Probabilistic Mu-Calculus: Decidability and Complete Axiomatization*

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Abstract

We introduce a version of the probabilistic mu-calculus (PMC) built on top of a probabilistic modal logic that allows encoding n-ary inequational conditions on transition probabilities. PMC extends previously studied calculi and we prove that, despite its expressiveness, it enjoys a series of good meta-properties. Firstly, we prove the decidability of satisfiability checking by establishing the small model property. An algorithm for deciding the satisfiability problem is developed. As a second major result, we provide a complete axiomatization for the alternation-free fragment of PMC. The completeness proof is innovative in many aspects combining various techniques from topology and model theory.

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1 Introduction

From the perspective of industrial practice, especially in the area of embedded and cyber-physical systems, an essential problem is how to deal with the high complexity of the systems, while still meeting the requirements of correctness, predictability, performance and also non-functional properties. In this respect, for embedded systems, specification and verification should not only consider functional properties but also non-functional properties. Particularly, effort has been put into formalisms and logics that address stochastic aspects of a system. The seminal work of Hansson and Jonsson [13] introduced pCTL, a probabilistic extension of CTL. In a number of recent work results related to decidability and complexity of model checking and satisfiability checking of (variants of) pCTL have been established [1, 14, 4, 22, 24, 5].

In parallel, various probabilistic modal μ-calculi have been considered. Typically, one characterizes the probabilistic bisimulation by using a probabilistic version of modal logic with the modality indexed by a subunital positive real: e.g. $\langle\rangle_{>p}\phi$ describes that the probability of reaching a next state satisfying ϕ is greater than p. Whereas the resulting logic does fully characterize probabilistic bisimulation, it is not sufficiently expressive with respect to

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decomposition of properties under static operators. To address this, in [21] an extended n-ary next-state modality (*(in-)equational modality*) was introduced: e.g. $[\langle x\rangle\phi_1, \langle y\rangle\phi_2 : x+y \leq 0.7]$ describes that the probabilities x and y of reaching next-states satisfying ϕ_1 and ϕ_2 respectively must satisfy the constraint $x+y \leq 0.7$. This modality allows one to encode complex linear constraints on probabilities.

In this paper we introduce a probabilistic μ-calculus (PMC) for specifying and reasoning about the Markov processes. PMC extends with block sequences (equation systems) the modal logic of [21]. As a first main result, we prove the decidability of satisfiability checking by establishing a small model property for this logic. As a second main result, we provide a sound and complete axiomatization for the alternation-free fragment of PMC.

Related Work. The satisfiability problem for the probabilistic logics with fixed points has been a hot topic for a number of years. While this is still an open problem for pCTL and pCTL*, various fragments have been solved. In [14, 4], it is shown that qualitative pCTL (expressing only whether a probability is bigger than 0 or equal to 1) has no finite model property and its satisfiability problem is ExpTime-complete. Moreover, it is proven that satisfiability checking for pCTL against models with bounded branching degree is highly undecidable; however, every satisfiable formula has a model with branching degree bounded by the size of the formula. More recently, in [1], pCTL satisfiability problem for bounded-size models is studied and proved to be decidable. In [22, 24], the qualitative fragment of pCTL* is proved to be decidable too. In recent works [23, 6], the satisfiability problem for an extension of the logic in [20] with fixed points is proven to be decidable. This logic only involves probabilistic next-state operator and it cannot express the (in-)equational modalities of [21].

The decidability of probabilistic μ-calculus of [23] also derives as a particular case of the more general results proven in [7], where it is shown that the decidability of coalgebraic mu-calculi parametrized by a tractable set of so-called one-step rules is in ExpTime; in [17] such a rule set has been exhibited for probabilistic modal logic with linear inequalities. However, all these works were done only for finite sets with discrete probability distributions.

In [26, 27, 25] probabilistic modal μ-calculus, Łukasiewicz μ-calculus, probabilistic modal μ-calculus with independent product are studied in the context of denotational semantics and game semantics, relying on a satisfiability relation that is not essentially boolean but rather quantitative.

Another fixed points probabilistic logic is proposed in [8]. Its syntax is divided into a probabilistic part (so called state formulas) and a non-probabilistic part involving fixed points (so called fuzzy formulas). This logic can encode the probabilistic modal logic and pCTL* and it is studied from the perspective of (finite) model checking and bisimulation checking.

Considering the axiomatization, complete axiomatizations for the qualitative fragment of pCTL* are shown in [22], but only for bounded finite systems.

Our Work. With respect to the related work described above, our probabilistic μ-calculus involves the equational modalities of [21], thus allowing us to encode (in-)equational conditions on probabilities. The logic is definitely more general than that of [23]. The semantics, with respect to the other related works such as [7, 17] is in term of general (analytical) measurable sets and not just finite spaces with discrete sigma-algebras; it has been repeatedly proven in literature, see e.g. [28], that going from discrete systems to continuous systems is far from being a trivial step, and complex topological and measure theoretical arguments applied to model theory must be invoked.

Our logic is incomparable with pCTL and pCTL* as it cannot express modalities such as “probabilistic Until”. The work here includes a modality extension but does not try to add this modality to the more complicated logics of [5, 26, 27, 25]. However, our logic can be used to approximate pCTL formulas with arbitrary precision when restricting to finite models. This is interesting as the satisfiability problem for quantitative pCTL is still open.

We prove that this logic enjoys the finite model property and its satisfiability problem is decidable. We develop an algorithm that checks the satisfiability of a formula and, if the formula is satisfiable, it constructs a finite model. Being the aforementioned state of the art in the field, these are important results presenting our logic as a good trade-off between expressiveness and decidability. Moreover, these results generalize the ones in [23] while our proof constructs on top of the classic tableau method [33, 15, 34].

Another key contribution of our paper is the complete axiomatization that we propose for the alternation-free fragment of PMC. At the best of our knowledge, the problem of axiomatizing probabilistic μ-calculus has not been previously approached at this level of generality. The completeness proof is a non-standard extension of the filtration method relying on topological facts such as Rasiowa-Sikorski lemma and its relation to Lindenbaum's lemma (following the technique developed by the first two authors in collaboration with Dexter Kozen and Prakash Panangaden [16]). The proof also applies the technique developed in [18] by the authors for proving the completeness of fixed points logics. These can be easily adapted to other versions of probabilistic μ-calculus.

Due to space limit, most of the results stated here are without proofs. For a detailed presentation with the proofs and some of the classical definitions and lemmas, the reader is referred to `http://people.cs.aau.dk/~bingt/probaMuCalc.pdf`

2 Probabilistic Mu-Calculus

Probabilistic μ-Calculus (PMC) that we develop in this paper encodes properties of Markov processes. As usual with μ-Calculus based on equation systems, the syntax is given in two stages: we firstly introduce the basic formulas and secondly use them to define blocks. The basic formulas are boolean formulas, constructed on top of a set $\mathcal{A}$ of atomic propositions and involving the following:

- *recursive-variables* range over the set $\mathcal{X}$; they are used to define simultaneous recursive equations in order to express maximal and minimal fixed points, in the style of [19, 9, 10, 2];
- *(in-)equational modalities* of type $\langle x_1\rangle\phi_1, \ldots, \langle x_n\rangle\phi_n \colon \Sigma_{i=1}^n a_i x_i \geq r$ where $x_1, \ldots, x_n$ are *probability variables* ranging over a set $\mathcal{V}$ and $a_1, \ldots, a_n, r \in \mathbb{Q}$.

▶ **Definition 1** (Basic formulas)**.** The *basic formulas* of PMC are defined by the following grammar, for arbitrary $p \in \mathcal{A}$, $X \in \mathcal{X}$, $a_1, \ldots, a_n, r \in \mathbb{Q}$, $x_1, \ldots, x_n \in \mathcal{V}$:

$$\mathcal{L}: \quad \phi := \ p \mid \neg\phi \mid \phi \vee \phi \mid \langle x_1\rangle\phi_1, \ldots, \langle x_n\rangle\phi_n \colon \Sigma_{i=1}^n a_i x_i \geq r \mid X \ .$$

Notation: For arbitrary $\overline{x} \in \mathcal{V}^n$, $\overline{a} \in \mathbb{Q}^n$, $r \in \mathbb{Q}$ and $\overline{\phi} \in \mathcal{L}^n$, instead of $\langle x_1\rangle\phi_1 \ldots \langle x_n\rangle\phi_n$, we simply write $\overline{\langle x\rangle\phi}$ and instead of $\Sigma_{i=1}^n a_i x_i \geq r$ we write $\overline{ax} \geq r$. This will simplify the syntax of the equational modalities and instead of $\langle x_1\rangle\phi_1 \ldots \langle x_n\rangle\phi_n \colon \Sigma_{i=1}^n a_i x_i \geq r$, we will write $\overline{\langle x\rangle\phi} \colon \overline{a}\cdot\overline{x} \geq r$. In this case, n is called the *length* of $\overline{\langle x\rangle\phi} \colon \overline{a}\cdot\overline{x} \geq r$. If $\overline{\langle x\rangle\phi} = \langle x_1\rangle\phi_1 \ldots \langle x_n\rangle\phi_n$ and $\overline{a}\cdot\overline{x} = \Sigma_{i=1}^n a_i x_i$, for $k < n$, let $\overline{\langle x\rangle\phi}\big|_k \stackrel{\text{def}}{=} \langle x_1\rangle\phi_1 \ldots \langle x_k\rangle\phi_k$ and $\overline{a}\cdot\overline{x}\big|_k \stackrel{\text{def}}{=} \Sigma_{i=1}^k a_i x_i$.

