Stabilisation of Highly Nonlinear Hybrid Stochastic Differential Delay Equations by Delay Feedback Control ⋆

Xiaoyue Li [a], Xuerong Mao [b]

[a] *School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P. R. China.*

[b] *Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, UK.*

Abstract

Given an unstable hybrid stochastic differential equation (SDDE, also known as an SDDE with Markovian switching), can we design a *delay* feedback control to make the controlled hybrid SDDE become asymptotically stable? If the feedback control is based on the current state, the stabilisation problem has been studied. However, there is little known when the feedback control is based on the past state. The problem becomes even harder when the coefficients of the underlying hybrid SDDE do not satisfy the linear growth condition (namely, the coefficients are highly nonlinear). The aim of this paper is to tackle the stabilisation problem for a given unstable highly nonlinear hybrid SDDE.

Key words: Brownian motion, SDDE, Markov chain, Asymptotic stability, Lyapunov functional.

1 Introduction

Hybrid stochastic differential delay equations (SDDEs) whose coefficients depend on the states of continuous-time Markov chains (also known as SDDEs with Markovian switching) appear in many branches of science and industry. One of the important issues in the study of hybrid SDDEs is the analysis of stability (see, e.g., [11,12,21,24,25,30,31,33]). In particular, the stability of highly nonlinear hybrid SDDEs has recently become one of the most popular topics (see, e.g., [8,11,17]).

Consider an unstable hybrid SDDE

$$\begin{aligned} dx(t) = &\, f(x(t), x(t-\delta), r(t), t)dt \\ &+ g(x(t), x(t-\delta), r(t), t)dB(t), \end{aligned} \tag{1.1}$$

where the state $x(t)$ takes values in R^n and the mode $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, \cdots, N\}$, $B(t)$ is a Brownian motion, δ is a positive constant which stands for the time delay of the system, and f and g are referred to as the drift and diffusion coefficient, respectively. In order to make this given unstable system become stable, it is classical to find a feedback control $u(x(t), r(t), t)$, based on the current state $x(t)$, for the controlled system

$$\begin{aligned} dx(t) = &\, [f(x(t), x(t-\delta), r(t), t) + u(x(t), r(t), t)]dt \\ &+ g(x(t), x(t-\delta), r(t), t)dB(t) \end{aligned} \tag{1.2}$$

to become stable. However, taking into account a time lag τ (> 0) between the time when the observation of the state is made and the time when the feedback control reaches the system, it is more realistic that the control depends on a past state $x(t-\tau)$. Accordingly, the control should be of the form $u(x(t-\tau), r(t), t)$. Hence, the stabilisation problem becomes to design a delay feedback control $u(x(t-\tau), r(t), t)$ for the controlled system

$$\begin{aligned} dx(t) = &\, [f(x(t), x(t-\delta), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ &+ g(x(t), x(t-\delta), r(t), t)dB(t) \end{aligned} \tag{1.3}$$

to be stable. When the given unstable system is a hybrid SDE (not SDDE), Mao et al. [23] were the first to study this stabilisation problem by the delay feedback control and there have been some further developments since then (see, e.g., [22,32]), although the method of delay feedback controls has been well used in the area of ordinary differential equations (see, e.g., [1,6,29]). The common stringent assumption imposed in these papers in the area of hybrid SDEs is that both drift and diffusion coefficients need to satisfy the linear growth condition. Only very recently have Lu et al. made a signifi-

⋆ This paper was not presented at any IFAC meeting. Corresponding author X. Mao. E-mail: x.mao@strath.ac.uk

Email addresses: `lixy209@nenu.edu.cn` (Xiaoyue Li), `x.mao@strath.ac.uk` (Xuerong Mao).

cant progress in [17] where they established a new theory on the stabilisation by delay feedback control for highly nonlinear hybrid SDEs. The aim of this paper is to develop their theory further for highly nonlinear hybrid SDDEs. This is necessary from practical point of view. In fact, many real world systems are described by highly nonlinear hybrid SDDEs, for example, population systems, neural networks, financial and economic systems (see, e.g.,[3,5,16,26]). Although there are some papers on the delay feedback controls (see, e.g., [1,6,23]), they are not applicable to highly nonlinear hybrid SDDEs. Mathematically speaking, the development from SDEs to SDDEs is no trivial at all due to the infinite-dimensional nature of SDDEs. We highlight a few significant features in comparison with [17]:

- Under some standing hypotheses we will propose a number of rules to stabilise the given SDDE. We will explain how to design the delay feedback control to satisfy these rules and these discussions will also reveal that there are many such delay feedback controls available. Such developments are totally different from the study in [17].
- The stabilisation of SDDEs discussed in this paper is an infinite-dimensional problem while that of SDEs in [17] is finite-dimensional.
- The mathematical analysis of the infinite-dimensional problem in this paper is much harder than that of a finite-dimensional one in [17].

Let us begin to develop our new theory on the stabilisation problem.

2 Notation and Standing Hypotheses

Throughout this paper, unless otherwise specified, we use the following notation. If A is a vector or matrix, its transpose is denoted by A^T. For $x \in R^n$, $|x|$ denotes its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm. If A is a symmetric real-valued matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. By $A \le 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. Let $R_+ = [0, \infty)$. For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions φ from $[-h, 0] \to R^n$ with the norm $\|\varphi\| = \sup_{-h\le u\le 0} |\varphi(u)|$. Denote by $C(R^n; R_+)$ the family of continuous functions from R^n to R_+. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If A is a subset of Ω, denote by I_A its indicator function; that is, $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a filtered complete probability space with a filtration satisfying the usual conditions (i.e., it is right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the same probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j|r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from i to j if $i \ne j$ while $\gamma_{ii} = -\sum_{j\ne i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$ under $\mathbb{P}$.

Suppose that the underlying system is described by the nonlinear hybrid SDDE (1.1) with the initial data

$$\{x(t) : -\delta \le t \le 0\} = \xi \in C([-\delta, 0]; R^n), \tag{2.1}$$

where the coefficients $f : R^n \times R^n \times S \times R_+ \to R^n$ and $g : R^n \times R^n \times S \times R_+ \to R^{n\times m}$ are Borel measurable functions. The classical conditions for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see, e.g., [18–20,25]). In this paper, we of course need the local Lipschitz condition. However, we will consider highly nonlinear hybrid SDDEs which, in general, do not satisfy the linear growth condition. We therefore impose the polynomial growth condition, instead of the linear growth condition.

Assumption 2.1 *Assume that for any real number $b > 0$, there exists a positive constant K_b such that*

$$\begin{aligned} |f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)| \vee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)| \\ \le K_b(|x - \bar{x}| + |y - \bar{y}|) \end{aligned} \tag{2.2}$$

for all $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le b$ and all $(i, t) \in S \times R_+$. Assume moreover that there exist constants $K > 0$, $q_1 > 1$ and $q_i \ge 1$ ($2 \le i \le 4$) such that

$$\begin{aligned} |f(x, y, i, t)| \le K(|x| + |y| + |x|^{q_1} + |y|^{q_2}), \\ |g(x, y, i, t)| \le K(|x| + |y| + |x|^{q_3} + |y|^{q_4}) \end{aligned} \tag{2.3}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

For the convenience of the study in this paper we let $q_1 > 1$ but essentially we need only $\max_{1\le i\le4} q_i > 1$ as we are here interested in hybrid SDDEs without the linear growth condition. We will refer to condition (2.3) as the polynomial growth condition.

It is known that Assumption 2.1 only guarantees that the hybrid SDDE (1.1) has a unique maximal local solution, which may explode to infinity at a finite time (see, e.g., [25]). To avoid such a possible explosion, we need to impose another Khasminskii-type condition.

Assumption 2.2 *Assume that there exist positive constants $p, q, \alpha_1, \alpha_2, \alpha_3$ such that $\alpha_2 > \alpha_3$ and*

$$q > (p + q_1 - 1) \vee (2(q_1 \vee q_2 \vee q_3 \vee q_4)), \tag{2.4}$$
$$p \geq 2(q_1 \vee q_2 \vee q_3 \vee q_4) - q_1 + 1, \tag{2.5}$$

(where $q_1, \cdots, q_4$ have been specified in Assumption 2.1) while for all $(x, i, t) \in R^n \times S \times R_+$,

$$\begin{aligned} &x^T f(x, y, i, t) + \frac{q-1}{2}|g(x, y, i, t)|^2 \\ &\quad \leq \alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p. \end{aligned} \tag{2.6}$$

It is useful to point out conditions (2.4) and (2.5) along with $q_1 > 1$ implies that both p and q are larger than 2. The following theorem does not only show the existence and uniqueness of the global solution but also the L^q-boundedness of the solution.

Theorem 2.3 *Under Assumptions 2.1 and 2.2, equation (1.1) with the initial data (2.1) has a unique global solution $x(t)$ on $[-\delta, \infty)$ which satisfies*

$$\sup_{-\delta \leq t < \infty} \mathbb{E}|x(t)|^q < \infty. \tag{2.7}$$

Proof. We will apply [11, Theorem 3.1] to show this theorem. Comparing the assumptions of [11, Theorem 3.1] with those in our Theorem 2.3, we see that all we need to do is to verify Assumption 2.2 in [11]. Following the notation used in [11], we define $V(x, i, t) = |x|^q$ for $(x, i, t) \in R^n \times S \times R_+$. Then the function $\mathcal{L}_1 V : R^n \times R^n \times S \times R_+ \to R$ with respect to the SDDE (1.1) defined there ([11] uses LV but we change it into $\mathcal{L}_1 V$ in order to show it differs from $\mathcal{L}_2 U$ etc. used later) has the form

$$\begin{aligned} &\mathcal{L}_1 V(x, y, i, t) = q|x|^{q-2} x^T f(x, y, i, t) \\ &+ \frac{q}{2}|x|^{q-2}|g(x, y, i, t)|^2 + \frac{q(q-2)}{2}|x|^{q-4}|x^T g(x, y, i, t)|^2 \end{aligned} \tag{2.8}$$

(namely, $\mathcal{L}_1$ may be regarded as the differential operator corresponding to the SDDE (1.1)). Then, by Assumption 2.2,

$$\begin{aligned} &\mathcal{L}_1 V(x, y, i, t) \\ &\quad \leq q|x|^{q-2}\Big[\alpha_1(|x|^2 + |y|^2) - \alpha_2|x|^p + \alpha_3|y|^p\Big]. \end{aligned} \tag{2.9}$$

