The asymptotic stability of hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise

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Abstract

The main aim of this paper is to investigate the asymptotic stability of hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise (HSSwPDLNs). Under the local Lipschitz condition and non-linear growth condition, we investigate the existence and uniqueness of the solution to HSSwPDLNs. By using the Lyapunov functions and M-matrix theory, we establish some sufficient conditions on the asymptotic stability and polynomial stability for HSSwPDLNs. Finally, two examples are provided to illustrate our results.

Key words: Hybrid stochastic systems, Pantograph delay, Non-Gaussian Lévy noise, Asymptotic stability, Polynomial stability.

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1 Introduction

It is well known that the following pantograph differential equations

$$\begin{aligned} x'(t) &= f(x(t), x(qt)),\ t \geq 0,\ q \in (0,1), \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

is a delay differential equations with unbounded delay. Pantograph differential equations (1.1) arises in different fields of pure and applied mathematics such as dynamical systems, probability, quantum mechanics and electrodynamics and it possess a wide range of applications. Due to these important applications, pantograph differential equations (1.1) has been widely studied by [9] and [11]. On the other hand, taking the environmental disturbances into account, the pantograph differential equations have been extended into stochastic pantograph differential equations (SPDEs)

$$dx(t) = f(x(t), x(qt))dt + g(x(t), x(qt))dw(t). \tag{1.2}$$

Such SPDEs was firstly introduced by Baker and Buckwar [3] and the existence and uniqueness of the solution have been discussed by [3, 6]. After that, the theory of SPDEs (1.2) has drawn increasing attention and we refer the reader to Appleby and Buckwar [2], Fan and song [7], Guo and Li [8], Milosvic [16], Zhang et al. [33] and the references therein.

Actually, SPDEs (1.2) can be regarded as pantograph differential equations perturbed by a Brownian motion. As a class of Gaussian noise, the Brownian motion is a continuous stochastic process, which only simulate fluctuations of the mean value in a very small range. In fact, due to the complexity of the external environment, the interference noise encountered in practical applications often have non-Gaussian characteristics, which may cause severe fluctuations. Besides the discrete stochastic factors mentioned above, many practical systems may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections. Then, hybrid stochastic system with markovian switching and non-Gaussian Lévy noise has been used to cover these types of perturbations which can provide a good mathematical model for describing such discontinuous processes. For a comprehensive and systematic study on hybrid system with markovian switching and non-Gaussian Lévy noise, we refer to Applebaum [1], Mao and Yuan [20], Yin and Zhu [31].

As we know, one of the important issues in the study of stochastic system is the analysis of stability. In many engineering and control problems, many systems are in operation for very long time, it is very important to determine whether these systems are stable. There is an intensive literature on the stability of stochastic hybrid system and we mention, for example, Mao et al. [21, 22, 23, 24, 27, 28], Xi and Yin [25, 32], You et al. [30], Zhu and Cao [35, 36, 37], Zong et al. [40]. It is worth noting that most existing works of research on the stability of stochastic hybrid system require that their coefficients are either linear or nonlinear but bounded by linear functions, which are somewhat restrictive for non-linear stochastic systems, such as stochastic Lotka–Volterra equation, stochastic interest rate models. Therefore, it is very interesting and challenging to study the stability of stochastic hybrid system when they do not satisfy the linear growth condition. In recent years, many scholars have obtained exponential stability of stochastic hybrid system where their coefficients are highly nonlinear. For example, Fei et al. [4, 5], Hu et al. [10], Li and Deng [14], Mao et al. [17], Zong et al. [39].

Motivated by the above discussions, there are some papers on the stability of hybrid stochastic pantograph differential systems (SPDSs) (see, e.g., [29, 34]). However, the existing stability research on hybrid SPDSs are about the exponential stability, while little is known on the moment asymptotic stability and almost sure asymptotic stability. In order to close this gap, we will make an attempt to investigate the asymptotic behavior of hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise

$$\begin{aligned} dx(t) &= f(x(t), x(qt), r(t))dt + g(x(t), x(qt), r(t))dw(t) \\ &+ \int_Z h(x(t^-), x(qt), r(t^-), v)N(dt, dv). \end{aligned} \tag{1.3}$$

Under non-linear growth condition, we show that (1.3) has a unique solution. By means of M-matrix theory, we establish the sufficient conditions for the moment asymptotic stability and almost sure asymptotic stability of the solution to (1.3).

On the other hand, the authors [29, 34] imposed the negative exponential function e^{-t} in the coefficients f and g to obtain the exponential decay of the convergence. However, not all stochastic systems are exponentially stable, there are also a lot of stochastic systems which are stable but subject to a lower decay rate other than exponential decay. Consequently, it appears to be necessary to study other stability, for instance, polynomial or logarithmic stability. Liu [13] and Mao [18, 19] studied the polynomial stability for stochastic differential equations (SDEs) and SDEs with bounded delay. Recently, Appleby and Buckwar [2] investigated the polynomial asymptotic stability of SPDSs with unbounded delay, but the equation they studied was linear one. In this paper, we will study the polynomial stability of (1.3) where the coefficients are highly nonlinear. By the Lyapunov functions and the nonnegative semimartingale convergence theorem, we obtain that the solution of (1.3) is polynomially stable in pth moment and almost sure polynomially stable. In particular, our results improve and generalize some works in the existing literature.

The paper is organized as follows. In Section 2, we introduce some notations and hypotheses concerning (1.3), meantime, we establish the existence and uniqueness of solutions to (1.3) under the nonlinear growth condition; In Section 3, by applying the Itô formula, stochastic inequality and M-matrix theory, we study the asymptotic behavior of the solution to (1.3), including moment asymptotic stability, almost sure asymptotic stability and the polynomial stability; While in Section 4 we give two examples to illustrate our theory.

2 Preliminaries and the global solution

Throughout this paper, unless otherwise specified, we use the following notation. Let $|.|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its norm $||A||$ is defined by $||A|| = \sup\{|Ax| : |x| = 1\}$. Let $t \geq t_0 > 0$ and $D([qt_0, t_0]; R^n)$ denote the family of functions φ from $[qt_0, t_0] \to R^n$ that are right-continuous and have limits on the left. $D([qt_0, t_0]; R^n)$ is equipped with the norm $||\varphi|| = \sup_{qt_0 \leq \theta \leq t_0} |\varphi(\theta)|$. Let $D^b_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$ be the family of all $\mathcal{F}_{t_0}$-measurable bounded $D([qt_0, t_0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : qt_0 \leq \theta \leq t_0\}$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq t_0}$ satisfying the usual conditions. Let $w = (w(t), t \geq t_0)$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$, and N be a Poisson random measure defined on $[t_0, \infty) \times \{R^n \backslash \{0\}\}$ with compensator $\tilde{N}$ and intensity measure π. We assume that π is a Leʹvy measure such that $\tilde{N}(dt, dv) = N(dt, dv) - \pi(dv)dt$ and $\pi(Z) < \infty$, where Z is a subset of $R^n \backslash 0$ that is the range space of the impulsive jumps. Let $r(t)$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \ldots N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$. We assume that Markov chain $r(.)$ is independent of the Brownian motion $w(.)$

and Poisson random measures $N(.,.)$. For $Z \in \mathcal{B}(R^n\backslash\{0\})$, $0 \notin \bar{Z}$, consider the nonlinear hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise

$$\begin{aligned} dx(t) &= f(x(t), x(qt), r(t))dt + g(x(t), x(qt), r(t))dw(t) \\ &+ \int_Z h(x(t^-), x(qt), r(t^-), v)N(dt, dv), \quad t \geq t_0, \end{aligned} \tag{2.1}$$

with the initial data $\{x(t) : qt_0 \leq t \leq t_0\} = \xi \in D^b_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$ and $r(t_0) = i_0$, where $x(t^-) = \lim_{s\uparrow t} x(s)$ and $0 < q < 1$. Here

$$f : R^n \times R^n \times S \to R^n,\ g : R^n \times R^n \times S \to R^{n\times m} \text{ and } h : R^n \times R^n \times S \times Z \to R^n.$$

In this paper, the following hypothesis are imposed on the coefficients f, g and h.

Assumption 2.1 *For each integer $d \geq 1$, there exists a positive constant k_d such that*

$$|f(x, y, i) - f(\bar{x}, \bar{y}, i)|^2 \vee |g(x, y, i) - g(\bar{x}, \bar{y}, i)|^2 \leq k_d(|x - \bar{x}|^2 + |y - \bar{y}|^2),$$
$$\int_Z |h(x, y, i, v) - h(\bar{x}, \bar{y}, i, v)|^2\pi(dv) \leq k_d(|x - \bar{x}|^2 + |y - \bar{y}|^2),$$

for all $i \in S$ and those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq d$. Moreover, assume that for all $i \in S$,

$$|f(0, 0, i)|^2 \vee |g(0, 0, i)|^2 \vee \int_Z |h(0, 0, i, v)|^2\pi(dv) < \infty.$$

