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STABILIZING PERIODIC ORBITS ABOVE THE ECLIPTIC PLANE IN THE SOLAR SAIL 3-BODY PROBLEM

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***Abstract*— We consider periodic orbits high above the ecliptic plane in the Elliptic Restricted Three-Body Problem where the third massless body is a solar sail. Periodic orbits above the ecliptic are of practical interest as they are ideally positioned for the year-round constant imaging of, and communication with, the poles. Initially we identify an unstable periodic orbit by using a numerical continuation from a known periodic orbit above the ecliptic in the circular case with the eccentricity as the varying parameter. This orbit is then used to construct a reference trajectory for the sail to track. In addition we illustrate an alternative method for constructing a periodic reference trajectory based on a time-delayed feedback mechanism. The reference trajectories are then tracked using a linear feedback regulator (LQR) where the control actuation is delivered by varying the solar sails orientation. Using this method it is shown that a 'near term' solar sail is capable of performing stable periodic motions high above the ecliptic.**

1. Introduction

A solar sail consists essentially of a large mirror, which uses the momentum change due to photons reflecting off the sail for its impulse. They are therefore of great interest as they do not require fuel for propulsion. In addition solar sails are capable of trajectories and orbits which are impossible for conventional spacecraft [1]. The solar sail is modeled in the context of the restricted 3-body problem with the Earth rotating the Sun in an elliptic orbit and the solar sail the third massless body. This formulation will be referred to as the Solar Sail Elliptical Three Body Problem (SSETBP) [2] a generalization of the Solar Sail Circular Three Body Problem (SSCTBP) [3], [4], [5]. The SSETBP takes into account the eccentricity of the orbit of the primaries, and is therefore a more accurate model than the SSCTBP.

The existence of equilibrium points out of the ecliptic plane [3], [4], [5] in the SSCTBP has lead to the identification of new and interesting orbits, which conventional spacecraft are not capable of. In Waters and McInnes [3] the authors identify families of periodic orbits above the ecliptic using the method of Lindstedt-Poincaré to find periodic approximations to the nonlinear solution. The initial conditions that yield these approximations were used as initial 'guesses' in a differential corrector to close the trajectory and give exact initial conditions that yield periodic orbits in the nonlinear system. These periodic orbits high above the ecliptic are ideally positioned for the constant viewing of the polar regions and high latitudes of the Earth [3]. Moreover, an orbit of period one year with appropriate timing can counter the seasonal effect of the variation of the Earth's axis of rotation.

The generalization to the SSETBP is considered in Baoyin and McInnes [2] where it is shown that an 'approximate' equilibrium point can be maintained above the ecliptic in the SSETBP using a linear feedback mechanism. In this paper we extend this analysis to the study of periodic orbits above the ecliptic in the SSETBP.

The method of Lindstedt-Poincaré is not applicable above the ecliptic in the SSETBP as there are no equilibrium points in this region. Consequently, we turn to numerical methods in the search for periodic orbits above the ecliptic plane. A numerical continuation, with the eccentricity e as the varying parameter is used to find a periodic orbit above the ecliptic, starting from a known orbit in the circular case ($e = 0$) [3]. Using Floquet theory this orbit is shown to be unstable and therefore requires active control to maintain. This unstable orbit will therefore be used as a reference trajectory for the solar sail to track using variations in its orientation. In addition we present an alternative method for determining reference trajectories based on a time-delayed feedback mechanism. Finally, a Linear Quadratic Regulator (LQR) control is implemented to track the periodic reference trajectories using the sail's orientation as the control. These methods illustrate that a 'near term' solar sail is capable of maintaining stable period motions of period 1 year, high above the ecliptic using variations in its orientation.

In the next section we describe the equations of motion for the SSETBP.

