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RAZUMIKHIN-TYPE THEOREMS ON POLYNOMIAL STABILITY OF HYBRID STOCHASTIC SYSTEMS WITH PANTOGRAPH DELAY

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Abstract. The main aim of this paper is to investigate the polynomial stability of hybrid stochastic systems with pantograph delay (HSSwPD). By using the Razumikhin technique and Lyapunov functions, we establish several Razumikhin-type theorems on the pth moment polynomial stability and almost sure polynomial stability for HSSwPD. For linear HSSwPD, sufficient conditions for polynomial stability are presented.

1. **Introduction.** Systems in many branches of science and industry do not only depend on the present state but also the past ones. Stochastic differential delay equations (SDDEs) have been widely used to model such systems. On the other hand, these systems may often experience abrupt changes in their structure and parameters and continuous time Markov chains have been used to model these abrupt changes. Hence, SDDEs with Markovian switching, known also as hybrid SDDEs, have appeared frequently in practice. The analysis and control of these systems involve investigating their stability, which is often regarded as one of the important issues of dynamical systems studied. There have been much work on the stability of hybrid SDDEs, we here mention Fei et al. [3], Hu et al. [8], Ji and Chizeck [11], Mao et al. [14, 16, 21, 32], Shaikhet [25], Wu et al. [30] and Zhang et al. [35] among others.

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It is well known that the classical and powerful techniques applied in the study of stability are based on the Lyapunov direct method. However, the greatest disadvantage of this approach is that it is difficult to construct the Lyapunov functionals effectively. In this case, the Razumikhin technique has been proposed and was used to study the stability of time delay systems. Razumikhin [24] was the first to develop this technique to study the stability of deterministic systems with delay. Mao [17, 18] extended this technique to stochastic systems with delay and employed the Razumikhin approach to study both pth moment and almost sure exponential stability for stochastic functional differential equations (SFDEs) and neutral SFDEs. After that, the Razumikhin technique has become very popular and there have appeared many results based on it to discuss the stability of stochastic systems with time delay, see for example, [9, 10, 15, 22, 23, 26, 28, 29, 33] and the many references therein.

On the other hand, it is noted that the literature cited above mainly focuses on pth moment and almost sure exponential stability. However, in many cases we may find that the equation is not exponentially stable, but the solution does tend to zero asymptotically. Consequently, it appears to be necessary to study other stability, for instance, polynomial stability. Mao [19, 20] considered the polynomial stability of stochastic system, which shows that solution tends to zero polynomially. Liu and Chen [12] studied the polynomial stability for stochastic differential equations (SDEs) with bounded delay. In recent years, Appleby and Buckwar [1] investigated the polynomial stability of SDEs with unbounded delay or pantograph delay. Milosevic [13] has established the sufficient conditions of almost sure polynomial stability of the solution for nonlinear SDEs with pantograph delay. Guo and Li [7] discuss the αth moment polynomial stability of the nonlinear SDEs with pantograph delay by using the Razumikhin-type technique.

Motivated by [19, 18, 12, 1, 13, 7], in this paper, we are concerned with the polynomial stability of hybrid stochastic system with pantograph delay

$$dx(t) = f(x(t), x(qt), t, r(t))dt + g(x(t), x(qt), t, r(t))dw(t), \tag{1}$$

where $0 < q < 1$, the coefficients $f : R^n \times R^n \times R_+ \times S \to R^n$ and $g : R^n \times R^n \times R_+ \times S \to R^{n\times m}$ are Borel-measurable, $S = \{1, 2, \cdots, N\}$, and $r(t)$ is a continuous-time Markov chain taking value in S, $w(t)$ is an m-dimensional Brownian motion. In fact, as a class of special stochastic delay systems, stochastic system with pantograph delay has been investigated by many scholars, we can refer to Baker and Buckwar [2], Fan et al. [5, 4], Xiao et al. [31], Guo and Li [6], Zhou and Xue [34], Shen et al. [27].

To the best of our knowledge, there are no results based on the Razumikhin technique referring to the polynomial stability of hybrid stochastic system with pantograph delay (1). The main aim of the present paper is to close this gap by extending the Razumikhin approach to the study of both the pth moment polynomial stability and almost sure polynomial stability for (1). Moreover, in [7], the authors only studied the polynomial stability in mean square, but in this paper, we shall also study the almost sure polynomial stability. On the other hand, since hybrid stochastic system (1) we are studied have continuous components as well as discrete components, their asymptotic behaviour is completely different from a single system of pantograph equation. Actually, (1) can be regarded as the result of the following N equations

$$dx(t) = f(x(t), x(qt), t, i)dt + g(x(t), x(qt), t, i)dw(t), \quad 1 \le i \le N \tag{2}$$

switching from one to the others according to the movement of the Markov chain. Due to the presence of the Markov chains, we will show that even if some subsystems (2) are not polymonially stable, the overall hybrid system (1) may still be polymonially stable.

In this paper, some preliminaries and notations on hybrid stochastic system with pantograph delay will be introduced in section 2. In sections 3, we will show the main results of our paper, where several useful criteria will be established on the polymonial stability in mean square as well as the almost sure polymonial stability for (1.1). In section 4, the general results established in sections 3 will be applied to deal with the polymonial stability of linear hybrid stochastic system with pantograph delay .

2. **Preliminaries.** Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions. Let $w(t)$ be an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. Let $t \geq t_0 > 0$ and $\mathbb{C}([qt, t]; R^n)$ denote the family of the continuous functions φ from $[qt, t] \to R^n$ with the norm $||\varphi|| = \sup\limits_{qt\leq\theta\leq t} |\varphi(\theta)|$, where $|.|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by $A^\top$. If A is a matrix, its norm $||A||$ is defined by $||A|| = \sup\{|Ax| : |x| = 1\}$. For $p > 0$, $L^p_{\mathcal{F}_t}([qt, t]; R^n)$ denote the family of all $(\mathcal{F}_t)$-measurable, $\mathbb{C}([qt; t], R^n)$-valued random variables $\varphi = \{\varphi(\theta) : qt \leq \theta \leq t\}$ such that $E||\varphi||^p < \infty$.