Choose a positive number $\alpha < \alpha_2 - \alpha_3$. By the well-known Young inequality, we have

$$\begin{aligned} &\alpha_1|x|^{q-2}|y|^2 \\ &\quad \leq \frac{q-2}{q}(0.5q\alpha)^{-2/(q-2)}\alpha_1^{q/(q-2)}|x|^q + \alpha|y|^q \\ &\quad \leq \frac{q-2}{q}(0.5q\alpha)^{-2/(q-2)}\alpha_1^{q/(q-2)}|x|^q + \alpha(1 + |y|^{q+p-2}) \end{aligned}$$

and

$$|x|^{q-2}|y|^p \leq \frac{q-2}{q+p-2}|x|^{q+p-2} + \frac{p}{q+p-2}|y|^{q+p-2}.$$

It therefore follows from (2.9) that

$$\begin{aligned} &\mathcal{L}_1 V(x, y, i, t) \leq q\alpha + \beta_1|x|^q - \beta_2|x|^{q+p-2} + \beta_3|y|^{q+p-2} \\ &\quad \leq c - 0.5(\beta_2 + \beta_3)(1 + |x|^{q+p-2}) + \beta_3(1 + |y|^{q+p-2}), \end{aligned}$$

where

$$\beta_1 = q\Big(\alpha_1 + \frac{q-2}{q}(0.5q\alpha)^{-2/(q-2)}\alpha_1^{q/(q-2)}\Big),$$
$$\beta_2 = q\Big(\alpha_2 - \frac{\alpha_3(q-2)}{q+p-2}\Big), \quad \beta_3 = q\Big(\alpha + \frac{\alpha_3 p}{q+p-2}\Big)$$

and

$$c = \sup_{u \geq 0}(q\alpha + 0.5(\beta_2 - \beta_3) + \beta_1 u^q - 0.5(\beta_2 - \beta_3)u^{q+p-2}) < \infty.$$

If we let $U_1(x, t) = |x|^q$ and $U_2(x, t) = 1 + |x|^{q+p-2}$ for $(x, t) \in R^n \times R_+$, we have verified Assumption 2.2 in [11]. The proof of the theorem is therefore complete. □

It is useful to point out that in some hybrid SDDEs, the constants p and q in Assumption 2.2 are different. In fact, q could be arbitrarily large sometimes. For example, consider the scalar hybrid SDDE

$$\begin{aligned} dx(t) &= f(x(t), x(t-\delta), r(t), t)dt \\ &\quad + g(x(t), x(t-\delta), r(t), t)dB(t), \end{aligned} \tag{2.10}$$

where the coefficients f and g are defined by

$$\begin{aligned} f(x, y, 1, t) &= x(1 - 3x^2 + y^2), \\ g(x, y, 1, t) &= |x|^{3/2} + 0.5y, \\ f(x, y, 2, t) &= x(1 - 2x^2 - y^2), \\ g(x, y, 2, t) &= 0.5|x|^{3/2} - 0.5y, \end{aligned} \tag{2.11}$$

$B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.12}$$

This is a simple version of hybrid SDDE models appeared frequently in population systems (see, e.g., [3,20]) and it is highly nonlinear (see, e.g., [8,11,17] for more on highly nonlinear hybrid SDDEs). It is easy to see that Assumption 2.1 is satisfied with $q_1 = 3,\ q_2 = 3,\ q_3 = 1.5$ and $q_4 = 1$. Moreover, for any $(x, y, i, t) \in R\times R\times S\times R_+$,

$$x^T f(x,y,i,t) + \frac{q-1}{2}|g(x,y,i,t)|^2 \le \begin{cases} x^2 - 2.5x^4 + 0.5y^4 + (q-1)(|x|^3 + 0.25y^2), & i = 1, \\ x^2 - 1.5x^4 + 0.5y^4 + (q-1)(0.25|x|^3 + 0.25y^2), & i = 2. \end{cases} \tag{2.13}$$

But

$$(q-1)|x|^3 \le |x|^4 + 0.25(q-1)^2x^2.$$

Hence

$$\begin{aligned} & x^T f(x,y,i,t) + \frac{q-1}{2}|g(x,y,i,t)|^2 \\ \le & (1 + 0.25(q-1)^2)(x^2 + y^2) - 1.25x^4 + 0.5y^4. \end{aligned} \tag{2.14}$$

That is, the hybrid SDDE (2.10) satisfies Assumption 2.2 for any $q > 6$ along with $p = 4$, $\alpha_1 = 1 + 0.25(q-1)^2$, $\alpha_2 = 1.25$ and $\alpha_3 = 0.5$.

3 Rules for Delay Feedback Controls

Although the solution of the hybrid SDDE (1.1) is bounded under Assumptions 2.1 and 2.2, the equation may not be stable. In this case, we are required to design a delay feedback control $u(x(t-\tau), r(t), t)$ for the controlled equation (1.3) to become stable. Here the control function $u : R^n \times S \times R_+ \to R^n$ is Borel measurable, while we shall assume $\tau \le \delta$ (it is possible to allow $\tau > \delta$ but the calculations will become more complicated). In this section, we shall propose a number of rules for the control function u to meet for the stabilisation purpose. Our first rule is:

Rule 3.1 *There exists a positive number β such that*

$$|u(x,i,t) - u(y,i,t)| \le \beta|x - y| \tag{3.1}$$

for all $x, y \in R^n$, $i \in S$ and $t \ge 0$. Moreover, for the stability purpose, we require that $u(0, i, t) \equiv 0$.

That is, the control function $u(x, i, t)$ is globally Lipschitz continuous in x. This assumption implies the linear growth condition

$$|u(x,i,t)| \le \beta|x|, \quad \forall (x,i,t) \in R^n \times S \times R_+. \tag{3.2}$$

This is a bit surprise as we would normally look for a highly nonlinear control function given that the coefficients of the given SDDE (1.1) are highly nonlinear. However, we will show that the globally Lipschitz continuous control function can work very well. Indeed, it is this rule that enables us to design the required control function much more easily. The following theorem forms a foundation for this paper.

Theorem 3.2 *Let Assumptions 2.1 and 2.2 hold. If the control function u satisfies Rule 3.1, then the controlled SDDE (1.3) with the initial data (2.1) has a unique global solution $x(t)$ on $[-\tau, \infty)$ which satisfies*

$$\sup_{-\tau \le t < \infty} \mathbb{E}|x(t)|^q < \infty. \tag{3.3}$$

This theorem can be proved in the same way as Theorem 2.3 so the proof is omitted. It is useful to point out that Theorem 3.2 along with conditions (2.3) and (2.4) guarantees, for example, $f(x(t), x(t-\delta), r(t), t)$ and $g(x(t), x(t-\delta), r(t), t)$ are bounded in L^2 on $t \in [0, \infty)$; $x(t)$ is bounded in $L^{\bar q}$ for any $\bar q \in (0, q]$. Theorem 3.2 shows that the controlled SDDE (1.3) preserves the boundedness of the given SDDE (1.1). However, in order for the stability of the controlled SDDE, the control function needs to meet more rules. The following one is more technical.

Rule 3.3 *Design the control function $u : R^n\times S\times R_+ \to R^n$ so that we can find real numbers a_i, $\bar a_i$, positive numbers c_i, $\bar c_i$ and nonnegative numbers b_i, $\bar b_i$, d_i, $\bar d_i$ ($i \in S$) such that*

$$\begin{aligned} & x^T[f(x,y,i,t) + u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2 \\ & \quad \le a_i|x|^2 + b_i|y|^2 - c_i|x|^p + d_i|y|^p \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & x^T[f(x,y,i,t) + u(x,i,t)] + \frac{q_1}{2}|g(x,y,i,t)|^2 \\ & \quad \le \bar a_i|x|^2 + \bar b_i|y|^2 - \bar c_i|x|^p + \bar d_i|y|^p \end{aligned} \tag{3.5}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$; while both

$$\begin{aligned} \mathcal{A}_1 &:= -2\mathrm{diag}(a_1, \cdots, a_N) - \Gamma, \\ \text{and } \mathcal{A}_2 &:= -(q_1 + 1)\mathrm{diag}(\bar a_1, \cdots, \bar a_N) - \Gamma \end{aligned} \tag{3.6}$$

are nonsingular M-matrices; and moreover,

$$1 > \gamma_1,\ \gamma_2 > \gamma_3,\ 1 > \gamma_4,\ \gamma_5 > \gamma_6, \tag{3.7}$$

where

$$\begin{aligned} (\theta_1, \cdots, \theta_N)^T &= \mathcal{A}_1^{-1}(1, \cdots, 1)^T, \\ (\bar\theta_1, \cdots, \bar\theta_N)^T &= \mathcal{A}_2^{-1}(1, \cdots, 1)^T, \end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\gamma_1 = \max_{i\in S} 2\theta_i b_i, \qquad \gamma_2 = \min_{i\in S} 2\theta_i c_i,\\
&\gamma_3 = \max_{i\in S} 2\theta_i d_i, \qquad \gamma_4 = \max_{i\in S}(q_1+1)\bar\theta_i \bar b_i, \\
&\gamma_5 = \min_{i\in S}(q_1+1)\bar\theta_i \bar c_i,\ \gamma_6 = \max_{i\in S}(q_1+1)\bar\theta_i \bar d_i.
\end{aligned} \tag{3.9}$$

We should point out that all θ_i and $\bar\theta_i$ defined by (3.8) are positive as both $\mathcal{A}_1$ and $\mathcal{A}_2$ are nonsingular M-matrices. Regarding the theory on M-matrices we refer the reader to [25, Section 2.6].

