expensive, so Mao [13] discretised the system observations and used a constant observation interval τ, which is a positive number. The system needs to be observed at time points $0, \tau, 2\tau, 3\tau, \cdots$, in [13]. Later this study has been developed by many researchers (see e.g. [18]-[24]).

However, a constant frequency of observations cannot make use of the time-varying property. For a non-autonomous system, whose coefficients depend on time explicitly, a time-varying observation frequency is more sensible than the constant one. Intuitively, when the system state or mode change rapidly, we should observe them very frequently and vice versa.

A particular interest for a time-varying system is its periodicity. Periodic phenomena are all around us, such as satellite orbit, seasons, wave vibration, etc. Stochastic models involving periodicity have been studied by researchers due to their wide applications in many areas. To name a few, periodic stochastic volatility, almost periodic solutions for SDEs, quantification of periodic, stochastic, and catastrophic environmental variation, almost periodic stochastic processes, etc. (see e.g. [25]-[32]). Control problem for periodic systems has also received increasing attentions. To name a few, output regulation problem for uncertain linear periodic systems, stabilization problem for periodic orbits of hybrid systems, control problem for periodic ETC (event-triggered control) systems and periodic piecewise linear systems, etc. (see e.g. [33]-[38]).

Since the existing techniques cannot be generalized to cope with the time-varying system observations, this paper uses a new method to investigate: how to stabilise a non-autonomous periodic (i.e., the system coefficients change with time explicitly periodically) hybrid SDE, by a periodic feedback control based on periodic discrete-time observations, and make the controlled system exponentially stable, almost surely and in pth moment for $p > 1$.

Define a periodic observation interval sequence to be $\{\tau_j\}_{j\geq 1}$ such that

$$\tau_{kM+j} = \tau_j$$

for a positive integer M, $\forall k = 0, 1, 2, \cdots$ and $j = 1, 2, \cdots, M$. In other words, the system will be observed at time points $0, \tau_1, \tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3, \cdots$. Note that for any $t \geq 0$, there is a positive integer k such that

$$\sum_{j=1}^{k} \tau_j \leq t < \sum_{j=1}^{k+1} \tau_j,$$

then we can define a step function

$$\delta_t := \sum_{j=1}^{k} \tau_j. \tag{1.2}$$

Consequently, the controlled system regarding to (1.1) has the form

$$\begin{aligned} dx(t) =& [f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)]dt \\ &+ g(x(t), r(t), t)dB(t). \end{aligned} \tag{1.3}$$

By making use of the time-varying property, our new results have two main advantages over the existing theory:
1) reducing the obseration frequency and hence the cost of control.
2) offering the flexibility to set part of the observation frequencies.

The remainder of this paper is organised as follows. Notations are explained in Section 2. In Section 3, we state the stabilisation problem, establish the new theory and provide a useful corollary. In Section 4, we explain how to calculate the observation interval sequence. Section 5 presents a numerical example and Section 6 concludes this paper.

2 Notation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is increasing and right continuous with $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathbb{R}_+$ denote the set of all non-negative real numbers $[0, \infty)$. We write the transpose of a matrix or vector A as A^T. Denote the m-dimensional Brownian motion defined on the probability space by $B(t) = (B_1(t), \cdots, B_m(t))^T$. For a vector x, $|x|$ means its Euclidean norm. For a matrix Q, its trace norm $|Q| = \sqrt{\text{trace}(Q^T Q)}$ and its operator norm $\|Q\| = \max\{|Qx| : |x| = 1\}$. For a real symmetric matrix Q, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ mean its smallest and largest eigenvalues respectively. There are some positive constants whose specific forms are not used for analysis. For simplicity, we denote those positive constants by C, regardless of their values.

Let $r(t)$ for $t \geq 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator matrix $\Gamma = (\gamma_{ij})_{N\times N}$, whose elements γ_{ij} are the transition rates from state i to j for $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. Define a positive number $\overline{\gamma} := -\min_{i\in\mathbb{S}} \gamma_{ii}$.

Define a step function ξ_t for $t \geq 0$ based on the observation interval sequence. Let $\xi_t := \tau_{k+1}$ for any $t \in [\sum_{j=1}^{k} \tau_j, \sum_{j=1}^{k+1} \tau_j)$. This means

$$\delta_t \leq t < \delta_t + \xi_t.$$

For example, when $t \in [0, \tau_1)$, we have $\delta_t = 0$ and $\xi_t = \tau_1$; when $t \in [\tau_1, \tau_1 + \tau_2)$, we have $\delta_t = \tau_1$ and $\xi_t = \tau_2$; when $t \in [\tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3)$, we have $\delta_t = \tau_1 + \tau_2$ and $\xi_t = \tau_3$; $\cdots$. The periodicity of function ξ_t follows from the periodicity of the sequence $\{\tau_j\}_{j\geq 1}$.

Define two positive parameters depending on the moment order p:

$$\zeta = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1, 2), \\ [\frac{p(p-1)}{2}]^{\frac{p}{2}} & \text{for } p \geq 2. \end{cases}$$

and

$$\theta = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1, 2), \\ \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}} & \text{for } p \geq 2. \end{cases}$$

3 Stabilisation Problem

Consider an n-dimensional periodic hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{3.1}$$

on $t \geq 0$, with initial values $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$. Here

$$f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}.$$

The given system may not be stable and our aim is to design a feedback control $u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ for stabilisation.

The controlled system corresponding to (3.1) has the form

$$\begin{aligned} dx(t) =& [f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)]dt \\ &+ g(x(t), r(t), t)dB(t). \end{aligned} \tag{3.2}$$

Assumption 3.1. *Assume that $f(x, i, t)$, $g(x, i, t)$ and $u(x, i, t)$ are all periodic with respect to time t. Assume f, g, u and ξ_t have a common period T.*

The assumption that T is a period of ξ_t means $\xi_t = \xi_{t+kT}$ for $k = 0, 1, 2, \cdots$ and $\sum_{j=1}^{M} \tau_j = T$.

Assumption 3.2. *Assume that the coefficients $f(x, i, t)$ and $g(x, i, t)$ are both locally Lipschitz continuous on x (see e.g. [7]). and satisfy the following linear growth condition*

$$|f(x, i, t)| \leq K_1(t)|x| \quad \textit{and} \quad |g(x, i, t)| \leq K_2(t)|x| \tag{3.3}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $K_1(t)$ and $K_2(t)$ are periodic non-negative continuous functions with period T.

Note (3.3) implies that

$$f(0, i, t) = 0 \quad \text{and} \quad g(0, i, t) = 0 \tag{3.4}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.3. *Assume*

$$|u(x, i, t) - u(y, i, t)| \leq K_3(t)|x - y| \tag{3.5}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $K_3(t)$ is a periodic non-negative continuous function with period T. Moreover, we assume

$$u(0, i, t) = 0 \tag{3.6}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.3 implies that the controller function $u(x,i,t)$ is globally Lipschitz continuous on x and satisfies

$$|u(x,i,t)| \leq K_3(t)|x| \tag{3.7}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Remark 3.4. *For linear controller of the form $u(x,i,t) = U_i(t)x$, where $U_i(t)$ are $n \times n$ real matrices with periodic time-varying elements for $t \geq 0$ and $i \in \mathbb{S}$, we can set $K_3(t) = \|U_i(t)\|$, if the operator norm $\|U_i(t)\|$ is a continuous function of time.*

Let

$$\overline{K_1} \geq \max_{0 \leq t \leq T} K_1(t), \quad \overline{K_2} \geq \max_{0 \leq t \leq T} K_2(t)$$
$$\text{and} \quad \overline{K_3} \geq \max_{0 \leq t \leq T} K_3(t).$$