Let $r(t), t \geq 0$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, P)$ taking values in a finite state space $S = \{1, 2 \ldots N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by:

$$P(r(t + \Delta) = j|r(t) = i) = \begin{cases} \gamma_{ij}\Delta + \circ(\Delta), & if \quad i \neq j, \\ 1 + \gamma_{ij}\Delta + \circ(\Delta), & if \quad i = j. \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j, $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(.)$ is independent of the Brownian motion $w(.)$. Let us consider the nonlinear hybrid stochastic systems with pantograph delay

$$dx(t) \quad = \quad f(x(t), x(qt), t, r(t))dt + g(x(t), x(qt), t, r(t))dw(t), \; t \geq t_0 \qquad (3)$$

with initial data $\{x(t) : qt_0 \leq t \leq t_0\} = \xi \in L^p_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$. Here

$$f : R^n \times R^n \times [t_0, \infty) \times S \to R^n, \text{ and } g : R^n \times R^n \times [t_0, \infty) \times S \to R^{n\times m}.$$

In this paper, the following hypothesis are imposed on the coefficients f and g .

Assumption 2.1. For each integer $d \geq 1$, there exists a positive constant k_d such that

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)|^2 \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)|^2 \leq k_d(|x - \bar{x}|^2 + |y - \bar{y}|^2), \quad (4)$$

for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq d$ and $(t, i) \in [t_0, \infty) \times S$.

It is known that Assumption 2.1 only guarantees that (3) has a unique maximal solution, which may explode to infinity at a finite time. To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions.

Let $C(R^n \times [qt_0, \infty) \times S; R_+)$ denote the family of continuous functions from $R^n \times [qt_0, \infty) \times S$ to R_+. Also denote by $C^{2,1} = C^{2,1}(R^n \times [qt_0, \infty) \times S; R_+)$ the family of all continuous non-negative functions $V(x,t,i)$ defined on $R^n \times [qt_0,\infty) \times S$ such that they are continuously differentiable twice in x and once in t. Given $V \in C^{2,1}$, we define the function $LV : R^n \times R^n \times [qt_0,\infty) \times S \to R$ by

$$\begin{aligned} LV(x,y,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x,y,t,i) \\ &+ \frac{1}{2}trace[g^\top(x,y,t,i)V_{xx}(x,t,i)g(x,y,t,i)] + \sum_{j=1}^{N}\gamma_{ij}V(x,t,j), \end{aligned}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, \quad V_x(x,t,i) = \Big(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\Big),$$

$$V_{xx}(x,t,i) = \Big(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

Assumption 2.2. There exist a Lyapunov function $V \in C^{1,2}$ and some positive constants c_1, c_2, α_1 and α_2 such that for any $x, y \in R^n$, $t \geq qt_0$, and $i \in S$,

$$c_1|x|^p \leq V(x,t,i) \leq c_2|x|^p, \tag{5}$$

and

$$LV(x,y,t,i) \leq -\alpha_1|x|^p + \alpha_2 q|y|^p. \tag{6}$$

Now, we present the definitions of the pth moment polynomial stability and almost surely polynomial stability of hybrid stochastic systems with pantograph delay (3).

Definition 2.3. The solution of (3) is said to be pth moment polynomially stable if there exists a constant γ such that

$$\limsup_{t\to\infty} \frac{logE|x(t)|^p}{log(1+t)} < -\gamma. \tag{7}$$

Definition 2.4. The solution of (3) is said to be almost surely polynomially stable if there exists a constant $\bar{\gamma}$ such that

$$\limsup_{t\to\infty} \frac{log|x(t)|}{log(1+t)} < -\bar{\gamma} \quad a.s. \tag{8}$$

3. **Main results.** In this section, we shall investigate the pth moment polynomial stability and almost surely polynomial stability of (3) by using the Razumikhin-type technique. Before analyzing the stability, we firstly show that (3) has a global solution.

Theorem 3.1. *Let Assumptions 2.1 and 2.2 hold. Then for any given initial data ξ, there is a unique global solution $x(t)$ to (3) on $t \in [qt_0,\infty)$. Moreover, the solution has the property that*

$$E|x(t)|^p < \infty \tag{9}$$

for any $t \geq qt_0$.

Proof. Since the coefficients of (2.1) are locally Lipschitz continuous, for any given initial data ξ, there is a maximal local solution $x(t)$ on $t \in [t_0, \sigma_\infty)$, where σ_∞ is the explosion time. Let $k_0 > 0$ be sufficiently large for $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = inf\{t \in [t_0,\sigma_\infty) : |x(t)| \geq k\}.$$

Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = lim_{k\to\infty}\tau_k$, whence $\tau_\infty \le \sigma_\infty$ a.s. Note if we can show that $\tau_\infty = \infty$ a.s., then $\sigma_\infty = \infty$ a.s. So we just need to show that $\tau_\infty = \infty$ a.s. We shall first show that $\tau_\infty > \frac{t_0}{q}$ a.s. By the Itô formula (see e.g. [14]) and condition (6), we can show that, for any $k \ge k_0$ and $t_1 \ge t_0$,

$$\begin{aligned} EV(x(\tau_k\wedge t_1),\tau_k\wedge t_1, r(\tau_k\wedge t_1)) &\le EV(x(t_0),t_0,r(t_0)) \\ &+ E\int_{t_0}^{\tau_k\wedge t_1}\Big(-\alpha_1|x(t)|^p+\alpha_2 q|x(qt)|^p\Big)dt. \end{aligned} \tag{10}$$

Let us now restrict $t_1 \in [t_0, \frac{t_0}{q}]$. By condition (2.3), we then get

$$c_1E|x(\tau_k\wedge t_1)|^p \le H_1 - \alpha_1 E\int_{t_0}^{\tau_k\wedge t_1}|x(t)|^p dt, \tag{11}$$

where

$$\begin{aligned} H_1 &= c_2E|x(t_0)|^p + \alpha_2 qE\int_{t_0}^{\frac{t_0}{q}}|x(qt)|^p dt \\ &\le c_2E||\xi||^p + \alpha_2 E\int_{qt_0}^{t_0}|x(t)|^p dt < \infty. \end{aligned}$$

It then follows that

$$E|x(\tau_k\wedge t_1)|^p \le \frac{H_1}{c_1}, \quad t_0 \le t_1 \le \frac{t_0}{q} \tag{12}$$

for any $k \ge k_0$. In particular, $E|x(\tau_k \wedge \frac{t_0}{q})|^p \le \frac{H_1}{c_1}$, $\forall k \ge k_0$. This implies $k^pP(\tau_k \le \frac{t_0}{q}) \le \frac{H_1}{c_1}$. Letting $k\to\infty$, we hence obtain that $P(\tau_\infty \le \frac{t_0}{q}) = 0$, namely

$$P(\tau_\infty > \frac{t_0}{q}) = 1. \tag{13}$$

Letting $k \to \infty$ in (12) yields

$$E|x(t_1)|^p \le \frac{H_1}{c_1}, \quad t_0 \le t_1 \le \frac{t_0}{q}. \tag{14}$$