It is known that Assumption 2.1 only guarantees that (2.1) has a unique maximal local solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions. Let $C(R^n \times S; R_+)$ denote the family of continuous functions from $R^n \times S$ to R_+. Also denote by $C^2(R^n \times S; R_+)$ the family of all continuous non-negative functions $V(x, i)$ defined on $R^n \times S$ such that for each $i \in S$, they are continuously twice differentiable in x. Given $V \in C^2(R^n \times S; R_+)$, we define the function $LV : R^n \times R^n \times S \to R$ by

$$\begin{aligned} LV(x, y, i) &= V_x(x, i)f(x, y, i) + \frac{1}{2}trace[g^\top(x, y, i)V_{xx}(x, i)g(x, y, i)] \\ &+ \int_Z [V(x + h(x, y, i, v), i) - V(x, i)]\pi(dv) + \sum_{j=1}^N \gamma_{ij}V(x, j), \end{aligned}$$

where

$$V_x(x, i) = \Big(\frac{\partial V(x, i)}{\partial x_1}, \cdots, \frac{\partial V(x, i)}{\partial x_n}\Big), \quad V_{xx}(x, i) = \Big(\frac{\partial^2 V(x, i)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Assumption 2.2 *There are two functions* $V \in C^2(R^n \times S; R_+)$ *and* $U \in C(R^n; R_+)$*, as well as positive constants* $p,\ K_1,\ K_2,\ c_1,\ c_2$*, such that*

$$c_1|x|^p \le V(x,i) \le c_2|x|^p, \quad \forall (x,i) \in R^n \times S \tag{2.2}$$

and

$$LV(x,y,i) \le K_1(1+|x|^p+|y|^p) - U(x) + K_2U(y) \tag{2.3}$$

for all $(x,y,i) \in R^n \times R^n \times S$*.*

Lemma 2.3 *Under Assumptions 2.1, there exists a unique maximal local solution* $x(t)$ *on* $t \in [t_0, \sigma_\infty)$ *to (2.1), where* σ_∞ *is the explosion time.*

Proof. Fix any initial data $\xi \in D^b_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$ and let k_0 be the bound for ξ. For each integer $k \ge k_0$, $x, y \in R^n$, $i \in S$ and $v \in Z$, define

$$f_k(x,y,i) = f\Big(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, i\Big), \quad g_k(x,y,i) = g\Big(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, i\Big),$$

and

$$h_k(x,y,i,v) = h\Big(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, i, v\Big),$$

where we set $(|x| \wedge k/|x|)x = 0$ when $x = 0$. Then, by Assumption 2.2, we observe that $f_k(x,y,i)$, $g_k(x,y,i)$ and $h_k(x,y,i,v)$ satisfy the global Lipschitz condition and the linear growth condition. Therefore, there exists a unique solution $x_k(t)$ on $t \ge t_0$ to the equation

$$\begin{aligned} dx_k(t) &= f_k(x_k(t), x_k(qt), r(t))dt + g_k(x_k(t), x_k(qt), r(t))dw(t) \\ &+ \int_Z h_k(x_k(t^-), x_k(qt), r(t^-), v)N(dt, dv) \end{aligned}$$

with the initial data $\{x_k(t) : qt_0 \le t \le t_0\} = \xi$. Define the stopping time

$$\sigma_k = \inf\{t \ge t_0 : |x_k(t)| > k\}.$$

It is not difficult to show that

$$x_k(t) = x_{k+1}(t) \quad if \quad t_0 \le t < \sigma_k.$$

This implies that σ_k is increasing in k. Let $\sigma_\infty = \lim_{k\to\infty} \sigma_k$. The property above also enables us to define $x(t)$ for $t \in [qt_0, \sigma_\infty)$ as follows

$$x(t) = x_k(t) \quad if \quad qt_0 \le t < \sigma_k.$$

It is clear that $x(t)$ is a unique solution to (2.1) for $t \in [qt_0, \sigma_\infty)$. The proof is therefore complete. □

Theorem 2.4 *Let Assumptions 2.1 and 2.2 hold. Then for any given initial data ξ, there is a unique global solution $x(t)$ to (2.1) on $t \in [t_0, \infty)$. Moreover, the solution has the properties that*

$$E|x(t)|^p < \infty \quad \textit{and} \quad E\int_{t_0}^{t} U(x(s))ds < \infty \tag{2.4}$$

for any $t \geq t_0$.

Proof. By Lemma 2.3, Assumption 2.1 guarantees the existence of the unique maximal local solution $x(t)$ on $t \in [t_0, \sigma_\infty)$, where σ_∞ is the explosion time. Let k_0 be the bound for ξ. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [t_0, \sigma_\infty) : |x(t)| > k\}.$$

Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = lim_{k\to\infty}\tau_k$, whence $\tau_\infty \leq \sigma_\infty$ a.s. Note if we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. So we just need to show that $\tau_\infty = \infty$ a.s. To complete the proof, we need to show that $P(\tau_\infty = \infty) = 1$. By the Itô formula and condition (2.3), we can derive that

$$\begin{aligned}
& V(x(t), r(t)) \\
= \;& V(x(t_0), r(t_0)) + \int_{t_0}^{t} LV(x(s), x(qs), r(s))ds \\
+ \;& \int_{t_0}^{t} V_x(x(s), r(s))g(x(s), x(qs), r(s))dw(s) \\
+ \;& \int_{t_0}^{t}\int_Z \Big[V(x(s^-) + h(x(s^-), x(qs), r(s^-), v), r(s^-)) - V(x(s^-), r(s^-))\Big]\tilde{N}(dt, dv) \\
\leq \;& V(x(t_0), r(t_0)) + \int_{t_0}^{t} \Big(K_1(1 + |x(s)|^p + |x(qs)|^p) - U(x(s)) + K_2U(x(qs))\Big)ds \\
+ \;& \int_{t_0}^{t} V_x(x(s), r(s))g(x(s), x(qs), r(s))dw(s) \\
+ \;& \int_{t_0}^{t}\int_Z \Big[V(x(s^-) + h(x(s^-), x(qs), r(s^-), v), r(s^-)) - V(x(s^-), r(s^-))\Big]\tilde{N}(dt, dv)
\end{aligned} \tag{2.5}$$

for $t \in [t_0, \tau_\infty)$. Now, we shall show that $\tau_\infty > \frac{t_0}{q}$ a.s. For any $k \geq k_0$ and $t \in [t_0, \frac{t_0}{q}]$, by taking expectations, we have

$$\begin{aligned}
& EV(x(\tau_k \wedge t), r(\tau_k \wedge t)) \\
= \;& EV(x(t_0), r(t_0)) + E\int_{t_0}^{\tau_k\wedge t} \Big(K_1(1 + |x(s)|^p + |x(qs)|^p) - U(x(s)) + K_2U(x(qs))\Big)ds
\end{aligned}$$

$$\begin{aligned} &+ \quad E\int_{t_0}^{\tau_k\wedge t} V_x(x(s),r(s))g(x(s),x(qs),r(s))dw(s) \\ &+ \quad E\int_{t_0}^{\tau_k\wedge t}\int_Z \Big[V(x(s^-)+h(x(s^-),x(qs),r(s^-),v),r(s^-)) - V(x(s^-),r(s^-))\Big]\tilde{N}(dt,dv). \end{aligned} \tag{2.6}$$

Since $|x(s)| \le k$ for all $s < t\wedge\tau_k$, by the continuity of V and the local linear growth condition of g, h, we can obtain that

$$E\int_{t_0}^{\tau_k\wedge t} |V_x(x(s),r(s))g(x(s),x(qs),r(s))|^2 ds < \infty$$

and

$$E\int_{t_0}^{\tau_k\wedge t}\int_Z |V(x(s^-)+h(x(s^-),x(qs),r(s^-),v),r(s^-)) - V(x(s^-),r(s^-))|^2\pi(dv)ds < \infty.$$

Therefore, both

$$\int_{t_0}^{\tau_k\wedge t} V(x(s),r(s))g(x_k(s),x_k(qs),r(s))dw(s)$$

and

$$\int_{t_0}^{\tau_k\wedge t}\int_Z \Big[V(x(s^-)+h_k(x(s^-),x(qs),r(s^-),v),r(s^-)) - V(x(s^-),r(s^-))\Big]\tilde{N}(dt,dv)$$

are martingales. Using condition (2.2), we then derive from (2.6) that

$$c_1E|x(\tau_k\wedge t)|^p \le H_1 + K_1E\int_{t_0}^{\tau_k\wedge t}|x(t)|^p dt - E\int_{t_0}^{\tau_k\wedge t} U(x(t))dt, \tag{2.7}$$

where

$$\begin{aligned} H_1 &= c_2E|x(t_0)|^p + \int_{t_0}^{\frac{t_0}{q}}\Big(K_1(1+E|x(qt)|^p) + K_2U(x(qt))\Big)dt \\ &\le c_2E|\xi|^p + \frac{1}{q}E\int_{qt_0}^{t_0}\Big(K_1(1+|x(t)|^p) + K_2U(x(t))\Big)dt < \infty. \end{aligned}$$

It then follows that

$$\begin{aligned} c_1E|x(\tau_k\wedge t)|^p &\le H_1 + K_1E\int_{t_0}^{\tau_k\wedge t}|x(t)|^p dt \\ &\le H_1 + K_1\int_{t_0} E|x(\tau_k\wedge t)|^p dt. \end{aligned}$$

Since this holds for any $t \in [t_0, \frac{t_0}{q}]$, the Gronwall inequality implies

$$E|x(\tau_k\wedge t)|^p \le \frac{H_1}{c_1}e^{\frac{K_1}{c_1}(\frac{1}{q}-1)t_0}, \quad t_0 \le t \le \frac{t_0}{q} \tag{2.8}$$

for any $k \geq k_0$. In particular,

$$E|x(\tau_k \wedge \frac{t_0}{q})|^p \leq \frac{H_1}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q}-1)t_0}, \ \forall k \geq k_0.$$

This implies $k^p P(\tau_k \leq \frac{t_0}{q}) \leq \frac{H_1}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q}-1)t_0}$. Letting $k \to \infty$, we hence obtain that $P(\tau_\infty \leq \frac{t_0}{q}) = 0$, namely

$$P(\tau_\infty > \frac{t_0}{q}) = 1. \tag{2.9}$$

Letting $k \to \infty$ in (2.8) yields

$$E|x(t)|^p \leq \frac{H_1}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q}-1)t_0}, \quad t_0 \leq t \leq \frac{t_0}{q}. \tag{2.10}$$