2. Equations of motion for the SSETBP

The classical Elliptical Restricted three body problem can be modeled using a pulsating-rotating frame [6], [7]. The pulsating-rotating frame is convenient as the true anomaly appears in the equations of motion as the independent variable and therefore we do not need to integrate Kepler's equation. In this paper we follow the same convention and use a pulsating-rotating frame to model the SSETBP [2]. Assume that an appropriate set of units is introduced so that the gravitational constant $G = 1$, the system has total unit mass and the semi-major axis of the earth's orbit about the sun is $a = 1$. Let μ be the dimensionless mass of the earth and then $1 - \mu$ is the mass of the sun. The equations of motion in pulsating-rotating coordinates x, y, z are then:

$$\begin{aligned} x'' - 2y' &= \tfrac{1}{1+e\cos f}\left(\tfrac{\partial\Omega}{\partial x} + acc_x\right) \\ y'' + 2x' &= \tfrac{1}{1+e\cos f}\left(\tfrac{\partial\Omega}{\partial y} + acc_y\right) \\ z'' + z &= \tfrac{1}{1+e\cos f}\left(\tfrac{\partial\Omega}{\partial z} + acc_z\right) \end{aligned} \tag{1}$$

where $(\cdot)'$ denotes differentiation with respect to the true anomaly f and

$$\Omega = \frac{1}{2}(x^2 + y^2 + z^2) + \frac{(1-\mu)}{\|r_1\|} + \frac{\mu}{\|r_2\|}$$

where e is the eccentricity and $acc = (acc_x, acc_y, acc_z)^T$ is the solar sail acceleration defined by:

$$acc = \frac{\beta(1-\mu)}{\|r_1^2\|}(\hat{r}_1 \cdot \hat{n})^2 \hat{n} \tag{2}$$

where β is the solar sail lightness number and is the ratio of the solar sail radiation pressure acceleration to the solar gravitational acceleration, the 'hat' notation denotes the unit vector and $\hat{n}$ is the unit normal of the sail with respect to the sun and describes the sails orientation. We define $\hat{n}$ in terms of two angles γ and δ in the rotating-pulsating frame:

$$\hat{n} = (\cos\gamma\cos\delta, \cos\gamma\sin\delta, \sin\gamma)^T \tag{3}$$

While a β value of order $0.3 - 0.4$ is considered within the realm of possibility, to put the analysis in this paper well within the near-term we will consider very modest β values of order 0.05. The values of the parameters in the Earth-Sun system are $e = 0.0167$, $\mu = 0.000003$. Note that the rotating-pulsating coordinates are related to the rotating coordinates X, Y, Z via the equations:

$$X = \rho x, \qquad Y = \rho y, \qquad Z = \rho z \tag{4}$$

with the semi-latus rectum $\rho = \frac{(1-e^2)}{1+e\cos f}$. It follows that when $e = 0$ the equations (1) reduce to the equations of motion for the SSCTBP.

3. Unstable periodic orbits above the ecliptic

In Waters and McInnes [3] initial conditions that yield periodic orbits high above the ecliptic have been found in the SSCTBP using the method of Lindstedt-Poincaré and a differential corrector. When the eccentricity of the earth's orbit about the sun is considered these initial conditions no longer yield periodic orbits. In this section we use a continuation method with the eccentricity (e) as the continuation parameter to find initial conditions that yield periodic orbits in the SSETBP. Firstly, we note that when $e \neq 0$ the equation (1) is non-autonomous and therefore any periodic orbit must be of the same period as the true anomaly dependent function, $\cos f$. Therefore, we choose an initial orbit above the ecliptic of period 1 year found in the SSCTBP [3] as a starter orbit in the numerical continuation and continue e until $e = 0.0167$. At each small increment of e the trajectory is closed using a monodromy variant of Newton's method [8]. Let $\mathrm{X}(t) = (x, y, z, x', y', z')$ be the solution of the nonlinear system (1). When $\mathrm{X}(t)$ is close to a natural periodic orbit $\Gamma(t)$ of the system (1) an iterative improvement to the choice of initial conditions for a periodic orbit is given by [8]:

$$\mathrm{X}^*(0) = \mathrm{X}(0) + (I - M)^{-1}[\mathrm{X}(T) - \mathrm{X}(0)] \qquad (5)$$

where $\mathrm{X}^*(0)$ is the improved initial condition, M the monodromy matrix and I the identity matrix. One of the problems encountered with this method is that the determinant of $(I - M)$ maybe zero and therefore its inverse is not well defined. However, this problem is resolved by using the Moore-Penrose pseudo inverse. The implementation of Newton's method relies on the computation of the monodromy matrix as follows:

Let $\Gamma(t)$ denote a periodic orbit with period $T = 2\pi$ which satisfies the condition $\Gamma(T) = \Gamma(0)$, by letting $\mathrm{x} = \mathrm{X}(t) - \Gamma(t)$, we may linearize the nonlinear system about this periodic orbit, resulting in the variational equations

$$\dot{\mathrm{x}} = A(t)\mathrm{x}$$

where

$$A(t) = A(t+T) = \left.\frac{\partial f}{\partial \mathrm{X}}\right|_{\mathrm{X}(t)=\Gamma(t)}$$

explicitly:

$$A(t) = \begin{pmatrix} 0 & I \\ J & \Omega \end{pmatrix}, \qquad J = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$
$$\Omega = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (6)$$

where

$$a = \left.\frac{\partial f_x}{\partial x}\right|_{\Gamma(t)}, \quad b = \left.\frac{\partial f_x}{\partial y}\right|_{\Gamma(t)}, \quad c = \left.\frac{\partial f_x}{\partial z}\right|_{\Gamma(t)}$$
$$d = \left.\frac{\partial f_y}{\partial x}\right|_{\Gamma(t)}, \quad e = \left.\frac{\partial f_y}{\partial y}\right|_{\Gamma(t)}, \quad f = \left.\frac{\partial f_y}{\partial z}\right|_{\Gamma(t)}$$
$$g = \left.\frac{\partial f_z}{\partial x}\right|_{\Gamma(t)}, \quad h = \left.\frac{\partial f_z}{\partial y}\right|_{\Gamma(t)}, \quad i = \left.\frac{\partial f_z}{\partial z}\right|_{\Gamma(t)}$$

here the partial derivatives a, b, c, d, e are time-dependent, with period 2π. Recasting the variational equations in terms of the state transition matrix (or principle fundamental matrix) $\Phi = \partial \mathrm{X}(t)/\partial \mathrm{X}(0)$, we have

$$\dot{\Phi} = A(t)\Phi, \qquad \Phi(0) = I$$

where Φ is a 6×6 matrix. The monodromy matrix M is then defined as $M = \Phi(T)$. The monodromy matrix M is computed at each iteration and Newton's method is successful in identifying 1 year period orbits with eccentricity as the continuation parameter. Two periodic orbits are illustrated in Figure 1 for $e = 0$ and $e = 0.0167$, where the solar sail angles are $\gamma = 0.809196$ and $\delta = 0$.

The stability of periodic orbits is determined using Floquet theory [9] and depends on the behavior of the eigenvalues

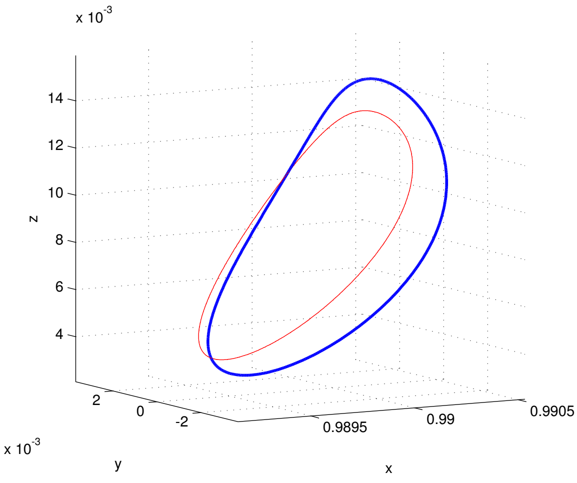


Fig. 1. Periodic Orbits in the rotating-pulsating frame: the thin line orbit is for $e = 0$ and the thick lined orbit is for $e = 0.0167$ and $\gamma = 0.809196$

of the monodromy matrix M. To these eigenvalues λ_i correspond the characteristic (Floquet) exponents α_i defined by

$$\lambda_i = e^{\alpha_i T}$$

The orbit is stable at linear order if the real parts of all the characteristic exponents are less than or equal to zero. The eigenvalues of the monodromy matrix are of the form:

$$\{\lambda_j, \bar{\lambda}_j, \lambda_i, \bar{\lambda}_i, \lambda_r, 1/\lambda_r\}$$

and the characteristic exponents

$$\{\alpha_j, \bar{\alpha}_j, \alpha_i, \bar{\alpha}_i, \pm\alpha_r\}$$

which is consistent with unstable periodic orbits in the the classical Elliptical Restricted 3-body Problem [10].