Observe that in the basic formulas we only allow one inequality using $\geq$ to specify the constraints on $\overline{x}$. However, we can, for instance, encode reversed inequalities since we are using all rationals; and we can encode a finite set of constraints by involving conjunctions of the equational modalities.

The dual of $\overline{\langle x\rangle\phi}$: $\overline{a}\cdot\overline{x} \geq r$ can be defined as $\overline{\langle x\rangle\phi}$: $\overline{a}\cdot\overline{x} < r$; for this reason, we write constraints freely using $\geq, \leq, >$ or $<$. We use both $\trianglelefteq$ and $\trianglerighteq$ to range over the set $\{\leq, \geq\}$ such that $\{\trianglelefteq, \trianglerighteq\} = \{\leq, \geq\}$. Similarly, we use $\triangleleft$ and $\triangleright$ to range over the set $\{<, >\}$ such that $\{\triangleleft, \triangleright\} = \{<, >\}$.

Now we introduce the equation blocks. Given $\phi, \psi_1, \ldots, \psi_h \in \mathcal{L}$ and $X_1, \ldots, X_h \in \mathcal{X}$, let $\phi\{\psi_1/X_1, \ldots,\ \psi_h/X_h\}$ be the formula obtained by substituting each occurrence of X_i in ϕ with ψ_i for $i = 1, \ldots, h$; denoted shortly $\phi\{\overline{\psi}/\overline{X}\}$, where $\overline{\psi} = (\psi_1, \ldots, \psi_h)$ and $\overline{X} = (X_1, \ldots, X_h)$. Following [9, 10, 2], we allow sets of the maximal or minimal *blocks* of mutually recursive equations in PMC.

▶ **Definition 2** (Equation Blocks). An equation block B over the set $\mathcal{X}_B = \{X_1, \ldots, X_N\} \subseteq \mathcal{X}$ of pairwise distinct variables has one of two forms – $\min\{E\}$ or $\max\{E\}$, where E is a system of (mutually recursive) equations such that for any $i, j \in \{1, \ldots, N\}$, ϕ_i is monotonic in X_j.

$$E : \quad \langle\ X_1 = \phi_1,\ \ldots,\ X_N = \phi_N\ \rangle\,.$$

If $B = \max\{E\}$ or $B = \min\{E\}$, the elements of $\mathcal{X}_B$ are called max-variables or min-variables respectively. Given the system E of equations in the previous definition, its *dual* is

$$\tilde{E} : \quad \langle\ X_1 = \neg\phi_1\{\neg X_1/X_1, \ldots, \neg X_N/X_N\},\ \ldots,\ X_N = \neg\phi_N\{\neg X_1/X_1, \ldots, \neg X_N/X_N\}\ \rangle\,.$$

If $B = \max\{E\}$ or $B = \min\{E\}$, then its *dual* is $\tilde{B} = \min\{\tilde{E}\}$ or $\tilde{B} = \max\{\tilde{E}\}$ respectively.

We say that a formula $\phi \in \mathcal{L}$ *depends on* B if it involves variables in $\mathcal{X}_B$. If $\mathcal{X}_B \cap \mathcal{X}_{B'} = \emptyset$, we say that B is *dependent* on B' if the right hand side formulas of the equations in B depend on B'.

▶ **Definition 3** (Block Sequence). A sequence $\mathcal{B} = B_1, \ldots, B_m$ of $m \geq 1$ pairwise-distinct equation blocks is a *block sequence* if $\mathcal{X}_{B_i} \cap \mathcal{X}_{B_j} = \emptyset$ for $i \neq j$. A block sequence $\mathcal{B} = B_1, \ldots, B_m$ of $m \geq 1$ is called *alternation-free* if B_i is not dependent on B_j whenever $i < j$.

A formula $\phi \in \mathcal{L}$ is *dependent* on $\mathcal{B}$ if it is dependent of each block in the sequence.

The semantics of our calculus is defined in terms of (probabilistic) Markov processes [28].

▶ **Definition 4** (Markov Process). A *(probabilistic) Markov process* (PMP) is a tuple $\mathcal{M} = (M, \Sigma, l, \theta)$ with (M, Σ) an analytic measurable space[1] of states, $l : M \to 2^{\mathcal{A}}$ a labeling function associating a set of state labels (i.e., atomic propositions) to each state and $\theta : M \to \Pi(M, \Sigma)$ the transition function associating a probability measure over (M, Σ) to each state.

Given a PMP $\mathcal{M} = (M, \Sigma, l, \theta)$, an *environment* is a function $\rho : \mathcal{X} \to 2^M$ that interprets the recursive-variables as sets of states. We use 0 as the empty environment that associates $\emptyset$ to all recursive-variables. Given an environment ρ and $S \subseteq M$, let $\rho[X \mapsto S]$ be the environment that interprets X as S and all the other recursive-variables as ρ does. Similarly, for a pairwise-disjoint tuple $\overline{X} = (X_1, \ldots, X_N) \in \mathcal{X}^N$ and $\overline{S} = (S_1, \ldots, S_N) \subseteq M^N$, let $\rho[\overline{X} \mapsto \overline{S}]$ be the environment that interprets X_i as S_i for all $i = 1, \ldots, N$ and all the other variables as ρ does.

[1] An analytic space1 is a continuous image of a Polish space in a Polish space; a Polish space is the topological space underlying a complete separable metric space.

Given a PMP $\mathcal{M} = (M, \Sigma, l, \theta)$ and an environment ρ, the semantics for the basic formulas in $\mathcal{L}$ is defined, on top of the classic semantics for Boolean logic, inductively as follows,

$\mathcal{M}, m, \rho \models p$ iff $p \in l(m)$;

$\mathcal{M}, m, \rho \models \neg\phi$ iff $\mathcal{M}, m, \rho \not\models \phi$;

$\mathcal{M}, m, \rho \models \phi_1 \vee \phi_2$ iff $\mathcal{M}, m, \rho \models \phi_1$ or $\mathcal{M}, m, \rho \models \phi_2$;

$\mathcal{M}, m, \rho \models X$ iff $m \in \rho(X)$;

$\mathcal{M}, m, \rho \models \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \geq r$ iff $\Sigma_{i=1}^{n} a_i\theta(m)([\![\phi_i]\!]_\rho^{\mathcal{M}}) \geq r$,

where $[\![\phi]\!]_\rho^{\mathcal{M}} = \{m \in M \mid \mathcal{M}, m, \rho \models \phi\}$.

Following [19, 9, 10, 2], we extend now the semantics to include the restrictions imposed by a sequence of blocks and obtain the so-called block-semantics.

Given a set of equations E with $\overline{X} = (X_1, \ldots, X_N)$, an environment ρ and $\overline{\Upsilon} = (\Upsilon_1, \ldots, \Upsilon_N) \subseteq M^N$, let the function $f_E^\rho : (2^M)^N \longrightarrow (2^M)^N$ be defined as: $f_E^\rho(\overline{\Upsilon}) = \langle [\![\phi_1]\!]_{\rho[\overline{X}\mapsto\overline{\Upsilon}]}, \ldots, [\![\phi_N]\!]_{\rho[\overline{X}\mapsto\overline{\Upsilon}]}\rangle$.

Observe that $(2^M)^N$ forms a complete lattice with the ordering, join and meet operations defined as the point-wise extensions of the set-theoretic inclusion, union and intersection, respectively. Moreover, for any E and ρ, f_E^ρ is monotonic with respect to the order of the lattice and therefore, it has a greatest fixed point denoted by $\nu\overline{X}.f_E^\rho$ and a least fixed point denoted by $\mu\overline{X}.f_E^\rho$ [9]. These can be characterized as: $\nu\overline{X}.f_E^\rho = \bigcup\{\overline{\Upsilon} \mid \overline{\Upsilon} \subseteq f_E^\rho(\overline{\Upsilon})\}$, $\mu\overline{X}.f_E^\rho = \bigcap\{\overline{\Upsilon} \mid f_E^\rho(\overline{\Upsilon}) \subseteq \overline{\Upsilon}\}$.

The blocks $\max\{E\}$ and $\min\{E\}$ define environments that satisfy all the equations in E; $\max\{E\}$ is the greatest fixed point and $\min\{E\}$ is the least fixed point. The environment defined by the block B is denoted by $[\![B]\!]_\rho$. Given a block sequence $\mathcal{B} = B_1, \ldots, B_m$ and an environment ρ_0, let $\rho_1, \ldots, \rho_m$ be defined by $\rho_i = [\![B_i]\!]_{\rho_{i-1}}$ for $i = 1, \ldots, m$. The semantics of $\mathcal{B}$ is then given by

$$[\![\mathcal{B}]\!]_{\rho_0} = \rho_m.$$

▶ **Definition 5** (Block-Semantics)**.** Given a block sequence $\mathcal{B}$, the $\mathcal{B}$-semantics of a formula $\phi \in \mathcal{L}$ that depends on $\mathcal{B}$ is given for a PMP $\mathcal{M} = (M, \Sigma, l, \theta)$ with $m \in M$ and an environment ρ, as follows,

$$\mathcal{M}, m, \rho \models_{\mathcal{B}} \phi \text{ iff } \mathcal{M}, m, [\![\mathcal{B}]\!]_\rho \models \phi.$$

We say that a formula ϕ is *$\mathcal{B}$-satisfiable* if there exists at least one PMP that satisfies it for the block sequence $\mathcal{B}$ in one of its states under some environment; ϕ is a *$\mathcal{B}$-validity*, written $\models_{\mathcal{B}} \phi$, if it is satisfied for $\mathcal{B}$ in all states of any PMP under any environment.