Moreover, setting $t = \frac{t_0}{q}$ in (2.7) yields

$$E\int_{t_0}^{\tau_k \wedge \frac{t_0}{q}} U(x(t))dt \leq H_1 + K_1 E\int_{t_0}^{\tau_k \wedge \frac{t_0}{q}} |x(t)|^p dt.$$

Letting $k \to \infty$, we have

$$E\int_{t_0}^{\frac{t_0}{q}} U(x(t))dt \leq H_1 + \frac{K_1 H_1}{c_1}(\frac{1}{q} - 1)t_0 e^{\frac{K_1}{c_1}(\frac{1}{q}-1)t_0} < \infty. \tag{2.11}$$

Let us now proceed to prove $\tau_\infty > \frac{t_0}{q^2}$ a.s. given that we have shown (2.9)-(2.11). For any $k \geq k_0$ and $t \in [t_0, \frac{t_0}{q^2}]$, it follows from (2.6) that

$$c_1 E|x(\tau_k \wedge t)|^p \leq H_2 + K_1 E\int_{t_0}^{\tau_k \wedge t} |x(t)|^p dt - E\int_{t_0}^{\tau_k \wedge t} U(x(t))dt, \tag{2.12}$$

where

$$\begin{aligned} H_2 &= c_2 E|x(t_0)|^p + E\int_{t_0}^{\frac{t_0}{q^2}} \Big(K_1(1 + |x(qt)|^p) + K_2 U(x(qt))\Big)dt \\ &= H_1 + E\int_{\frac{t_0}{q}}^{\frac{t_0}{q^2}} \Big(K_1(1 + |x(qt)|^p) + K_2 U(x(qt))\Big)dt \\ &= H_1 + \frac{1}{q} E\int_{t_0}^{\frac{t_0}{q}} \Big(K_1(1 + |x(t)|^p) + K_2 U(x(t))\Big)dt < \infty. \end{aligned}$$

Consequently

$$c_1 E|x(\tau_k \wedge t)|^p \leq H_2 + K_1 \int_{t_0}^{t} E|x(\tau_k \wedge t)|^p dt.$$

The Gronwall inequality implies

$$E|x(\tau_k \wedge t)|^p \le \frac{H_2}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q^2}-1)t_0}, \quad t_0 \le t \le \frac{t_0}{q^2}. \tag{2.13}$$

In particular,

$$E|x(\tau_k \wedge \frac{t_0}{q^2})|^p \le \frac{H_2}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q^2}-1)t_0}, \ \forall k \ge k_0.$$

This implies

$$k^p P(\tau_k \le \frac{t_0}{q^2}) \le \frac{H_2}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q^2}-1)t_0}.$$

Letting $k \to \infty$, we then obtain that $P(\tau_\infty \le \frac{t_0}{q^2}) = 0$, namely $P(\tau_\infty > \frac{t_0}{q^2}) = 1$. Letting $k \to \infty$ in (2.13) yields

$$E|x(t)|^p \le \frac{H_2}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q^2}-1)t_0}, \ t_0 \le t \le \frac{t_0}{q^2}.$$

Moreover, setting $t = \frac{t_0}{q^2}$ in (2.12) yields

$$E\int_{t_0}^{\tau_k \wedge \frac{t_0}{q^2}} U(x(t))dt \le H_2 + K_1 E\int_{t_0}^{\tau_k \wedge \frac{t_0}{q^2}} |x(t)|^p dt.$$

Letting $k \to \infty$, we have

$$E\int_{t_0}^{\frac{t_0}{q^2}} U(x(t))dt \le H_2 + \frac{K_1 H_2}{c_1}(\frac{1}{q^2} - 1)t_0 e^{\frac{K_1}{c_1}(\frac{1}{q^2}-1)t_0} < \infty.$$

Repeating this procedure, we can show that, for any integer $i \ge 1$, $\tau_\infty > \frac{t_0}{q^i}$ a.s.,

$$E|x(t)|^p \le \frac{H_i}{c_1} e^{\frac{K_1}{c_1}(\frac{1}{q^i}-1)t_0}, \ t_0 \le t_1 \le \frac{t_0}{q^i},$$

and

$$E\int_{t_0}^{\frac{t_0}{q^i}} U(x(t))dt \le H_i + \frac{K_1 H_i}{c_1}(\frac{1}{q^i} - 1)t_0 e^{\frac{K_1}{c_1}(\frac{1}{q^i}-1)t_0} < \infty,$$

where

$$\begin{aligned} H_i &= c_2 E|x(t_0)|^p + E\int_{t_0}^{\frac{t_0}{q^i}} \Big(K_1(1 + |x(qt)|^p) + K_2 U(x(qt))\Big)dt \\ &= H_{i-1} + \frac{1}{q} E\int_{\frac{t_0}{q^{i-2}}}^{\frac{t_0}{q^{i-1}}} \Big(K_1(1 + |x(t)|^p) + K_2 U(x(t))\Big)dt < \infty. \end{aligned}$$

We must therefore have $\tau_\infty = \infty$ a.s. and the required assertion (2.4) holds as well. □

Remark 2.5 *In [16], the author proved that SPDSs has a unique solution $x(t)$ under the local Lipschitz condition and the Khasminskii-type condition. However, the author added an additional condition $\alpha_2 > \frac{\alpha_1}{q}$ into the assumption $\mathcal{A}_2$ which has played an important role in their proof. In fact, if we follow the idea of [16, 34], we also need an additional condition $K_2 < 1$ to prove Theorem 2.4. But, our new proof shows that we do not require this condition. Hence, we improve and generalize the corresponding existence results of [16, 34].*

Theorem 2.6 *Let Assumptions 2.1 and 2.2 hold except (2.3) which is replaced by*

$$LV(x, y, i) \leq -\alpha_1|x|^p + \alpha_2 q|y|^p - \alpha_3 U(x) + \alpha_4 q U(y) \tag{2.14}$$

for all $(x, y, i) \in R^n \times R^n \times R_+ \times S$, where $\alpha_1 > \alpha_2 \geq 0$ and $\alpha_3 > \alpha_4 \geq 0$. Then for any given initial data ξ, there is a unique global solution $x(t)$ to (2.1) on $t \in [t_0, \infty)$. Moreover, the solution has the property that

$$\int_{t_0}^{\infty} EU(x(t))dt < \infty. \tag{2.15}$$

Proof. We first observe that (2.14) is stronger than (2.3). So, by Theorem 2.4, for any given initial data ξ, (2.1) has a unique global solution $x(t)$ on $t \geq t_0$. Let k_0 be the bound for ξ. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \geq t_0 : |x(t)| \geq k\}. \tag{2.16}$$

For any $t \geq t_0$, by the Itô formula, we obtain that

$$\begin{aligned}
& V(x(\tau_k \wedge t), r(\tau_k \wedge t)) \\
= \;& V(x(t_0), r(t_0)) + \int_{t_0}^{\tau_k \wedge t} LV(x(s), x(qs), r(s))ds \\
+ \;& \int_{t_0}^{\tau_k \wedge t} V_x(x(s), r(s))g(x(s), x(qs), r(s))dw(s) \\
+ \;& \int_{t_0}^{\tau_k \wedge t} \int_Z \Big[V(x(s^-) + h(x(s^-), x(qs), r(s^-), v), r(s^-)) - V(x(s^-), r(s^-))\Big]\tilde{N}(dt, dv).
\end{aligned} \tag{2.17}$$

Note $V(x, i) \geq 0$ and the last two terms of (2.17) are martingales. By conditions (2.2) and

(2.14), we then compute

$$\begin{aligned} 0 &\le EV(x(\tau_k\wedge t), r(\tau_k\wedge t)) = EV(x(t_0), r(t_0)) + E\int_{t_0}^{\tau_k\wedge t} LV(x(s), x(qs), r(s))ds \\ &\le c_2E|x(t_0)|^p + E\int_{t_0}^{t\wedge\tau_k}\Big[-\alpha_1|x(s)|^p + \alpha_2 q|x(qs)|^p - \alpha_3 U(x(s)) + q\alpha_4 U(x(qs))\Big]ds \\ &\le c_2E||\xi||^p + \int_{qt_0}^{t_0} E\Big[\alpha_2|x(s)|^p + \alpha_4 U(x(s))\Big]ds - (\alpha_3-\alpha_4)E\int_{t_0}^{t\wedge\tau_k} U(x(s))ds. \end{aligned}$$

This implies

$$E\int_{t_0}^{t\wedge\tau_k} U(x(s))ds \le \frac{1}{\alpha_3-\alpha_4}\Big(c_2E||\xi||^p + \int_{qt_0}^{t_0} E\Big[\alpha_2|x(s)|^p + \alpha_4 U(x(s))\Big]ds\Big).$$

Letting $k\to\infty$ and then applying the Fubini theorem, we get

$$\int_{t_0}^{t} EU(x(s))ds \le \frac{1}{\alpha_3-\alpha_4}\Big(c_2E||\xi||^p + \int_{qt_0}^{t_0} E\Big[\alpha_2|x(s)|^p + \alpha_4 U(x(s))\Big]ds\Big).$$

Letting $t\to\infty$ yields the desired assertion (2.15). □

Remark 2.7 *Likewise, without conditions $\alpha_1 > \alpha_2 \ge 0$ and $\alpha_3 > \alpha_4 \ge 0$, we can show the existence and uniqueness of the solution to (2.1) by following the proof of Theorem 2.4. In fact, these conditions are used to obtain the property (2.15).*

Remark 2.8 *Let us point out that the assertion $\int_{t_0}^{\infty} EU(x(t))dt < \infty$ obtained in Theorem 2.6 is useful. For example, if we further have $U(x) \ge c|x|^\gamma$ for some positive constant c, then this assertion implies that $\int_{t_0}^{\infty} E|x(t)|^\gamma dt < \infty$, which is known as the H_∞-stability. Moreover, this stablity will be discussed in Theorem 3.6 again.*

3 Asymptotic Stability of the solution

In previous section, we obtain that (2.1) admits a unique global solution. In this section, we will discuss the asymptotic behaviour of (2.1) by means of the M-matrix theory.