Let us explain that there are lots of such control functions available under Assumption 2.2. For example, in the case when the state $x(t)$ of the given SDDE (1.1) is observable in any mode $i \in S$ (otherwise it is more complicated and we will explain later), we could, for example, design the control function $u(x,i,t) = Ax^T$, where A is a symmetric $n\times n$ real-valued negative-definite matrix such that $\lambda_{\max}(A) \le -(\kappa+1)\alpha_1$ with $\kappa > 1$. Then

$$x^T u(x,i,t) \le -(\kappa+1)\alpha_1 |x|^2, \quad \forall (x,i,t) \in R^n\times S\times R_+.$$

By Assumption 2.2, in particular, noting $q-1 \ge q_1 > 1$, we further have

$$\begin{aligned}
&x^T[f(x,y,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,i,t)|^2\\
&\quad \le -\kappa\alpha_1|x|^2 + \alpha_1|y|^2 - \alpha_2|x|^p + \alpha_3|y|^p,\\
&x^T[f(x,i,t)+u(x,i,t)] + \frac{q_1}{2}|g(x,i,t)|^2\\
&\quad \le -\kappa\alpha_1|x|^2 + \alpha_1|y|^2 - \alpha_2|x|^p + \alpha_3|y|^p.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{A}_1 &= 2\kappa\,\mathrm{diag}(\alpha_1,\cdots,\alpha_1) - \Gamma,\\
\mathcal{A}_2 &= \kappa(q_1+1)\,\mathrm{diag}(\alpha_1,\cdots,\alpha_1) - \Gamma.
\end{aligned}$$

By the theory of M-matrices (see, e.g., [25, Theorem 2.10]), we see easily that both are nonsingular M-matrices. Moreover, when κ is sufficiently large, $\theta_i \approx 1/(2\kappa\alpha_1)$ and $\bar\theta_i \approx 1/(\kappa\alpha_1(q_1+1))$ for all $i \in S$. It then easy to see (3.7) is satisfied. In other words, for a sufficiently large number κ, the control function $u(x,i,t) = Ax^T$ meets Rule 3.3 as long as $\lambda_{\max}(A) \le -(\kappa+1)\alpha_1$. Of course, in application, we need to make full use of the special forms of both coefficients f and g to design the control function u more wisely.

Let us now explain why we propose Rule 3.3. Define a function $U: R^n\times S \to R_+$ by

$$U(x,i) = \theta_i|x|^2 + \bar\theta_i|x|^{q_1+1}, \quad (x,i)\in R^n\times S \tag{3.10}$$

while define a function $\mathcal{L}_2U : R^n\times R^n\times S\times R_+ \to R$ with respect to (1.2) by

$$\begin{aligned}
&\mathcal{L}_2U(x,y,i,t)\\
&= 2\theta_i\Big[x^T[f(x,y,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2\Big]\\
&+ (q_1+1)\bar\theta_i|x|^{q_1-1}x^T[f(x,y,i,t)+u(x,i,t)]\\
&+ \frac{(q_1+1)\bar\theta_i|x|^{q_1-1}}{2}|g(x,y,i,t)|^2 + \sum_{j=1}^N \gamma_{ij}\bar\theta_j|x|^{q_1+1}\\
&+\sum_{j=1}^N \gamma_{ij}\theta_j|x|^2 + \frac{(q_1+1)(q_1-1)\bar\theta_i}{2}|x|^{q_1-3}|x^Tg(x,y,i,t)|^2.
\end{aligned} \tag{3.11}$$

Please note that $\mathcal{L}_2U$ is only a function (not $\mathcal{L}_2$ acting on U) and it is associated with the the diffusion operator of the controlled SDDE (1.2) (where the control is non-delay one). By (3.4), (3.5) and (3.8), (3.9), we have

$$\begin{aligned}
&\mathcal{L}_2U(x,y,i,t)\\
&\le -|x|^2 + \gamma_1|y|^2 - \gamma_2|x|^p + \gamma_3|y|^p\\
&- |x|^{q_1+1} + \gamma_4|x|^{q_1-1}|y|^2 - \gamma_5|x|^{p+q_1-1} + \gamma_6|x|^{q_1-1}|y|^p\\
&\le -|x|^2 + \gamma_1|y|^2 - \gamma_2|x|^p + \gamma_3|y|^p + \frac{2\gamma_4}{q_1+1}|y|^{q_1+1}\\
&- \left(1 - \frac{\gamma_4(q_1-1)}{q_1+1}\right)|x|^{q_1+1} + \frac{\gamma_6 p}{p+q_1-1}|y|^{p+q_1-1}\\
&- \left(\gamma_5 - \frac{\gamma_6(q_1-1)}{p+q_1-1}\right)|x|^{p+q_1-1}.
\end{aligned} \tag{3.12}$$

By [11, Theorem 3.1] and condition (3.7), we know that the controlled SDDE (1.2) is asymptotically stable. In other words, the control function $u(x,i,t)$ satisfying Rules 3.1 and 3.3 will stabilise the given SDDE if the feedback control is of non-delay, i.e., $u(x(t),r(t),t)$. However, as explained in Section 1, it is better to use the delay state feedback control $u(x(t-\tau),r(t),t)$. That is, the controlled SDDE should be of the form (1.3) instead of (1.2). Comparing (1.3) with (1.2), we observe that if τ, the time lag between the time when the state is observed and that when the feedback control reaches the system, is sufficiently small, equation (1.3) should behave similarly to what equation (1.2) performs (i.e., stable). To describe "sufficiently small" more precisely while to cope with the highly nonlinear nature of the underlying SDDE, we now propose one more rule.

Rule 3.4 *Find eight positive constants ρ_j $(1 \le j \le 8)$ with $\rho_4 > \rho_5$ and $\rho_6 \in (0,1)$, and a function $W \in C(R^n;R_+)$, such that*

$$\begin{aligned}
&\mathcal{L}_2U(x,y,i,t) + \rho_1\big(2\theta_i|x| + (q_1+1)\bar\theta_i|x|^{q_1}\big)^2\\
&\quad + \rho_2|f(x,y,i,t)|^2 + \rho_3|g(x,y,i,t)|^2\\
&\qquad \le -\rho_4|x|^2 + \rho_5|y|^2 - W(x) + \rho_6W(y),
\end{aligned} \tag{3.13}$$

and $\rho_7|x|^{p+q_1-1} \le W(x) \le \rho_8(1+|x|^{p+q_1-1})$ (3.14)

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$.

Let us now explain why it is always possible to meet this rule. In fact, by Assumption 2.1 and (3.12), we have that

$$\begin{aligned}&\text{the left-hand-side terms of (3.13)}\\ &\le \mathcal{L}_2U(x,y,i,t) + 8\rho_1\theta_i^2|x|^2 + 2\rho_1(q_1+1)^2\bar\theta_i^2|x|^{2q_1}\\ &+ 4\rho_2K^2(|x|^2+|y|^2+|x|^{2q_1}+|x|^{2q_2})\\ &+ 4\rho_3K^2(|x|^2+|y|^2+|x|^{2q_3}+|x|^{2q_4}). \qquad (3.15)\end{aligned}$$

Recalling (2.5), we have $p+q_1-1 \ge 2(q_1\vee q_2\vee q_3\vee q_4)$ and hence

$$u^{2q_i} \le u^2 + u^{p+q_1-1}, \quad \forall u \ge 0,\ 1 \le j \le 4.$$

Making use of this inequality and (3.12), we can always choose ρ_1, ρ_2 and ρ_3 sufficiently small such that

$$\begin{aligned}&\text{the left-hand-side terms of (3.13)}\\ &\le -\rho_4|x|^2 + \rho_5|y|^2 - \xi_1|x|^p + \xi_2|y|^p - \xi_3|x|^{q_1+1}\\ &+ \xi_4|y|^{q_1+1} - \xi_5|x|^{p+q_1-1} + \xi_6|y|^{p+q_1-1}, \qquad (3.16)\end{aligned}$$

where ρ_4, ρ_5 and ξ_j $(1 \le j \le 6)$ are all positive numbers such that $\rho_4 > \rho_5$ and $\xi_{2k-1} > \xi_{2k}$ for $1 \le k \le 3$. Letting

$$W(x) = \xi_1|x|^p + \xi_3|x|^{q_1+1} + \xi_5|x|^{p+q_1-1} \quad \text{for } x \in R^n$$

and $\rho_6 = \max_{1\le k\le 3}\xi_{2k}/\xi_{2k-1}$, $\rho_7 = \xi_5$, $\rho_8 = \xi_1+\xi_3+\xi_5$, we see that $\rho_6 \in (0,1)$,

$$\begin{aligned}&\text{the left-hand-side terms of (3.13)}\\ &\le -\rho_4|x|^2 + \rho_5|y|^2 - W(x) + \rho_6W(y), \qquad (3.17)\end{aligned}$$

$$\text{and} \quad \rho_7|x|^{p+q_1-1} \le W(x) \le \rho_8(1+|x|^{p+q_1-1}).$$

We have therefore shown that it is always possible to meet Rule 3.4. Of course, in application, we need to make full use of the special forms of both coefficients f and g to choose ρ_1 - ρ_5 more wisely in order to have a larger bound on τ as stated in our final rule.

Rule 3.5 *The time lag* τ *satisfies*

$$\tau < \frac{\sqrt{(\rho_4-\rho_5)\rho_1}}{2\beta^2}, \quad \tau \le \frac{\sqrt{\rho_1\rho_2}}{\sqrt{2}\beta} \wedge \frac{\rho_1\rho_3}{\beta^2} \wedge \frac{1}{4\beta}. \qquad (3.18)$$

4 Stabilisation

4.1 H_∞ stabilisation

We can now form our first theorem on the stabilisation by the delay feedback control.

Theorem 4.1 *Under Assumptions 2.1 and 2.2, we can design a control function u to satisfy Rules 3.1 and 3.3 and then find eight positive constants ρ_j $(1 \le j \le 8)$ and a function $W \in C(R^n; R_+)$ to satisfy Rule 3.4. If we further make sure τ to be sufficiently small for Rule 3.5 to hold, then the solution of the controlled SDDE (1.3) with the initial data (2.1) has the property that*

$$\int_0^\infty \mathbb{E}|x(t)|^{\bar q}dt < \infty, \quad \forall \bar q \in [2, p+q_1-1]. \qquad (4.1)$$

That is, the controlled system (1.3) is H_∞-stable in $L^{\bar q}$ for any $\bar q \in [2, p+q_1-1]$.

Proof. To make the proof more understandable, we divide it into a number of steps.