▶ **Example 6.** Suppose a file is divided into n blocks that are distributed among several peers in a peer-to-peer network. When a user wants to get the complete file from the network, he needs to download all n blocks. When the user tries to download a block, there are three possibilities: (1) he gets the block successfully and he will try to download the next block (with probability 0.6); (2) the block is not available anymore, in which case it is not possible to get the complete file (with probability 0.1); (3) the peer did not response within a time limit and the user retries (with probability 0.3). To simplify the example, we assume that only one block can be downloaded at one time. The system (1) in Figure 1 is one of this type.

Consider the safety property that "it will never FAIL to get the file" as shown in the system (2) in Figure 1. In the non-probabilistic case, this can be specified by the *mu*-calculus

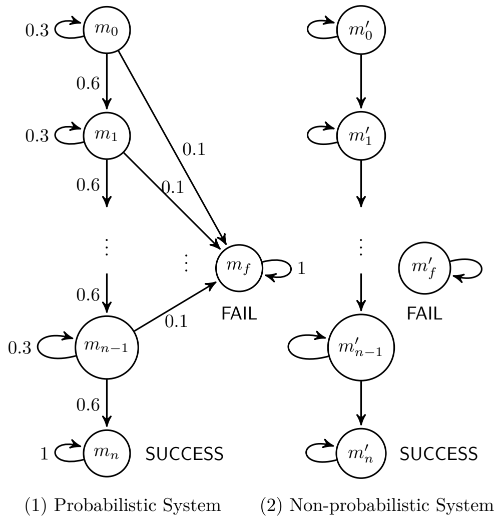

Figure 1 Peer-to-peer file sharing network.

formula ϕ:

$$\phi = \mathsf{SUCCESS} \vee X$$
$$B = \max \left\{ \begin{matrix} X & = & \neg\mathsf{FAIL} \wedge \neg\mathsf{SUCCESS} \\ & & \wedge(\langle\rangle\mathsf{SUCCESS} \vee \langle\rangle X) \end{matrix} \right\},$$

where ϕ is satisfied by $m'_0, \ldots, m'_n$ and X is satisfied by $m'_0, \ldots, m'_{n-1}$.

Consider the probabilistic safety property that "at any moment, the probability of FAIL to get the file is less than or equal to 0.1". This requirements can be expressed in PMC as:

$$\phi = \mathsf{SUCCESS} \vee X$$
$$B = \max \left\{ \begin{matrix} X \quad = \quad \neg\mathsf{FAIL} \wedge \neg\mathsf{SUCCESS} \wedge \\ (\langle (x_1 \rangle \mathsf{SUCCESS}, \langle x_2 \rangle X : x_1 + x_2 \geq 0.9) \end{matrix} \right\},$$

where ϕ is satisfied by $m'_0, \ldots, m'_n$ and X by $m'_0, \ldots, m'_{n-1}$. Notice that, they still hold when the system is infinite, i.e., n goes to $+\infty$.

3 Decidability and finite model property

In this section, we prove that the $\mathcal{B}$-satisfiability problem of PMC is decidable, i.e., it is decidable whether a given formula ϕ of PMC which is closed w.r.t. a block sequence $\mathcal{B}$ is satisfiable. We do this by involving the tableau construction [33, 15, 34] that will eventually help us constructing a model for ϕ. We show that PMC enjoys the finite model property and present a decision procedure. This work is done for the entire PMC and not only for the alternation-free fragment.

Given a formula ϕ dependent on $\mathcal{B}$, the construction of the model follows 4 steps (in brief):

1. Find the so-called co-prime formula ϕ^c for ϕ, which has a special format and admits the same models as ϕ. Similarly, we construct a co-prime block sequence $\mathcal{B}^c$. Both ϕ^c and $\mathcal{B}^c$ only involve integer inequalities with co-prime coefficients in the equational modalities.
2. Construct a set of formulas which is vital in constructing the tableau for ϕ^c. In contrast to that of the classical μ-calculus, this set not only contains the subformulas of ϕ^c and $\mathcal{B}^c$

but also it is still a finite set of formulas. This construction involves complex continuity arguments on rationals and this makes it particularly different from any similar techniques used previously with μ-calculi. The basic idea behind it is that every rational inequality (system) has (at least) one rational solution.

3. Construct a tableau for ϕ^c by adapting the classical tableau method. They key here is to use maximal sets as nodes, in order to get the probability distributions over the state space.
4. The tableau provides a PMP, which is also a model for ϕ^c, hence also for ϕ.

▶ **Definition 7** (Co-Prime). A block sequence $\mathcal{B}$ (a formula $\phi \in \mathcal{L}$ dependent on $\mathcal{B}$) is said to be *co-prime* iff for any $\overline{\langle x\rangle\psi\colon \overline{a}\cdot\overline{x} \geq r}$ that appears in $\mathcal{B}$ (in ϕ or $\mathcal{B}$), $a_1, \ldots, a_n$ are co-prime integers.

For any inequality $\Sigma_{i=1}^{n} a_i x_i \geq r$, one can divide both sides of the inequality by the greatest common divisor of $a_1, \ldots, a_n$ to get an inequality that has the same solution. Hence, for any block sequence (formula), one can get its *co-prime block sequence* (*co-prime formula*) by changing all inequalities in it by the above mentioned method.

Properties:

1. For any block sequence $\mathcal{B}$, there exists a unique co-prime block sequence denoted by $\mathcal{B}^c$; for any formula ϕ, there exists a unique co-prime formula denoted by ϕ^c.
2. For any formula $\langle x\rangle\phi : ax \geq r$, $(\langle x\rangle\phi : ax \geq r)^c \in \{\langle x\rangle\phi : x \leq \frac{r}{a}, \langle x\rangle\phi : x \geq \frac{r}{a}\}$.

▶ **Proposition 8.** *For any $\phi \in \mathcal{L}$ dependent on $\mathcal{B}$ and its co-prime formula ϕ^c, any model satisfying one also satisfies the other, i.e., for any PMP $\mathcal{M} = (M, \Sigma, l, \theta)$ with $m \in M$ and any environment ρ,*

$$\mathcal{M}, m, \rho \models_{\mathcal{B}} \phi \text{ iff } \mathcal{M}, m, \rho \models_{\mathcal{B}^c} \phi^c.$$

Therefore, for solving the satisfiability problem of a formula, it is sufficient to solve the satisfiability problem of its co-prime formula.

Consider $\phi \in \mathcal{L}$ dependent on $\mathcal{B}$. The set of all the recursive-variables in ϕ and $\mathcal{B}$ is denoted $\mathcal{X}[\phi, \mathcal{B}]$. Let $R[\phi, \mathcal{B}] \subseteq \mathbb{Q}$ be the set of all rationals in ϕ or $\mathcal{B}$; let $R^*[\phi, \mathcal{B}] \subseteq \mathbb{Q}$ be the set of all $\frac{r}{a_i}$ s.t. $\overline{\langle x\rangle\psi\colon (a_1, .., a_n)\cdot\overline{x} \geq r}$ appears in ϕ or $\mathcal{B}$ and $a_i \neq 0$. Obviously, $R[\phi, \mathcal{B}]$ and $R^*[\phi, \mathcal{B}]$ are both finite.

- The *granularity* of ϕ dependent on $\mathcal{B}$, denoted by $gr(\phi, \mathcal{B})$, is the least common denominator of the elements of $R^*[\phi, \mathcal{B}]$. Let $I[\phi, \mathcal{B}]$ be the set of all rationals of type $\frac{p}{gr(\phi,\mathcal{B})}$ in the interval $[\min(R^*[\phi, \mathcal{B}]), \max(R^*[\phi, \mathcal{B}])]$, for $p \in \mathbb{Z}$. Notice that $I[\phi, \mathcal{B}] = \emptyset$ whenever $R^*[\phi, \mathcal{B}] = \emptyset$.
- The *modal depth* of ϕ dependent on $\mathcal{B}$, denoted by $md(\phi, \mathcal{B})$, is defined inductively by

$$md(\phi, \mathcal{B}) = \begin{cases} 0, & \text{if } \phi = p \text{ or } \phi = X \\ md(\psi, \mathcal{B}), & \text{if } \phi = \neg\psi \\ \max\{md(\psi), md(\psi')\}, & \text{if } \phi = \psi \vee \psi' \\ \max\{md(\psi_i) \mid i = 1, \ldots, n\} + 1, & \text{if } \phi = \overline{\langle x\rangle\psi\colon \overline{a}\cdot\overline{x} \geq r} \end{cases}$$

- The *modality length* of ϕ dependent on $\mathcal{B}$, denoted by $ml(\phi, \mathcal{B})$, is largest length of the sub-formula $\overline{\langle x\rangle\psi\colon \overline{a}\cdot\overline{x} \geq r}$ that appears in ϕ or $\mathcal{B}$.