Let $U(x,i,t)$ be a Lyapunov function periodic with respect to t, and we require $U \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$. Then based on the controlled system, we define $\mathcal{L}U : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}U(x,i,t) =& U_t(x,i,t) + U_x(x,i,t)[f(x,i,t) + u(x,i,t)] \\ &+ \frac{1}{2}\text{trace}[g^T(x,i,t)U_{xx}(x,i,t)g(x,i,t)] \\ &+ \sum_{k=1}^{N} \gamma_{ik} U(x,k,t).\end{aligned} \tag{3.8}$$

Assumption 3.5. *For a fixed moment order $p > 1$, we assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^p \leq U(x,i,t) \leq c_2|x|^p \tag{3.9}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Remark 3.6. *For Lyapunov functions of the form*

$$U(x(t), r(t), t) = (x^T(t)Q_{r(t)}x(t))^{\frac{p}{2}},$$

where $Q_{r(t)}$ are positive-definite symmetric $n \times n$ matrices, Assumption 3.5 holds and we can set

$$c_1 = \min_{i \in \mathbb{S}} \lambda_{\min}^{\frac{p}{2}}(Q_i) \text{ and } c_2 = \max_{i \in \mathbb{S}} \lambda_{\max}^{\frac{p}{2}}(Q_i).$$

Assumption 3.7. *Assume that there is a Lyapunov function $U(x,i,t)$ and a positive continuous function $\lambda(t)$ which have a common period T, constants $l > 0$ and $p > 1$ such that*

$$\mathcal{L}U(x,i,t) + l|U_x(x,i,t)|^{\frac{p}{p-1}} \leq -\lambda(t)|x|^p \tag{3.10}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times [0,T]$.

Let us divide $[0,T]$ into $Z-1$ subintervals, where $Z \geq 2$ is an arbitrary integer, by choosing a partition $\{T_j\}_{1 \leq j \leq Z}$ with $T_1 = 0$ and $T_Z = T$. Then we define the following three step functions on $t \geq 0$ with periodic T:

$$\begin{aligned}\hat{K}_{1t} &= \sup_{T_j \leq s \leq T_{j+1}} K_1(s) \quad \text{for } T_j \leq t < T_{j+1}, \\ \hat{K}_{2t} &= \sup_{T_j \leq s \leq T_{j+1}} K_2(s) \quad \text{for } T_j \leq t < T_{j+1}, \\ \hat{K}_{3t} &= \sup_{T_j \leq s \leq T_{j+1}} K_3(s) \quad \text{for } T_j \leq t < T_{j+1},\end{aligned} \tag{3.11}$$

where $j = 1, \cdots, Z-1$.

Before proposing our theorem, let us define two periodic step functions:

$$\begin{aligned}\varphi_{1t} :=& 8^{p-1}\xi_t^p \hat{K}_{3t}^p + 16^{p-1}\xi_t^{\frac{p}{2}}(1 + \xi_t^p \hat{K}_{3t}^p)(2^{p-1}\xi_t^{\frac{p}{2}}\hat{K}_{1t}^p + \zeta\hat{K}_{2t}^p) \\ &\times \exp(4^{p-1}\xi_t^p \hat{K}_{1t}^p + 4^{p-1}\xi_t^{\frac{p}{2}}\theta\hat{K}_{2t}^p);\end{aligned}$$

and

$$\varphi_{2t} := 8^{p-1}\xi_t^p \hat{K}_{3t}^p + \frac{16^{p-1}\xi_t^{\frac{p}{2}}(1 + \xi_t^p \hat{K}_{3t}^p)(2^{p-1}\xi_t^{\frac{p}{2}}\hat{K}_1^p + \zeta\hat{K}_2^p)}{1 - 4^{p-1}\xi_t^{\frac{p}{2}}(\xi_t^{\frac{p}{2}}\hat{K}_{1t}^p + \theta\hat{K}_{2t}^p)},$$

for sufficiently small ξ_t such that $4^{p-1}\xi_t^{\frac{p}{2}}(\xi_t^{\frac{p}{2}}\hat{K}_{1t}^p + \theta\hat{K}_{2t}^p) < 1$.

3.1 Main Result

Theorem 3.8. *Let the system satisfies Assumptions 3.1 and 3.2. Design the feedback control such that Assumptions 3.3, 3.5 and 3.7 hold. Divide $[0,T]$ into $Z-1$ subintervals with $T_1 = 0$ and $T_Z = T$. Choose the observation interval sequence $\{\tau_j\}_{1 \leq j \leq M}$ sufficiently small such that $\xi_t \leq T_{j+1} - T_j$ for $t \in [T_j, T_{j+1})$ where $j = 1, 2, \cdots, Z-1$ and the following two conditions hold:*
1) for $\forall t \in [0,T)$,
either

$$\varphi_t = \varphi_{1t} < 1, \tag{3.12}$$

or

$$\varphi_t = \varphi_{2t} < 1 \text{ and } 4^{p-1}\xi_t^{\frac{p}{2}}(\xi_t^{\frac{p}{2}}\hat{K}_{1t}^p + \theta\hat{K}_{2t}^p) < 1; \tag{3.13}$$

2)

$$\int_0^T \beta(t)dt > 0, \tag{3.14}$$

where

$$\begin{aligned}\beta(t) := \beta(\xi_t, t) =& \frac{\lambda(t)}{c_2} - \frac{1}{c_2 p(1-\varphi_t)}(\frac{p-1}{pl})^{p-1}K_3^p(t) \\ &\times \Big[2^{3p-2}(1 - e^{-\overline{\gamma}\xi_t}) + 2^{p-1}\varphi_t\Big].\end{aligned} \tag{3.15}$$

Then the solution of the controlled system (3.2) satisfies

$$\limsup_{t \to \infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \leq -\frac{v}{T} \tag{3.16}$$

and

$$\limsup_{t \to \infty} \frac{1}{t}\log(|x(t)|) \leq -\frac{v}{pT} \quad a.s. \tag{3.17}$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$, where

$$v = \int_0^T \beta(t)dt.$$

Remark 3.9. *Notice that T is a period of φ_t, then T is also a period of $\beta(t)$. For φ_t defined in either (3.12) or (3.13), we have the following discussion: when $\xi_t = 0$, $\varphi_t = 0$, then $\beta(t) = \lambda(t)/c_2 > 0$; if ξ_t increases, both φ_t and $\frac{\varphi_t}{1-\varphi_t}$ increases, then $\beta(t)$ will decrease. So there exists $\xi_t > 0$ for $0 \leq t < T$ such that $\int_0^T \beta(t)dt > 0$.*

We will use the same observation frequency in one subinterval of $[0,T]$.

Remark 3.10. *Notice that ξ_t is a right-continuous step function. Since we use the same observation frequency within the same subinterval $[T_j, T_{j+1})$ where $j = 1, \cdots, Z-1$, ξ_t is constant for $t \in [T_j, T_{j+1})$. Notice that $\hat{K}_{1t}$, $\hat{K}_{2t}$ and $\hat{K}_{3t}$ are also right-continuous step functions which are constant for $t \in [T_j, T_{j+1})$. So is φ_t. Therefore, $\beta(t)$ is a right-continuous step function which only jumps at T_1, T_2, $\cdots$.*

We can calculate the observation interval sequence using both conditions (3.12) and (3.13) respectively, then choose the one that yields less frequent observations.