4. Reference Orbits

One of the obstacles in implementing an LQR control is in the generation of a 'good' reference trajectory. In our case a 'good' reference trajectory is characterized by it being periodic and requiring as little control as possible to track. In this respect a natural candidate for a reference trajectory would be the unstable periodic orbit in Figure 1 for $e = 0.0167$. To measure how 'good' this reference orbit is, a comparison will be made with the periodic orbit in Figure 1 for $e = 0$.

In addition to this we propose a novel method for designing reference orbits based on a time-delayed feedback mechanism [11], [12]. This is a robust method, in that, even if it does not converge to a natural unstable orbit, it will provide a closed orbit of pre-specified period that requires a minimum amount of control to maintain. Therefore, it is particularly useful when the initial trajectory cannot be closed with a Newton method or a differential corrector. For illustration we use a monte-carlo simulation of initial conditions close to the region that yield periodic orbits in the circular case to find a trajectory as a close as possible to a periodic orbit with $e = 0.0167$. This trajectory is illustrated in Figure 2. It is noted that Newton's method fails to close this trajectory.

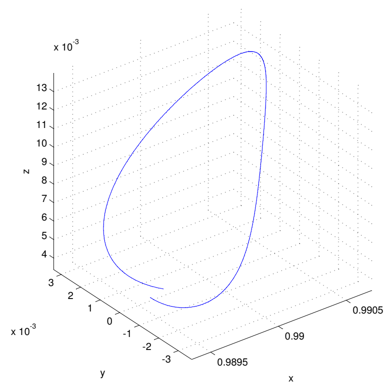


Fig. 2. A near periodic orbit found using monte-carlo simulation.

Our approach is then to use a time-delayed feedback mechanism to obtain a periodic reference trajectory. For the nonlinear system $\dot{\mathrm{X}}(t) = f(\mathrm{X}(t), t)$ we assign a time delayed feedback mechanism $v(t)$ such that:

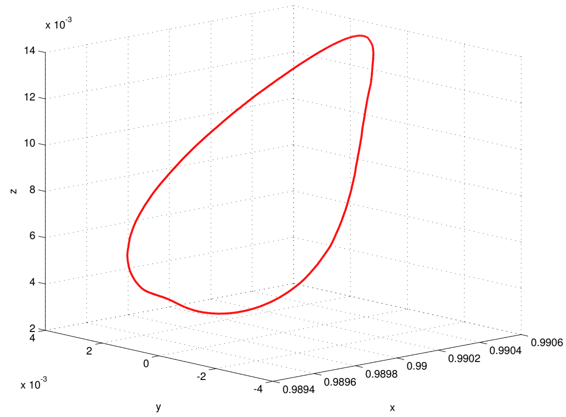
$$\begin{aligned} \dot{\mathrm{X}}(t) &= f(\mathrm{X}(t), t) + v(t) \\ v(t) &= -K(\mathrm{X}(t) - \mathrm{X}(t-\tau)) \end{aligned} \quad (7)$$

where τ is the delay time, which will be 2π in order to obtain a 1 year orbit and K a 6×6 matrix which is computed experimentally. A reasonable choice for K is a scalar multiple of the identity matrix $I_{6\times6}$. By inspection of the time-delayed feedback function (7) it can be seen that when the trajectory $X(t)$ is almost periodic (which is the case for numerically determined orbits) i.e. $\|X(t) - X(t-\tau)\| = \varepsilon$ where ε is small then the feedback mechanism $v(t)$ will also be small and equal to $-K\varepsilon$. Therefore, if the time-delayed feedback mechanism causes the trajectory to converge to an approximate periodic orbit the feedback will converge to approximately zero. Hence, the final periodic trajectory is the one that requires the least amount of feedback to maintain. This effect is illustrated in simulation in Figure 3. Figure 3 (i) shows the final periodic orbit that will be used as our reference trajectory. This orbit corresponds to the minimum feedback requirement which can be seen in Figure 3 (ii).