For the convenience of the reader, let us cite some useful results on M-matrix. For more detailed information please see e.g. [20]. We will need a few more notations. If B is a vector or matrix, by $B \gg 0$ we mean all elements of B are positive. If B_1 and B_2 are vectors or matrices with same dimensions we write $B_1 \gg B_2$ if and only if $B_1 - B_2 \gg 0$. Moreover, we also adopt here the traditional notation by letting

$$Z^{N\times N} = \{A = [a_{ij}]_{N\times N} : a_{ij} \le 0,\ i \ne j\}.$$

Definition 3.1 *A square matrix* $A = [a_{ij}]_{N\times N}$ *is called a nonsingular M-matrix if* A *can be expressed in the form* $A = sI - B$ *with* $s > \rho(B)$ *while all the elements of* B *are nonnegative, where* I *is the identity matrix and* $\rho(B)$ *the spectral radius of* B*.*

It is easy to see that a nonsingular M-matrix A has non-positive off-diagonal and positive diagonal entries, that is

$$a_{ii} > 0 \text{ while } a_{ij} \le 0,\ i \ne j.$$

In particular, $A \in Z^{N\times N}$. There are many conditions which are equivalent to the statement that A is a nonsingular M-matrix and we now cite some of them for the use of this paper (see e.g. [20, 22, 23]).

Lemma 3.2 *If* $A \in Z^{N\times N}$*, then the following statements are equivalent:*

(1) A *is a nonsingular M-matrix.*

(2) A *is semi-positive; that is, there exists* $x \gg 0$ *in* R^N *such that* $Ax \gg 0$*.*

(3) A^{-1} *exists and its elements are all nonnegative.*

(4) All the leading principal minors of A *are positive; that is*

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0 \quad \textit{for every } k = 1, 2, \cdots, N.$$

Lemma 3.3 *(see [15]) Let* $A(t)$ *be an* $\mathcal{F}_t$*-adapted increasing processes on* $t \ge 0$ *with* $A(0) = 0$ *a.s. Let* $M(t)$ *be a real-valued local martingale with* $M(0) = 0$ *a.s. Let* ζ *be a nonnegative* $\mathcal{F}_0$*-measurable random variable. Assume that* $x(t)$ *is nonnegative semi-martingale and*

$$x(t) = \zeta + A(t) + M(t) \quad \textit{for} \quad t \ge 0.$$

If $\lim_{t\to\infty} A(t) < \infty$ *a.s. then for almost all* $\omega \in \Omega$, $\lim_{t\to\infty} x(t) < \infty$,, *that is,* $x(t)$ *converges to finite random variables.*

Let us now state our hypothesis in terms of an M-matrix.

Assumption 3.4 *Let* $\gamma > p \geq 2$ *and assume that for each* $i \in S$*, there are nonnegative numbers* $\alpha_{2i}, \alpha_{3i}, \alpha_{4i}, \beta_{1i}, \beta_{2i}, \beta_{3i}, \beta_{4i}$ *and a real number* α_{1i} *as well as bounded functions* $h_i(.)$ *such that*

$$x^\top f(x,y,i) + \frac{p-1}{2}|g(x,y,i)|^2 \leq \alpha_{1i}|x|^2 + \alpha_{2i}q|y|^2 - \alpha_{3i}|x|^{\gamma-p+2} + \alpha_{4i}q|y|^{\gamma-p+2}$$

and

$$|x + h(x,y,i,v)|^p \leq h_i(v)\Big(\beta_{1i}|x|^p + \beta_{2i}q|y|^p + \beta_{3i}|x|^\gamma + \beta_{4i}q|y|^\gamma\Big)$$

for any $x, y \in R^n$*,* $v \in Z$*.*

Assumption 3.5 *Let* $C_{h_i} = \int_Z h_i(v)\pi(dv) < \infty,\ \eta_i = p\alpha_{1i} + \beta_{1i}C_{h_i}$ *and assume that*

$$\mathcal{A}_p := -\mathrm{diag}(\eta_1, \cdots, \eta_N) - \Gamma \tag{3.1}$$

is a nonsingular M-matrix.

In fact, by lemma 3.2 and Assumption 3.5, it follows that

$$\theta = (\theta_1, \cdots, \theta_N)^\top := \mathcal{A}_p^{-1}\overrightarrow{1} > 0 \tag{3.2}$$

for all $i \in S$, where $\overrightarrow{1} = (1, \cdots, 1)^\top$.

Theorem 3.6 *Let Assumptions 2.1, 3.4 and 3.5 hold. Assume that*

$$\max_{i\in S}(p\alpha_{2i} + \beta_{2i}C_{h_i})\theta_i < 1 \tag{3.3}$$

and

$$\min_{i\in S} p(\alpha_{3i} - \alpha_{4i})\theta_i > \max_{i\in S}(\beta_{3i} + \beta_{4i})C_{h_i}\theta_i. \tag{3.4}$$

Then for any given initial data ξ*, there is a unique global solution* $x(t)$ *to (2.1) on* $t \in [t_0, \infty)$*. Moreover, the solution has the property that*

$$\int_{t_0}^{\infty} E|x(t)|^\gamma dt < \infty \tag{3.5}$$

for any $t \geq t_0$*.*

Proof. Let us define the function $V(x,i)=\theta_i|x|^p$. Clearly, V obeys condition (2.2) with $c_1=\min_{i\in S}\theta_i$ and $c_2=\max_{i\in S}\theta_i$. To verify condition (2.14), we compute the operator LV as follows

$$\begin{aligned} LV(x,y,i) &= p\theta_i|x|^{p-2}x^\top f(x,y,i)+\frac{p}{2}\theta_i|x|^{p-2}|g(x,y,i)|^2+\frac{p(p-2)}{2}\theta_i|x|^{p-4}|x^\top g(x,y,i)|^2 \\ &+ \sum_{j=1}^N\gamma_{ij}\theta_j|x|^p+\int_Z[\theta_i|x+h(x,y,i,v))|^p-\theta_i|x|^p]\pi(dv). \end{aligned}$$

By Assumption 3.4, it follows that

$$\begin{aligned} LV(x,y,i) &\le \Big(\eta_i\theta_i+\sum_{j=1}^N\gamma_{ij}\theta_j\Big)|x|^p+p\alpha_{2i}\theta_i|x|^{p-2}|y|^2+\beta_{2i}C_{h_i}\theta_i|y|^p \\ &- p\alpha_{3i}\theta_i|x|^\gamma+\beta_{3i}C_{h_i}\theta_i|x|^\gamma+\beta_{4i}C_{h_i}\theta_i|y|^\gamma+p\alpha_{4i}\theta_i|x|^{p-2}|y|^{\gamma-p+2}. \end{aligned}$$

By the definition of θ_i, we have $\eta_i\theta_i+\sum_{j=1}^N\gamma_{ij}\theta_j=-1$. Hence,

$$\begin{aligned} LV(x,y,i) &\le -|x|^p+\delta_2q|x|^{p-2}|y|^2+\hat\delta_2q|y|^p-\delta_3|x|^\gamma+\hat\delta_3|x|^\gamma \\ &+ \delta_4q|x|^{p-2}|y|^{\gamma-p+2}+\hat\delta_4q|y|^\gamma, \end{aligned} \tag{3.6}$$

where $\delta_2=\max_{i\in S}p\alpha_{2i}\theta_i,\ \delta_3=\min_{i\in S}p\alpha_{3i}\theta_i,\ \delta_4=\max_{i\in S}p\alpha_{4i}\theta_i,\ \hat\delta_k=\max_{i\in S}\beta_{ki}C_{h_i}\theta_i,\ k=2,3,4$. By the elementary inequality $a^rb^{1-r}\le ar+b(1-r),$ for any $a,b\ge 0$ and $r\in[0,1]$, we have

$$|x|^{p-2}|y|^2\le\frac{p-2}{p}|x|^p+\frac{2}{p}|y|^p,$$

and

$$|x|^{p-2}|y|^{\gamma-p+2}\le\frac{p-2}{\gamma}|x|^\gamma+\frac{\gamma-p+2}{\gamma}|y|^\gamma.$$

Inserting these two inequalities into (3.6), we get

$$LV(x,y,i)\ \le\ -\alpha_1|x|^p+\alpha_2q|y|^p-\alpha_3|x|^\gamma+\alpha_4q|y|^\gamma, \tag{3.7}$$

where $\alpha_1=1-\delta_2q\frac{p-2}{p},\ \alpha_2=\delta_2\frac{2}{p}+\hat\delta_2,\ \alpha_3=\delta_3-\hat\delta_3-\delta_4q\frac{p-2}{\gamma},\ \alpha_4=\delta_4\frac{\gamma-p+2}{\gamma}+\hat\delta_4$. Recalling (3.3) and (3.4), we can obtain that

$$\begin{aligned} \alpha_1-\alpha_2 &= 1-\delta_2q\frac{p-2}{p}-\delta_2\frac{2}{p}-\hat\delta_2>1-\delta_2-\hat\delta_2>0, \\ \alpha_3-\alpha_4 &= \delta_3-\hat\delta_3-\delta_4q\frac{p-2}{\gamma}-\delta_4\frac{\gamma-p+2}{\gamma}-\hat\delta_4>\delta_3-\hat\delta_3-\delta_4-\hat\delta_4>0. \end{aligned}$$

That is, condition (2.14) is fulfilled. By Theorem 2.6, we can conclude that for any given initial data ξ, there is a unique global solution $x(t)$ to (2.1) on $t \in [t_0, \infty)$. Moreover, we have $\int_{t_0}^{\infty} E|x(t)|^\gamma dt < \infty$ for any $t \geq t_0$. The proof is therefore complete. □

Remark 3.7 *In Theorem 3.6, we impose some conditions on the coefficients f, g, h and establish sufficient criterions (3.3) and (3.4) on H_∞ stability. In fact, the conditions (3.3) and (3.4) reveal the impact of the jumps term h on the stability of (2.1). Compared with the condition (2.14), it is very convenient to check the conditions (3.3) and (3.4) of Theorem 3.6 since Assumption 3.4 is explicitly related to the coefficients f, g and h. And this will be fully illustrated by Examples 4.1 and 4.2.*

By using the M-matrix theory, the above theorem gives a criterion on H_∞-stability in L^γ. However, it does not follow from (3.5) that $\lim_{t\to\infty} E|x(t)|^\gamma = 0$. To show this result, we will need some additional conditions.