3.2 Proof of the Main Result

Proof.
Step 1. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. By the generalized Itô formula, we have

$$\mathbb{E}U(x(t), r(t), t) = U_0 + \int_0^t \mathbb{E}LU(x(s), r(s), s)ds, \qquad (3.18)$$

where $U_0 = U(x(0), r(0), 0)$ and

$$\begin{aligned} &LU(x(s), r(s), s) \\ =&U_s(x(s), r(s), s) + \sum_{k=1}^{N} \gamma_{ik} U(x, k, s) \\ &+ U_x(x(s), r(s), s)[f(x(s), r(s), s) + u(x(\delta_s), r(\delta_s), s)] \\ &+ \frac{1}{2}\text{trace}[g^T(x(s), r(s), s)U_{xx}(x(s), r(s), s)g(x(s), r(s), s)]. \end{aligned} \qquad (3.19)$$

Notice that $LU(x(s), r(s), s)$ can be rewritten as

$$\begin{aligned} LU(x(s), r(s), s) =&\mathcal{L}U(x(s), r(s), s) - U_x(x(s), r(s), s) \\ &\times [u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)]. \end{aligned} \qquad (3.20)$$

By the Young inequality, we can derive that

$$\begin{aligned} &- U_x(x(s), r(s), s)[u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)] \\ \leq&\Big[\varepsilon|U_x(x(s), r(s), s)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} \\ &\times \Big[\varepsilon^{1-p}|u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)|^p\Big]^{\frac{1}{p}} \\ \leq& l|U_x(x(s), r(s), s)|^{\frac{p}{p-1}} \\ &+ \frac{1}{p}(\frac{p-1}{pl})^{p-1}|u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)|^p, \end{aligned} \qquad (3.21)$$

where $l = \frac{p-1}{p}\varepsilon$ for $\forall \varepsilon > 0$.

According to Lemma 1 in [21], for any $t \geq t_0$, $v > 0$ and $i \in \mathbb{S}$,

$$\mathbb{P}(r(s) \neq i \text{ for some } s \in [t, t+v] \Big| r(t) = i) \leq 1 - e^{-\overline{\gamma}v}. \qquad (3.22)$$

By Assumption 3.3, we have

$$\begin{aligned} &\mathbb{E}|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^p \\ =&\mathbb{E}\Big[\mathbb{E}\big(|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^p \Big| \mathcal{F}_{\delta_s}\big)\Big] \\ \leq& 2^{2p-1}K_3^p(s)(1 - e^{-\overline{\gamma}\xi_s})[\mathbb{E}|x(s)|^p + \mathbb{E}|x(\delta_s) - x(s)|^p]. \end{aligned} \qquad (3.23)$$

Then by the elementary inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $a, b \in \mathbb{R}$ and $p > 1$, we have

$$\begin{aligned} &\mathbb{E}|u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)|^p \\ \leq& 2^{p-1}\mathbb{E}|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^p \\ &+ 2^{p-1}\mathbb{E}|u(x(\delta_s), r(s), s) - u(x(s), r(s), s)|^p \\ \leq& 2^{3p-2}K_3^p(s)(1 - e^{-\overline{\gamma}\xi_s})\mathbb{E}|x(s)|^p \\ &+ [2^{3p-2}K_3^p(s)(1 - e^{-\overline{\gamma}\xi_s}) + 2^{p-1}K_3^p(s)]\mathbb{E}|x(\delta_s) - x(s)|^p. \end{aligned} \qquad (3.24)$$

Substitute (3.24) into (3.21). Then by (3.20) and Assumption 3.7, we obtain that

$$\begin{aligned} &\mathbb{E}LU(x(s), r(s), s) \\ \leq& - [\lambda(s) - \frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1 - e^{-\overline{\gamma}\xi_s})]\mathbb{E}|x(s)|^p \\ &+ \frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)[2^{3p-2}(1 - e^{-\overline{\gamma}\xi_s}) + 2^{p-1}]\mathbb{E}|x(\delta_s) - x(s)|^p. \end{aligned} \qquad (3.25)$$

Note that $t - \delta_t \leq \xi_t$ for all $t \geq 0$. By the Itô formula, Hölder's inequality, the Burkholder-Davis-Gundy inequality (see e.g. [5, p.40]) and [5, Theorem 7.1 on page 39], we obtain that (see e.g. [23])

$$\begin{aligned} &\mathbb{E}|x(t) - x(\delta_t)|^p \\ \leq& 2^{p-1}\xi_t^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t \Big[\xi_t^{\frac{p}{2}}|f(x(s), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^p \\ &+ \zeta|g(x(s), r(s), s)|^p\Big]ds. \end{aligned} \qquad (3.26)$$

Let

$$\hat{U}(x(t), r(t), t) = e^{\int_0^t \beta(s)ds}U(x(t), r(t), t).$$

We can obtain from the generalized Itô formula that

$$\begin{aligned} &\mathbb{E}\hat{U}(x(t), r(t), t) \\ =&\mathbb{E}U_0 + \mathbb{E}\int_0^t L\hat{U}(x(s), r(s), s)ds \\ \leq&\mathbb{E}U_0 + \int_0^t e^{\int_0^s \beta(z)dz}\big[\mathbb{E}LU(x(s), r(s), s) \\ &+ \beta(s)\mathbb{E}U(x(s), r(s), s)\big]ds, \end{aligned} \qquad (3.27)$$

where $LU(x(s), r(s), s)$ has been defined in (3.19).

By (3.26), Assumptions 3.2 and 3.3, we have that for any $s \in [\delta_s, \delta_s + \xi_s)$,

$$\begin{aligned} &\mathbb{E}|x(s) - x(\delta_s)|^p \\ \leq& 4^{p-1}\xi_s^{p-1}\int_{\delta_s}^s K_3^p(z)dz\mathbb{E}|x(\delta_s)|^p \\ &+ 2^{p-1}\xi_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s [2^{p-1}\xi_s^{\frac{p}{2}}K_1^p(z) + \zeta K_2^p(z)]|x(z)|^p dz \\ \leq& 4^{p-1}\xi_s^p\hat{K}_{3s}^p\mathbb{E}|x(\delta_s)|^p \\ &+ 2^{p-1}\xi_s^{\frac{p}{2}}[2^{p-1}\xi_s^{\frac{p}{2}}\hat{K}_{1s}^p + \zeta\hat{K}_{2s}^p]\mathbb{E}\Big(\sup_{\delta_s \leq t \leq s}|x(t)|^p\Big). \end{aligned} \qquad (3.28)$$

Step 2. We will prove that under either condition (3.12) or (3.13), we have

$$\mathbb{E}|x(s)-x(\delta_s)|^p \le \frac{\varphi_s}{1-\varphi_s}\mathbb{E}|x(s)|^p, \tag{3.29}$$

for the corresponding φ_s.

Firstly, we prove it using condition (3.12).

By the elementary inequality $|\sum_{i=1}^k x_i|^p \le k^{p-1}\sum_{i=1}^k |x_i|^p$ for $p\ge 1$ and $x_i\in\mathbb{R}$ (see e.g. [7]), Hölder's inequality and the Burkholder-Davis-Gundy inequality (see e.g. [5, page 40]), we have that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p + 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t f(x(z),r(z),z)dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t u(x(\delta_z),r(\delta_z),z)]dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t g(x(z),r(z),z)dB(z)\Big|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p+(4\xi_s)^{p-1}\\
&\times\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t[K_1^p(z)|x(z)|^p+K_3^p(z)|x(\delta_s)|^p]dz\Big)\\
&+4^{p-1}\xi_s^{\frac{p-2}{2}}\theta\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t K_2^p(z)|x(z)|^p dz\Big)\\
\le&\Big[4^{p-1}+(4\xi_s)^{p-1}\int_{\delta_s}^s K_3^p(z)dz\Big]\mathbb{E}|x(\delta_s)|^p\\
&+\Big[(4\xi_s)^{p-1}\hat{K}_{1s}^p+4^{p-1}\xi_s^{\frac{p-2}{2}}\theta\hat{K}_{2s}^p\Big]\int_{\delta_s}^s\mathbb{E}\Big(\sup_{\delta_s\le z\le t}|x(z)|^p\Big)dt
\end{aligned}$$

Then the Gronwall inequality implies

$$\begin{aligned}
\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\le&\Big[4^{p-1}+(4\xi_s)^{p-1}\int_{\delta_s}^s K_3^p(z)dz\Big]\mathbb{E}|x(\delta_s)|^p\\
&\times\exp(4^{p-1}\xi_s^p\hat{K}_{1s}^p+4^{p-1}\xi_s^{\frac{p}{2}}\theta\hat{K}_{2s}^p).
\end{aligned}\tag{3.30}$$