Let us now proceed to prove $\tau_\infty > \frac{t_0}{q^2}$ a.s. given that we have shown (13)-(14). For any $k \ge k_0$ and $t_1 \in [t_0, \frac{t_0}{q^2}]$, it follows from (10) that

$$c_1E|x(\tau_k\wedge t_1)|^p \le H_2 - \alpha_1 E\int_{t_0}^{\tau_k\wedge t_1}|x(t)|^p dt, \tag{15}$$

where

$$\begin{aligned} H_2 &= c_2E|x(t_0)|^p + \alpha_2 qE\int_{t_0}^{\frac{t_0}{q^2}}|x(qt)|^p dt \\ &\le H_1 + \alpha_2 E\int_{\frac{t_0}{q}}^{\frac{t_0}{q^2}}|x(qt)|^p dt = H_1 + \frac{\alpha_2}{q}E\int_{t_0}^{\frac{t_0}{q}}|x(t)|^p dt < \infty. \end{aligned}$$

Consequently

$$E|x(\tau_k\wedge t_1)|^p \le \frac{H_2}{c_1}, \quad t_0 \le t_1 \le \frac{t_0}{q^2}. \tag{16}$$

In particular, $E|x(\tau_k \wedge \frac{t_0}{q^2})|^p \leq \frac{H_2}{c_1}$, $\forall k \geq k_0$. This implies $k^p P(\tau_k \leq \frac{t_0}{q^2}) \leq \frac{H_2}{c_1}$. Letting $k \to \infty$, we then obtain that $P(\tau_\infty \leq \frac{t_0}{q^2}) = 0$, namely $P(\tau_\infty > \frac{t_0}{q^2}) = 1$. Letting $k \to \infty$ in (16) yields

$$E|x(t_1)|^p \leq \frac{H_2}{c_1},\ t_0 \leq t_1 \leq \frac{t_0}{q^2}.$$

Repeating this procedure, we can show that, for any integer $i \geq 1$, $\tau_\infty > \frac{t_0}{q^i}$ a.s.,

$$E|x(t_1)|^p \leq \frac{H_i}{c_1},\ t_0 \leq t \leq \frac{t_0}{q^i},$$

where

$$H_i = c_2 E|x(t_0)|^p + \alpha_2 E\int_{t_0}^{\frac{t_0}{q^i}} |x(qt)|^p dt < \infty.$$

We must therefore have $\tau_\infty = \infty$ a.s. and the required assertion (9) holds as well. The proof is therefore complete.

Remark 3.2. In this theorem, we removed an additional condition $\alpha_1 > \alpha_2$ which has played an important role in the proof of the existence results [6, 7, 13, 27, 34]. Hence, we improve and generalize the corresponding existence results of [6, 7, 13, 27, 34].

Theorem 3.3. *Let $p > 0$ and let Assumptions 2.1, 2.2 hold. Assume that there exists a constant $\lambda > 0$ such that*

$$E\Big[\max_{1\leq i\leq N} LV(x(t), x(qt), t, i)\Big] \leq -\lambda E\Big[\max_{1\leq i\leq N} V(x(t), t, i)\Big] \tag{17}$$

for all $t \geq t_0$ and those V satisfying

$$E\Big[\min_{1\leq i\leq N} V(x(qt), qt, i)\Big] \leq q^{-\lambda} E\Big[\max_{1\leq i\leq N} V(x(t), t, i)\Big]. \tag{18}$$

Then (3) is pth moment polynomially stable, that is, for any initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$,

$$E|x(t)|^p \leq \frac{c_2}{c_1}(1+t_0)^\lambda E||\xi||^p (1+t)^{-\lambda}, \quad on\ t \geq t_0. \tag{19}$$

Proof. Let $\eta \in (0, \lambda)$ be arbitrary. Define

$$U(t) = (1+t)^\eta EV(x(t), t, r(t)).$$

Since for any $t \geq qt_0$, $E|x(t)|^p < \infty$ and by (5), we have

$$U(t) \leq c_2(1+t)^\eta E|x(t)|^p < \infty. \tag{20}$$

Hence, $U(t)$ is well defined and is right continuous on $t \geq qt_0$. For any $\xi \in L^p_{\mathcal{F}_{t_0}}([qt_0, t_0); R^n)$, if we prove that

$$U(t) \leq c_2(1+t_0)^\eta E||\xi||^p =: \beta, \tag{21}$$

then (19) follows straightforwardly. It is obvious that

$$U(t_0) = (1+t_0)^\eta EV(x(t_0), t_0, r(t_0)) \leq c_2(1+t_0)^\eta E|x(t_0)|^p \leq c_2(1+t_0)^\eta E||\xi||^p = \beta.$$

We claim that (21) holds for all $t > t_0$. Otherwise, by the right continuity of $U(t)$, there exists a smallest $\bar{t} \geq 0$ such that for any $t \in [t_0, t_0 + \bar{t}]$, $U(t) \leq \beta$ and

$U(t_0+\bar{t}) = \beta$ as well as $U(t_0+\bar{t}+\delta) > U(t_0+\bar{t})$ for all sufficiently small δ. Then, if $q(t_0+\bar{t}) < t_0$, by condition (5),

$$\begin{aligned} & EV(x(q(t_0+\bar{t})), q(t_0+\bar{t}), r(q(t_0+\bar{t}))) \\ \le\ & c_2 E|q(t_0+\bar{t})|^p \le c_2 E||\xi||^p = (1+t_0)^{-\eta}\beta = (1+t_0)^{-\eta}U(t_0+\bar{t}) \\ \le\ & [1+q(t_0+\bar{t})]^{-\eta}[1+(t_0+\bar{t})]^{\eta} EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})) \\ \le\ & q^{-\eta} EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})) \\ \le\ & q^{-\lambda} EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})). \end{aligned}$$