Theorem 3.8 *Let the conditions of Theorem 3.6 hold. Assume that there exists a constant $L > 0$ such that*

$$x^\top f(x,y,i) + \frac{\gamma-1}{2}|g(x,y,i)|^2 \leq L(|x|^2 + q|y|^2) \tag{3.8}$$

and

$$|x + h(x,y,i,v)|^\gamma \leq Lh_i(v)(|x|^\gamma + q|y|^\gamma). \tag{3.9}$$

Then the solution of (2.1) satisfies that

$$\lim_{t\to\infty} E|x(t)|^\gamma = 0 \tag{3.10}$$

for any initial data ξ.

Proof. Fix any initial data $\xi \in D^b_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$. If (3.10) is not true, then there is some $\varepsilon > 0$ and a sequence of positive numbers $\{t_n\}_{n\geq 1}$ such that $t_n \to \infty$ as $n \to \infty$ and

$$E|x(t_n)|^\gamma \geq 2\varepsilon, \quad \forall\ n \geq 1. \tag{3.11}$$

Without loss of generality, we may set $t_1 > q^{-2}t_0$ and $t_{n+1} > q^{-2}t_n$. By (3.5), we obtain

$$\sum_{n=1}^{\infty} \int_{q^2 t_n}^{t_n} E|x(s)|^\gamma ds \leq \int_{t_0}^{\infty} E|x(s)|^\gamma ds < \infty,$$

Consequently, there exists a n_0 such that

$$\int_{q^2 t_n}^{t_n} E|x(s)|^\gamma ds \le \frac{q\varepsilon}{4L\gamma + 2LC_{h_i}}, \quad \forall\ n \ge n_0. \tag{3.12}$$

For any $k \ge k_0$, $n \ge n_0$ and $t \in [qt_n, t_n]$, by the generalized Itô formula, we have

$$\begin{aligned} & |x(t_n \wedge \tau_k)|^\gamma - |x(t \wedge \tau_k)|^\gamma \\ \le\ & \int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \gamma|x(s)|^{\gamma-2}\Big(x(s)^\top f(x(s), x(qs), r(s)) + \frac{\gamma-1}{2}|g(x(s), x(qs), r(s))|^2\Big)ds \\ +\ & \int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \int_Z \Big(|x(s^-) + h(x(s^-), x(qs), r(s^-), v))|^\gamma - |x(s^-)|^\gamma\Big)\pi(dv)ds \\ +\ & \int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \gamma|x(s)|^{\gamma-2}\Big(x(s)^\top f(x(s), x(qs), r(s))\Big)dw(s) \\ +\ & \int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \int_Z \Big(|x(s^-) + h(x(s^-), x(qs), r(s^-), v))|^\gamma - |x(s^-)|^\gamma\Big)\tilde{N}(dt, dv), \end{aligned}$$

where τ_k is defined as (2.15). Note that the last two terms are martingales. Take the expectation, we get

$$\begin{aligned} & E|x(t_n \wedge \tau_k)|^\gamma - E|x(t \wedge \tau_k)|^\gamma \\ \le\ & E\int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \gamma|x(s)|^{\gamma-2}\Big(x(s)^\top f(x(s), x(qs), r(s)) + \frac{\gamma-1}{2}|g(x(s), x(qs), r(s))|^2\Big)ds \\ +\ & E\int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \int_Z \Big(|x(s^-) + h(x(s^-), x(qs), r(s^-), v))|^\gamma - |x(s^-)|^\gamma\Big)\pi(dv)ds. \end{aligned}$$

By the conditions (3.8) and (3.9), we obtain

$$\begin{aligned} & E|x(t_n \wedge \tau_k)|^\gamma - E|x(t \wedge \tau_k)|^\gamma \\ \le\ & E\int_{t\wedge\tau_k}^{t_n\wedge\tau_k} L\gamma|x(s)|^{\gamma-2}(|x(s)|^2 + q|x(qs)|^2)ds \\ +\ & E\int_{t\wedge\tau_k}^{t_n\wedge\tau_k} \int_Z \Big(Lh_i(v)(|x(s^-)|^\gamma + q|x(qs)|^\gamma) - |x(s^-)|^\gamma\Big)\pi(dv)ds \\ \le\ & L(2\gamma + C_{h_i})\int_{t\wedge\tau_k}^{t_n\wedge\tau_k} E(|x(s)|^\gamma + |x(qs)|^\gamma)ds \\ \le\ & L(2\gamma + C_{h_i})\int_{qt_n\wedge\tau_k}^{t_n\wedge\tau_k} E|x(s)|^\gamma ds + \frac{L}{q}(2\gamma + C_{h_i})\int_{q^2 t_n\wedge\tau_k}^{qt_n\wedge\tau_k} E|x(s)|^\gamma ds \\ \le\ & \frac{2L}{q}(2\gamma + C_{h_i})\int_{q^2 t_n\wedge\tau_k}^{t_n\wedge\tau_k} E|x(s)|^\gamma ds. \end{aligned} \tag{3.13}$$

Letting $k \to \infty$, we obtain from (3.12) and (3.13) that

$$\begin{aligned} E|x(t_n)|^\gamma \ &\le\ E|x(t)|^\gamma + \frac{2L}{q}(2\gamma + C_{h_i})\int_{q^2 t_n}^{t_n} E|x(s)|^\gamma ds \\ &\le\ E|x(t)|^\gamma + \varepsilon. \end{aligned} \tag{3.14}$$

Hence, for any $t \in [qt_n, t_n]$, it follows from (3.11) and (3.14) that

$$E|x(t)|^\gamma \geq E|x(t_n)|^\gamma - \varepsilon \geq \varepsilon.$$

Thus

$$\int_{t_0}^{\infty} E|x(s)|^\gamma ds \geq \sum_{n=n_0}^{\infty} \int_{qt_n}^{t_n} E|x(s)|^\gamma ds \geq \sum_{n=n_0}^{\infty} \varepsilon(1-q)t_n \geq \varepsilon(1-q)t_0 \sum_{n=n_0}^{\infty} (\frac{1}{q})^{2n} = \infty.$$

That is to say, (2.1) is asymptotically stable in γth moment. Then the proof of Theorem 3.8 is completed. □

Remark 3.9 *As is known to all, most of the literatures focus on the moment asymptotic stability of SDEs driven by Brownian motion, but there are few works on the moment stability of SDEs with jumps. In order to fill this gap, Zhu [38] first extended some results on the moment asymptotic stability of SDEs driven by Brownian motion [20] to the case of SDEs with jumps. Mao et al. [17] studied the moment asymptotic stability of hybrid SDEs with jumps under highly nonlinear growth condition. Now, by Theorem 3.8, we obtain that hybrid stochastic systems with pantograph delay and jumps (2.1) is asymptotically stable in γth moment under highly nonlinear growth condition.*

In general, we cannot imply $\lim_{t\to\infty} |x(t)| = 0$ a.s. from $\lim_{t\to\infty} E|x(t)|^\gamma = 0$. But in our case, this is possible. Next, we will show this result under the same conditions of Theorem 3.6 without any additional condition.

Theorem 3.10 *Let all the conditions of Theorem 3.6 hold. For any initial data ξ, the solution of (2.1) satisfies that*

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.15}$$

Proof. We divide the proof into two steps.