Substituting this into (3.28) gives

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p\\
\le&4^{p-1}\xi_s^{\frac{p}{2}}\Big[\xi_s^{\frac{p}{2}}\hat{K}_{3s}^p+2^{p-1}(1+\xi_s^p\hat{K}_{3s}^p)(2^{p-1}\xi_s^{\frac{p}{2}}\hat{K}_{1s}^p+\zeta\hat{K}_{2s}^p)\\
&\times\exp(4^{p-1}\xi_s^p\hat{K}_{1s}^p+4^{p-1}\xi_s^{\frac{p}{2}}\theta\hat{K}_{2s}^p)\Big]\mathbb{E}|x(\delta_s)|^p.
\end{aligned}$$

Noticing that

$$\mathbb{E}|x(\delta_s)|^p\le 2^{p-1}\mathbb{E}|x(s)|^p+2^{p-1}\mathbb{E}|x(s)-x(\delta_s)|^p$$

for all $p>1$, we have

$$\mathbb{E}|x(s)-x(\delta_s)|^p\le\varphi_s[\mathbb{E}|x(s)|^p+\mathbb{E}|x(s)-x(\delta_s)|^p],$$

where φ_s was been defined in (3.12). Rearranging it gives (3.29).

Alternatively, we prove it under condition (3.13).

By the elementary inequality, Hölder's inequality and the Burkholder-Davis-Gundy inequality, we have that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p + 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t f(x(z),r(z),z)dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t u(x(\delta_z),r(\delta_z),z)]dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t g(x(z),r(z),z)dB(z)\Big|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p+(4\xi_s)^{p-1}\\
&\times\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t[K_1^p(z)|x(z)|^p+K_3^p(z)|x(\delta_s)|^p]dz\Big)\\
&+4^{p-1}\xi_s^{\frac{p-2}{2}}\theta\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t K_2^p(z)|x(z)|^p dz\Big)\\
\le&4^{p-1}(1+\xi_s^p\hat{K}_{3s}^p)\mathbb{E}|x(\delta_s)|^p\\
&+4^{p-1}\xi_s^{\frac{p}{2}}(\xi_s^{\frac{p}{2}}\hat{K}_{1s}^p+\theta\hat{K}_{2s}^p)\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big).
\end{aligned}\tag{3.31}$$

The condition in (3.13) requires that $4^{p-1}\xi_t^{\frac{p}{2}}(\xi_t^{\frac{p}{2}}\hat{K}_{1t}^p+\theta\hat{K}_{2t}^p)<1$. So we can rearrange (3.31) and get

$$\mathbb{E}\Big(\sup_{\delta_s\le z\le s}|x(z)|^p\Big)\le\frac{4^{p-1}(1+\xi_s^p\hat{K}_{3s}^p)}{1-4^{p-1}\xi_s^{\frac{p}{2}}(\xi_s^{\frac{p}{2}}\hat{K}_{1s}^p+\theta\hat{K}_{2s}^p)}\mathbb{E}|x(\delta_s)|^p. \tag{3.32}$$

Substituting this into (3.28) gives

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p\\
\le&\Big[4^{p-1}\xi_s^p\hat{K}_{3s}^p+\frac{8^{p-1}\xi_s^{\frac{p}{2}}(2^{p-1}\xi_s^{\frac{p}{2}}\hat{K}_1^p+\zeta\hat{K}_2^p)(1+\xi_s^p\hat{K}_{3s}^p)}{1-4^{p-1}\xi_s^{\frac{p}{2}}(\xi_s^{\frac{p}{2}}\hat{K}_{1s}^p+\theta\hat{K}_{2s}^p)}\Big]\\
&\times\mathbb{E}|x(\delta_s)|^p\\
\le&\varphi_s(\mathbb{E}|x(s)|^p+\mathbb{E}|x(s)-x(\delta_s)|^p),
\end{aligned}\tag{3.33}$$

where φ_s has been defined in (3.13).

Since condition (3.13) requires $\varphi_t<1$ for all $t>0$, we can rearrange (3.33) and obtain (3.29).

Step 3. Substitute (3.29) into (3.25). Then by (3.15), we have

$$\begin{aligned}
&\mathbb{E}LU(x(s),r(s),s)\\
\le&-[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1-e^{-\overline{\gamma}\xi_s})]\mathbb{E}|x(s)|^p\\
&+\frac{1}{p}(\frac{p-1}{pl})^{p-1}\frac{\varphi_s}{1-\varphi_s}K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\xi_s})+2^{p-1}]\mathbb{E}|x(s)|^p\\
\le&-c_2\beta(s)\mathbb{E}|x(s)|^p.
\end{aligned}\tag{3.34}$$

Substitute (3.34) into (3.27). Then by Assumption 3.5, we have

$$\begin{aligned}
&\mathbb{E}\hat{U}(x(t),r(t),t)\\
\le&\mathbb{E}U_0+\int_0^t e^{\int_0^s\beta(z)dz}[\mathbb{E}LU(x(s),r(s),s)+c_2\beta(s)\mathbb{E}|x(s)|^p]ds\\
\le&\mathbb{E}U_0.
\end{aligned}\tag{3.35}$$

Assumption 3.5 indicates that

$$c_1e^{\int_0^t\beta(s)ds}\mathbb{E}|x(t)|^p\le\mathbb{E}\hat{U}(x(t),r(t),t)\le\mathbb{E}U_0.$$

Then

$$\mathbb{E}|x(t)|^p \le Ce^{-\int_0^t \beta(s)ds}.$$

Recall that C's denote positive constants.

So we have

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \le \limsup_{t\to\infty} \frac{-1}{t}\int_0^t \beta(s)ds = -\frac{v}{T}.$$

Hence we have obtained assertion (3.16).

Let $\epsilon \in (0, \frac{v}{2T})$ be arbitrary. Then (3.16) implies that there exists a constant $C > 0$ such that

$$\mathbb{E}|x(t)|^p \le Ce^{-(v/T-\epsilon)t} \quad \text{for } \forall t \ge 0. \tag{3.36}$$

Notice that

$$4^{p-1}(1+\xi_t^p \hat{K}_{3t}^p)[1-4^{p-1}\xi_t^{\frac{p}{2}}(\xi_t^{\frac{p}{2}}\hat{K}_{1t}^p+\theta\hat{K}_{2t}^p)]^{-1}$$

in (3.32) is bounded. It follows from (3.30) and (3.32) that

$$\mathbb{E}\Big(\sup_{\delta_t\le s\le \delta_t+\xi_t}|x(s)|^p\Big) \le C\mathbb{E}|x(\delta_t)|^p \le Ce^{-(v/T-\epsilon)\delta_t} \tag{3.37}$$

for $\forall t \ge 0$.

Then by the Chebyshev inequality, we have

$$\mathbb{P}\Big(\sup_{\delta_t\le s\le \delta_t+\xi_t}|x(s)| \ge \exp[\frac{\delta_t}{p}(2\epsilon-\frac{v}{T})]\Big) \le Ce^{-\epsilon\delta_t}.$$

The Borel-Cantelli lemma indicates that, there is a $t^* = t^*(\omega) > 0$ for almost all $\omega \in \Omega$ such that

$$\sup_{\delta_t\le s\le \delta_t+\xi_t}|x(s)| < \exp[\frac{\delta_t}{p}(2\epsilon-\frac{v}{T})] \quad \text{for } \forall t \ge t^*.$$

So

$$\log\frac{1}{t}(|x(t)|) < -(\frac{v}{T}-2\epsilon)\frac{\delta_t}{pt}.$$

As $t \to \infty$,

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t,\omega)|) \le -\frac{1}{p}(\frac{v}{T}-2\epsilon) \;\; a.s.$$

Letting $\epsilon \to 0$ gives assertion (3.17). The proof is complete. □

3.3 Corollary

For Lyapunov functions of the form

$$U(x(t), r(t), t) = (x^T(t)Q_{r(t)}x(t))^{\frac{p}{2}}$$

where $Q_{r(t)}$ are positive-definite symmetric $n\times n$ matrices for $p \ge 2$, we propose the following corollary.