If $q(t_0+\bar{t}) \ge t_0$, we have

$$\begin{aligned} & EV(x(q(t_0+\bar{t})), q(t_0+\bar{t}), r(q(t_0+\bar{t}))) \\ =\ & [1+q(t_0+\bar{t})]^{-\eta} U(q(t_0+\bar{t})) \\ \le\ & [1+q(t_0+\bar{t})]^{-\eta} U(t_0+\bar{t}) \\ \le\ & [1+q(t_0+\bar{t})]^{-\eta}[1+(t_0+\bar{t})]^{\eta} EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})) \\ \le\ & q^{-\eta} EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})) \\ \le\ & q^{-\lambda} EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})). \end{aligned}$$

Consequently,

$$E\Big[\min_{1\le i\le N} V(x(q(t_0+\bar{t})), q(t_0+\bar{t}), i)\Big] \le q^{-\lambda} E\Big[\max_{1\le i\le N} V(x(t_0+\bar{t}), t_0+\bar{t}, i)\Big].$$

Thus, by condition (17), we have

$$\begin{aligned} & E\Big[\max_{1\le i\le N} LV(x(t_0+\bar{t}), x(q(t_0+\bar{t})), t_0+\bar{t}, i)\Big] \\ \le\ & -\lambda E\Big[\max_{1\le i\le N} V(x(t_0+\bar{t}), t_0+\bar{t}, i)\Big] \\ <\ & -\eta E\Big[\max_{1\le i\le N} V(x(t_0+\bar{t}), t_0+\bar{t}, i)\Big], \end{aligned} \tag{22}$$

which implies

$$ELV(x(t_0+\bar{t}), x(q(t_0+\bar{t})), t_0+\bar{t}, r(t_0+\bar{t})) < -\eta EV(x(t_0+\bar{t}), t_0+\bar{t}, r(t_0+\bar{t})).$$

By the right continuity of V, $t \in [t_0+\bar{t}, t_0+\bar{t}+\delta]$ for sufficiently small δ

$$ELV(x(t), x(qt), t, r(t)) \le -\eta EV(x(t), t, r(t)).$$

Applying the Itô formula to $U(t)$ yields

$$\begin{aligned} U(t_0+\bar{t}+\delta) \ =\ & U(t_0+\bar{t}) + \int_{t_0+\bar{t}}^{t_0+\bar{t}+\delta} (1+t)^{\eta}\Big(ELV(x(t), x(qt), t, r(t)) \\ & + \frac{\eta}{1+t} EV(x(t), t, r(t))\Big) dt \le U(t_0+\bar{t}). \end{aligned}$$

However, this contradicts $U(t_0+\bar{t}+\delta) > U(t_0+\bar{t})$, so (21) must hold. The proof is therefore complete.

Now, we use this theorem to establish a useful result on pth moment polynomial stability.

Theorem 3.4. *Let $p > 0$, $\alpha_1 > \alpha_2 \ge 0$ and let Assumptions 2.1 and 2.2 hold. Assume that, for all $x, y \in R^n$, $i \in S$ and $t \ge t_0$,*

$$\max_{1\le i\le N} LV(x, y, t, i) \le -\alpha_1 \max_{1\le i\le N} V(x, t, i) + \alpha_2 \min_{1\le i\le N} V(y, qt, i). \tag{23}$$

Then (3) is pth moment polynomially stable, that is, for any initial data $\xi \in L^p_{\mathcal{F}_{t_0}}([qt_0, t_0]; R^n)$,

$$E|x(t)|^p \leq \frac{c_2}{c_1}(1+t_0)^{\lambda} E||\xi||^p (1+t)^{-\lambda} \tag{24}$$

where λ *is the unique root to the following equation* $\lambda = \alpha_1 - \alpha_2 q^{-\lambda}$.

Proof. If the solution $x(t)$ of (3) satisfies that for all $t \geq t_0$,

$$E\Big[\min_{1\leq i\leq N} V(x(qt), qt, i)\Big] \leq q^{-\lambda} E\Big[\max_{1\leq i\leq N} V(x(t), t, i)\Big], \tag{25}$$

then by (23) and (25), we have

$$\begin{aligned} & E\Big[\max_{1\leq i\leq N} LV(x(t), x(qt), t, i)\Big] \\ \leq\ & -\alpha_1 E\Big[\max_{1\leq i\leq N} V(x(t), t, i)\Big] + \alpha_2 E\Big[\min_{1\leq i\leq N} V(x(qt), qt, i)\Big] \\ \leq\ & -(\alpha_1 - \alpha_2 q^{-\lambda}) E\Big[\max_{1\leq i\leq N} V(x(t), t, i)\Big]. \end{aligned}$$

This shows that condition (17) is satisfied and (24) follows from Theorem 3.3.

The following theorem gives the sufficient conditions for the almost sure polynomial stability of (3).

Theorem 3.5. *Let* $p \geq 2$, $\eta \in (1, \lambda)$ *and let all the conditions of Theorem 3.4 hold. Assume that there exists a constant* $L > 0$ *such that*

$$|g(x, y, t, i)| \leq L(|x| + |y|). \tag{26}$$

Then (24) implies

$$\limsup_{t\to\infty} \frac{log|x(t)|}{log(1+t)} < -\frac{\lambda - \eta}{p}, \quad a.s. \tag{27}$$

In other words, the pth moment polynomial stability implies almost sure polynomial stability.

Proof. For any integer $n \geq 0$. Applying the Itô formula to $V(x, t, i) = |x|^p$, we can obtain that

$$\begin{aligned} & E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p\Big) \\ =\ & E|x(q^{-n}t_0)|^p + E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} LV(x(s), x(qs), s, r(s))ds\Big) \\ +\ & E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} V_x(x(s), s, r(s)) g(x(s), x(qs), s, r(s)) dw(s)\Big). \end{aligned} \tag{28}$$

By condition (6), we have

$$\begin{aligned} & E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} LV(x(s), x(qs), s, r(s))ds\Big) \\ \leq\ & \alpha_2 q E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} |x(qs)|^p ds\Big) \\ \leq\ & \alpha_2 q E\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(qt)|^p dt = \alpha_2 E\int_{q^{-(n-1)}t_0}^{q^{-n}t_0} |x(t)|^p dt. \end{aligned} \tag{29}$$