Step 1. We will use the method of Lyapunov functionals to prove the theorem (please see, e.g., [4,7–10,14] for more details on the method). For this purpose, we define two segments $\hat x_t := \{x(t+s) : -2\delta \le s \le 0\}$ and $\hat r_t := \{r(t+s) : -2\delta \le s \le 0\}$ for $t \ge 0$. For $\hat x_t$ and $\hat r_t$ to be well defined for $0 \le t < 2\delta$, we set $x(s) = x_0$ and $r(s) = r_0$ for $s \in [-2\delta, 0)$. The Lyapunov functional used in this proof has the form

$$V(\hat x_t, \hat r_t, t) = U(x(t), r(t)) + I(t) \qquad (4.2)$$

for $t \ge 0$, where U has been defined by (3.10), ζ is a positive constant to be determined later and

$$\begin{aligned}I(t) = \zeta\int_{-\tau}^0\int_{t+s}^t \Big[&\tau|f(x(v),x(v-\delta),r(v),v)\\ &+ u(x(v-\tau),r(v),v)|^2\\ &+ |g(x(v),x(v-\delta),r(v),v)|^2\Big]dvds. \qquad (4.3)\end{aligned}$$

Here we set $f(x,y,i,v) = f(x,y,i,0)$, $u(x,i,v) = u(x,i,0)$, $g(x,y,i,v) = g(x,y,i,0)$ for $(x,i,y,v) \in R^n \times R^n \times S \times [-2\delta, 0)$. We claim that $V(\hat x_t, \hat r_t, t)$ is an Itô process on $t \ge 0$. In fact, by the generalised Itô formula (see, e.g., [25]), we have

$$\begin{aligned}dU(x(t),r(t)) = dM(t) + \Big(&\mathcal{L}_3U(x(t),x(t-\delta),r(t),t)\\ &- (2\theta_i + (q_1+1)\bar\theta_i|x|^{q_1-1})x^T(t)[u(x(t),r(t),t)\\ &- u(x(t-\tau),r(t),t)]\Big)dt \qquad (4.4)\end{aligned}$$

for $t \ge 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper so we do not state it here but it can be found in [25, Theorem 1.45 on page 48]) and $\mathcal{L}_3U : R^n \times R^n \times$

$S \times R_+ \to R$ with respect to (1.3) is defined by

$$\begin{aligned}
&\mathcal{L}_3 U(x,y,i,t)\\
&= 2\theta_i \Big[x^T[f(x,y,i,t)+u(x,i,t)] + \frac{1}{2}|g(x,y,i,t)|^2\Big]\\
&+ (q_1+1)\bar{\theta}_i |x|^{q_1-1} x^T [f(x,y,i,t)+u(x,i,t)]\\
&+ \frac{q_1(q_1+1)\bar{\theta}_i}{2}|x|^{q_1-1}|g(x,y,i,t)|^2\\
&+ \frac{(q_1+1)(q_1-1)}{2}\bar{\theta}_i |x|^{q_1-3}|x^T g(x,y,i,t)|^2\\
&+ \sum_{j=1}^N \gamma_{ij}(\theta_j|x|^2 + \bar{\theta}_j|x|^{q_1+1}).
\end{aligned}$$

On the other hand, the fundamental theory of calculus shows

$$\begin{aligned}
dI(t) = \Big(&\zeta\tau\Big[\tau|f(x(t),x(t-\delta),r(t),t)\\
&+ u(x(t-\tau),r(t),t)|^2 + |g(x(t),x(t-\delta),r(t),t)|^2\Big]\\
&- \zeta\int_{t-\tau}^t \Big[\tau|f(x(v),x(v-\delta),r(v),v)\\
&\quad + u(x(v-\tau),r(v),v)|^2\\
&\quad + |g(x(v),x(v-\delta),r(v),v)|^2\Big]dv\Big)dt. \qquad (4.5)
\end{aligned}$$

Summing (4.4) and (4.5), we see that $V(\hat{x}_t,\hat{r}_t,t)$ is an Itô process as claimed, while also noting $\mathcal{L}_3 U(x,y,i,t) \le \mathcal{L}_2 U(x,y,i,t)$ (the function $\mathcal{L}_2 U$ has been defined by (3.11)), we get

$$dV(\hat{x}_t,\hat{r}_t,t) \le \mathcal{L}_3 V(\hat{x}_t,\hat{r}_t,t)dt + dM(t), \qquad (4.6)$$

where

$$\begin{aligned}
&\mathcal{L}_3 V(\hat{x}_t,\hat{r}_t,t) = \mathcal{L}_2 U(x(t),x(t-\delta),r(t),t)\\
&- [2\theta_{r(t)} + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1-1}]x^T(t)\\
&\quad \times [u(x(t),r(t),t) - u(x(t-\tau),r(t),t)]\\
&+ \zeta\tau\Big[\tau|f(x(t),x(t-\delta),r(t),t) + u(x(t-\tau),r(t),t)|^2\\
&\quad + |g(x(t),x(t-\delta),r(t),t)|^2\Big]\\
&- \zeta\int_{t-\tau}^t \Big[\tau|f(x(v),x(v-\delta),r(v),v)\\
&+ u(x(v-\tau),r(v),v)|^2 + |g(x(v),x(v-\delta),r(v),v)|^2\Big]dv. \qquad (4.7)
\end{aligned}$$

Moreover, by Theorem 3.2 and Assumptions 2.1 and 2.2 as well as Rules 3.1 and 3.4, it is straightforward to see that

$$\sup_{0\le t<\infty} \mathbb{E}|\mathcal{L}_3 V(\hat{x}_t,\hat{r}_t,t)| < \infty. \qquad (4.8)$$

Step 2. In this step we will estimate $\mathcal{L}_3 V(\hat{x}_t,\hat{r}_t,t)$. Let $\zeta = \beta^2/\rho_1$, (please recall that ζ is a free parameter in the definition of the Lyapunov functional). By Rule 3.1, we have

$$\begin{aligned}
&- [2\theta_{r(t)} + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1-1}]x^T(t)\\
&\times [u(x(t),r(t),t) - u(x(t-\tau),r(t),t)]\\
&\quad \le \rho_1\big[2\theta_{r(t)}|x(t)| + (q_1+1)\bar{\theta}_{r(t)}|x(t)|^{q_1}\big]^2\\
&\qquad + \frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2. \qquad (4.9)
\end{aligned}$$

By Rule 3.5, we also have

$$2\zeta\tau^2 \le \rho_2 \quad \text{and} \quad \zeta\tau \le \rho_3. \qquad (4.10)$$

It then follows from (4.7) along with Rule 3.4 and inequality (3.2) that

$$\begin{aligned}
&\mathcal{L}_3 V(\hat{x}_t,\hat{r}_t,t)\\
&\le -\rho_4|x(t)|^2 + \rho_5|x(t-\delta)|^2 - W(x(t)) + \rho_6 W(x(t-\delta))\\
&+ \frac{2\tau^2\beta^4}{\rho_1}|x(t-\tau)|^2 + \frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2\\
&- \frac{\beta^2}{\rho_1}\int_{t-\tau}^t \Big[\tau|f(x(v),x(v-\delta),r(v),v)\\
&+ u(x(v-\tau),r(v),v)|^2 + |g(x(v),x(v-\delta),r(v),v)|^2\Big]dv.
\end{aligned}$$

But, noting $\beta\tau \le 1/4$ from Rule 3.5, we have

$$\frac{2\tau^2\beta^4}{\rho_1}|x(t-\tau)|^2 \le \frac{4\tau^2\beta^4}{\rho_1}|x(t)|^2 + \frac{\beta^2}{4\rho_1}|x(t)-x(t-\tau)|^2.$$

Consequently,

$$\begin{aligned}
&\mathcal{L}_3 V(\hat{x}_t,\hat{r}_t,t)\\
&\le -\Big(\rho_4 - \frac{4\tau^2\beta^4}{\rho_1})|x(t)|^2 + \rho_5|x(t-\delta)|^2 - W(x(t))\\
&+ \rho_6 W(x(t-\delta)) + \frac{\beta^2}{2\rho_1}|x(t)-x(t-\tau)|^2\\
&- \frac{\beta^2}{\rho_1}\int_{t-\tau}^t \Big[\tau|f(x(v),x(v-\delta),r(v),v)\\
&\quad + u(x(v-\tau),r(v),v)|^2\\
&\quad + |g(x(v),x(v-\delta),r(v),v)|^2\Big]dv. \qquad (4.11)
\end{aligned}$$

Step 3. Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \ge k_0$, define the stopping time

$$\zeta_k = \inf\{t \ge 0 : |x(t)| \ge k\},$$

where throughout this paper we set $\inf\emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). By Theorem 3.2, we see that ζ_k is increasing to infinity with probability 1 as $k \to \infty$.

By the generalised Itô formula (see, e.g., [25, Lemma 1.9 on page 49]), we obtain from (4.6) that

$$\mathbb{E}V(\hat{x}_{t\wedge\zeta_k},\hat{r}_{t\wedge\zeta_k},t\wedge\zeta_k) \le V(\hat{x}_0,\hat{r}_0,0)+\mathbb{E}\int_0^{t\wedge\zeta_k}\mathcal{L}_3V(\hat{x}_s,\hat{r}_s,s)ds \tag{4.12}$$

for any $t\ge 0$ and $k\ge k_0$. Recalling (4.8), we can let $k\to\infty$ and then apply the dominated convergence theorem as well as the Fubini theorem to get

$$\mathbb{E}V(\hat{x}_t,\hat{r}_t,t)\le V(\hat{x}_0,\hat{r}_0,0)+\int_0^t\mathbb{E}\mathcal{L}_3V(\hat{x}_s,\hat{r}_s,s)ds \tag{4.13}$$

for any $t\ge 0$. By (4.11), we have

$$\begin{aligned}&\mathbb{E}\mathcal{L}_3V(\hat{x}_s,\hat{r}_s,s)\\ &\le -\Big(\rho_4-\frac{4\tau^2\beta^4}{\rho_1}\Big)\mathbb{E}|x(s)|^2+\rho_5\mathbb{E}|x(s-\delta)|^2\\ &-\mathbb{E}W(x(s))+\rho_6\mathbb{E}W(x(s-\delta))+\frac{\beta^2}{2\rho_1}\mathbb{E}|x(s)-x(s-\tau)|^2\\ &-\frac{\beta^2}{\rho_1}\mathbb{E}\int_{s-\tau}^s\Big[\tau|f(x(v),x(v-\delta),r(v),v)\\ &+u(x(v-\tau),r(v),v)|^2+|g(x(v),x(v-\delta),r(v),v)|^2\Big]dv.\end{aligned} \tag{4.14}$$