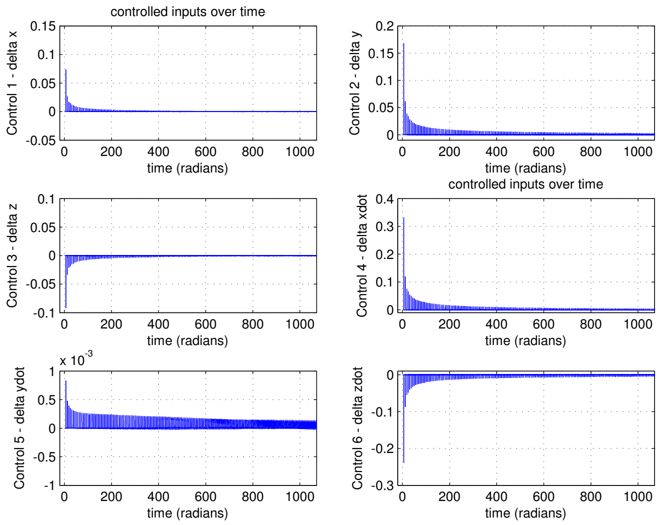
It is also noted that the time-delayed feedback mechanism works both for forward and backward integrations. For forward integrations it appears that the initial part of the trajectory is corrected, more so, than the final part and using backward integration the opposite is true. In the following section we compare each of the reference orbits illustrated in Figures 1 and 3 (i) by tracking them with an LQR control. The control actuation is delivered via variations in the sail's orientation.

5. Tracking periodic trajectories above the ecliptic using the solar sail

In this section we propose using LQR to track the periodic reference trajectories in Figures 1 and 3 (i) using variations in the sails orientation (angles γ and δ provide the control actuation). Specifically, we aim to stabilize the motion of the sail about the periodic orbit over 10 years. The constraints on the solar sail is its maximum deflection (rads) and maximum rate of deflection (rads/sec) which are taken to be $-\pi/2 \leq \gamma, \delta \leq \pi/2$ and $-5\times10^{-6} < \frac{d\lambda}{dt}, \frac{d\delta}{dt} < 5\times10^{-6}$ (approx. $\pm$ 1 degree per hour) respectively. These constraints are imposed on our controls in Simulation. Firstly, to define the



(i)



(ii)

Fig. 3. The Time-delay feedback mechanism is used to compute a periodic reference trajectory: (i) The final orbit that corresponds to the minimum feedback requirement that will be used as a reference trajectory. (ii) the feedback required to close the initial orbit is high but this reduces with time during simulation to a minimum value.

reference orbits we fit a Fourier function to the numerical data describing the orbits in Figures 1 and 3 (i) of the form:

$$X_{1j}(t) = a_0 + a_1\cos t + b_1 \sin t + a_2\cos 2t + b_2\sin 2t + a_3\cos 3t + b_3\sin 3t \quad (8)$$

where $j = 1, ..., 6$. This function provides a good fit to the numerical data and ensures that the reference orbits are exactly periodic. Following this we linearize the nonlinear equations $\dot{X} = f(X, u(t))$ about the reference trajectory Γ.

Writing $\mathrm{x} = \mathrm{X} - \Gamma$ and $u = u(t) - u_e$ yields:

$$\dot{\mathrm{x}} = A\mathrm{x} + Bu \qquad (9)$$

where $A = \frac{\partial f}{\partial \mathrm{X}}\Big|_{\Gamma}$ and $B = \frac{\partial f}{\partial u(t)}\Big|_{u_e}$ where $u_e = (\gamma_e, \delta_e)^T = (0.809196, 0)^T$. From control systems theory the gain matrix K for the linear state feedback control law $u = -K\mathrm{x}$ which minimizes the quadratic cost function

$$J = \int_0^{\infty} \mathrm{x}^T Q \mathrm{x} + u^T R u \mathrm{d}t$$

where Q, R are symmetric positive semi-definite weighting matrices respectively is given by:

$$K = R^{-1} B^T P \qquad (10)$$

where P is the unique, positive semidefinite solution to the algebraic Riccati equation:

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \qquad (11)$$

In simulation we find that defining a constant gains matrix evaluated at the initial point of the reference orbit is sufficient to stabilize the orbit using small variations in the solar sail angles. The weighting matrices Q and R are chosen so that the periodic orbit is stabilized about the reference orbit over 10 years with minimum control effort. The ability for the solar sail to track the reference trajectories is illustrated in Figure 4 with the thin line the reference trajectory and the thicker line the actual trajectory. Figure 4 (i) shows that the solar sail does not track the reference trajectory obtained in the SSCTBP well. However, Figures 4 (ii) and (iii) illustrate that the sail can track the reference trajectory obtained using the continuation and the time delayed-feedback mechanism closely.