By the Burkholder-Davis-Gundy inequality, condition (26) and the basic inequality $|a+b|^{\frac{1}{2}} \leq |a|^{\frac{1}{2}} + |b|^{\frac{1}{2}}$, we have

$$\begin{aligned}
& E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} V_x(x(s),s,r(s))g(x(s),x(qs),s,r(s))dw(s)\Big) \\
\leq\ & C_p E\Big(\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |V_x(x(t),t,r(t))g(x(t),x(qt),t,r(t))|^2 dt\Big)^{\frac{1}{2}} \\
\leq\ & pC_pLE\Big(\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{2p-2}(|x(t)|+|x(qt)|)^2 dt\Big)^{\frac{1}{2}} \\
\leq\ & \sqrt{2}pC_pLE\Big(\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{2p} dt\Big)^{\frac{1}{2}} \\
+\ & \sqrt{2}pC_pLE\Big(\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{2p-2}|x(qt)|^2 dt\Big)^{\frac{1}{2}} \\
\leq\ & \sqrt{2}pC_pLE\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt\Big)^{\frac{1}{2}} \\
+\ & \sqrt{2}pC_pLE\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p \int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{p-2}|x(qt)|^2 dt\Big)^{\frac{1}{2}}.
\end{aligned}$$

For any $\epsilon > 0$, the Young inequality implies that

$$\begin{aligned}
& E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} V_x(x(s),s,r(s))g(x(s),x(qs),s,r(s))dw(s)\Big) \\
\leq\ & \sqrt{2}pC_pLE\Big[\Big(\epsilon \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p\Big)^{\frac{1}{2}}\Big(\frac{1}{\epsilon}\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt\Big)^{\frac{1}{2}}\Big] \\
+\ & \sqrt{2}pC_pLE\Big[\Big(\epsilon \sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p\Big)^{\frac{1}{2}}\Big(\frac{1}{\epsilon}\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{p-2}|x(qt)|^2 dt\Big)^{\frac{1}{2}}\Big] \\
\leq\ & \sqrt{2}pC_pL\epsilon E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p\Big) + \frac{\sqrt{2}pC_pL}{2\epsilon} E\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt \\
+\ & \frac{\sqrt{2}pC_pL}{2\epsilon} E\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^{p-2}|x(qt)|^2 dt.
\end{aligned} \tag{30}$$

Letting $\epsilon = \frac{1}{2\sqrt{2}pC_pL}$, it follows from (30) that

$$\begin{aligned}
& E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} \int_{q^{-n}t_0}^{t} V_x(x(s),s,r(s))g(x(s),x(qs),s,r(s))dw(s)\Big) \\
\leq\ & \frac{1}{2}E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p\Big) + 2p^2C_p^2L^2E\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt \\
+\ & 2p^2C_p^2L^2E\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} (|x(t)|^p + |x(qt)|^p) dt \\
\leq\ & \frac{1}{2}E\Big(\sup_{q^{-n}t_0 \leq t \leq q^{-(n+1)}t_0} |x(t)|^p\Big) + 4p^2C_p^2L^2E\int_{q^{-n}t_0}^{q^{-(n+1)}t_0} |x(t)|^p dt
\end{aligned}$$

$$+2p^2C_p^2L^2E\int_{q^{-n-1}t_0}^{q^{-n}t_0}|x(t)|^pdt. \tag{31}$$

Hence,

$$\begin{aligned} E\Big(\sup_{q^{-n}t_0\le t\le q^{-(n+1)}t_0}|x(t)|^p\Big) &\le 2E|x(q^{-n}t_0)|^p+8p^2C_p^2L^2\int_{q^{-n}t_0}^{q^{-(n+1)}t_0}E|x(t)|^pdt \\ &+ (\alpha_2+2p^2C_p^2L^2)\int_{q^{-n-1}t_0}^{q^{-n}t_0}E|x(t)|^pdt \\ &\le K_1(1+q^{-n}t_0)^{-\lambda}+K_2\int_{q^{-n}t_0}^{q^{-(n+1)}t_0}(1+t)^{-\lambda}dt, \\ &+ K_3\int_{q^{-(n-1)}t_0}^{q^{-n}t_0}(1+t)^{-\lambda}dt, \end{aligned} \tag{32}$$

where $K_1=2\frac{c_2}{c_1}(1+t_0)^{\lambda}E||\xi||^p$, $K_2=12p^2C_p^2L^2\frac{c_2}{c_1}(1+t_0)^{\lambda}E||\xi||^p$, $K_3=2(\alpha_2+2p^2C_p^2L^2)\frac{c_2}{c_1}(1+t_0)^{\lambda}E||\xi||^p$. Now, by the Chebyshev inequality, we have

$$\begin{aligned} & P\Big\{\sup_{q^{-n}t_0\le t\le q^{-(n+1)}t_0}|x(t)|^p>(1+q^{-n}t_0)^{-(\lambda-\eta)}\Big\} \\ \le\ & (1+q^{-n}t_0)^{\lambda-\eta}E\Big(\sup_{q^{-n}t_0\le t\le q^{-(n+1)}t_0}|x(t)|^p\Big) \\ \le\ & K_1(1+q^{-n}t_0)^{-\eta}+K_2(1+q^{-n}t_0)^{-\eta}(q^{-(n+1)}t_0-q^{-n}t_0) \\ +\ & K_3(1+q^{-n}t_0)^{\lambda-\eta}(1+q^{-(n-1)}t_0)^{-\lambda}(q^{-n}t_0-q^{-(n-1)}t_0) \\ \le\ & K_1(1+q^{-n}t_0)^{-\eta}+K_2(1+q^{-n}t_0)^{-\eta}(q^{-(n+1)}t_0-q^{-n}t_0) \\ +\ & K_3q^{-\lambda}(1+q^{-n}t_0)^{-\eta}(q^{-n}t_0-q^{-(n-1)}t_0). \end{aligned}$$

Clearly, when $\eta>1$,

$$\begin{aligned} & K_1\sum_{n=0}^{\infty}(1+q^{-n}t_0)^{-\eta}+K_2\sum_{n=0}^{\infty}(1+q^{-n}t_0)^{-\eta}(q^{-(n+1)}t_0-q^{-n}t_0) \\ +\ & K_3q^{-\lambda}\sum_{n=0}^{\infty}(1+q^{-n}t_0)^{-\eta}(q^{-n}t_0-q^{-(n-1)}t_0) \\ \le\ & [K_1+K_2(q^{-1}-1)t_0+K_3q^{-\lambda}(1-q)t_0]\sum_{n=0}^{\infty}\frac{(1+q^{-n}t_0)^{-\eta}}{q^n}<\infty. \end{aligned}$$

In view of the Borel-Cantelli lemma, there exists an $\Omega_0\subseteq\Omega$ with $P(\Omega_0)=1$ such that for any $w\in\Omega$, there exists an integer $n_0=n_0(w)>0$, when $n\ge n_0$ and $q^{-n}t_0\le t\le q^{-(n+1)}t_0$,

$$|x(t)|^p\le(1+q^{-n}t_0)^{-(\lambda-\eta)}.$$

Therefore,

$$\limsup_{t\to\infty}\frac{log|x(t)|}{log(1+t)}<-\frac{\lambda-\eta}{p},\quad a.s.$$

The proof is therefore completed.