Assumption 3.11. *Assume that there exist positive-definite symmetric matrices $Q_i \in \mathbb{R}^{n\times n}$ ($i \in \mathbb{S}$) and a periodic positive continuous function $b(t)$ such that*

$$\begin{aligned}
&p(x^TQ_ix)^{\frac{p}{2}-1}\Big(x^TQ_i[f(x,i,t)+u(x,i,t)] \\
&+\frac{1}{2}\text{trace}[g^T(x,i,t)Q_ig(x,i,t)]\Big) \\
&+p(\frac{p}{2}-1)[x^TQ_ix]^{\frac{p}{2}-2}|g^TQ_ix|^2+\sum_{j=1}^N\gamma_{ij}[x^TQ_jx]^{\frac{p}{2}} \\
\le& -b(t)|x|^p,
\end{aligned} \tag{3.38}$$

for all $(x,i,t) \in \mathbb{R}^n\times\mathbb{S}\times[0,T]$.

We can see that T is a period of $b(t)$.

Corollary 3.12. *If Assumptions 3.7 and 3.5 are replaced by Assumption 3.11, then Theorem 3.8 still holds for $p \ge 2$ with $c_2 = \max_{i\in\mathbb{S}}\lambda_{\max}^{\frac{p}{2}}(Q_i)$, $\lambda(t) = b(t) - ld$ where $d = (pc_2)^{\frac{p}{p-1}}$ and $0 < l < \min_{0\le t\le T} b(t)/d$.*

Proof: Calculate condition (3.10) in Assumption 3.7 for $U(x,i,t) = (x^TQ_ix)^{\frac{p}{2}}$. Firstly, calculate the partial derivative

$$U_x(x,i,t) = p(x^TQ_ix)^{\frac{p}{2}-1}x^TQ_i.$$

Then we have

$$|U_x(x,i,t)| \le p\lambda_{\max}^{\frac{p}{2}-1}(Q_i)\|Q_i\||x|^{p-1} = pc_2|x|^{p-1}.$$

Secondly, calculate the partial derivative $U_t(x,i,t) = 0$ and

$$U_{xx}(x,i,t) = p(p-2)[x^TQ_ix]^{\frac{p}{2}-2}Q_ixx^TQ_i + p[x^TQ_ix]^{\frac{p}{2}-1}Q_i.$$

So $\mathcal{L}U(x,i,t)$ is equivalent to the left-hand-side of (3.38). This means

$$\mathcal{L}U(x,i,t) \le -b(t)|x|^p.$$

Substitute these into (3.10), we get

$$\mathcal{L}U(x,i,t) + l|U_x(x,i,t)|^{\frac{p}{p-1}} \le (-b(t)+ld)|x|^p = -\lambda(t)|x|^p.$$

The condition $l < \min_{0\le t\le T} b(t)/d$ guarantees $\lambda(t)$ is positive.
Consequently, Assumption 3.7 can be guaranteed by Assumption 3.11. The proof is complete. □

4 Computation and Discussion

4.1 Computation Procedure

Now we discuss how to divide $[0,T]$ and how to calculate the observation interval sequence. We can either use even division or divide according to the shape of an auxiliary function. We use the same observation frequency in one subinterval of $[0,T]$. Notice that β can be negative at some time points, we only need to guarantee that its integral over $[0,T]$ is positive. This gives flexibility on the setting of ξ_t. For example, we can choose to increase the shortest observation interval to avoid high frequency observations by reducing the large observation intervals in some time intervals, or choose to make the large observation intervals even larger. This will be illustrated in the example.

Here we show one method to find an observation interval sequence that satisfies the conditions in Theorem 3.8, although there are other ways. We can find an observation interval sequence that satisfies the conditions in Theorem 3.8 by the following four steps:

Step 1. Choose to satisfy condition (3.12) or (3.13).
Suppose we choose condition (3.12).
Firstly, find a positive number $\overline{\xi}$ such that

$$\begin{aligned}
&8^{p-1}\overline{\xi}^p\overline{K_3}^p + 16^{p-1}\overline{\xi}^{\frac{p}{2}}(1+\overline{\xi}^p\overline{K_3}^p)(2^{p-1}\overline{\xi}^{\frac{p}{2}}\overline{K_1}^p+\zeta\overline{K_2}^p) \\
&\times\exp(4^{p-1}\overline{\xi}^p\overline{K_1}^p + 4^{p-1}\overline{\xi}^{\frac{p}{2}}\theta\overline{K_2}^p) \\
&\le 1.
\end{aligned} \tag{4.1}$$

Noticing that the left-hand-side is an increasing function of $\overline{\xi}$, in practice, we can find $\overline{\xi}$ by solving the equality in (4.1) numerically by computer and then choosing $\overline{\xi}$ smaller than the approximate solution.

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Secondly, let ξ be a positive number to be determined. Define

$$\begin{aligned}\tilde{\varphi}(t) =& 8^{p-1}\xi^p K_3^p(t) + 16^{p-1}\xi^{\frac{p}{2}}[1+\xi^p K_3^p(t)] \\ &\times [2^{p-1}\xi^{\frac{p}{2}}K_1^p(t) + \zeta K_2^p(t)] \\ &\times \exp[4^{p-1}\xi^p K_1^p(t) + 4^{p-1}\xi^{\frac{p}{2}}\theta K_2^p(t)]\end{aligned}$$

and

$$\beta_a(t) = \frac{\lambda(t)}{c_2} - \frac{2^{3p-2}(1-e^{-\overline{\gamma}\xi}) + 2^{p-1}\tilde{\varphi}(t)}{c_2 p(1-\tilde{\varphi}(t))}(\frac{p-1}{pl})^{p-1}K_3^p(t).$$

Alternatively, suppose we choose (3.13).
Firstly, find a positive number $\overline{\xi}$ such that

$$4^{p-1}\overline{\xi}^{\frac{p}{2}}(\overline{\xi}^{\frac{p}{2}}\overline{K_1}^p + \theta\overline{K_2}^p) < 1$$

and

$$8^{p-1}\overline{\xi}^p\overline{K_3}^p + \frac{16^{p-1}\overline{\xi}^{\frac{p}{2}}(1+\overline{\xi}^p\overline{K_3}^p)(2^{p-1}\overline{\xi}^{\frac{p}{2}}\overline{K_1}^p + \zeta\overline{K_2}^p)}{1-4^{p-1}\overline{\xi}^{\frac{p}{2}}(\overline{\xi}^{\frac{p}{2}}\overline{K_1}^p + \theta\overline{K_2}^p)} < 1.$$

Secondly, let ξ be a positive number to be determined. Define

$$\begin{aligned}\tilde{\varphi}(t) =& 8^{p-1}\xi^p K_3^p(t) \\ &+ \frac{16^{p-1}\xi^{\frac{p}{2}}[1+\xi^p K_3^p(t)][2^{p-1}\xi^{\frac{p}{2}}K_1^p(t) + \zeta K_2^p(t)]}{1-4^{p-1}\xi^{\frac{p}{2}}[\xi^{\frac{p}{2}}K_1^p(t) + \theta K_2^p(t)]}\end{aligned}$$

and $\beta_a(t)$ has the same form as above.
For choice of either (3.12) or (3.13), using corresponding definitions above, choose a positive number $\xi < \overline{\xi}$ such that $\int_0^T \beta_a(t)dt > 0$.