In the following, we fix a co-prime formula $\phi^c \in \mathcal{L}$ dependent on a co-prime block sequence $\mathcal{B}^c$ and we construct a model for it. Let

$$(\circledast) \qquad \mathcal{L}[\phi^c, \mathcal{B}^c] = \{\phi \in \mathcal{L} \mid \mathcal{X}[\phi, \mathcal{B}^c] \subseteq \mathcal{X}[\phi^c, \mathcal{B}^c], R[\phi, \mathcal{B}] \subseteq R[\phi^c, \mathcal{B}^c], \\ I[\phi, \mathcal{B}] \subseteq I[\phi^c, \mathcal{B}^c], md(\phi, \mathcal{B}^c) \leq md(\phi^c, \mathcal{B}^c), ml(\phi, \mathcal{B}^c) \leq ml(\phi^c, \mathcal{B}^c)\}.$$

The classical construction will take sets of formulas from the set $\mathcal{L}[\phi^c, \mathcal{B}^c]$, which are propositional maximal as defined in the next definition. However, in our setting, the set $\mathcal{L}[\phi^c, \mathcal{B}^c]$ does not contain enough quantitative information for constructing the model yet. Therefore, there are two extension steps to gather all the quantitative information to get the right candidate for the states of the model, which are quantitative maximal and quantitative complete as defined in Definition 10 and Definition 11. This information will make sure that we are able to find the rational solutions for all the inequalities, which will be used to define the probabilities on the transitions.

▶ **Definition 9** (Propositional Maximal Set). A set $\Lambda \subseteq \mathcal{L}[\phi^c, \mathcal{B}^c]$ is *(propositional) maximal* iff:

1. if $\phi \in \Lambda$, then $\neg\phi \notin \Lambda$; if $\phi \vee \psi \in \Lambda$, then $\phi \in \Lambda$ or $\psi \in \Lambda$; if $X \in \Lambda$ and $X = \phi \in \mathcal{B}^c$, then $\phi \in \Lambda$;
2. for all $\phi \in \mathcal{L}[\phi^c, \mathcal{B}^c]$, $\langle x\rangle\phi : x \geq 0 \in \Lambda$ and $\langle x\rangle\phi : x \leq 1 \in \Lambda$;
3. if $\overline{\langle x\rangle\psi}\colon \overline{a}\cdot\overline{x} \lhd r \in \Lambda$, then $\overline{\langle x\rangle\psi}\colon \overline{a}\cdot\overline{x} \unlhd r \in \Lambda$.

Let $\Pi[\phi^c, \mathcal{B}^c]$ the set of all the (propositional) maximal sets of $\mathcal{L}[\phi^c, \mathcal{B}^c]$. Since $\mathcal{L}[\phi^c, \mathcal{B}^c]$ is finite, $\Pi[\phi^c, \mathcal{B}^c]$ is finite and any $\Lambda \in \Pi[\phi^c, \mathcal{B}^c]$ is finite. As we mentioned earlier, $\mathcal{L}[\phi^c, \mathcal{B}^c]$ is not sufficient for constructing the model, so we will extend $\mathcal{L}[\phi^c, \mathcal{B}^c]$ and $\Pi[\phi^c, \mathcal{B}^c]$ in two steps. Firstly, $\Lambda\in \Pi[\phi^c, \mathcal{B}^c]$ is not quantitative maximized defined as follows:

▶ **Definition 10** (Quantitatively Maximized Set). A set $A \subseteq \mathcal{L}$ is *quantitatively maximized* iff

1. if $\langle x\rangle\phi : x \unlhd r \in A$, then $\langle x\rangle\neg\phi : x \unrhd 1 - r \in A$;
2. if $\langle x\rangle(\phi \wedge \psi) : x \unlhd r_1 \in A$ and $\langle x\rangle(\phi \wedge \neg\psi) : x \unlhd r_2 \in A$, then $\langle x\rangle\phi : x \unlhd r_1 + r_2 \in A$;
3. if $\langle x_n\rangle\phi_n : x_n \unrhd r_n \in A$, $\overline{\langle x\rangle\phi}\colon \overline{ax} \unlhd r \in A$ and $a_n \geq 0$, then $\overline{\langle x\rangle\phi}\big|_{n-1}\colon \overline{ax}\big|_{n-1} \unlhd r - a_n r_n \in A$;
4. if $\langle x_n\rangle\phi_n : x_n \unrhd r_n \in A$, $\overline{\langle x\rangle\phi}\colon \overline{ax} \lhd r \in A$ and $a_n \geq 0$, then $\overline{\langle x\rangle\phi}\big|_{n-1}\colon \overline{ax}\big|_{n-1} \lhd r - a_n r_n \in A$;
5. if $\langle x_n\rangle\phi_n : x_n \unlhd r_n \in A$, $\overline{\langle x\rangle\phi}\colon \overline{ax} \unlhd r \in A$ and $a_n \leq 0$, then $\overline{\langle x\rangle\phi}\big|_{n-1}\colon \overline{ax}\big|_{n-1} \unlhd r - a_n r_n \in A$;
6. if $\langle x_n\rangle\phi_n : x_n \unlhd r_n \in A$, $\overline{\langle x\rangle\phi}\colon \overline{ax} \lhd r \in A$ and $a_n \leq 0$, then $\overline{\langle x\rangle\phi}\big|_{n-1}\colon \overline{ax}\big|_{n-1} \lhd r - a_n r_n \in A$.

The quantitative maximization extends the lower bound and upper bound of all the rationals considered. This makes sure that all the numbers related to the given formula are included. These numbers are needed in order to find all the solutions for the inequalities in ϕ^c and $\mathcal{B}^c$.

Extension Step I: Let $\frac{p_{max}}{gr(\phi^c, \mathcal{B}^c)}, \frac{p_{min}}{gr(\phi^c, \mathcal{B}^c)}$ with $p_{max}, p_{min} \in \mathbb{Z}$ and $\max', \min' \in \mathbb{Q}$ be such that if the conditions 1 – 4 below are satisfied, then for any $\Lambda \in \Pi[\phi^c, \mathcal{B}^c]$, there exists $\Lambda' \in \Pi'[\phi^c, \mathcal{B}^c]$ such that $\Lambda \subseteq \Lambda'$ and Λ' is quantitatively maximized,

1. $I'[\phi^c, \mathcal{B}^c]$ be the set of all $\frac{p}{gr(\phi^c, \mathcal{B}^c)}$ in the interval $[\frac{p_{max}}{gr(\phi^c, \mathcal{B}^c)}, \frac{p_{min}}{gr(\phi^c, \mathcal{B}^c)}]$ for any $p \in \mathbb{Z}$;
2. $R'[\phi^c, \mathcal{B}^c] = \{r \in \mathbb{Q} \mid \min' \leq r \leq \max'\}$;
3. $\mathcal{L}'[\phi^c, \mathcal{B}^c] \supseteq \mathcal{L}[\phi^c, \mathcal{B}^c]$ is the set of formulas defined as ($\circledast$) based on $I'[\phi^c, \mathcal{B}^c]$ and $R'[\phi^c, \mathcal{B}^c]$;
4. $\Pi'[\phi^c, \mathcal{B}^c]$ the set of the propositional maximal sets of $\mathcal{L}'[\phi^c, \mathcal{B}^c]$.

In order to find the maximal number related to 2 in Definition 10, one can start with adding the $\langle x\rangle\phi : x \unlhd r_1 + r_2$ and its negation which are not in $\mathcal{L}[\phi^c, \mathcal{B}^c]$ to get $\mathcal{L}'[\phi^c, \mathcal{B}^c]$ and continue doing the same to the new $\mathcal{L}'[\phi^c, \mathcal{B}^c]$. Since $\mathcal{L}[\phi^c, \mathcal{B}^c]$ is finite, this procedure will terminate. Similarly one can do the same for the others and find the numbers. It is obvious that $\mathcal{L}'[\phi^c, \mathcal{B}^c]$ and $\Pi'[\phi^c, \mathcal{B}^c]$ are still finite. For any $\Lambda \in \Pi[\phi^c, \mathcal{B}^c]$, choose $\Lambda' \in \Pi'[\phi^c, \mathcal{B}^c]$

Table 1 Tableau Rules.

$$(\wedge)\ \frac{\{\phi_1,\phi_2,\Delta\}\subseteq\Lambda^+}{\{\phi_1\wedge\phi_2,\Delta\}\subseteq\Lambda^+}\qquad(\vee)\ \frac{\{\phi_i,\Delta\}\subseteq\Lambda^+}{\{\phi_1\vee\phi_2,\Delta\}\subseteq\Lambda^+}\ \ \phi_i\in\Lambda^+, i=1\text{ or }2\qquad(\text{Reg})\ \frac{\{\phi_X,\Delta\}\subseteq\Lambda^+}{\{X,\Delta\}\subseteq\Lambda^+}\ \ X=\phi_X\in\mathcal{B}$$

$$(\text{Mod})\ \frac{\Delta_1\subseteq\Lambda_1^+\ \cdots\ \Delta_k\subseteq\Lambda_k^+}{\Delta\subseteq\Lambda^+}\quad \emptyset\neq\Delta_j\subseteq\bigcup_{\overline{\langle x\rangle\phi}:\ \overline{a}\cdot\overline{x}\geq r\in\Delta}\{\phi_1,\ldots,\phi_n\}\subseteq\bigcup_{j=1,\ldots,k}\Delta_j$$

s.t. $\Lambda\subseteq\Lambda'$ and Λ' is quantitatively maximized. Let the set of the chosen Λ' be $\Omega'[\phi^c,\mathcal{B}^c]$, which is finite.