Step 1. By (3.5) and the Funibi theorem, it follows that $\int_{t_0}^{\infty} |x(t)|^\gamma dt < \infty$ *a.s.* This implies

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{3.16}$$

Now, we will claim that the assertion (3.15). If this is false, then

$$P\Big\{ \limsup_{t\to\infty} |x(t)| > 0 \Big\} > 0.$$

Hence, there is a number $\varepsilon > 0$ such that

$$P(\Omega_1) > \varepsilon, \tag{3.17}$$

where $\Omega_1 = \{\limsup_{t\to\infty} |x(t)| > 2\varepsilon\}$. On the other hand, by (3.7), we can obtain that

$$c_1 E|x(t\wedge\tau_k)|^p \le c_2 E||\xi||^p + \int_{qt_0}^{t_0} E\Big[\alpha_2|x(s)|^p + \alpha_4|x(s)|^\gamma)\Big]ds, \quad \forall\ t \ge t_0.$$

where τ_k is defined as (2.16). So

$$k^p P(\tau_k \le t) \le c_2 E||\xi||^p + \int_{qt_0}^{t_0} E\Big[\alpha_2|x(s)|^p + \alpha_4|x(s)|^\gamma)\Big]ds.$$

Letting $t \to \infty$ and choosing k sufficiently large, we have $P(\tau_k < \infty) \le \varepsilon$. This means that

$$P(\Omega_2) > 1 - \varepsilon, \tag{3.18}$$

where $\Omega_2 = \{|x(t)| < k\ for\ all\ t \ge qt_0\}$. It then follows easily from (3.16) and (3.17) that

$$P(\Omega_1 \cap \Omega_2) > \varepsilon. \tag{3.19}$$

Step 2. Let us now define the stopping process $X(t) = x(t\wedge\tau_k)$ for $t \ge qt_0$. Obviously, $X(t)$ is an Itô process of the form

$$\begin{aligned} dX(t) &= f(x(t), x(qt), r(t)) I_{[t_0,\tau_k)}(t)dt + g(x(t), x(qt), r(t)) I_{[t_0,\tau_k)}(t)dw(t) \\ &+ \int_Z h(x(t^-), x(qt), r(t^-), v) I_{[t_0,\tau_k)}(t) N(dt, dv). \end{aligned} \tag{3.20}$$

By Assumption 2.1, we see that

$$|f(x(t), x(qt), r(t)) I_{[t_0,\tau_k)}(t)|^2 \vee |g(x(t), x(qt), r(t)) I_{[t_0,\tau_k)}(t)|^2 \le M_1 \tag{3.21}$$

and

$$\int_Z |h(x(t^-), x(qt), r(t^-), v) I_{[t_0,\tau_k)}(t)|^2 \pi(dv) \le M_1 \tag{3.22}$$

for all $i \in S$ and $t \ge t_0$. Define a sequence of stopping times

$$\begin{aligned} \rho_1 &= \inf\{t \ge t_0 : |X(t)|^2 \ge 2\varepsilon\}, \\ \rho_{2i} &= \inf\{t \ge \rho_{2i-1} : |X(t)|^2 \le \varepsilon\}, \quad i = 1, 2, \cdots, \\ \rho_{2i+1} &= \inf\{t \ge \rho_{2i} : |X(t)|^2 \ge 2\varepsilon\}, \quad i = 1, 2, \cdots. \end{aligned}$$

By (3.16) and the definitions of Ω_1 and Ω_2, we have

$$\Omega_1 \cap \Omega_2 \subset \cap_{j=1}^{\infty}\{\rho_j < \infty\}. \tag{3.23}$$

Choose a positive number λ and a positive integer j_0 such that

$$3M_1[12\lambda + (1+2\pi(Z))\lambda^2] \leq \varepsilon^3 \quad \text{and} \quad M_2 < \lambda \varepsilon^{\frac{\gamma}{2}+1} j_0 \tag{3.24}$$

where $M_2 = E\int_{t_0}^{\infty} |x(t)|^{\gamma} dt$. By (3.19) and (3.23), we can further choose a sufficiently large number T for

$$P(\rho_{2j_0} \leq T) \geq 2\varepsilon. \tag{3.25}$$

In fact, if $\rho_{2j_0} \leq T$, $X(\rho_{2j_0}) = \varepsilon$. By the definition of $X(t)$, we have $\rho_{2j_0} < \tau_k$. For any $t_0 \leq t \leq \rho_{2j_0}$ and $\omega \in \{\rho_{2j_0} \leq T\}$, we get

$$X(t,\omega) = x(t,\omega). \tag{3.26}$$

Now, by the Holder inequality and the Burkholder-Davis-Gundy inequality, we can obtain that, for $1 \leq j \leq j_0$,

$$\begin{aligned}
& E\Big(\sup_{t_0\leq t\leq\lambda} |X(\rho_{2j-1}\wedge T + t) - X(\rho_{2j-1}\wedge T)|^2\Big) \\
\leq\ & 3E\Big(\sup_{t_0\leq t\leq\lambda} |\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+t} f(x(s),x(qs),r(s)) I_{[t_0,\tau_k)}(s) ds|^2\Big) \\
+\ & 3E\Big(\sup_{t_0\leq t\leq\lambda} |\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+t} g(x(s),x(qs),r(s)) I_{[t_0,\tau_k)}(s) dw(s)|^2\Big) \\
+\ & 3E\Big(\sup_{t_0\leq t\leq\lambda} |\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+t} \int_Z h(x(s^-),x(qs),r(s^-),v) I_{[t_0,\tau_k)}(s) N(ds,dv)|^2\Big) \\
\leq\ & 3\lambda E\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda} |f(x(s),x(qs),r(s)) I_{[t_0,\tau_k)}(s)|^2 ds \\
+\ & 12E\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda} |g(x(s),x(qs),r(s)) I_{[t_0,\tau_k)}(s)|^2 ds + Q.
\end{aligned} \tag{3.27}$$

By the Doob-Meyer's decomposition theorem and the basic inequality $|a+b|^2 \leq 2|a|^2 + 2|b|^2$, we have

$$\begin{aligned}
Q &\le 6E\Big(\sup_{t_0\le t\le\lambda}|\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+t}\int_Z h(x(s^-),x(qs),r(s^-),v)I_{[t_0,\tau_k)}(s)\tilde N(ds,dv)|^2\Big)\\
&+ 6E\Big(\sup_{t_0\le t\le\lambda}|\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+t}\int_Z h(x(s^-),x(qs),r(s^-),v)I_{[t_0,\tau_k)}(s)\pi(dv)ds|^2\Big)\\
&\le 24E\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda}\int_Z |h(x(s^-),x(qs),r(s^-),v)I_{[t_0,\tau_k)}(s)|^2\pi(dv)ds\\
&+ 6\lambda E\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda}|\int_Z h(x(s^-),x(qs),r(s^-),v)I_{[t_0,\tau_k)}(s)\pi(dv)|^2ds\\
&\le [24+6\lambda\pi(Z)]E\int_{\rho_{2j-1}\wedge T}^{\rho_{2j-1}\wedge T+\lambda}\int_Z |h(x(s^-),x(qs),r(s^-),v)I_{[t_0,\tau_k)}(s)|^2\pi(dv)ds. \qquad (3.28)
\end{aligned}$$

Inserting (3.28) into (3.27), it follows from (3.21) and (3.22) that

$$E\Big(\sup_{t_0\le t\le\lambda}|X(\rho_{2j-1}\wedge T+t)-X(\rho_{2j-1}\wedge T)|^2\Big)\le 3M_1[12\lambda+(1+2\pi(Z))\lambda^2].$$

By the Chebyshev inequality and (3.24), we have

$$\begin{aligned}
&P\Big(\sup_{t_0\le t\le\lambda}|X(\rho_{2j-1}\wedge T+t)-X(\rho_{2j-1}\wedge T)|\ge\varepsilon\Big)\\
\le\ &\frac{1}{\varepsilon^2}E\Big(\sup_{t_0\le t\le\lambda}|X(\rho_{2j-1}\wedge T+t)-X(\rho_{2j-1}\wedge T)|^2\Big)\le\varepsilon. \qquad (3.29)
\end{aligned}$$

Noting that $\rho_{2j-1}\le T$ if $\rho_{2j_0}\le T$, it follows from (3.24) and (3.28) that

$$\begin{aligned}
&P\Big(\{\rho_{2j_0}\le T\}\cap\{\sup_{t_0\le t\le\lambda}|X(\rho_{2j-1}+t)-X(\rho_{2j-1})|<\varepsilon\}\Big)\\
=\ &P(\rho_{2j_0}\le T)-P\Big(\sup_{t_0\le t\le\lambda}|X(\rho_{2j-1}+t)-X(\rho_{2j-1})|\ge\varepsilon\Big)\ge\varepsilon.
\end{aligned}$$

This implies that

$$P\Big(\{\rho_{2j_0}\le T\}\cap\{\rho_{2j}-\rho_{2j-1}\ge\lambda\}\Big)\ge\varepsilon. \qquad (3.30)$$

Finally, by (3.25) and (3.30), we have

$$\begin{aligned}
M_2 &\ge \sum_{j=1}^{j_0}E\Big(\int_{\rho_{2j-1}}^{\rho_{2j}}|X(t)|^\gamma dt I_{\{\rho_{2j_0}\le T\}}\Big)\ge\varepsilon^{\frac{\gamma}{2}}\sum_{j=1}^{j_0}E\Big((\rho_{2j}-\rho_{2j-1})I_{\{\rho_{2j_0}\le T\}}\Big)\\
&\ge \varepsilon^{\frac{\gamma}{2}}\sum_{j=1}^{j_0}E\Big((\rho_{2j}-\rho_{2j-1})I_{\{\rho_{2j}-\rho_{2j-1}\ge\lambda\}}I_{\{\rho_{2j_0}\le T\}}\Big)\\
&\ge \lambda\varepsilon^{\frac{\gamma}{2}}\sum_{j=1}^{j_0}P\Big(\{\rho_{2j}-\rho_{2j-1}\ge\lambda\}\cap\{\rho_{2j_0}\le T\}\Big)\\
&\ge \lambda\varepsilon^{\frac{\gamma}{2}+1}j_0.
\end{aligned}$$

But this contradicts (3.24). Hence, (3.15) must hold. The proof is therefore complete. □

In the previous argument, we have discussed two kinds of asymptotic stabilities of the solution to (2.1). However, these two stabilities do not reveal the rate at which the solution tends to zero. Next, we will discuss the polynomial stablity under conditions of Theorem 3.6.