Remark 3.6. In general, the pth moment stability and almost sure stability of the solution do not imply each other. In [17, 18, 26, 10, 23], the authors proved that the

pth moment stability imply the almost sure stability when the diffusion term and the drift term of stochastic systems obey the linear growth condition. It should be mentioned that the drift term of stochastic systems (3) does not satisfy the linear growth condition in this paper. Therefore, the proposed method in [17, 18, 26, 10, 23] can not be used here.

Example 3.7. Let $w(t)$ is a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -8 & 8 \\ \gamma & -\gamma \end{pmatrix}.$$

Of course, $w(t)$ and $r(t)$ are assumed to be independent. Consider the following scalar hybrid stochastic systems with pantograph delay

$$dx(t) = f(x(t), t, r(t))dt + g(x(0.5t), t, r(t))dw(t), \ t \geq 1, \tag{33}$$

with $q = 0.5$, the initial data $\xi(t) = x_0$ $(0.5 \leq t \leq 1)$ and $r(1) = 1$. Moreover, for $(x, y, t, i) \in R \times R \times [0.5, \infty) \times S$,

$$f(x, t, 1) = x, \quad g(y, t, 1) = 0.5y,$$
$$f(x, t, 2) = -5x - x^3, \quad g(y, t, 2) = \tfrac{\sqrt{2}}{2}y.$$

We note that (33) can be regarded as the result of the two equations

$$dx(t) \quad = \quad x(t)dt + 0.5x(0.5t)dw(t) \tag{34}$$

and

$$dx(t) \quad = \quad (-5x(t) - x^3(t))dt + \frac{\sqrt{2}}{2}x(0.5t)dw(t), \tag{35}$$

switching among each other according to the movement of the Markov chain $r(t)$. It is easy to see that (35) is polynomially stable but (34) is unstable. However, we shall see that due to the Markovian switching, the overall system (33) will be polynomially stable. In fact, the coefficients f and g satisfy the local Lipschitz condition but the coefficient f do not satisfy the linear growth condition. To find out whether hybrid stochastic systems with pantograph delay (33) is polynomial stability in mean square, we use the Lyapunov function

$$V(x, t, i) = \begin{cases} x^2, & if \ i = 1, \\ 0.4x^2, & if \ i = 2. \end{cases}$$

It is easy to see that the operator LV from $R \times R \times [0.5, \infty) \times S$ to R has the form

$$LV(x, y, t, 1) \quad \leq \quad -2.8x^2 + 0.25y^2,$$

and

$$LV(x, y, t, 2) \quad \leq \quad -(4 - 0.6\gamma)x^2 + 0.2y^2.$$

According to (23), we require $(4 - 0.6\gamma) \wedge 2.8 > 0.625$.

Case I. If $4 - 0.6\gamma \geq 2.8$, namely $0 < \gamma \leq 2$, then

$$\begin{aligned} \max_{i=1,2} LV(x, y, t, i) \quad &\leq \quad -2.8x^2 + 0.25y^2 \\ &\leq \quad -2.8 \max_{i=1,2} V(x, t, i) + 0.625 \min_{i=1,2} V(y, qt, i). \end{aligned}$$

By Theorem 3.4, we can conclude that if the transition rate $\gamma \in (0, 2]$, then hybrid stochastic systems with pantograph delay (33) will be mean square polynomially stable and the 2th moment Lyapunov exponent is not greater than the unique root

$\lambda > 0$ of the equation $\lambda = 2.8 - 0.625 \times 0.5^{-\lambda}$. On the other hand, it is obvious that condition (26) is satisfied. According to Theorem 3.5, hybrid stochastic systems with pantograph delay (33) is almost surely polynomially stable.

Case II. If $0.625 < 4 - 0.6\gamma < 2.8$, namely $2 < \gamma < 5.625$, then

$$\max_{i=1,2} LV(x,y,t,i) \leq -(4-0.6\gamma)\max_{i=1,2} V(x,t,i) + 0.625 \min_{i=1,2} V(y,qt,i).$$

By Theorem 3.4, we can conclude that if the transition rate $\gamma \in (2, 5.625)$, then for any initial data x_0, (33) will be mean square polynomially stable. On the other hand, by Theorem 3.5, we can conclude that if the transition rate $\gamma \in (2, 2.917)$, (33) will be almost surely polynomially stable. But, if the transition rate $\gamma \in (2.917, 5.625)$, (33) will not be almost surely polynomially stable. For example, if we choosing $\gamma = 3$, we can conclude that (33) will be mean square polynomially stable but not be almost surely polynomially stable. In fact, the root λ of the following equation $\lambda = 2.2 - 0.625 \times 0.5^{-\lambda}$ is not great than 1 which contradicts the fact $\lambda > 1$.

4. **Linear Hybrid Stochastic Systems with Pantograph delay.** As an application of Theorem 3.4 and 3.5, we consider the linear hybrid stochastic systems with pantograph delay of the form

$$dx(t) = [A(r(t))x(t) + B(r(t))x(qt)]dt + [C(r(t))x(t) + D(r(t))x(qt)]dw(t), \quad (36)$$

on $t \geq t_0$. Here $A, B, C, D : S \to R^{n\times n}$ and we shall write $A(i) = A_i, B(i) = B_i, C(i) = C_i, D(i) = D_i$.

Theorem 4.1. *Assume that there exist positive constants θ_i, $1 \leq i \leq N$ and $\lambda > 1$, such that $\lambda_3 > 0$ and $q^{-\lambda} < \frac{\lambda_1\lambda_3}{\lambda_2\lambda_4}$, where $\lambda_1 = \min_{1\leq i\leq N}\theta_i, \lambda_2 = \max_{1\leq i\leq N}\theta_i$,*

$$\lambda_3 = -\max_{1\leq i\leq N} \lambda_{max}\Big([A_i + A_i^\top + 2C_i^\top C_i]\theta_i + (\theta_i + \sum_{j=1}^N \gamma_{ij}\theta_j)I]\Big), \quad (37)$$

and

$$\lambda_4 = \max_{1\leq i\leq N} ||(B_i^\top B_i + 2D_i^\top D_i)\theta_i||. \quad (38)$$

Then (36) is mean square polynomially stable and almost surely polynomially stable.