Step 2. The second step is to divide $[0,T]$ into $Z-1$ subintervals. There is no restriction on the partition. We can simply set even division or divide according to the shape of $\beta_a(t)$, in which case we want the maximum and minimum of $\beta_a(t)$ in each subinterval are relatively close. Then set a sequence of $Z-1$ numbers $\{\underline{\beta}_j\}_{1\le j\le Z-1}$ such that

$$\underline{\beta}_j \le \min_{T_j\le t\le T_{j+1}} \beta_a(t) \quad \text{and} \quad \sum_{j=1}^{Z-1}\underline{\beta}_j(T_{j+1}-T_j) \ge 0.$$

If

$$\sum_{j=1}^{Z-1}\min_{T_j\le t\le T_{j+1}}\beta_a(t)(T_{j+1}-T_j) \ge 0,$$

then we can simply set $\underline{\beta}_j = \min_{T_j\le t\le T_{j+1}}\beta_a(t)$ for $j = 1,\cdots,Z-1$.

Step 3. Find the solution $\tilde{\tau}(t)$ for $t\in[0,T)$ to the following equation

$$\beta(\tilde{\tau}(t),t) = \underline{\beta}_j \quad \text{for } j=1,2,\cdots,Z-1. \tag{4.2}$$

An approximate solution by computer is enough. Then let $\tilde{\tau_j} \le \inf_{t\in[T_j,T_{j+1})}\tilde{\tau}(t)$, i.e. the infimum of $\tilde{\tau}$ over the jth subinterval, for $j=1,\cdots,Z-1$.

Find a function $\tilde{\tau}(t)$ with $\inf_{t\in[0,T)}\tilde{\tau}(t) > 0$ such that

$$\beta(\tilde{\tau}(t),t) \ge \underline{\beta}_j \quad \text{for } j=1,2,\cdots,Z-1. \tag{4.3}$$

This can be done by solving Then let $\tilde{\tau_j} = \inf_{t\in[T_j,T_{j+1})}\tilde{\tau}(t)$, i.e. the infimum of $\tilde{\tau}$ over the jth subinterval, for $j=1,\cdots,Z-1$.

Step 4. For the jth subinterval, choose a positive integer N_j such that $\frac{T_{j+1}-T_j}{N_j} < \min(\tilde{\tau_j},\overline{\xi})$, then let $\underline{\xi}_j = \frac{T_{j+1}-T_j}{N_j}$.
Find N_j and $\underline{\xi}_j$ for all $1\le j\le Z-1$. Then over the jth subinterval ($t\in[T_j,T_{j+1})$), the observation interval is $\underline{\xi}_j$ and we observe the system N_j times.

This means, for the first subinterval, $\tau_1 = \cdots = \tau_{N_1} = \underline{\xi}_1$;
for the second subinterval, $\tau_{N_1+1} = \cdots = \tau_{N_1+N_2} = \underline{\xi}_2$;
for the third subinterval, $\tau_{N_1+N_2+1} = \cdots = \tau_{N_1+N_2+N_3} = \underline{\xi}_3$; $\cdots$.

In other words, in one period $[0,T)$, the system is observed at:

$0(=T_1),\ \underline{\xi}_1,\ 2\underline{\xi}_1,\ \cdots,\ (N_1-1)\underline{\xi}_1;$

$N_1\underline{\xi}_1(=T_2),\ N_1\underline{\xi}_1+\underline{\xi}_2,\ N_1\underline{\xi}_1+2\underline{\xi}_2,\ \cdots,\ N_1\underline{\xi}_1+(N_2-1)\underline{\xi}_2;$

$N_1\underline{\xi}_1+N_2\underline{\xi}_2(=T_3),\ N_1\underline{\xi}_1+N_2\underline{\xi}_2+\underline{\xi}_3,\ N_1\underline{\xi}_1+N_2\underline{\xi}_2+2\underline{\xi}_3,\ \cdots$
$N_1\underline{\xi}_1+N_2\underline{\xi}_2+(N_3-1)\underline{\xi}_3;\cdots$

4.2 *Discussion*

Now let us explain why the observation interval sequence founded above can satisfy the conditions in Theorem 3.8. Notice $\underline{\xi}_j < \tilde{\tau_j} \le \tilde{\tau}(t)$ for $t\in[T_j,T_{j+1})$, $j=1,\cdots,Z-1$ and $\beta(\xi_t,t)$ defined in (3.15) is negatively related to ξ_t. Then we have

$$\begin{aligned}\int_0^T\beta(\xi_t,t)dt &= \sum_{j=1}^{Z-1}\int_{T_j}^{T_{j+1}}\beta(\underline{\xi}_j,t)dt > \sum_{j=1}^{Z-1}\int_{T_j}^{T_{j+1}}\beta(\tilde{\tau_j},t)dt \\ &\ge \sum_{j=1}^{Z-1}\int_{T_j}^{T_{j+1}}\beta(\tilde{\tau}(t),t)dt = \sum_{j=1}^{Z-1}\underline{\beta}_j(T_{j+1}-T_j) \ge 0.\end{aligned}$$

So condition (3.14) can be guaranteed if we follow the above four steps. Step 4 gives $\max_{j=1}^{Z-1}\underline{\xi}_j < \overline{\xi}$, which guarantees condition (3.12) or (3.13) as chosen in Step 1.

Inequality (3.14) is a condition on the integral over one period instead of on every time point. This gives flexibility to the setting of observation frequencies. The flexibility comes from the settings of partition of $[0,T]$ and $\{\underline{\beta}_j\}_{1\le j\le Z-1}$. By adjusting the partition of $[0,T]$ and $\underline{\beta}_j$'s for some $j\in[1,Z-1]$, we can change or set the observation frequency for a specific time interval, to some extent.

Parameter $\beta(t)$ is negative related to $\varphi_t, K_3, \overline{\gamma}$ and ξ_t. φ_t defined in either (3.12) or (3.13) is positive related to K_1, K_2, K_3 and ξ_t. So when K_1, K_2, K_3 or ξ_t increases, $\beta(t)$ will decrease. Therefore, large $K_1(t), K_2(t), K_3(t)$ and $\overline{\gamma}$ tend to yield small ξ_t. Notice that: small observation intervals indicate high observation frequencies; large values of $K_1(t), K_2(t)$ and $K_3(t)$ imply rapid change of the system state $x(t)$; and a large $\overline{\gamma}$ is corresponding to frequent switching of the system mode. So our conditions tend to require frequent observations when the system changes quickly, which is in accordance with our intuition and experience. However, the integral condition allows for some exceptions, as long as the negative values of $\beta(t)$ in some time intervals can be compensated by its positive values in some time intervals and its integral over $[0,T]$ is positive. In other words, although some corrections to the system are delayed, as long as it can be compensated by prompt corrections in other time intervals, the controlled system (3.2) can still achieve exponential stability.

For exponential stabilisation, our observations can be less frequently than the constant observation frequency obtained in the existing studies. To give an extreme example, let the periodic system coefficients $f(x,i,t) = g(x,i,t) = 0$ for a time interval, say $[t_1,t_2]$. Then we can stop controlling and let $u(x,i,t) = 0$ in this interval. Thus, we can stop monitoring the system in (t_1, t_2) and we only need observations at $t = t_1$ and $t = t_2$. This benefit comes from our consideration of the time-varying property.

5 Example

Let us design a feedback control to make the following 2-dimensional periodic nonlinear hybrid SDE mean square exponentially stable.