On the other hand, it follows from the SDDE (1.3) that

$$\begin{aligned}&\mathbb{E}|x(s)-x(s-\tau)|^2\\ &\le 2\mathbb{E}\int_{s-\tau}^s\Big[\tau|f(x(v),x(v-\delta),r(v),v)\\ &+u(x(v-\tau),r(v),v)|^2+|g(x(v),x(v-\delta),r(v),v)|^2\Big]dv.\end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.14) and then putting the result into (4.13) we get

$$\begin{aligned}&\mathbb{E}V(\hat{x}_t,\hat{r}_t,t)\\ &\le V(\hat{x}_0,\hat{r}_0,0)-\Big(\rho_4-\frac{4\tau^2\beta^4}{\rho_1}\Big)\int_0^t\mathbb{E}|x(s)|^2ds\\ &+\rho_5\int_0^t\mathbb{E}|x(s-\delta)|^2ds-\int_0^t\mathbb{E}W(x(s))ds\\ &+\rho_6\int_0^t\mathbb{E}W(x(s-\delta))ds.\end{aligned} \tag{4.16}$$

This implies easily that

$$\begin{aligned}\mathbb{E}V(\hat{x}_t,\hat{r}_t,t)\le C_1-\Big(\rho_4-\rho_5-\frac{4\tau^2\beta^4}{\rho_1}\Big)\int_0^t\mathbb{E}|x(s)|^2ds\\ -(1-\rho_6)\int_0^t\mathbb{E}W(x(s))ds,\end{aligned} \tag{4.17}$$

where

$$C_1=V(\hat{x}_0,\hat{r}_0,0)+\tau\sup_{-\delta\le s\le 0}[\rho_5\mathbb{E}|x(s)|^2+\rho_6\mathbb{E}W(x(s))]<\infty.$$

By Rule 3.5, $\rho_4-\rho_5-4\tau^2\beta^4/\rho_1>0$ and $1-\rho_6>0$. Hence

$$\int_0^t\mathbb{E}|x(s)|^2ds\le\frac{C_1}{\rho_4-\rho_5-4\tau^2\beta^4/\rho_1},$$
$$\int_0^t\mathbb{E}W(x(s))ds\le\frac{C_1}{1-\rho_6}.$$

Letting $t\to\infty$ and recalling (3.14), we obtain that

$$\int_0^\infty\mathbb{E}|x(s)|^2ds<\infty\quad\text{and}\quad\int_0^\infty\mathbb{E}|x(s)|^{p+q_1-1}ds<\infty. \tag{4.18}$$

The required assertion (4.1) follows immediately as $\mathbb{E}|x(s)|^{\bar q}\le\mathbb{E}|x(s)|^2ds+\mathbb{E}|x(s)|^{p+q_1-1}$ for any $\bar q\in[2,p+q_1-1]$. The proof is therefore complete. □

The next slightly weaker result follows directly from the above proof but under a weaker condition.

Corollary 4.2 *Under the same conditions of Theorem 4.1 except that $\rho_6\in(0,1)$ is replaced by $\rho_6=1$ in Rule 3.4, then the solution of the controlled SDDE (1.3) with the initial data (2.1) has the property that*

$$\int_0^\infty\mathbb{E}|x(t)|^2dt<\infty. \tag{4.19}$$

4.2 Asymptotic stabilisation

In general, it does not follow from (4.19) that $\lim_{t\to\infty}\mathbb{E}|x(t)|^2=0$. However, this is possible in our situation. In fact, we can even show a stronger result as described in the following theorem.

Theorem 4.3 *Under the same conditions of Theorem 4.2, the solution of the controlled hybrid SDDE (1.3) with the initial data (2.1) has the property that*

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^{\bar q}=0,\quad\forall\bar q\in[2,q) \tag{4.20}$$

That is, the controlled system (1.3) is asymptotically stable in $L^{\bar q}$ for any $\bar q\in[2,q)$.

Proof. By Theorem 3.2,

$$C_2:=\sup_{0\le t<\infty}\mathbb{E}|x(t)|^q<\infty. \tag{4.21}$$

For any $0 \le t_1 < t_2 < \infty$, using the Itô formula, by Assumption 2.1 and Rule 3.1, we see

$$\begin{aligned}
&\big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\big| \\
\le &\mathbb{E}\int_{t_1}^{t_2} \Big(2K|x(t)|(|x(t)| + |x(t-\delta)| + |x(t)|^{q_1}\Big) \\
&+ 2K|x(t)||x(t-\delta)|^{q_1} + 2\beta|x(t)||x(t-\tau)| \\
&+ K^2\big[|x(t)| + |x(t-\delta)| + |x(t)|^{q_3} + |x(t-\delta)|^{q_4})\big]^2\Big)dt \\
\le &\int_{t_1}^{t_2} C_3\big(1 + \mathbb{E}|x(t)|^q + \mathbb{E}|x(t-\delta)|^q\big)dt,
\end{aligned}$$

where C_3 is a constant independent of t_1 and t_2. This, together with (4.21), implies $\mathbb{E}|x(t)|^2$ is uniformly continuous in t on R_+. It then follows from (4.19) that

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0. \tag{4.22}$$

Let us now fix any $\bar{q} \in (2, q)$. For a constant $\nu \in (0,1)$, the Hölder inequality shows

$$\mathbb{E}|x(t)|^{\bar{q}} \le \big(\mathbb{E}|x(t)|^2\big)^{\nu}\big(\mathbb{E}|x(t)|^{(\bar{q}-2\nu)/(1-\nu)}\big)^{1-\nu}.$$

In particular, letting $\nu = (q-\bar{q})/(q-2)$, we get

$$\begin{aligned}
\mathbb{E}|x(t)|^{\bar{q}} &\le \big(\mathbb{E}|x(t)|^2\big)^{(q-\bar{q})/(q-2)}\big(\mathbb{E}|x(t)|^q\big)^{(\bar{q}-2)/(q-2)} \\
&\le C_2^{(\bar{q}-2)/(q-2)}\big(\mathbb{E}|x(t)|^2\big)^{(q-\bar{q})/(q-2)}. \qquad (4.23)
\end{aligned}$$

This, along with (4.22), implies the required assertion (4.19). The proof is complete. □

4.3 Exponential stabilisation

Asymptotic stabilisation discussed above shows the solution of the controlled SDDE (1.3) will tend to zero in $L^{\bar{q}}$ asymptotically but does not show the rate of decay. In this subsection, we will take a further step to show how the delay feedback control can stabilise the given SDDE exponentially (namely, the solution of the controlled SDDE (1.3) will tend to zero exponentially fast). The following theorem shows that under slightly stronger conditions than those in Theorem 4.1 (condition (4.24) below is stronger than Rule 3.5), the delay feedback control can stabilise the given SDDE exponentially in the sense of $L^{\bar{q}}$.

Theorem 4.4 *Under Assumptions 2.1 and 2.2, we can design a control function u to satisfy Rules 3.1 and 3.3 and then find eight positive constants ρ_j ($1 \le j \le 8$) and a function $W \in C(R^n; R_+)$ to satisfy Rule 3.4. If we further make sure*

$$\tau < \frac{\sqrt{(\rho_4-\rho_5)\rho_1}}{2\beta^2} \quad \text{and} \quad \tau \le \frac{\sqrt{\rho_1\rho_2}}{\sqrt{2}\beta} \wedge \frac{\rho_1\rho_3}{\beta^2} \wedge \frac{1}{4\sqrt{2}\beta}, \tag{4.24}$$

then the solution of the controlled SDDE (1.3) with the initial data (2.1) has the property that for any initial value $x(0) = x_0 \in R^n$,

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^{\bar{q}}) < 0, \quad \forall \bar{q} \in [2, q). \tag{4.25}$$

That is, the controlled SDDE (1.3) is exponentially stable in $L^{\bar{q}}$.

Proof. We will use the same Lyapunov functional $V(\hat{x}_t, \hat{r}_t, t)$ as defined by (4.2) with the same $\zeta = \beta^2/\rho_1$. By the method of stopping times as we did in Step 3 of the proof of Theorem 4.1, we can show that

$$\begin{aligned}
&e^{\varepsilon t}\mathbb{E}V(\hat{x}_t, \hat{r}_t, t) \le V(\hat{x}_0, \hat{r}_0, 0) \\
&+ \int_0^t e^{\varepsilon s}\mathbb{E}\Big(\varepsilon V(\hat{x}_s, \hat{r}_s, s) + \mathcal{L}V(\hat{x}_s, \hat{r}_s, s)\Big)ds \qquad (4.26)
\end{aligned}$$

for all $t \ge 0$, where ε is a sufficiently small positive number to be determined later. Let

$$h_1 = \min_{i\in S}\theta_i, \quad h_2 = \max_{i\in S}\theta_i, \quad h_3 = \max_{i\in S}\bar{\theta}_i,$$

we then have

$$\begin{aligned}
&h_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le V(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\beta^2}{\rho_1}J_1(t) \\
&+ \int_0^t e^{\varepsilon s}\Big(\varepsilon h_2\mathbb{E}|x(s)|^2 + \varepsilon h_3\mathbb{E}|x(s)|^{q_1+1}\Big)ds \\
&+ \int_0^t e^{\varepsilon s}\mathbb{E}\mathcal{L}V(\hat{x}_s, \hat{r}_s, s)ds, \qquad (4.27)
\end{aligned}$$

where

$$\begin{aligned}
J_1(t) = \mathbb{E}\int_0^t e^{\varepsilon s}\Big(\int_{-\tau}^0\int_{s+u}^s \Big[&\tau|f(x(v), x(v-\delta), r(v), v) \\
&+ u(x(v-\tau), r(v), v)|^2 \\
&+ |g(x(v), x(v-\delta), r(v), v)|^2\Big]dvdu\Big)ds.
\end{aligned}$$

As we did in Step 2 of the proof of Theorem 4.4, we can show that

$$\begin{aligned}
&\mathcal{L}_3V(\hat{x}_s, \hat{r}_s, s) \\
&\le -\Big(\rho_4 - \frac{4\tau^2\beta^4}{\rho_1})|x(s)|^2 + \rho_5|x(s-\delta)|^2 \\
&- W(x(s)) + \rho_6 W(x(s-\delta)) + \frac{3\beta^2}{8\rho_1}|x(s) - x(s-\tau)|^2 \\
&- \frac{\beta^2}{\rho_1}\int_{s-\tau}^s \Big[\tau|f(x(v), x(v-\delta), r(v), v) \\
&+ u(x(v-\tau), r(v), v)|^2 + |g(x(v), x(v-\delta), r(v), v)|^2\Big]dv.
\end{aligned} \tag{4.28}$$