Proof. Let $V(x(t),t,i) = \theta_i|x(t)|^2$, where $\theta_i > 0$ for all $i \in S$. Clearly,

$$\lambda_1|x|^2 \leq V(x,t,i) \leq \lambda_2|x|^2.$$

The operator LV has the form

$$\begin{aligned} LV(x(t),x(qt),t,i) &= 2\theta_i x^\top(t)[A_i x(t) + B_i x(qt)] \\ &+ \theta_i[C_i x(t) + D_i x(qt)]^\top[C_i x(t) + D_i x(qt)] \\ &+ \sum_{j=1}^N \gamma_{ij}\theta_j x^\top(t)x(t). \end{aligned} \quad (39)$$

Note that

$$\begin{aligned} 2\theta_i x^\top(t)A_i x(t) &= \theta_i x^\top(t)(A_i + A_i^\top)x(t), \\ 2\theta_i x^\top(t)B_i x(qt) &\leq \theta_i x^\top(t)x(t) + \theta_i x^\top(qt)B_i^\top B_i x(qt) \end{aligned}$$

and

$$\begin{aligned} \theta_i[C_i x(t) + D_i x(qt)]^\top[C_i x(t) + D_i x(qt)] &\leq 2\theta_i x^\top(t)C_i^\top C_i x(t) \\ &+ 2\theta_i x^\top(qt)D_i^\top D_i x(qt). \end{aligned}$$

It follows that

$$\begin{aligned} LV(x(t),x(qt),t,i) &\leq \theta_i x^\top(t)\Big(A_i + A_i^\top + 2C_i^\top C_i + (1+\frac{1}{\theta_i}\sum_{j=1}^N \gamma_{ij}\theta_j)I\Big)x(t) \\ &+ \theta_i x^\top(qt)\Big(B_i^\top B_i + 2D_i^\top D_i\Big)x(qt) \\ &\leq -\lambda_3|x(t)|^2 + \lambda_4|x(qt)|^2. \end{aligned} \tag{40}$$

For any $t \geq t_0$ and V satisfying

$$E[\min_{1\leq i\leq N} V(x(qt),qt,i)] \leq q^{-\lambda}E[\max_{1\leq i\leq N} V(x(t),t,i)],$$

we have

$$\lambda_1 E|x(qt)|^2 \leq q^{-\lambda}\lambda_2 E|x(t)|^2 \quad on\ t \geq t_0.$$

Hence, by (40), we obtain

$$\begin{aligned} E[\max_{1\leq i\leq N} LV(x(t),x(qt),t,i)] &\leq -\lambda_3 E|x(t)|^2 + \lambda_4 E|x(qt)|^2 \\ &\leq -\Big(\lambda_3 - q^{-\lambda}\frac{\lambda_2\lambda_4}{\lambda_1}\Big)E|x(t)|^2. \end{aligned}$$

Recalling $q^{-\lambda} < \frac{\lambda_1\lambda_3}{\lambda_2\lambda_4}$, we see all assumptions of Theorem 3.3 are satisfied with $p = 2$, therefore, linear hybrid stochastic systems with pantograph delay (36) is mean square polynomially stable. On the other hand, for any $i \in S$, we have

$$\begin{aligned} |C(i)x(t) + D(i)x(qt)| &\leq ||C(i)||x(t)| + ||D(i)|||x(qt)| \\ &\leq L\Big(|x(t)| + |x(qt)|\Big), \end{aligned}$$

where $L = \max_{1\leq i\leq N}(||C(i)||+||D(i)||)$. Hence, by Theorem 3.5, (36) is also almost surely polynomially stable.

Example 4.2. Let $w(t)$ is a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}.$$

Assume that $w(t)$ and $r(t)$ are independent. Consider a two-dimensional hybrid stochastic systems with pantograph delay

$$dx(t) = A(r(t))x(t)dt + B(r(t))x(0.4t)dt + D(r(t))x(0.4t)dw(t),\ t \geq 1, \tag{41}$$

with $q = 0.4$, the initial data $\xi(t) = x_0$ $(0.4 \leq t \leq 1)$ and $r(1) = 1$. Here

$$A_1 = \begin{pmatrix} -8 & 1 \\ 2 & -4 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.3 & 0.2 \\ -0.1 & 0.4 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.5 & 0.2 \\ -0.5 & 0.1 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} -5 & -1 \\ 1 & -6 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0.2 \\ 0.1 & 0.4 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -0.5 & 0.3 \\ 0.1 & 0.1 \end{pmatrix}.$$

Define the Lyapunov function $V(x,t,1) = 0.4|x|^2$ and $V(x,t,2) = 0.3|x|^2$. Throught a straight computation, it is not difficult to obtain $\lambda_1 = 0.3$, $\lambda_2 = 0.4$, $\lambda_3 = 2.3$, $\lambda_4 = 0.4886$. Obviously, $\lambda_3 = 2.3 > 0$. If $0.4^{-\lambda} < \frac{\lambda_1\lambda_3}{\lambda_2\lambda_4} = 3.5305$, i.e., $\lambda \in (1, 2.4694)$, then by Theorem 4.1, we obtain that hybrid stochastic systems with pantograph delay (41) is polynomially stable in mean square and is also almost surely polynomially stable.

Conclusion. This paper is devoted to the pth moment polynomial stability and almost sure polynomial stability of hybrid stochastic systems with pantograph delay. The Razumikhin technique and Lyapunov functions is used to derive sufficient conditions for stabilities of nonlinear hybrid stochastic systems with pantograph delay, which further helps to derive easily verifiable conditions for linear hybrid stochastic systems with pantograph delay. Finally, two examples are provided to verify the effectiveness of the main results.

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REFERENCES

[1] J. Appleby, E. Buckwar, Sufficient conditions for polynomial asymptotic behaviour of the stochastic pantograph equation, *Electron. J. Qual. Theo.*, **2016** (2016), 1–32.