In order to define the distribution on the model correctly, we need to obtain more information about the maximal sets, which is the quantitative completeness defined as follows.

▶ **Definition 11** (Quantitatively Complete Set)**.** Given any finite set $\mathcal{L}^*\subseteq\mathcal{L}$. A propositional maximal set Λ^* of $\mathcal{L}^*$ is called quantitatively complete iff $ul^{\phi}_{\Lambda^*}=ur^{\phi}_{\Lambda^*}$ for any $\phi\in\mathcal{L}^*$, where

$$ul^{\phi}_{\Lambda^*}\ =\ \max\{r\in\mathbb{Q}\mid\langle x\rangle\phi: x\geq r\in\Lambda^*\},\qquad ur^{\phi}_{\Lambda^*}\ =\ \min\{s\in\mathbb{Q}\mid\langle x\rangle\phi: x\leq s\in\Lambda^*\}.$$

The above notion captures the accuracy of the rationals, which states how precise we can express in the logic. This makes sure that we include (at least) one rational solution for every inequality.

▶ **Lemma 12.** *For any $\phi\in\mathcal{L}'[\phi^c,\mathcal{B}^c]$ ($\mathcal{L}[\phi^c,\mathcal{B}^c]$) and any $\Lambda'\in\Omega'[\phi^c,\mathcal{B}^c]$ ($\Lambda\in\Pi[\phi^c,\mathcal{B}^c]$),*
1. *$ul^{\phi}_{\Lambda'},ur^{\phi}_{\Lambda'}\in[0,1]\cap\mathbb{Q}$;*
2. *either $ul^{\phi}_{\Lambda'}=ur^{\phi}_{\Lambda'}$ or $ul^{\phi}_{\Lambda'}+\frac{1}{gr(\phi^c,\mathcal{B}^c)}=ur^{\phi}_{\Lambda'}$.*

Extension Step II: Let $h\in\mathbb{N}$ be such that if the conditions $1-3$ below are satisfied, then for any $\Lambda'\in\Omega'[\phi^c,\mathcal{B}^c]$, there exists $\Lambda^+\in\Pi^+[\phi^c,\mathcal{B}^c]$ such that $\Lambda'\subseteq\Lambda^+$ and Λ^+ is quantitatively complete.
1. $gr^+(\phi^c,\mathcal{B}^c)=gr(\phi,\mathcal{B})\cdot 2^h$;
2. $\mathcal{L}^+[\phi^c,\mathcal{B}^c]\supseteq\mathcal{L}'[\phi^c,\mathcal{B}^c]$ is the set of formulas defined as (⊛) based on $gr^+(\phi^c,\mathcal{B}^c)$;
3. $\Pi^+[\phi^c,\mathcal{B}^c]$ the set of the propositional maximal sets of $\mathcal{L}^+[\phi^c,\mathcal{B}^c]$.

Since $\mathcal{L}'[\phi^c,\mathcal{B}^c]$ is finite and all the numbers in the constraints on the quantitative variables are rationals, there exist rational solutions for the inequality systems. Hence, we can find such an h in finitely many steps by multiplying the granularity by 2 every time. Obviously $\mathcal{L}^+[\phi^c,\mathcal{B}^c]$ and $\Pi^+[\phi^c,\mathcal{B}^c]$ are finite. For any $\Lambda'\in\Omega'[\phi^c,\mathcal{B}^c]$, choose $\Lambda^+\in\Pi^+[\phi^c,\mathcal{B}^c]$ s.t. $\Lambda'\subseteq\Lambda^+$ and Λ^+ is quantitatively complete. Let the set of the chosen Λ^+ be $\Omega^+[\phi^c,\mathcal{B}^c]$.

▶ **Lemma 13.** *For any $\phi\in\mathcal{L}^+[\phi^c,\mathcal{B}^c]$ and any $\Lambda^+\in\Omega^+[\phi^c,\mathcal{B}^c]$,*

$$ul^{\phi}_{\Lambda^+}=ur^{\phi}_{\Lambda^+}\in[0,1]\cap\mathbb{Q}.$$

In what follows, let $u^{\phi}_{\Lambda^+}=ul^{\phi}_{\Lambda^+}=ur^{\phi}_{\Lambda^+}$. Now we are ready to construct a model for ϕ^c dependent on $\mathcal{B}^c$. We construct a tableau $\mathcal{T}[\phi^c,\mathcal{B}^c]$ for ϕ^c with $\Lambda^+\subseteq\Omega^+[\phi^c,\mathcal{B}^c]$ as the nodes. The reason for here, unlike in the standard construction [33, 15, 34], we consider Λ^+ as a node is because we need to derive information about probabilities from the nodes. The tableau rules are listed in Table 1, where $\Delta\subseteq\Lambda^+$ denotes Λ^+ including Δ and $\{\phi,\Delta\}$ denotes $\{\phi\}\cup\Delta$.

If (Mod) is applied at node t, the nodes $\Delta_j\subseteq\Lambda^+_j$ obtained from $\overline{\langle x\rangle\phi}$: $\overline{a}\cdot\overline{x}\geq r$ s.t. $\phi_i\in\Delta_j$ are called *ϕ_i-sons* of t. The tableaux may be infinite. However, because $\Omega^+[\phi^c,\mathcal{B}^c]$

and any $\Lambda^+ \in \Omega^+[\phi^c, \mathcal{B}^c]$ are both finite, the nodes of the type $\Delta \subseteq \Lambda^+$ appear in $\mathcal{T}[\phi^c, \mathcal{B}^c]$ are finitely many.

As in the classic method for μ-calculus [33, 15, 34], we use max-*trace*, min-*trace* to capture the idea of a history of the regeneration of a formula (similar to the classic definitions and presented in the full version of the paper). We adapt the notions of *markings*, *consistent markings* to the probability case to characterize $\mathcal{B}$-satisfiability of a formula in a state of a PMP.

▶ **Definition 14** (Marking). For a tableau $\mathcal{T}$, we define its *marking* with respect to a PMP $\mathcal{M} = (M, \Sigma, l, \theta)$ and state $m_0 \in M$ to be a relation $\mathfrak{M} \subseteq M \times \mathcal{T}$ satisfying the following conditions:

(i) $(m_0, t_0) \in \mathfrak{M}$, where t_0 is the root of $\mathcal{T}$;
(ii) if $(m, t) \in \mathfrak{M}$ and a rule other than (Mod) was applied at t, then for the son t' of t, $(m, t') \in \mathfrak{M}$;
(iii) if $(m, t) \in \mathfrak{M}$ with $t = (\Delta \subseteq \Lambda^+)$ and rule (Mod) was applied at t, then for any $\overline{\langle x \rangle \phi}\colon \overline{a} \cdot \overline{x} \geq r \in \Delta$, there exists $F_1, \ldots, F_n \subseteq M$ s.t. for any $i = 1, \ldots, n$:
  (a) for every ϕ_i-son t' of t, there exists a state $m' \in F_i$ s.t. $(m', t') \in \mathfrak{M}$, and
  (b) for every state $m' \in F_i$, there exists a ϕ_i-son t' of t s.t. $(m', t') \in \mathfrak{M}$, and
  (c) $u^{\phi_i}_{\Lambda^+} = \theta(m)(F_i)$.

▶ **Definition 15** (Consistent Marking). A marking $\mathfrak{M}$ of $\mathcal{T}$ is *consistent* with respect to $\mathcal{M} = (M, \Sigma, l, \theta)$ and $m_0 \in M$, if and only if $\mathfrak{M}$ satisfies the following conditions:

- *local consistency*: for any node $t = (\Delta \subseteq \Lambda^+) \in \mathcal{T}$ and state $m \in M$, if $(m, t) \in \mathfrak{M}$ then for any $\psi \in \Delta$, $\mathcal{M}, m, 0 \models_{\mathcal{B}} \psi$;
- *global consistency*: for every path $\mathcal{P} = t_0, t_1, \ldots$ of $\mathcal{T}$ s.t. there exist π_i with $(\pi_i, t_i) \in \mathfrak{M}$ for $i = 0, 1, \ldots$, there is no min-trace on $\mathcal{P}$.