Making use of (4.15), we get

$$\begin{aligned}
&\mathcal{L}_3 V(\hat{x}_s, \hat{r}_s, s) \\
&\le -\Big(\rho_4 - \frac{4\tau^2\beta^4}{\rho_1}\big)|x(s)|^2 + \rho_5|x(s-\delta)|^2 - W(x(t)) \\
&+ \rho_6 W(x(t-\delta)) - \frac{\beta^2}{4\rho_1}\int_{s-\tau}^{s}\Big[\tau|f(x(v), x(v-\delta), r(v), v) \\
&+ u(x(v-\tau), r(v), v)|^2 + |g(x(v), x(v-\delta), r(v), v)|^2\Big]dv.
\end{aligned} \tag{4.29}$$

Moreover, we clearly have

$$\begin{aligned}
\mathbb{E}|x(s)|^{q_1+1} &\le \mathbb{E}|x(s)|^2 + \mathbb{E}|x(s)|^{p+q_1-1} \\
&\le \mathbb{E}|x(s)|^2 + \rho_7^{-1}\mathbb{E}W(x(s)).
\end{aligned} \tag{4.30}$$

Substituting (4.29) and (4.30) into (4.27) yields

$$\begin{aligned}
&h_1 e^{\varepsilon t}\mathbb{E}|x(t)|^2 \le V(\hat{x}_0, \hat{r}_0, 0) + \frac{\varepsilon\beta^2}{\rho_1}J_1(t) - \frac{\beta^2}{4\rho_1}J_2(t) \\
&- \Big(\rho_4 - \frac{4\tau^2\beta^4}{\rho_1} - \varepsilon h_2 - \varepsilon h_3\Big)\int_0^t e^{\varepsilon s}\mathbb{E}|x(s)|^2 ds \\
&+ \rho_5\int_0^t e^{\varepsilon s}\mathbb{E}|x(s-\delta)|^2 ds + \rho_6\int_0^t e^{\varepsilon s}\mathbb{E}W(x(s-\delta))ds \\
&- \Big(1 - \frac{\varepsilon h_3}{\rho_7}\big)\int_0^t e^{\varepsilon s}\mathbb{E}W(x(s))ds,
\end{aligned} \tag{4.31}$$

where

$$\begin{aligned}
J_2(t) = \mathbb{E}\int_0^t e^{\varepsilon s}\Big(\int_{s-\tau}^{s}\Big[&\tau|f(x(v), x(v-\delta), r(v), v) \\
&+ u(x(v-\tau), r(v), v)|^2 \\
&+ |g(x(v), x(v-\delta), r(v), v)|^2\Big]dv\Big)ds.
\end{aligned}$$

On the other hand, it is easy to see that

$$J_1(t) \le \tau J_2(t).$$

We can now choose a sufficiently small $\varepsilon > 0$ such that

$$\varepsilon\tau \le \frac{1}{4},\ \ \varepsilon(h_2+h_3)+\rho_5 e^{\varepsilon\delta} \le \rho_4 - \frac{4\tau^2\beta^4}{\rho_1},\ \ \rho_6 e^{\varepsilon\delta} + \frac{\varepsilon h_3}{\rho_7} \le 1.$$

We can then easily show from (4.31) that

$$\mathbb{E}|x(t)|^2 \le C_4 e^{-\varepsilon t}, \quad \forall t \ge 0, \tag{4.32}$$

where

$$\begin{aligned}
C_4 = h_1^{-1}\big[&V(\hat{x}_0, \hat{r}_0, 0) \\
&+ \tau(\rho_5 \vee \rho_6)\sup_{-\delta\le s\le 0}(\mathbb{E}|x(s)|^2 + \mathbb{E}W(x(s)))\big].
\end{aligned}$$

Finally, for any $\bar{q} \in [2, q)$, by (4.23) and (4.32), we get

$$\mathbb{E}|x(t)|^{\bar{q}} \le C_2^{(\bar{q}-2)/(q-2)} C_4^{(q-\bar{q})/(q-2)} e^{-\varepsilon t(q-\bar{q})/(q-2)}. \tag{4.33}$$

This implies the required assertion (4.25). The proof is complete.

In general, it is not possible to imply the almost surely exponential stability from the $\bar{q}$th moment exponential stability. However, in our situation, this is possible as described in the following theorem.

Theorem 4.5 *Let all the conditions of Theorem 4.4 hold. Then the solution of the controlled system (1.3) with the initial data (2.1) has the property that*

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) < 0 \quad a.s. \tag{4.34}$$

That is, the controlled SDDE (1.3) is almost surely exponentially stable.

Proof. Let k be any nonnegative integer. By the Hölder inequality and the Doob martingale inequality (see, e.g., [25]), we can obtain that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{k\le t\le k+1}|x(t)|^2\Big) \\
&\le 3\mathbb{E}|x(k)|^2 + 3\mathbb{E}\int_k^{k+1}\Big(|f(x(t), x(t-\delta), r(t), t) \\
&\quad + u(x(t-\tau), r(t), t)|^2\Big)dt \\
&\quad + 12\mathbb{E}\int_k^{k+1}|g(x(t), x(t-\delta), r(t), t)|^2 dt.
\end{aligned}$$

By Assumption 2.1, it is then straightforward to show that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{k\le t\le k+1}|x(t)|^2\Big) \le 3\mathbb{E}|x(k)|^2 \\
&+ C_5\int_k^{k+1}\mathbb{E}\Big(|x(t)|^2 + |x(t-\delta)|^2 + |x(t-\tau)|^2\Big)dt \\
&+ C_5\int_k^{k+1}\mathbb{E}\Big(|x(t)|^{\bar{q}} + |x(t-\delta)|^{\bar{q}}\Big)dt,
\end{aligned}$$

where $\bar{q} = 2(q_1 \vee q_2 \vee q_3 \vee q_4)$ and C_5 is a positive constant. Noting that $\bar{q} \in [2, q)$ by Assumption 2.2, we can apply (4.32) and (4.33) (both of them hold for $t \in [-\delta, 0]$ as well) to get

$$\mathbb{E}\Big(\sup_{k\le t\le k+1}|x(t)|^2\Big) \le C_6 e^{-\bar{\varepsilon}k},$$

where $\bar{\varepsilon} = \varepsilon(q - \bar{q})/(q - 2)$ and C_6 is another positive constant. Consequently

$$\sum_{k=0}^{\infty} \mathbb{P}\Big(\sup_{k \le t \le k+1} |x(t)| > e^{-0.25\bar{\varepsilon}k} \Big) \le \sum_{k=0}^{\infty} C_6 e^{-0.5\bar{\varepsilon}k} < \infty.$$

The well-known Borel-Cantelli lemma (see, e.g., [25, p.10]) shows that for almost all $\omega \in \Omega$, there is positive integer $k_0 = k_0(\omega)$ such that

$$\sup_{k \le t \le k+1} |x(t)| \le e^{-0.25\bar{\varepsilon}k}, \quad k \ge k_0.$$

Hence, for almost all $\omega \in \Omega$,

$$\frac{1}{t} \log(|x(t)|) \le -\frac{0.25\bar{\varepsilon}k}{(k+1)}, \quad t \in [k, k+1], \ k \ge k_0.$$

This implies

$$\limsup_{t \to \infty} \frac{1}{t} \log(|x(t)|) \le -0.25\bar{\varepsilon} < 0 \quad a.s.$$

which is the required assertion. The proof is complete. □

Let us make a number of comments to close this section. The results established in this paper are all independent of δ, which is the time lag of the given SDDE, but very much dependent on τ, which is the time lag between the time when the state is observed and the time when the feedback control reaches the system.

Rule 3.3 describes a way how to find positive numbers θ_i and $\bar{\theta}_i$ $(i \in S)$ and then further to find positive numbers ρ_j $(1 \le j \le 8)$ in Rule 3.4. On the other hands, if one can find all these positive numbers for Rule 3.4 to be satisfied, then all of our results hold without Rule 3.3.

The control function u used in this paper is allowed to depend on mode i, namely we use $u(x, i, t)$. This enables us to make use of different system structure in different mode to design the control function more wisely. It is possible to use a simpler control function which depends on the state x only, namely $u(x)$, for example, $u(x) = Ax$ as shown in the paragraph below Rule 3.3. Of course, this is applicable only in the situation where the state is observable and the feedback control can be input in every mode. In some situation where the state of the underlying system is not observable in some modes, we have to design the feedback control function only on those modes which state is observable and put no control on the other modes. The examples discussed in the next section illustrate these situations fully.

5 Examples

To illustrate our theoretical results, we will discuss a couple of examples.

Example 5.1 Let us return to the hybrid SDE (2.10), where the coefficients f and g are defined by (2.11), $B(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on $S = \{1, 2\}$ with the generator Γ defined by (2.12). As we mentioned in Section 1, this is a simple version of hybrid SDE models appeared frequently in finance and population systems (see, e.g., [3,20]).

Recalling the last paragraph in Section 2, we know that the SDDE (2.10) satisfies Assumptions 2.1 and 2.2 with any $q > 6$ and $p = 4$, $q_1 = q_2 = 3$, $q_3 = 1.5$, $q_4 = 1$, $\alpha_1 = (1+0.25(q-1)^2) \vee (q-1)$, $\alpha_2 = 1.25$ and $\alpha_3 = 0.5$.