Theorem 3.11 *Let conditions of Theorem 3.6 hold. Then for any given initial data ξ, the unique solution $x(t)$ to (2.1) has the properties that*

$$\limsup_{t\to\infty} \frac{\log(E|x(t)|^p)}{\log t} \le -\varepsilon \tag{3.31}$$

and

$$\limsup_{t\to\infty} \frac{\log|x(t)|}{\log t} \le -\frac{\varepsilon}{p} \quad a.s. \tag{3.32}$$

where $\varepsilon = \varepsilon_1 \wedge \varepsilon_2$ while $\varepsilon_1 = -\log\frac{\alpha_3}{\alpha_4}/\log q$ and $\varepsilon_2 > 0$ is the unique root to the following equation $\alpha_1 = c_2\varepsilon_2 + \alpha_2 q^{-\varepsilon_2}$.

Proof. By Theorem 3.6, for any given initial data ξ, (2.1) has a unique global solution $x(t)$ on $t \ge t_0$. Let the stopping time τ_k be the same as defined in the proof of Theorem 2.6. Define the function $V(x,i) = \theta_i|x|^p$. By the generalized Itô formula, we have that, for any $t \ge t_0$,

$$\begin{aligned}(1+t\wedge\tau_k)^\varepsilon\theta_{r(\tau_k\wedge t)}|x(\tau_k\wedge t)|^p &= (1+t_0)^\varepsilon\theta_{r(t_0)}|x(t_0)|^p + \int_{t_0}^{\tau_k\wedge t}\Big(\varepsilon(1+s)^{\varepsilon-1}\theta_{r(s)}|x(s)|^p \\ &+ (1+s)^\varepsilon LV(x(s),x(qs),r(s))\Big)ds + M(t\wedge\tau_k), \end{aligned}\tag{3.33}$$

where

$$\begin{aligned}M(t\wedge\tau_k) &= \int_{t_0}^{\tau_k\wedge t}(1+s)^\varepsilon p\theta_{r(s)}|x(s)|^{p-2}x(s)^\top g(x(s),x(qs),r(s))dw(s) \\ &+ \int_{t_0}^{\tau_k\wedge t}\int_Z (1+s)^\varepsilon\Big(\theta_{r(s^-)}|x(s^-)+h(x(s^-),x(qs),r(s^-),v)|^p - \theta_{r(s^-)}|x(s^-)|^p\Big)\tilde{N}(ds,dv)\end{aligned}$$

is a martingale with the initial value $M(t_0) = 0$. Taking expectation on both sides of (3.33), we have

$$\begin{aligned}E\Big[(1+t\wedge\tau_k)^\varepsilon\theta_{r(\tau_k\wedge t)}|x(\tau_k\wedge t)|^p\Big] &= (1+t_0)^\varepsilon\theta_{r(t_0)}E|x(t_0)|^p + E\int_{t_0}^{\tau_k\wedge t}\Big(\varepsilon(1+s)^{\varepsilon-1}\theta_{r(s)}|x(s)|^p \\ &+ (1+s)^\varepsilon LV(x(s),x(qs),r(s))\Big)ds,\end{aligned}\tag{3.34}$$

where $EM(t\wedge\tau_k)=EM(t_0)=0$. In the same way as (3.7) was proved, we can show that

$$\begin{aligned} c_1E\Big((1+t\wedge\tau_k)^{\varepsilon}|x(\tau_k\wedge t)|^p\Big) &\leq \theta_{r(t_0)}(1+t_0)^{\varepsilon}E||\xi||^p+E\int_{t_0}^{\tau_k\wedge t}\Big[\varepsilon(1+s)^{\varepsilon-1}c_2|x(s)|^p \\ &+ (1+s)^{\varepsilon}\Big(-\alpha_1|x(s)|^p+\alpha_2q|x(qs)|^p \\ &- \alpha_3|x(s)|^{\gamma}+\alpha_4q|x(qs)|^{\gamma}\Big)\Big]ds. \end{aligned} \tag{3.35}$$

Now, we compute

$$\begin{aligned} E\int_{t_0}^{\tau_k\wedge t}(1+s)^{\varepsilon}q|x(qs)|^pds &\leq \frac{1}{q^{\varepsilon}}\int_{qt_0}^{q(\tau_k\wedge t)}(1+s)^{\varepsilon}E|x(s)|^pds \\ &\leq \frac{1}{q^{\varepsilon}}(1+t_0)^{\varepsilon}\int_{qt_0}^{t_0}E|x(s)|^pds+\frac{1}{q^{\varepsilon}}\int_{t_0}^{\tau_k\wedge t}(1+s)^{\varepsilon}E|x(s)|^pds \end{aligned}$$

and, similarly

$$E\int_{t_0}^{\tau_k\wedge t}(1+s)^{\varepsilon}q|x(qs)|^{\gamma}ds \leq \frac{1}{q^{\varepsilon}}(1+t_0)^{\varepsilon}E\int_{qt_0}^{t_0}E|x(s)|^{\gamma}ds+\frac{1}{q^{\varepsilon}}E\int_{t_0}^{\tau_k\wedge t}(1+s)^{\varepsilon}|x(s)|^{\gamma}ds.$$

Substituting these into (3.35) gives

$$\begin{aligned} c_1E\Big((1+t\wedge\tau_k)^{\varepsilon}|x(\tau_k\wedge t)|^p\Big) &\leq C-(\alpha_1-\varepsilon c_2-\frac{\alpha_2}{q^{\varepsilon}})E\int_{t_0}^{\tau_k\wedge t}(1+s)^{\varepsilon}|x(s)|^pds \\ &- (\alpha_3-\frac{\alpha_4}{q^{\varepsilon}})E\int_{t_0}^{\tau_k\wedge t}(1+s)^{\varepsilon}|x(s)|^{\gamma}ds, \end{aligned} \tag{3.36}$$

where $C=c_2(1+t_0)^{\varepsilon}E||\xi||^p+(1+t_0)^{\varepsilon}E\int_{qt_0}^{t_0}(\alpha_2|x(s)|^p+\alpha_4|x(s)|^{\gamma})ds$. By the definitions of ε_1 and ε_2, we have

$$\alpha_1-\varepsilon c_2-\frac{\alpha_2}{q^{\varepsilon}}\geq 0 \quad\text{and}\quad \alpha_3-\frac{\alpha_4}{q^{\varepsilon}}\geq 0.$$

Therefore,

$$c_1E\Big((1+t\wedge\tau_k)^{\varepsilon}|x(\tau_k\wedge t)|^p\Big) \leq C. \tag{3.37}$$

Letting $k\to\infty$, we obtain that

$$(1+t)^{\varepsilon}E|x(t)|^p \leq \frac{C}{c_1}, \quad \forall\, t\geq t_0.$$

Dividing both sides by $(1+t)^{\varepsilon}$ and letting $t\to\infty$, we get the assertion (3.31). Similar to (3.36), we can show in the same way as before that

$$c_1(1+t)^{\varepsilon}|x(t)|^p\leq C+M(t),$$

for any $t \geq t_0$, where $M(t)$ is the same as defined in (3.33). By lemma 3.3, we obtain that

$$\limsup_{t\to\infty} c_1(1+t)^{\varepsilon}|x(t)|^p \leq \infty \ a.s.$$

which implies the assertion (3.32). The proof is therefore complete. □

Remark 3.12 *In particular, when $p=2$, we have that (2.1) are polynomially stable in mean-square and almost sure polynomially stable. Compared with Appleby and Buckwar [2], we study the polynomial stability of hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise (2.1) under nonlinear growth conditions. Due to the nonlinearity of the coefficients to (2.1), some results of [2] on the polynomial stability are improved and generalized.*

4 Examples

In this section, we will discuss two examples to illustrate our results.

Example 4.1 *Let $r(t)$ be a right-continuous Markov chain taking values in $S=\{1,2,3\}$ with the generator*

$$\Gamma = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -4 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$

Let $N(dt,dv)$ be a Poisson random measures and σ-finite measure $\pi(dv)$ is given by $\pi(dv) = \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv,\ -\infty < v < +\infty$. Assume that $N(dt,dv)$ and $r(t)$ are independent.

Consider the following scalar hybrid stochastic systems with pantograph delay and pure Lévy jumps

$$dx(t) = f(x(t),r(t))dt + \int_0^{\infty} h(x(0.2t^-),r(t^-),v)N(dt,dv), \tag{4.1}$$

with initial data $\xi(t) = x_0$ $(0.2 \leq t \leq 1)$ and $r(1)=1$. Here

$$f(x,1) = -3x-2x^3, \quad h(y,1,v) = \rho_1\sqrt{v}y^2,$$
$$f(x,2) = 0.5x-3x^3, \quad h(y,2,v) = \rho_2 v y^2,$$
$$f(x,3) = -2x-x^3, \quad h(y,3,v) = \rho_3 v^2 y^2,$$

for $x \in R$, but ρ_1, ρ_2 and ρ_3 are unknown parameters. Obviously, the coefficients f,h satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a *straight computation, we can have*

$$x^\top f(x,1) \le -3|x|^2 - 2|x|^4,\ \ x^\top f(x,2) \le 0.5|x|^2 - 3|x|^4, \tag{4.2}$$

$$x^\top f(x,3) \le -2|x|^2 - |x|^4,\ \ |x + h(x,1,v)|^2 \le (2+v)(0.5|x|^2 + \rho_1^2|y|^4), \tag{4.3}$$

$$|x + h(x,2,v)|^2 \le (1.25 + 0.5v^2)(0.8|x|^2 + 2\rho_2^2|y|^4), \tag{4.4}$$

$$|x + h(x,3,v)|^2 \le (1 + 0.2v^4)(|x|^2 + 5\rho_3^2|y|^4) \tag{4.5}$$

where

$$\begin{aligned}
&\alpha_{11} = -3, \alpha_{21} = 0, \alpha_{31} = 2, \alpha_{41} = 0, \alpha_{12} = 0.5, \alpha_{22} = 0, \alpha_{32} = 3, \alpha_{42} = 0,\\
&\alpha_{13} = -2, \alpha_{23} = 0, \alpha_{33} = 1, \alpha_{43} = 0, \beta_{11} = 0.5, \beta_{21} = 0, \beta_{31} = 0, \beta_{41} = 5\rho_1^2,\\
&\beta_{12} = 0.8, \beta_{22} = 0, \beta_{32} = 0, \beta_{42} = 10\rho_2^2, \beta_{13} = 1, \beta_{23} = 0, \beta_{33} = 0, \beta_{43} = 25\rho_3^2
\end{aligned}$$

and

$$\gamma = 4, h_1(v) = 2 + v, h_2(v) = 1.25 + 0.5v^2, h_3(v) = 1 + 0.2v^4.$$

So the inequalities (4.2)-(4.5) show that Assumption 3.4 holds. Moreover, we can compute that

$$\begin{aligned}
C_{h_1} &= \int_0^\infty (2+v)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 1.3989,\\
C_{h_2} &= \int_0^\infty (1.25+0.5v^2)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 0.875,\\
C_{h_3} &= \int_0^\infty (1+0.2v^4)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 0.8.
\end{aligned} \tag{4.6}$$