▶ **Lemma 16.** *ϕ^c is satisfied at state m_0 in a PMP $\mathcal{M} = (M, \Sigma, l, \theta)$ if and only if there is a consistent marking of $\mathcal{T}[\phi^c, \mathcal{B}^c]$ with respect to $\mathcal{M}$ and m_0.*

The proof of Lemma 16 relies on notion of *signature*, similar to that considered by Streett and Emerson [33]. These notions come from the characterization of fixed point formulas by means of transfinite chains of approximations, which have been extended to the setting with fixed points defined with blocks in [9, 10]. Involving these, the previous lemma is proven similarly to the case of classic μ-calculus [33, 15, 34]. The correctness of the cases with probability is guaranteed by the quantitative maximatization and quantitative completeness defined in Definition 10 and 11.

This lemma allows us to prove the finite model property for PMC, by following the classic proof strategy of [15]; the only difference consists in managing the probability modalities.

▶ **Theorem 17** (Finite Model Property). *Let $\phi_0 \in \mathcal{L}$ be a formula that depends of $\mathcal{B}_0$. If ϕ_0 is $\mathcal{B}_0$-satisfiable, then there exists a finite PMP $\mathcal{M}^f = (M^f, \Sigma^f, \theta^f)$ with $m^f \in M^f$ and an environment ρ^f such that $\mathcal{M}^f, m^f, \rho^f \models_{\mathcal{B}_0} \phi_0$.*

According to Proposition 8, Lemma 16 and Theorem 17, we can obtain an algorithm to decide the satisfiability of a given PMC formula.

▶ **Algorithm.** *Given a PMC formula $\phi_0 \in \mathcal{L}$ dependent on the block sequence $\mathcal{B}_0$, the algorithm constructs a finite PMP $\mathcal{M}^f = (M^f, \Sigma^f, l^f, \theta^f)$ and an environment ρ^f such that $\mathcal{M}^f, m_f, \rho^f \models_{\mathcal{B}_0} \phi_0$ in the following steps:*

1. *Construct the co-prime block sequence $\mathcal{B}^c_0$ of $\mathcal{B}_0$ and the co-prime formula ϕ^c_0 of ϕ_0.*
2. *Construct $\mathcal{L}[\phi^c_0, \mathcal{B}^c_0]$ and $\Pi[\phi^c_0, \mathcal{B}^c_0]$, which are finite.*

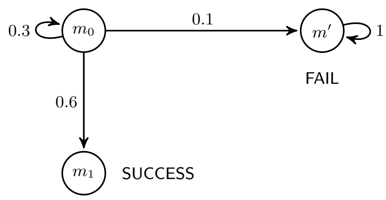

Figure 2 Small model Construction.

3. *Construct $\mathcal{L}'[\phi_0^c, \mathcal{B}_0^c]$ and $\Omega'[\phi_0^c, \mathcal{B}_0^c]$ by **Extension Step I**. $\mathcal{L}'[\phi_0^c, \mathcal{B}_0^c]$ and $\Omega'\phi_0^c, \mathcal{B}_0^c]$ are finite.*
4. *Construct $\mathcal{L}^+[\phi_0^c, \mathcal{B}_0^c]$ and $\Omega^+[\phi_0^c, \mathcal{B}_0^c]$ by **Extension Step II**. $\mathcal{L}^+[\phi_0^c, \mathcal{B}_0^c]$ and $\Omega^+[\phi_0^c, \mathcal{B}_0^c]$ are finite.*
5. *Construct the tableau $\mathcal{T}[\phi_0^c, \mathcal{B}_0^c]$ according to the rules in Table 1.*
6. *Construct $\mathcal{M}^f = (M^f, \Sigma^f, \theta^f)$ as follows:*
   - *M^f is the set of the nodes of $t \in \mathcal{T}[\phi_0^c, \mathcal{B}_0^c]$ such that either (Mod) is appiled at t or no rules are applicable at t (t is a leaf).*
   - *Let $(\!|\phi_i|\!) = \{\Lambda_i^+ \mid \phi_i \in \Lambda_i^+ \in \mathcal{M}^f\}$, $N = \{(\!|\psi|\!) \mid$ (Mod) is applied in $\mathcal{T}[\phi_0^c, \mathcal{B}_0^c]$ for $\overline{\langle x \rangle \phi}^n : \overline{a}$ $\overline{x} \geq r, \psi = \phi_i$ for some $i\}$. Then $\Sigma^f = \sigma(N)$.*
   - *l^f is defined as: for any $t = \Lambda^+ \in M^f$, $l^f(t) = \{p \in \mathcal{A} \mid p \in \Lambda^+\}$.*
   - *For $t = \Lambda^+ \in M^f$ where (Mod) is applied, let $\theta^f(t)((\!|\phi|\!)) = u^{\phi}_{\Lambda^+}$ for any $(\!|\phi|\!) \in N$.*

   Let $\rho^f(X) = \{\Lambda^+ \mid X \in \Lambda^+\}$ for $X \in \mathcal{X}$. By Theorem 17 $\mathcal{M}^f, t, \rho^f \models_{\mathcal{B}_0^c} \phi_0^c$ for $t = \Lambda^+$ s.t. $\phi_0^c \in \Lambda^+$.
7. *Therefore, $\mathcal{M}^f, t, \rho^f \models_{\mathcal{B}_0} \phi_0$, by Theorem 8.*

▶ **Example 18.** Consider the property in Example 6:

$$\phi = \mathsf{SUCCESS} \vee X$$
$$B = \max \left\{ \begin{array}{l} X \quad = \quad \neg\mathsf{FAIL} \wedge \neg\mathsf{SUCCESS} \\ \quad \wedge (\langle x_1 \rangle \mathsf{SUCCESS}, \langle x_2 \rangle X : x_1 + x_2 \geq 0.9) \end{array} \right\},$$

As discussed in Example 6, ϕ is satisfiable. We can use the above algorithm to construct a model for it (the smallest one), as shown in Figure 2. The detailed steps of construction is omitted here.

▶ **Theorem 19** (Decidability of $\mathcal{B}$-Satisfiability). *The $\mathcal{B}$-satisfiability problem for PMC is decidable.*

PMC can be used to approximate pCTL formulas with arbitrary precision when restricting to finite models. Our approximation is based on a partition $P : 0 < \pi_1 < \cdots < \pi_k < 1$ of $[0, 1]$. To (under-)approximate the pCTL formula $\phi_i = \mathbb{P}_{\geq \pi_i}(\phi_1 U \phi_2)$ in PMC, we define recusively:

$$B_P = \min\{X_i^u = \phi_2^u \vee (\phi_1^u \wedge \overline{\langle x_j \rangle X_j^u} : \overline{(x_j - x_{j+1})\pi_j} \geq \pi_i \mid i = 1 \ldots k\}.$$

Let S_i be the set of states satisfying the pCTL formula ϕ_i. Then the vector $\langle S_i : i = 1 \ldots k \rangle$ is a fixed point to the block B above, and it follows (from minimal fixed point semantics of B) that $X_i^u \Rightarrow \phi_i$. Thus successful application of our finite-model property construction to X_i^u will provide a model for ϕ_i as well. We conjecture that if there is a finite model $\mathcal{M}$ satisfying $\phi_i = \mathbb{P}_{\geq \pi}(\phi_1 U \phi_2)$, then for any $\epsilon > 0$ we can find a partitioning P such that $\mathcal{M} \models_{B_P} X_i^u$ where $\pi_i \geq \pi - \epsilon$. This will be an alternative to the construction for pCTL

Table 2 Axiomatic System of PMC basic formulas.

$$\begin{array}{ll}
\text{(A1):} & \vdash \langle x\rangle\phi : x \geq 0 \wedge \langle x\rangle\phi : x \leq 1 \\
\text{(A2):} & \vdash \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \geq r \vee \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \leq r \\
\text{(A3):} & \vdash \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \leq r \rightarrow \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} < s,\ r < s \\
\text{(A4):} & \vdash \neg(\overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \geq r) \leftrightarrow \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} < r \\
\text{(A5):} & \vdash \langle x\rangle\phi\colon a\cdot x \geq r \leftrightarrow \langle x\rangle\neg\phi\colon a\cdot x < a - r \\
\text{(A6):} & \vdash \langle x_1\rangle(\phi\wedge\psi) : x_1 \trianglelefteq r_1 \wedge \langle x_2\rangle(\phi\wedge\neg\psi) : x_2 \trianglelefteq r_2 \rightarrow \langle x\rangle\phi : x \trianglelefteq r_1 + r_2 \\
\text{(A7):} & \vdash \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \geq r \rightarrow \overline{\langle x\rangle\phi}\colon \alpha\cdot(\overline{a}\cdot\overline{x}) \geq \alpha r,\ \alpha \in \mathbb{Q}_{\geq 0} \\
\text{(A8):} & \vdash \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \geq r \wedge \overline{\langle x\rangle\phi}\colon \overline{b}\cdot\overline{x} \geq s \rightarrow \overline{\langle x\rangle\phi}\colon (\overline{a}+\overline{b})\cdot\overline{x} \geq r + s \\
\text{(A9):} & \text{if } a_n = 0, \text{ then } \vdash \overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \geq r \rightarrow \overline{\langle x\rangle\phi}\big|_{n-1}\colon \overline{a}\cdot\overline{x}\big|_{n-1} \geq r \\
\text{(R1):} & \text{if } \vdash \phi \leftrightarrow \psi, \text{ then } \vdash \langle x\rangle\phi : x \trianglelefteq r \leftrightarrow \langle x\rangle\psi : x \trianglelefteq r \\
\text{(R2):} & \{C[\overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \trianglelefteq r] \mid r \rhd s\} \vdash C[\overline{\langle x\rangle\phi}\colon \overline{a}\cdot\overline{x} \trianglelefteq s]
\end{array}$$

satisfiability problem in [1]. This also shows that, when restricting to finite models, even though we could not encode pCTL in PMC (e.g., the until operator), we could use a PMC theory (a (infinite) set of formulas) to approximate it.