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{5.1}$$

on $t \geq 0$, where $B(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with the generator matrix

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The system coefficients are

$$f(x, 1, t) = k_1(t) \begin{bmatrix} 0 & \sin(x_1) \\ \cos(x_2) & 0 \end{bmatrix} x,$$

$$g(x, 1, t) = k_2(t) \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} x,$$

$$f(x, 2, t) = k_3(t) \begin{bmatrix} \sin(x_2)x_1 \\ \cos(x_1)x_2 \end{bmatrix},$$

and

$$g(x, 2, t) = \frac{1}{2\sqrt{2}} \; k_4(t) \begin{bmatrix} \sqrt{3x_1^2 + x_2^2} \\ \sqrt{x_1^2 + 3x_2^2} \end{bmatrix},$$

where

$$k_1(t) = 1.5 + \cos(\frac{\pi}{6}t), \;\; k_2(t) = 1 + \sin(\frac{\pi}{6}t - 2.8),$$
$$k_3(t) = 1.5 + \sin(\frac{\pi}{6}t), \;\; k_4(t) = 1 + \cos(\frac{\pi}{6}t + 2.8).$$

The upper plot in Fig. 1 shows that the original system (5.1) is not mean square exponentially stable. The system coefficients $f(x, i, t)$ and $g(x, i, t)$ have common period $T = 12$.

Let us calculate $K_1(t)$ and $K_2(t)$. Since $|f(x, 1, t)| \leq k_1(t)|x|$ and $|f(x, 2, t)| \leq [1.5 + \sin(\frac{\pi}{6}t)]|x|$, we get

$$K_1(t) = 1.5 + \max\{\cos(\frac{\pi}{6}t), \sin(\frac{\pi}{6}t)\}.$$

Similarly, $K_2(t) = \max\{k_2(t), \frac{1}{\sqrt{2}}[1 + \cos(\frac{\pi}{6}t + 2.8)]\}$. Then $K_1(t) \leq \overline{K_1} = 2.5$ and $K_2(t) \leq \overline{K_2} = 2$. So Assumption 3.2 holds.

Then we can design a feedback control according to Corollary 3.12, and find an observation interval sequence, to make the controlled system

$$dx(t) = \big[f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big]dt + g(x(t), r(t), t)dB(t) \tag{5.2}$$

achieve mean square exponential stability.

Suppose the controller has form $u(x, i, t) = A(x, i, t)x$ and our need to design the function $A : \mathbb{R}^2 \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ with bounded norm. Let us choose the Lyapunov function of the simplest form $U(x, i, t) = x^T x$ for two modes. In other words, we choose Q_i to be the 2×2 identity matrix, for $i = 1, 2$. Then $c_2 = 1$ and $d = 4$. The left-hand-side of (3.38) in Assumption 3.11 becomes

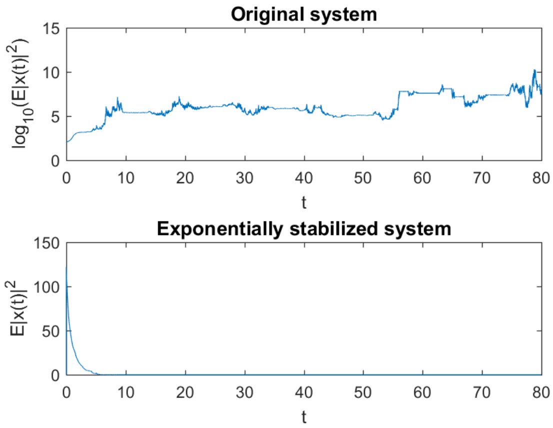

Fig. 1: Sample averages of $|x(t)|^2$ from 500 simulated paths by the Euler-Maruyama method with step size $1e-5$ and random initial values. Upper plot shows original system (5.1); lower plot shows controlled system (5.2) with calculated observation intervals.

$$2x^T(f(x, i, t) + u(x, i, t)) + g^T(x, i, t)g(x, i, t). \tag{5.3}$$

For mode 1, to keep the notation simple, define two matrices F and G by letting $f(x, 1, t) = F(x, t)x$ and $g(x, 1, t) = G(t)x$. Then (5.3) for mode 1 becomes

$$2x^T[F(x, t) + A(x, 1, t)]x + x^T G^T(t)G(t)x = x^T \tilde{Q} x, \tag{5.4}$$

where

$$\tilde{Q} = F(x, t) + F^T(x, t) + A(x, 1, t) + A^T(x, 1, t) + G^T(t)G(t).$$

We design $A(x, t)$ to make $\tilde{Q}$ negative definite, then Assumption 3.11 can hold. Calculate the matrix

$$\tilde{Q} = \begin{bmatrix} 0.5k_2^2(t) & k_1(t)G_1(x) - 0.5k_2^2(t) \\ k_1(t)G_1(x) - 0.5k_2^2(t) & 0.5k_2^2(t) \end{bmatrix} + A(x, 1, t) + A^T(x, 1, t),$$

where $G_1(x) = \sin(x_1) + \cos(x_2)$. Let

$$A(x, 1, t) = \begin{bmatrix} a_1(t) & a_2(x, t) \\ a_2(x, t) & a_1(t) + 0.1\sin(\frac{\pi}{6}t) \end{bmatrix},$$

where $a_1(t) = -0.25k_2^2(t) - 0.5$ and $a_2(x, t) = -0.5k_1(t)G_1(x) - 0.25k_2^2(t)$. Then

$$\tilde{Q} = \begin{bmatrix} -1 & 0 \\ 0 & -1 + 0.2\sin(\frac{\pi}{6}t) \end{bmatrix}$$

is negative definite.

For mode 2, (5.3) is

$$\begin{aligned}&2x^T[f(x,2,t)+u(x,2,t)]+g^T(x,2,t)g(x,2,t)\\=&[2k_3(t)\sin(x_2)+0.5k_4^2(t)]x_1^2\\&+[2k_3(t)\cos(x_1)+0.5k_4^2(t)]x_2^2+2x^TA(x,2,t)x.\end{aligned}$$

For simplicity, let $A(x,2,t)$ be a diagonal matrix. We set

$$A(x,2,t)=\begin{bmatrix}-k_3(t)\sin(x_2)-1.4 & 0\\ 0 & -k_3(t)\cos(x_1)-1.4\end{bmatrix},$$

Then (5.3) for mode 2 is

$$\big(-2.8+0.5k_4^2(t)\big)|x|^2\le -0.8|x|^2.$$

Therefore

$$\begin{aligned}b(t)&=\min\{-\lambda_{\max}(\tilde{Q}),2.8-0.5k_4^2(t)\}\\&=\min\{1-0.2\sin(\frac{\pi}{6}t),1,2.8-0.5k_4^2(t)\}\\&\ge 0.8.\end{aligned}$$

Then set $l=0.1$ and we have $\lambda(t)=b(t)-0.4$. Assumption 3.3 holds with $K_3(t)=\max_{x\in\mathbb{R}^2,i\in\{1,2\}}\|A(x,i,t)\|$. $T=12$ is a period of u. So we have designed a feedback control for stabilisation.

Then let us calculate the observation interval sequence. We choose condition (3.12) and get the integral of the auxiliary function $\int_0^1 2\beta_a(t)dt=0.1218>0$. Based on its shape, we divide $[0,12]$ into 20 subintervals, which is shown in Fig. 2. When $\beta_a(t)$ change fast, we divide that time period into narrow subintervals; when $\beta_a(t)$ change slowly, our partition is wide. Specifically, the partition $T_1=0$, $T_2=1$, $T_3=1.85,\cdots,T_{20}=11$, $T_{21}=12$. Then we use the lower bound $\{\underline{\beta}_j\}_{1\le j\le 20}$ to calculate $\tilde{\tau}(t)$, which is shown in Fig. 3. Based on $\tilde{\tau}(t)$, we calculate the observation interval $\{\underline{\xi}_j\}_{1\le j\le 20}$ that leads to an integer time of observations in each subinterval. For example, in the first subinterval, $0\le t<1$, the observation interval is 0.00035 and the system would be observed for 2822 times. The largest observation interval is 0.00041, which is for the third subinterval $[1.85,2.9)$. The shortest observation interval is 0.00028 for the fourth subinterval $[2.9,4.7)$.