[2] C. T. H. Baker and E. Buckwar, Continuous θ-methods for the stochastic pantograph equation, *Electron. T. Numer. Anal.*, **11** (2000), 131–151.

[3] W. Fei, L. Hu, X. Mao, M. Shen, Structured robust stability and boundedness of nonlinear hybrid delay systems, *SIAM J. Control. Optim.*, **56** (2018), 2662–2689.

[4] Z. Fan, M. Song, M. Liu, The αth moment stability for the stochastic pantograph equation,*J. Comput. Appl. Math*, **233** (2009), 109–120.

[5] Z. Fan, M. Liu, and W. Cao, Existence and uniqueness of the solutions and convergence of semi-implicit Euler methods for stochastic pantograph equations, *J. Math. Anal. Appl.*, **35** (2007), 1142–1159.

[6] P. Guo, C. Li, Almost sure exponential stability of numerical solutions for stochastic pantograph differential equations, *J. Math. Anal. Appl.*, **460** (2018), 411–424.

[7] P. Guo, C. Li, Razumikhin-type technique on stability of exact and numerical solutions for the nonlinear stochastic pantograph differential equations, *BIT Numer. Math.*, **59** (2019), 77–96.

[8] L. Hu, X. Mao, L. Zhang, Robust stability and boundedness of nonlinear hybrid stochastic differential delay equations,*IEEE Trans. Automa. Control.*, **58** (2013), 2319–2332.

[9] L. Huang, F. Deng, Razumikhin-type theorems on stability of neutral stochastic functional differential equations, *IEEE Trans. Automa. Control.*, **53** (2008), 1718–1723.

[10] S. Jankovic, J. Randjelovic, M. Jovanovic, Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations, *J. Math. Anal. Appl.*, **355** (2009), 811–820.

[11] Y. Ji and H. Chizeck, Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automa. Control.*, **35** (1990), 777–788.

[12] K. Liu, A. Chen, Moment decay rates of solutions of stochastic differential equations, *Tohoku Math. J.*, **53** (2001), 81–93.

[13] M. Milosevic, Existence, uniqueness, almost sure polynomial stability of solution to a class of highly nonlinear pantograph stochastic differential equations and the Euler-Maruyama approximation, *Appl. Math. Comput.*, **237** (2014), 672–685.

[14] X. Mao, C. Yuan, Stochastic Differential Equations with Markovian Switching, *Imperial College Press*, 2006.

[15] X. Mao, J. Lam , S. Xu , H. Gao, Razumikhin method and exponental stability of hybrid stochastic delay interval systems, *J. Math. Anal. Appl.*, **314** (2006), 45–66.

[16] X. Mao, A. Matasov, A. B. Piunovskiy, Stochastic differential delay equations with Markovian switching, *Bernoulli.*, **5** (2000), 73–90.

[17] X. Mao, Razumikhin-type theorems on exponential stability of stochastic functional differential equations, *Stoch. Proc. Appl.*, **65** (1996), 233–250.

[18] X. Mao, Razumikhin-type theorems on exponential stability of neutral stochastic functional differential equations, *SIAM J. Math. Anal.*, **28** (1997), 389–401.

[19] X. Mao, Almost sure polynomial stability for a class of stochastic differential equations, *Quart. J. Math. Oxford Ser.*, **43** (1992), 339–348.

[20] X. Mao, Polynomial stability for perturbed stochastic differential equations with respect to semimartingales, *Stoch. Proc. Appl.*, **41** (1992), 101–116.

[21] X. Mao, J. Lam, L. Huang, Stabilisation of hybrid stochastic differential equations by delay feedback control, *Syst. Control. Lett.*, **57** (2008), 927–935.

[22] S. Peng, Y. Zhang, Razumikhin-type theorems on pth moment exponential stability of impulsive stochastic delay differential equations, *IEEE Trans. Automa. Control.*, **55** (2010), 1917–1922.

[23] G. Pavlovic, S. Jankovic, Razumikhin-type theorems on general decay stability of stochastic functional differential equations with infinite delay, *J. Comput. Appl. Math.*, **236** (2012), 1679–1690.

[24] B. S. Razumikhin, On the stability of systems with a delay, *Prikl. Mat. Mekh.*, **20** (1956), 500–512.

[25] L. Shaikhet, Stability of stochastic hereditary systems with Markov switching, *Theory. Stoch. Proc.*, **2** (1996), 180–184.

[26] Y. Shen, X. Liao, Razumikhin-type theorems on exponential stability of neutral stochastic functional differential equations, *Chinese Science Bulletin.*, **44** (1999), 2225–2228.

[27] M. Shen, W. Fei, X. Mao, S. Deng, Exponential Stability of Highly Nonlinear Neutral Pantograph Stochastic Differential Equations, *Asian. J. Control.*, **22** (2020), 1–13.

[28] F. Wu, G. Yin, L. Wang, Razumikhin-type theorems on moment exponential stability of functional differential equations involving two-time-scale Markovian switching, *Math. Control. Related. Fields.*, **5** (2015), 697–719.

[29] F. Wu, S. Hu, Razumikhin type theorems on general decay stability and robustness for stochastic functional differential equations, *I. J. Robust. Nonlinear. Control.*, **22** (2012), 763–777.

[30] X. Wu, W. Zhang, Y. Tang, pth Moment stability of impulsive stochastic delay differential systems with Markovian switching, *Commun. Nonlinear. Sci. Numer. Simulation.*, **18** (2013), 1870–1879.

[31] Y. Xiao, M. Song, M. Liu, Convergence and stability of the semi-implicit Euler method with variable step size for a linear stochastic pantograph differential equation, *Int. J. Numer. Anal. Model.*, **8** (2011), 214–225.

[32] C. Yuan, X. Mao, Robust stability and controllability of stochastic differential delay equations with Markovian switching, *Automatica.*, **40** (2004), 343–354.

[33] Q. Zhu,Razumikhin-type theorem for stochastic functional differential equations with lévy noise and markov switching, *Int. J. Control.*, **90** (2017), 1703–1712.

[34] S. Zhou, M. Xue,Exponential stability for nonlinear hybrid stochastic pantograph equations and numerical approximation,*Acta. Math. Sci.*, **34** (2014), 1254–1270.

[35] T. Zhang, H. Chen, C. Yuan, T. Caraballo, On the asymptotic behavior of highly nonlinear hybrid stochastic delay differential equations, *Discrete Contin. Dyn. Syst. Ser. B.*, (2019), 364–380.

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