On the one hand, the matrix $\mathcal{A}_3$ defined by (3.1) is

$$\mathcal{A}_3 = \begin{pmatrix} 7.3005 & -1 & -1 \\ -3 & 2.3 & -1 \\ -1 & -1 & 5.2 \end{pmatrix}.$$

It is easy to compute

$$\mathcal{A}_3^{-1} = \begin{pmatrix} 0.188596 & 0.106687 & 0.056785 \\ 0.285648 & 0.636042 & 0.177248 \\ 0.091201 & 0.142832 & 0.237315 \end{pmatrix}.$$

By lemma 3.2, we see that $\mathcal{A}_3$ *is a nonsingular M-matrix. Compute*

$$(\theta_1, \theta_2, \theta_3)^\top = \mathcal{A}_3^{-1}\overrightarrow{1} = (0.352068, 1.098938, 0.471348)^\top.$$

Conditions (3.3) and (3.4) become

$$\min\{1.408272, 6.593628, 0.942696\} > \max\{2.462539\rho_1^2, 9.615707\rho_2^2, 9.42696\rho_3^2\}$$

i.e.,

$$\rho_1^2 < 0.382814,\ \ \rho_2^2 < 0.098037,\ \ \rho_3^2 < 0.1. \tag{4.7}$$

By Theorems 3.6 and 3.10, we can conclude that if the parameters $\rho_i, i = 1, 2, 3$ *satisfy (4.7), then for any initial data* x_0*, there is a unique global solution* $x(t)$ *to (4.1) on* $t \in [1, \infty)$*. Moreover, the solution has the properties that* $\int_1^\infty E|x(t)|^4 dt < \infty$ *and* $\lim_{t\to\infty} |x(t)| = 0$ *a.s..*

Example 4.2 *Let* $w(t)$ *is a scalar Brownian motion. Let* $r(t)$ *be a right-continuous Markov chain taking values in* $S = \{1, 2\}$ *with the generator*

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

Let $N(dt, dv)$ *be a Poisson random measures and* σ*-finite measure* $\pi(dv)$ *is given by* $\pi(dv) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} dv,\ -\infty < v < +\infty$*. Of course,* $w(t)$*,* $N(dt, dv)$ *and* $r(t)$ *are assumed to be independent.*

Consider the following scalar hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise

$$dx(t) = f(x(t), r(t))dt + g(x(0.5t), r(t))dw(t) + \int_0^\infty vh(x(0.5t^-), r(t^-))N(dt, dv), \tag{4.8}$$

with initial data $\xi(t) = x_0$ $(0.5 \le t \le 1)$ *and* $r(1) = 1$*. Here*

$$\begin{aligned} &f(x,1) = -2x - 3x^3, \quad g(y,1) = 0.5(y + y^2), \quad h(y,1) = 0.2y^2, \\ &f(x,2) = 0.25x - 2x^3, \qquad g(y,2) = \tfrac{1}{\sqrt{3}}y, \qquad h(y,2) = 0.1y \end{aligned}$$

for any $x, y \in R$*. We note that (4.8) can be regarded as the result of the two equations*

$$\begin{aligned} dx(t) &= (-2x(t) - 3x^3(t))dt + 0.5\Big(x(0.5t) + x^2(0.5t)\Big)dw(t) \\ &+ 0.2\int_0^\infty vx^2(0.5t^-)N(dt, dv), \end{aligned} \tag{4.9}$$

$$\begin{aligned} dx(t) &= (0.25x(t) - 2x^3(t))dt + \frac{1}{\sqrt{3}}x(0.5t)dw(t) \\ &+ 0.1\int_0^\infty vx(0.5t-)N(dt, dv), \end{aligned} \tag{4.10}$$

switching among each other according to the movement of the Markov chain $r(t)$. It is easy to see that subsystem (4.9) is polynomially stable but subsystem (4.10) is unstable. However, we shall see that due to the Markovian switching, the overall system (4.8) will be polynomially stable. In fact, the coefficients f, g and h satisfy the local Lipschitz condition but they do not satisfy the linear growth condition. Through a straight computation, we can obtain

$$\begin{aligned}
& x^\top f(x,1) + \frac{1}{2}|g(y,1)|^2 \leq -2|x|^2 + 0.15625|y|^2 - 3|x|^4 + 0.625|y|^4 && (4.11)\\
& x^\top f(x,2) + \frac{1}{2}|g(y,2)|^2 \leq 0.25|x|^2 + \frac{1}{6}|y|^2 - 4|x|^4, && (4.12)\\
& |x + vh(y,1)|^2 \leq (1 + 0.04v^2)(|x|^2 + |y|^4), && (4.13)\\
& |x + vh(y,2)|^2 \leq (1 + 0.05v^2)(|x|^2 + 0.2|y|^2) && (4.14)
\end{aligned}$$

where

$$\alpha_{11} = -2,\ \alpha_{21} = 0.3125,\ \alpha_{31} = 3,\ \alpha_{41} = 1.25,\ \alpha_{12} = 0.25,\ \alpha_{22} = \tfrac{1}{3},\ \alpha_{32} = 4,\ \alpha_{42} = 0,$$
$$\beta_{11} = 1,\ \beta_{21} = 0,\ \beta_{31} = 0,\ \beta_{41} = 2,\ \beta_{12} = 1,\ \beta_{22} = 0.4,\ \beta_{32} = 0,\ \beta_{42} = 0$$

and

$$\gamma = 4, \quad h_1(v) = 1 + 0.04v^2, \quad h_2(v) = 1 + 0.05v^2.$$

So the inequalities (4.11)-(4.14) show that the Assumption 3.4 holds. Moreover, by the property of normal distribute, we can obtain that

$$\begin{aligned}
C_{h_1} &= \int_0^\infty (1 + 0.04v^2)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 0.52,\\
C_{h_2} &= \int_0^\infty (1 + 0.05v^2)\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}dv = 0.525.
\end{aligned}$$

By (3.1), we get the matrix $\mathcal{A}_2$

$$\begin{aligned}
\mathcal{A}_2 &= -\mathrm{diag}(2\alpha_{11} + \beta_{11}C_{h_1}, 2\alpha_{12} + \beta_{12}C_{h_2}) - \Gamma\\
&= \begin{pmatrix} 4.48 & -1 \\ -4 & 2.975 \end{pmatrix}.
\end{aligned}$$

It is easy to compute

$$\mathcal{A}_2^{-1} = \begin{pmatrix} 0.318932 & 0.107204 \\ 0.428816 & 0.480274 \end{pmatrix}.$$

By Lemma 3.2, we see that $\mathcal{A}_2$ *is a non-singular M-matrix. Compute*

$$(\theta_1, \theta_2)^T = \mathcal{A}_2^{-1}\vec{1} = (0.426136, 0.90909)^T$$

and

$$\alpha_2 = \max_{i=1,2}\Big(2\alpha_{2i}\theta_i + \beta_{2i}C_{h_i}\theta_i\Big) = 0.796963,\ \alpha_3 = \min_{i=1,2}\Big([2\alpha_{3i}\theta_i - \beta_{3i}C_{h_i}\theta_i\Big) = 2.556816,$$

$$\alpha_4 = \max_{i=1,2}\Big(2\alpha_{4i}\theta_i + \beta_{4i}C_{h_i}\theta_i\Big) = 1.508521.$$

Hence, we conclude that the conditions (3.3) and (3.4) hold. By Theorem 3.11, we can obtain that

$$\limsup_{t\to\infty}\frac{\log(E|x(t)|^2)}{\log t} \le -\varepsilon$$

and

$$\limsup_{t\to\infty}\frac{\log|x(t)|}{\log t} \le -\frac{\varepsilon}{2}\quad a.s.$$

where $\varepsilon = 0.136399$ *is the unique root of* $1 = 0.90909\varepsilon + 0.796963 \times 0.5^{-\varepsilon}$. *That is to say, the solution of (4.8) decays at the polynomial rate of at least* 0.068199.

5 Conclusion

This paper is devoted to the asymptotic stability and polynomial stability of hybrid stochastic systems with pantograph delay and non-Gaussian Lévy noise (HSSwPDLNs). The Lyapunov functions and M-matrix theory are used to derive sufficient conditions for stabilities of nonlinear HSSwPDLNs. Moreover, as illustrated by two examples, it has been shown that even if some subsystems are not stable, the overall hybrid system may still be stable as long as certain conditions are satisfied.

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