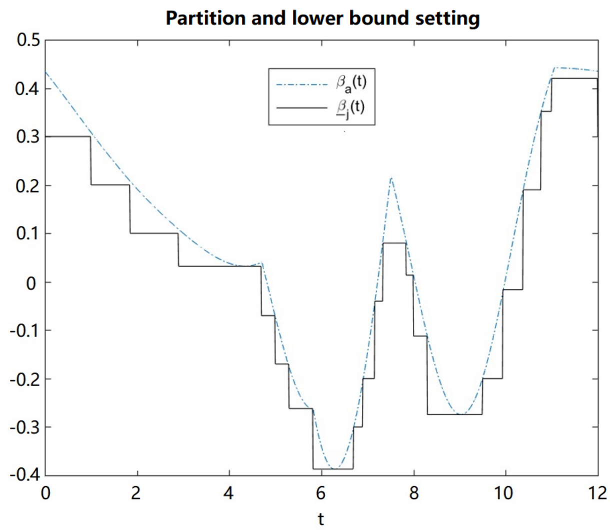

Fig. 2: Partition of one period and lower bound setting for calculation of observation intervals. The blue dash-dot line is the auxiliary function $\beta_a(t)$. The black solid line is $\{\underline{\beta}_j\}_{1\le j\le 20}$.

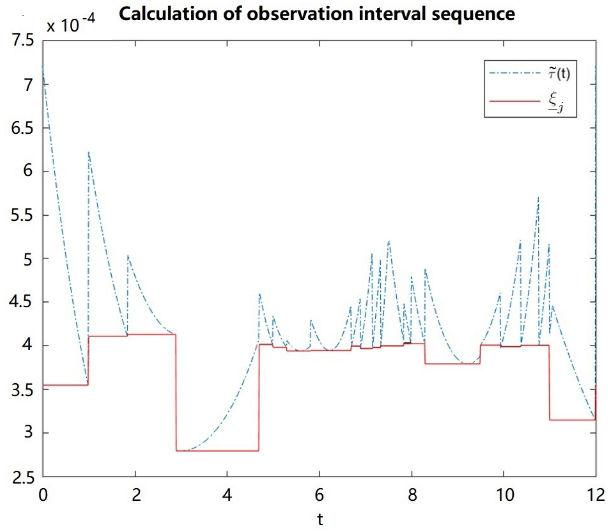

Fig. 3: Calculation of observation interval sequence for each subinterval. The blue dash-dot line is the function $\tilde{\tau}(t)$. The red solid line is the calculated observation interval $\{\underline{\xi}_j\}_{1\le j\le 20}$.

We substitute the results into Theorem 3.8 and calculate. We find $\varphi\in(0.0001,0.0065)\in(0,1)$ and $\int_0^{12}\beta_a(t)dt=0.0877>0$. So all the conditions are satisfied, the system is stabilised. The lower plot in Fig. 1 shows that the controlled system (5.2) is indeed mean square exponentially stable.

In addition, we calculate observation intervals using condition (3.13). This gives better result, as shown in Fig. 4. The black and red lines almost coincide, the blue and red lines also almost coincide when $t>8$. The largest and smallest observation intervals we get are 0.0012 and 0.00037 respectively. When $t\in[8.3,9.95)$, the system is set to be observed 1375 times with interval 0.0012. The highest observation frequency is required for $t\in[11,12)$.

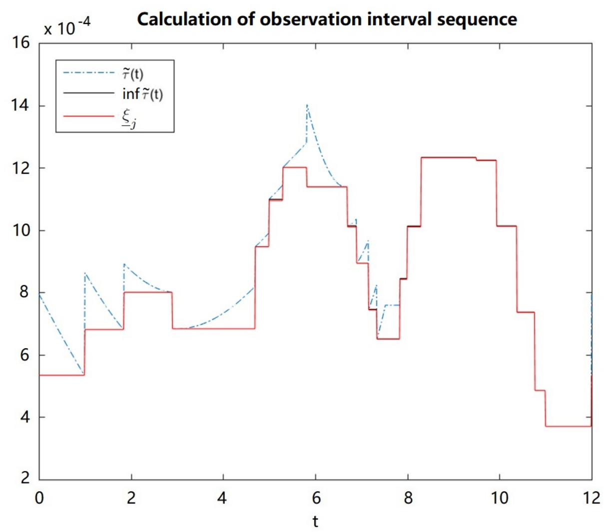

Fig. 4: Calculation of observation interval sequence for each subinterval. The blue dash-dot line is the function $\tilde{\tau}(t)$. The black line is $\inf_{t\in[T_j,T_{j+1})}\tilde{\tau}(t)$. The red solid line is the calculated observation interval $\{\underline{\xi}_j\}_{1\le j\le 20}$.

Moreover, existing theory yields the constant observation interval $\tau \leq 0.00026$, calculated with the same controller and same Lyapunov function, according to [23] with observation of system mode discretised. Previously frequent observations were required for all times. Clearly, both conditions (3.12) and (3.13) give better results than this. Our shortest observation interval is still wider than the constant one given by existing theory. This benefit comes from our consideration of system's time-varying property.

Another advantage of our new results is the flexibility of observation frequency setting. On one hand, we can reduce the lowest observation frequency. There are two ways to make it. One is by dividing some certain subintervals into several shorter intervals, without changing the setting of lower bound $\underline{\beta}_j$. This will not affect the observation frequencies in other subintervals. The result is shown as a red dashed line in Fig. 5. Over time $[0, 0.1)$, the system can be observed once every 0.00075 time units. The other way is to reduce $\underline{\beta}_j$ for the corresponding subinterval. However, this would increase the observation frequencies in some other subintervals. On the other hand, the flexibility brought by the integral condition enables us to reduce the high observation frequencies. By dividing the period into 24 subintervals with narrower partition and changing the lower bound $\underline{\beta}_j$, we increased the shortest observation interval from 0.00028 to 0.00032. The result is shown in Fig. 5 as a blue dash-dot line.

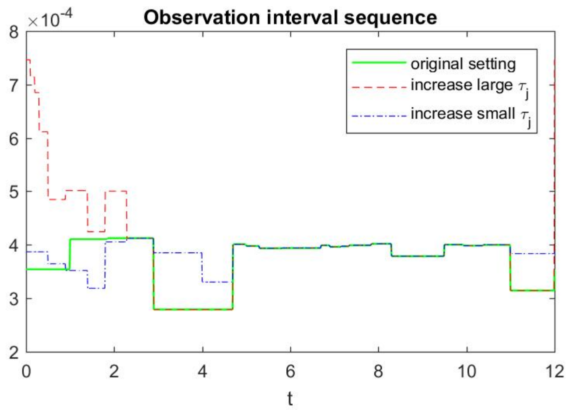

Fig. 5: Three settings of observation intervals. The green solid line shows original setting. The red dashed line and the blue dash-dot line respectively show settings to increase the large and small observation intervals.

6 Conclusion

This paper provides sufficient conditions for exponential stabilisation of periodic hybrid SDEs, by feedback control based on periodic discrete-time observations. The stabilities analyzed include exponential stability in almost sure and pth moment for $p > 1$. We point out that, since inequality plays an important role in derivation of the new results, using less conservative inequalities would reduce observation frequencies.

The main contributions of this paper are: (1) using time-varying observation frequencies for stabilization of periodic SDEs; (2) improving the observational efficiency by reducing the observation frequencies dramatically; (3) allowing to set observation frequencies over some time intervals flexibly without a lower bound, as long as it can be compensated by relatively high frequencies over other time intervals.

These three contributions update existing theories by improving the observational efficiency and providing flexibility. This paper provides theoretical foundation for stabilization of SDEs using time-varying system observations.

Acknowledgements

The first author would like to thank the University of Strathclyde for the PhD studentship. The second author would like to thank the EPSRC (EP/K503174/1), the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship) for their financial support.

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