We first consider the case where the system is fully observable and controllable in both mode 1 and 2. That is, we could use a feedback control in both modes to stabilise the given unstable hybrid SDDE (2.10). In our notation, we will use the control function $u : R \times S \times R_+ \to R$ defined by

$$u(x, 1, t) = -5x, \quad u(x, 2, t) = -4x. \tag{5.1}$$

Obviously, Rule 3.1 is satisfied with $\beta = 5$. By Theorem 3.2, the controlled system

$$\begin{aligned} dx(t) = [f(x(t), x(t-\delta), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ + g(x(t), x(t-\delta), r(t), t)dB(t) \end{aligned} \tag{5.2}$$

has a unique global solution on $t \ge -\delta$ for any initial data $\xi \in C([-\delta, 0]; R)$ and the solution has the property that

$$\sup_{-\delta \le t < \infty} \mathbb{E}|x(t)|^q < \infty \quad \forall q > 6. \tag{5.3}$$

Let us now verify Rule 3.3. It is straightforward to show that, for $(x, y, i, t) \in R \times R \times S \times R_+$,

$$\begin{aligned} & x[f(x, y, i, t) + u(x, t, i)] + \frac{1}{2}|g(x, y, t, i)|^2 \\ \le & \begin{cases} -3.5x^2 + 0.25y^2 - 2x^4 + 0.5y^4, & i = 1, \\ -2.875x^2 + 0.25y^2 - 1.375x^4 + 0.5y^4, & i = 2, \end{cases} \end{aligned}$$

and

$$\begin{aligned} & x[f(x, y, i, t) + u(x, t, i)] + \frac{q_1}{2}|g(x, y, t, i)|^2 \\ \le & \begin{cases} -2.5x^2 + 0.75y^2 - x^4 + 0.5y^4, & i = 1, \\ -2.625x^2 + 0.75y^2 - 1.125x^4 + 0.5y^4, & i = 2. \end{cases} \end{aligned}$$

Namely, (3.4) and (3.5) hold with

$$\begin{aligned} & a_1 = -3.5, \ b_1 = 0.25, \ c_1 = 2, \ d_1 = 0.5, \\ & a_2 = -2.875, \ b_2 = 0.25, \ c_2 = 1.375, \ d_2 = 0.5, \\ & \bar{a}_1 = -2.5, \ \bar{b}_1 = 0.75, \ \bar{c}_1 = 1, \ \bar{d}_1 = 0.5, \\ & \bar{a}_2 = -2.625, \ \bar{b}_2 = 0.75, \ \bar{c}_2 = 1.125, \ \bar{d}_2 = 0.5. \end{aligned}$$

Moreover,

$$\mathcal{A}_1 = \begin{pmatrix} 8 & -1 \\ -1 & 6.75 \end{pmatrix} \quad \text{and} \quad \mathcal{A}_2 = \begin{pmatrix} 11 & -1 \\ -1 & 11.5 \end{pmatrix},$$

which are both M-matrices. By (3.8), we then have

$$\theta_1 = 0.1462,\ \theta_2 = 0.1698,\ \bar{\theta}_1 = 0.0996,\ \bar{\theta}_2 = 0.0956.$$

Consequently,

$$\gamma_1 = 0.0849,\ \gamma_2 = 0.4670,\ \gamma_3 = 0.1698,$$

$$\gamma_4 = 0.2988,\ \gamma_5 = 0.3984,\ \gamma_6 = 0.1992.$$

Hence (3.7) holds as well. We have therefore verified Rule 3.3. To verify Rule 3.4, we note that the function U defined by (3.10) has the form

$$U(x,i) = \begin{cases} 0.1462x^2 + 0.0996x^4, & i = 1, \\ 0.1698x^2 + 0.0956x^4, & i = 2. \end{cases}$$

By (3.12), we also have

$$\begin{aligned} \mathcal{L}_2 U(x,y,i,t) \le & -x^2 + 0.0849y^2 - 1.3176x^4 \\ & + 0.3192y^4 - 0.3320x^6 + 0.1328y^6. \end{aligned}$$

Moreover, we can show (by elementary calculations) that

$$\begin{aligned} \big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 & \le 0.1153x^2 + 0.2597x^4 \\ & \quad + 0.1587x^6, \\ |f(x,y,i,t)|^2 & \le x^2 - 4x^4 + y^4 + 9.3333x^6 + 2y^6, \\ |g(x,y,i,t)|^2 & \le 0.5x^2 + 0.5y^2 + 2x^4. \end{aligned}$$

Choosing $\rho_1 = 0.4$, $\rho_2 = 0.01$ and $\rho_3 = 0.45$, we then obtain

$$\begin{aligned} & \mathcal{L}_2 U(x,y,i,t) + \rho_1\big(2\theta_i|x| + (q_1+1)\bar{\theta}_i|x|^{q_1}\big)^2 \\ & \quad + \rho_2|f(x,y,i,t)|^2 + \rho_3|g(x,y,i,t)|^2 \\ & \le -0.7189x^2 + 0.3099y^2 - W(x) + 0.9307W(y) \quad (5.4) \end{aligned}$$

where $W(x) = 0.3537x^4 + 0.1752x^6$. That is, Rule 3.5 is satisfied with additional $\rho_4 = 0.7189$, $\rho_5 = 0.3099$, $\rho_6 = 0.9307$, $\rho_7 = 0.1752$ and $\rho_8 = 0.8419$. Consequently, Condition (4.24) becomes $\tau < 0.0071$. By Theorems 4.4 and 4.5, we can therefore conclude that the controlled system (2.10) with the control function (5.1) is not only exponentially stable in $L^{\bar{q}}$ for any $\bar{q} \ge 2$ but also almost surely exponentially stable provided $\tau < 0.0071$.

We perform a computer simulation with $\delta = 0.2$, $\tau = 0.005$, the initial data $x(t) = 1 + \cos(t)$ for $t \in [-0.2, 0]$ and $r(0) = 1$. The sample paths of the Markov chain and the solution of the controlled SDDE (5.2) are plotted in Figure 5.1. The simulation supports our theoretical results clearly.

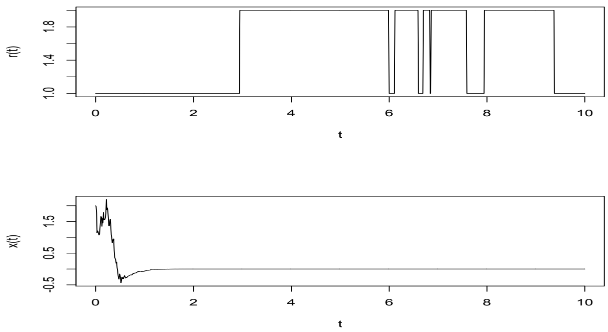

Figure 5.1: The computer simulation of the sample paths of the Markov chain and the solution of the controlled SDDE (5.2) using the Euler–Maruyama method with step size 10^{-4}.

Example 5.2 Consider the scalar hybrid SDDE

$$\begin{aligned} dx(t) = & f(x(t), x(t-\delta), r(t), t)dt \\ & + g(x(t), x(t-\delta), r(t), t)dB(t), \end{aligned} \tag{5.5}$$

where the coefficients f and g are defined by

$$\begin{aligned} f(x,y,1,t) &= x(-2x^2 + 2y), \quad g(x,y,1,t) = 0.3|y|^{3/2}, \\ f(x,y,2,t) &= x(-1.5x^2 + y), \quad g(x,y,2,t) = 0.1|y|^{3/2}, \end{aligned} \tag{5.6}$$

$B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator $\Gamma = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}$. It is easy to see that Assumptions 2.1 and 2.2 hold with $q_1 = 3$, $q_2 = 2$, $q_3 = 1$, $q_4 = 1.5$, $p = 4$ and any $q > 6$.

Assume that the system is observable only in mode 1 but not in mode 2 so we could only use a feedback control in mode 1 (namely we have to set the control function to be 0 in mode 2). Accordingly, we let the control function

$$u(x,1,t) = -3x, \quad u(x,2,t) = 0. \tag{5.7}$$

Obviously, Rule 3.1 is satisfied with $\beta = 3$. By Theorem 3.2, the controlled system

$$\begin{aligned} dx(t) = & [f(x(t), x(t-\delta), r(t), t) + u(x(t-\tau), r(t), t)]dt \\ & + g(x(t), x(t-\delta), r(t), t)dB(t) \end{aligned} \tag{5.8}$$

has a unique global solution on $t \ge -\delta$ for any initial data $\xi \in C([-\delta, 0]; R)$ and the solution has the property that

$$\sup_{-\delta \le t < \infty} \mathbb{E}|x(t)|^q < \infty \quad \forall q > 4. \tag{5.9}$$

It is straightforward to show that, for $(x, y, i, t) \in R \times R \times S \times R_+$,

$$x[f(x,y,i,t)+u(x,t,i)]+\frac{1}{2}|g(x,y,t,i)|^2 \leq \begin{cases} -3x^2+1.0225y^2-x^4+0.0225y^4, & i=1, \\ 0.5025y^2-x^4+0.0025y^4, & i=2, \end{cases}$$

and

$$x[f(x,y,i,t)+u(x,t,i)]+\frac{q_1}{2}|g(x,y,t,i)|^2 \leq \begin{cases} -3x^2+1.0675y^2-x^4+0.0675y^4, & i=1, \\ 0.5075y^2-x^4+0.0075y^4, & i=2. \end{cases}$$

Namely, (3.4) and (3.5) hold with

$$\begin{aligned} &a_1=-3,\ b_1=1.0225,\ c_1=1,\ d_1=0.0225,\\ &a_2=0,\ b_2=0.5025,\ c_2=1,\ d_2=0.0025,\\ &\bar a_1=-3,\ \bar b_1=1.0675,\ \bar c_1=1,\ \bar d_1=0.0675,\\ &\bar a_2=0,\ \bar b_2=0.5075,\ \bar c_2=1,\ \bar d_2=0.0075. \end{aligned}$$

Moreover, $\mathcal{A}_1 = \begin{pmatrix} 7 & -1 \\ -6 & 6 \end{pmatrix}$ and $\mathcal{A}_2 = \begin{pmatrix} 13 & -1 \\ -6 & 6 \end{pmatrix}$, which are both M-matrices. By (3.8), we then have

$$\theta_1=0.1944,\ \theta_2=0.3611,\ \bar\theta_1=0.0972,\ \bar\theta_2=0.2639.$$

Consequently,

$$\begin{aligned} &\gamma_1=0.3975,\ \gamma_2=0.3888,\ \gamma_3=0.0087,\\ &\gamma_4=0.5357,\ \gamma_5=0.3888,\ \gamma_6=0.0262. \end{aligned}$$

Hence (3.7) holds as well. We have therefore verified Rule 3.3. To verify Rule 3.4, we note that the function U defined by (3.10) has the form

$$U(x,i)=\begin{cases} 0.1944x^2+0.0972x^4 & \text{if } i=1, \\ 0.3611x^2+0.2639x^4 & \text{if } i=2. \end{cases}$$