4 Axiomatization for Alternation-free PMC

In this section, we propose an axiomatization for the validities of alternation-free fragment of PMC with respect to the PMP-semantics and prove it sound and (weak-)complete.

4.1 Sound axiomatization

In order to state the axioms for PMC we need to establish some notions. Let X be a metavariable quantifying over $\mathcal{L}$ and $\overline{\langle x\rangle\phi}(\mathbb{X}) = \langle x_1\rangle\phi_1, \ldots, \langle x_i\rangle[\mathbb{X}], \ldots, \langle x_n\rangle\phi_n$. For arbitrary sequences $\overline{\phi_j} = \phi_{j1}\ldots\phi_{jk_j}$ and $\overline{x_j} = x_{j1}\ldots x_{jk_j}$, $j = 1,\ldots,l$, we construct the following generic formula involving $\mathbb{X}$:

$$C[\mathbb{X}] = \overline{\langle x_1\rangle\phi_1}(\overline{\langle x_2\rangle\phi_2}(\cdots(\overline{\langle x_l\rangle\phi_l}(\mathbb{X}) : \overline{a_l}\cdot\overline{x_l} \geq r_l)\cdots) : \overline{a_2}\cdot\overline{x_2} \geq r_2) : \overline{a_1}\cdot\overline{x_1} \geq r_1.$$

We call $C[X]$ a *context*; it can be instantiated to a PMC formula $C[\phi]$ for $\phi \in \mathcal{L}$. Also $\epsilon[X]$ is a context - the empty one - and for $\phi \in \mathcal{L}$, $\epsilon[\phi] = \phi$. Notice that the metavariable $\mathbb{X}$ only appears once in the syntax of the context, i.e., we only consider contexts with one hole.

The axiomatization of PMC is given in two phases. Firstly, we provide axioms for deriving the validities that do not depend on sequences of blocks; and secondly, we extend the axiomatization to recursive constructs. The axioms and rules presented in Table 2 together with the axioms and the rules of propositional logic axiomatize a classic deducibility relation (see [12]) for the non-recursive validities of PMC denoted by $\vdash$. The axioms and the rules are stated for arbitrary $\phi,\psi \in \mathcal{L}$, $r,s \in \mathbb{Q}$, $x,y \in \mathcal{V}$ and arbitrary context $C[\mathbb{X}]$, where $\{\trianglelefteq,\trianglerighteq\} = \{\leq,\geq\}$ and $\rhd \in \{<,>\}$.

The axiom (A1) states that x is a probability. The axioms (A2)-(A3) state simple arithmetic facts. (A4) states that the dual of the equatioanl modality is itself. (A5) and (A6) state that the probability of reaching ϕ or $\neg\phi$ is 1. The axioms (A7)-(A9) show the arithmetic transformation of inequalities. The rule (R1) states that the probabilities of reaching two equivalent formulas are the same. (R2) is infinitary and encode the Archimedean properties of rational numbers.

Table 3 Axiomatic System of Maximal Equation Blocks.

(max-R1):	If $\vdash^* \phi$, then $\vdash^*_B \phi$
(max-A1):	$\vdash^*_B \bigwedge_{i=1,\dots,N}(X_i \to \phi_i)$
(max-R2):	If $\vdash^*_B \bigwedge_{i=1,\dots,N}(\psi_i \to \phi_i\{\overline{\Psi}/\overline{\mathcal{X}}\})$, then $\vdash^*_B \bigwedge_{i=1,\dots,N}(\psi_i \to X_i)$

Table 4 Axiomatic System of Minumum Equation Blocks.

(min-R1):	If $\vdash^* \phi$, then $\vdash^*_B \phi$
(min-A1):	$\vdash^*_B \bigwedge_{i=1,\dots,N}(\phi_i \to X_i)$
(min-R2):	If $\vdash^*_B \bigwedge_{i=1,\dots,N}(\phi_i\{\overline{\Psi}/\overline{\mathcal{X}}\} \to \psi_i)$, then $\vdash^*_B \bigwedge_{i=1,\dots,N}(X_i \to \psi_i)$

▶ **Theorem 20** (Soundness)**.** *The axiomatic system of* $\vdash$ *is sound, i.e., for arbitrary* $\phi \in \mathcal{L}$*,*

$\vdash \phi$ *implies* $\models \phi$.

Now we can proceed with the recursive constructs.

Given a maximal equation block $B = \max\{X_1 = \phi_1, \dots, X_N = \phi_N\}$ and an arbitrary classical deducibility relation $\vdash^*$, we define the deducibility relation $\vdash^*_B$ as the extension of $\vdash^*$ given by the axioms and rules in Table 3, which are the equation-version of the classic fixed points axioms of μ-calculus [15, 32, 29]. These are stated for arbitrary $\phi \in \mathcal{L}$ and $\overline{\Psi} = (\psi_1, \dots, \psi_N) \in \mathcal{L}^N$, where $\overline{X} = (X_1, \dots, X_N)$. Similarly, we define a classical deducibility relation $\vdash^*_B$ for a minimal equation block $B = \min\{X_1 = \phi_1, \dots, X_N = \phi_N\}$ based on $\vdash^*$ by using the axioms and rules in Table 4.

Given an alternation-free block sequence $\mathcal{B} = B_1, \dots, B_m$, we define the classical deducibility relations $\vdash_0, \vdash_1, \dots, \vdash_m$ as follows and consequently get $\vdash_{\mathcal{B}} = \vdash_m$.

$$\vdash_0 = \vdash; \qquad \vdash_i = \vdash^{i-1}_{B_i} \quad \text{for } i = 1, \dots, m$$

As usual, we say that a formula ϕ (or a set Φ of formulas) is $\mathcal{B}$-*provable*, denoted by $\vdash_{\mathcal{B}} \phi$ (respectively $\vdash_{\mathcal{B}} \Phi$), if it can be proven from the given axioms and rules of $\vdash_{\mathcal{B}}$. We denote by $\overline{\Psi} = \{\phi \in \mathcal{L} \mid \Psi \vdash_{\mathcal{B}} \phi\}$. An induction on the structure of the alternation-free blocks shows that all the theorems of $\vdash_{\mathcal{B}}$ are sound in the PMC-semantics.

▶ **Theorem 21** (Extended Soundness)**.** *The axiomatic system of* $\vdash_{\mathcal{B}}$ *is sound, i.e., for any* $\phi \in \mathcal{L}$*,*

$\vdash_{\mathcal{B}} \phi$ *implies* $\models_{\mathcal{B}} \phi$.

4.2 Completeness

In the rest of this section we prove that the axiomatic system of $\vdash_{\mathcal{B}}$ is not only sound, but also (weak-) complete, meaning that all the $\mathcal{B}$-validities can be proved, as theorems, from the proposed axioms and rules, i.e., for arbitrary $\phi \in \mathcal{L}$, $\models_{\mathcal{B}} \phi$ implies $\vdash_{\mathcal{B}} \phi$. To complete this proof it is sufficient to show that any $\mathcal{B}$-consistent formula has a model.

For some set $S \subseteq \mathcal{L}$, Φ is $(S, \mathcal{B})$-*maximally consistent* if it is $\mathcal{B}$-consistent and no formula in S can be added to Φ without making it inconsistent. Φ is $\mathcal{B}$-*maximally-consistent* if it is $(\mathcal{L}, \mathcal{B})$-maximally-consistent.

In the following we fix a consistent formula ϕ_0 depending on a fixed alternation-free sequence $\mathcal{B}_0$ and we construct a model. Let

$$(\circledast) \qquad \mathcal{L}[\phi_0, \mathcal{B}_0] \;=\; \{\phi \in \mathcal{L} \mid \mathcal{X}[\phi, \mathcal{B}_0] \subseteq \mathcal{X}[\phi_0, \mathcal{B}_0], R[\phi, \mathcal{B}] \subseteq R[\phi_0, \mathcal{B}_0], I[\phi, \mathcal{B}] \subseteq I[\phi_0, \mathcal{B}_0], md(\phi, \mathcal{B}_0) \leq md(\phi_0, \mathcal{B}_0), ml(\phi, \mathcal{B}_0) \leq ml(\phi_0, \mathcal{B}_0)\}$$

and $\Pi[\phi_0, \mathcal{B}_0]$ be the set of all the maximal consistent sets of $\mathcal{L}[\phi_0, \mathcal{B}_0]$. Similar to the arguments in Section 3, $\mathcal{L}[\phi_0, \mathcal{B}_0]$ and $\Pi[\phi_0, \mathcal{B}_0]$ are finite. Let Π be the set of the $\mathcal{L}$-maximal consistent sets.

Different from the model construction in Section 3, we take $\mathcal{L}$-maximally consistent sets as states. However, we don't take all the $\mathcal{L}$-maximally consistent sets as the state space, which are countably many. We will develop a finite state space as follows.

Since the set of instances of the infinitary rule in Table 2 is countable, we can use the *Rasiowa-Sikorski Lemma* [30, 12] to prove Lindenbaum's Lemma [11, 12] for PMC, following the technique in [16]. These lemmas are presented in the full version of the paper. Suppose that for each $\Lambda \in \Pi[\phi_0, \mathcal{B}_0]$ we chose one $\Gamma \in \Pi$ such that $\Lambda \subseteq \Gamma$ (Lindenbaum's Lemma); to identify it, we denote this Γ by Λ^e. Let $\Theta = \{\Lambda^e \in \Pi \mid \Lambda \in \Pi[\phi_0, \mathcal{B}_0]\}$. Since $\Pi[\phi_0, \mathcal{B}_0]$ is finite, Θ is obviously finite as well.

In what follows we will construct a PMP $\mathcal{M} = (\Theta, \Sigma, l, \theta)$ that satisfies ϕ_0 in one of its states. To do this, we have to properly define l, Σ and θ. l is defined as: $l(\Gamma) = \{p \in \mathcal{A} \mid p \in \Gamma\}$ for any $\Gamma \in \Theta$.

For defining Σ and θ, we firstly observe that given a $\mathcal{B}_0$-maximally-consistent set of formulas, the information contained about the resource-variable for a given formula is complete, in the sense that we can really identify its value, since any real number can be seen as the limit of some sequences of rational numbers. This is exactly what the next lemma states.

▶ **Lemma 22.** *For arbitrary* $\Gamma \in \Theta$ *and* $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$,

$$\sup\{r \in \mathbb{Q}_{\geq 0} \mid \langle x \rangle \phi : x \geq r \in \Gamma\} = \inf\{s \in \mathbb{Q} \mid \langle x \rangle \phi : x \leq s \in \Gamma\} \in \mathbb{R} \cup [0, 1].$$

▶ **Lemma 23.** *Let* $(\!|\phi|\!) = \{\Gamma \in \Theta \mid \phi \in \Gamma\}$ *and* $N = \{(\!|\phi|\!) \mid \phi \in \mathcal{L}[\phi_o, \mathcal{B}_0]\}$. *Then* $2^\Theta = N$.

Then let $\Sigma = \sigma(N)$, where $\sigma(N)$ is the least σ-algebra generated by N. Then the previous lemmas allow us to define, for any $\Gamma \in \Theta$ and $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$,

$$\theta(\Gamma)((\!|\phi|\!)) = \sup\{r \in \mathbb{Q}_{\geq 0} \mid \langle x \rangle \phi : x \geq r \in \Gamma\}.$$

$\theta(\Gamma)$ is a set function defined on the field N. According to Theorem 11.3 of [3] [2], $\theta(\Gamma)$ can be uniquely extended to a measure on Σ if it is finitely additive and countably subadditive on $(\!|\phi|\!)$. Since Θ is finite, we only need to prove that $\theta(\Gamma)$ is finitely additive, as stated in the following lemma. Notice also that since Θ is finite, (Θ, Σ) is an analytic space.

▶ **Lemma 24.** *For any* $\Gamma \in \Theta$*, the function* $\theta(\Gamma)$ *is finitely additive, i.e., for any* $(\!|\phi_1|\!)$ *and* $(\!|\phi_2|\!)$ *s.t.* $(\!|\phi_1|\!) \cap (\!|\phi_2|\!) = \emptyset$, $\theta(\Gamma)((\!|\phi_1|\!) \cup (\!|\phi_2|\!)) = \theta(\Gamma)((\!|\phi_1|\!)) + \theta(\Gamma)((\!|\phi_2|\!))$.

▶ **Theorem 25.** $\mathcal{M} = (\Theta, \Sigma, l, \theta)$ *is a probabilistic Markov process.*

Let ρ_0 be the environment defined as: for any $X \in \mathcal{X}$, by $\rho_0(X) = \{\Gamma \mid X \in \Gamma\}$.

Firstly, we prove the restricted truth lemma that does not consider recursive constructs.

▶ **Lemma 26** (Restricted Truth Lemma)**.** *For* $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$ *and* $\Gamma \in \Theta$,

$$\mathcal{M}, \Gamma, \rho_0 \models \phi \textit{ iff } \phi \in \Gamma.$$

On the restricted truth lemma we can base the following two results that indicate how we can extend the results to include the recursive cases, as developed in [18].

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[2] If $\mathcal{F} \subseteq 2^M$ is a field of sets and $\mu : \mathcal{F} \to \mathbb{R}_{\geq 0}$ is finitely additive and countably subadditive, then μ extends uniquely to a measure on $\sigma(\mathcal{F})$

▶ **Lemma 27.** *Let* $B = \max\{X_1 = \phi_1, \dots, X_N = \phi_N\}$ *be an equation block in the sequence* $\mathcal{B}_0$ *and* ρ *an environment such that* $\rho(X_i) = \{\Gamma \mid X_i \in \Gamma\}$ *for any* $i = 1, .., N$. *For any* $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$ *and* $\Gamma \in \Theta$,

$$\text{if } [\mathcal{M}, \Gamma, \rho \models \phi \text{ iff } \phi \in \Gamma], \quad \text{then } [\mathcal{M}, \Gamma, [\![B]\!]_\rho \models \phi \text{ iff } \phi \in \Gamma].$$

Since the minimal blocks are dual of the maximal ones, we have a similar lemma for minimal blocks.

▶ **Lemma 28.** *Let* $B = \min\{X_1 = \phi_1, \dots, X_N = \phi_N\}$ *be an equation block in the sequence* $\mathcal{B}_0$ *and* ρ *an environment such that* $\rho(X_i) = \{\Gamma \mid X_i \in \Gamma\}$ *for any* $i = 1, ..N$. *For any* $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$ *and* $\Gamma \in \Theta$,

$$\text{if } [\mathcal{M}, \Gamma, \rho \models \phi \text{ iff } \phi \in \Gamma], \quad \text{then } [\mathcal{M}, \Gamma, [\![B]\!]_\rho \models \phi \text{ iff } \phi \in \Gamma].$$

These lemmas allow us to prove the stronger version of the truth lemma.

▶ **Theorem 29** (Extended Truth Lemma)**.** *For* $\phi \in \mathcal{L}[\phi_0, \mathcal{B}_0]$ *and* $\Gamma \in \Theta$,

$$\mathcal{M}, \Gamma, \rho_0 \models_{\mathcal{B}} \phi \text{ iff } \phi \in \Gamma.$$

A direct consequence of Theorem 29 is the completeness[3] of the axiomatic system.

▶ **Theorem 30** (Completeness)**.** *The axiomatic system of* $\vdash_{\mathcal{B}}$ *is complete, i.e., for arbitrary* $\phi \in \mathcal{L}$,

$$\models_{\mathcal{B}} \phi \text{ implies } \vdash_{\mathcal{B}} \phi.$$

5 Conclusions

In this paper we have extended the probabilistic modal logic of [21], which is a modal logic allowing (in-)equational conditions on probabilities, with fixed point constructions in the form of block sequences, thus obtaining the probabilistic μ-calculus (PMC).

We prove that PMC enjoys the finite model property and its satisfiability problem is decidable. In order to do this, we involved the classic tableau construction that had to be adapted to the more challenging probabilistic settings. These results generalize previous results from [23] and recommend our logic as a good trade-off between expressiveness and decidability. The second key contribution of our paper is the sound-complete axiomatization that we propose for the alternation-free fragment of PMC. At the best of our knowledge, the problem of axiomatizing probabilistic μ-calculus has not been previously approached. The completeness proof is a non-standard extension of the filtration method, which can be easily adapted to other versions of probabilistic μ-calculus.

Unlike for the standard μ-calculus, the complexity of our algorithm is not clear. This is because for every formula, we only know that there exists the number h in **Extension Step II** such that all the inequalities in the given formula have rational solutions that can be expressed according to the accuracy defined, but we do not know how big h would be. The complexity of the satisfiability algorithm will be studied in the future work.

One might wonder whether there exists a finite axiomatization, as the model construction here is similar to that in Section 3 and the rules there are all finite. However, how we

[3] In this context by completeness we mean the weak-completeness. Since PMC is not compact, the weak- and strong-completeness do not coincide.

define the probability on the transition in Section 3 is using the truth that there is always rational solution(s) for any rational inequality. In [35], Zhou proved that there exists a finite axiomatization for Markov Logic by involving a finitary Archimedean rule (similar to our Rule (R2)). The idea there is similar to our satisfiability algorithm. We believe that similar arguments for finite axiomatization can be made for our logic as well by applying the Fourier–Motzkin elimination method [31] as in [35]. However, it is difficult to formalize this finite axiomatization. As we discussed in the last paragraph, we cannot know how precise we need to be in the logic in order to specify the solutions for the inequalities. Whether one can axiomatize and if yes how to will be interesting to look into.

Moreover, axiomatization for the full logic will also be considered. In the axiomatization here, the axioms and rules for fixed points are the same as those for the μ-calculus. Hence, for the full PMC, we believe that the axiomatization would look the same. However, the difficulty for proving the completeness will be at least that for the full modal μ-calculus [34].

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