By (3.12), we also have

$$\begin{aligned} \mathcal{L}_2U(x,y,i,t) \leq{}& -x^2+0.3975y^2-1.1210x^4 \\ &+0.2766y^4-0.3801x^6+0.0175y^6. \end{aligned}$$

Moreover, we can show that

$$\big(2\theta_i|x|+(q_1+1)\bar\theta_i|x|^{q_1}\big)^2 \leq 0.5215x^2+1.5247x^4+1.1143x^6,$$

$$\begin{aligned} |f(x,y,i,t)|^2 &\leq 8x^4+4y^4+8x^6, \\ |g(x,y,i,t)|^2 &\leq 0.045y^2+0.045y^4. \end{aligned}$$

Choosing $\rho_1 = 0.25$, $\rho_2 = 0.01$ and $\rho_3 = 1$, we then obtain

$$\begin{aligned} &\mathcal{L}_2U(x,y,i,t)+\rho_1\big(2\theta_i|x|+(q_1+1)\bar\theta_i|x|^{q_1}\big)^2 \\ &\quad+\rho_2|f(x,y,i,t)|^2+\rho_3|g(x,y,i,t)|^2 \\ &\leq -0.8696x^2+0.4425y^2-W(x)+0.8139W(y) \end{aligned} \tag{5.10}$$

where $W(x) = 0.6598x^4 + 0.0215x^6$. That is, Rule 3.5 is satisfied with additional $\rho_4 = 0.8283$, $\rho_5 = 0.4425$, $\rho_6 = 0.8139$, $\rho_7 = 0.0215$ and $\rho_8 = 0.4614$. Consequently, Condition (4.24) becomes $\tau < 0.01178$. By Theorems 4.4 and 4.5, we can therefore conclude that the controlled system (5.8) with the control function (5.7) is not only exponentially stable in $L^{\bar q}$ for any $\bar q \geq 2$ but also almost surely exponentially stable provided $\tau < 0.01178$.

We perform a computer simulation with $\delta = 0.2$, $\tau = 0.01$, the initial data $x(t) = 1 + \cos(t)$ for $t \in [-0.2, 0]$ and $r(0) = 2$. The sample paths of the Markov chain and the solution of the controlled SDDE (5.8) are plotted in Figure 5.2. The simulation supports our theoretical results clearly.

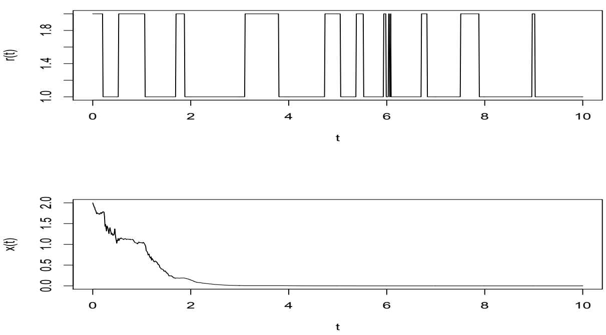

Figure 5.2: The computer simulation of the sample paths of the Markov chain and the solution of the controlled SDDE (5.8) using the Euler–Maruyama method with step size 10^{-4}.

6 Conclusion

In this paper we have discussed the stabilisation of highly nonlinear hybrid SDDEs by the delay feedback controls. We pointed out that there is little known on this stabilisation problem when the feedback control is based on the past state although the feedback control based on the current state has been well studied. We also pointed out that the problem becomes even harder when the coefficients of the underlying hybrid SDDE do not satisfy the linear growth condition (namely, the coefficients are highly nonlinear). In this paper we consider a class of hybrid SDDEs which are not stable but their solutions are bounded in qth moment. We then propose four rules for the control functions such that the controlled SDDEs become stable. These rules, to a very much degree, also describe a way how to design the control functions. The stability discussed in this paper include the H_∞-stable in $L^{\bar{q}}$, asymptotic stability in $\bar{q}$th moment, qth moment exponential stability and almost surely exponential stability. The key technique used in this paper is the method of Lyapunov functionals. A couple of examples and computer simulations have been used to illustrate our theory.

Acknowledgements

The authors would like to thank the associate editor and reviewers for their helpful comments. The authors would also like to thank the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the EPSRC (EP/K503174/1), the Natural Science Foundation of China (11971096), the Natural Science Foundation of Jilin Province (No.20170101044JC), the Education Department of Jilin Province (No.JJKH20170904KJ), the Fundamental Research Funds for the Central Universities for their financial support.

References

[1] Ahlborn, A. and Parlitz, U., Stabilizing unstable steady states using multiple delay deedback control, *Phys. Review Lett.* **93** (2004), 264101.

[2] Bao, J., Bottcher, B., Mao, X., Yuan, C., Convergence rate of numerical solutions to SFDEs with jumps. *J. Comput. Appl. Math.* **236** (2011), 119-131.

[3] Bahar, A. and Mao, X., Stochastic delay population dynamics, *J. Int. Appl. Math.* **11(4)** (2004), 377–400.

[4] Barbu, V., Cordoni, F. and Di Persio, L. Optimal control of stochastic FitzHugh-Nagumo equation, *Int. J. Control* **89(4)** (2016), 746–756.

[5] Blythe, S., Mao, X. and Shah, A., Razumikhin-type theorems on stability of stochastic neural networks with delays, *Sto. Anal. Appl.* **19(1)** (2001), 85–101.

[6] Cao, J., Li, H.X. and Ho, D.W.C., Synchronization criteria of Lur's systems with time-delay feedback control, *Chaos, Solitons and Fractals* **23** (2005) 1285–1298.

[7] Cordoni, F. and Di Persio, L., Optimal control for the stochastic fitzhugh-nagumo model with recovery variable, *Evolution Equations and Control Theory* **7(4)** (2018), 571–585.

[8] Fei, W., Hu, L., Mao, X. and Shen, M., Delay dependent stability of highly nonlinear hybrid stochastic systems, *Automatica* **82** (2017), 65–70.

[9] Fei, W., Hu, L., Mao, X. and Shen, M., Structured robust stability and boundedness of nonlinear hybrid delay systems, *SIAM J. Cont. Optim.* **56 (4)** (2018), 2662–2689.

[10] Fei, C., Shen, M., Fei, W., Mao, X. and Yan, L., Stability of highly nonlinear hybrid stochastic integro-differential delay equations, *Nonlinear Anal.: Hybrid Systems* **31** (2019), 180–199.

[11] Hu, L., Mao, X. and Shen, Y., Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Syst. Cont. Lett.* **62** (2013), 178–187.

[12] Ji, Y. and Chizeck, H.J., Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Automat. Control* **35** (1990), 777–788.

[13] Kolmanovskii, V.B. and Nosov, V.R., *Stability of Functional Differential Equations*, Academic Press, 1986.

[14] Kolmanovskii, V.B., Koroleva, N., Maizenberg, T., Mao, X. and Matasov, A., Neutral stochastic differential delay equations with Markovian switching, *Stoch. Anal. Appl.* **21(4)**(2003), 819-847.

[15] Ladde, G.S. and Lakshmikantham, V., *Random Differential Inequalities*, Academic Press, 1980.

[16] Lewis A.L., *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press, 2000.

[17] Lu, Z., Hu, J. and Mao, X., Stabilisation by delay feedback control for highly nonlinear hybrid stochastic differential equations, to appear in *Discrete Contin. Dyn. Syst. Ser. B.* (2018).

[18] Mao, X., *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, 1991.

[19] Mao, X., *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, 1994.

[20] Mao, X., *Stochastic Differential Equations and Their Applications*, 2nd Edition, Chichester: Horwood Pub., 2007.

[21] Mao, X., Stability of stochastic differential equations with Markovian switching, *Stochastic Process. Appl.* **79** (1999), 45–67.

[22] Mao, X., Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, *Automatica* **49(12)** (2013), 3677–3681.

[23] Mao, X., Lam, J. and Huang, L., Stabilisation of hybrid stochastic differential equations by delay feedback control, *Syst. Cont. Lett.* **57** (2008), 927–935.

[24] Mao, X., Matasov, A. and Piunovskiy, A.B., Stochastic differential delay equations with Markovian switching, *Bernoulli* **6(1)** (2000), 73–90.

[25] Mao, X. and Yuan, C., *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[26] Mao, X., Yuan, C. and Zou, J., Stochastic differential delay equations of population dynamics, *J. Math. Anal. Appl.* **304** (2005), 296–320.

[27] Mariton, M., *Jump Linear Systems in Automatic Control*, Marcel Dekker, 1990.

[28] Mohammed, S.E.A., *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1984.

[29] Pyragas, K., Control of chaos via extended delay feedback, *Physics Lett. A* **206** (1995), 323–330.

[30] Shaikhet, L., Stability of stochastic hereditary systems with Markov switching, *Theory of Stochastic Processes* **2(18)** (1996), 180–184.

[31] Wu, L., Su, X. and Shi, P., Sliding mode control with bounded L_2 gain performance of Markovian jump singular time-delay systems, *Automatica* **48(8)** (2012), 1929–1933.

[32] You, S., Liu, W., Lu, J., Mao, X. and Qiu, Q., Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM J. Cont. Optim.* **53(2)** (2015), 905–925.

[33] Yue, D. and Han, Q., Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, *IEEE Trans. Automat. Control* **50** (2005), 217–222.

[ ]Xiaoyue Li received her M. S. degree in 2001 and Ph. D. degree in 2005 in mathematics from Northeast Normal University, China. She joined in its department of applied mathematics in 2001 as a tutor, was made to a lecture in 2004, was promoted to an associate professor in 2007, and was made Professor in 2012 which post she still holds. Now her research interests lie in the theory of stochastic differential equations and their applications.

[ ]Xuerong Mao received the Ph.D. degree from Warwick University, Coventry, U.K., in 1989. He was SERC (Science and Engineering Research Council, U.K.) Post-Doctoral Research Fellow from 1989 to 1992. Moving to Scotland, he joined the University of Strathclyde, Glasgow, U.K., as a Lecturer in 1992, was promoted to Reader in 1995, and was made Professor in 1998 which post he still holds. He was elected as a Fellow of the Royal Society of Edinburgh (FRSE) in 2008. He has authored five books and over 300 research papers. His main research interests lie in the field of stochastic analysis including stochastic stability, stabilization, control, numerical solutions and stochastic modelling in finance, economic and population systems. He is a member of the editorial boards of several international journals.