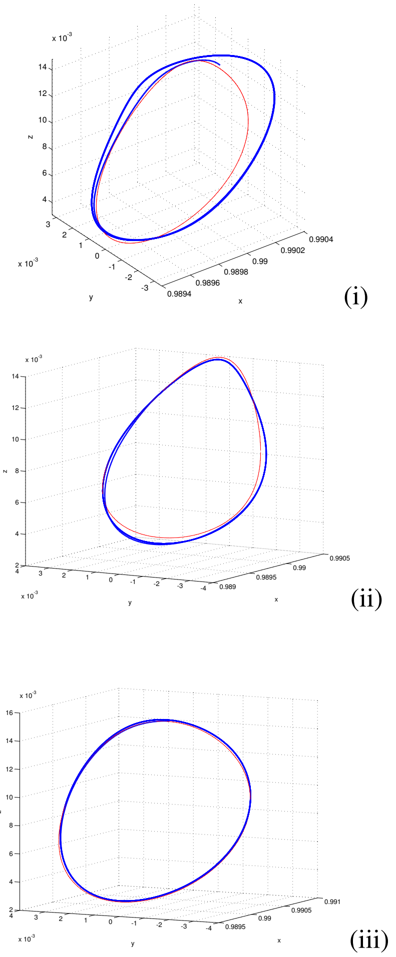


Fig. 4. LQR control: the thick line is the controlled trajectory and the thin line is the reference orbit. Each figure shows the actual trajectory tracking a reference orbit defined by (i) a periodic reference from the SSCTBP (ii) a periodic reference computed using a time-delayed feedback machanism. (iii) a periodic reference from an unstable orbit in the SSETBP

6. Conclusions

We consider periodic orbits above the ecliptic plane in the Solar Sail Elliptic Three Body Problem (SSETBP). It is shown that stable periodic motions high above the ecliptic can be achieved by manipulating the orientation of the solar sail. Such periodic orbits are of particular interest as they can be used to constantly view the polar regions and high latitudes of the Earth. To show this we initially use a numerical continuation from a previously known orbit above the ecliptic in the circular case with the eccentricity e as the varying parameter to obtain a periodic orbit at $e = 0.0167$. This orbit is shown to be unstable and therefore active control is required to maintain the sail on this orbit. This orbit is used as a reference trajectory and tracked using an

LQR control. In addition we present an alternative method for constructing a suitable reference trajectory based on a time-delayed feedback mechanism. Finally, we use a Linear Quadratic Regulator (LQR) to track the reference trajectories using small variations in the sails orientation. In summary it is shown that a 'near' term solar sail can successfully track suitably defined 1 year periodic orbits high above the ecliptic.

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REFERENCES

[1] McInnes, C. R., 'Solar sailing: technology, dynamics and mission applications'. Springer Praxis, 1999.

[2] Baoyin, H., McInnes, C.R., 'Solar sail equilibria in the elliptical restricted three-body problem'. Journal of Guidance, Control and Dynamics, Vol. 29, No. 3, pp. 538-543, 2006.

[3] Waters, T. J., McInnes, C. R., 'Periodic Orbits above the Ecliptic in the Solar sail Restricted Three-body problem'. Journal of Guidance, Control and Dynamics, Vol. 30, No. 3, May-June, 2007.

[4] McInnes, C. R., 'Artificial Lagrange points for a non-perfect solar sail'. Journal of Guidance, Control and Dynamics, Vol. 22, No. 1, pp. 185-187, 1999.

[5] McInnes, C.R.,McDonald, A.J.C., Simmons, J.F.C., and McDonald, E.W., 'Solar sail parking in restricted three-body systems'. Journal of Guidance, Control and Dynamics, Vol. 17, No. 2, pp. 399-406, 1994.

[6] Gurfil, P., and Meltzer, D., 'Stationkeeping on Unstable Orbits: Generalization to the Elliptic Restricted Three-Body Problem'. The Journal of Astronautical Sciences, Vol. 54, No. 1, pp. 29-51, 2006.

[7] Szebehely, V., 'Theory of Orbits: The restricted problem of three bodies'. Academic Press, New York, 1967.

[8] Marcinek, R., Pollak, E., 'Numerical methods for locating stable periodic orbits embedded in a largely chaotic system'. Journal of Chemical Physics, 100, 8, pp. 5894-5904, 1994.

[9] Grimshaw, R., 'Nonlinear ordinary differential equations'. Blackwell Scientific Publications, 1990.

[10] Broucke, R., 'Stability of Periodic Orbits in the Elliptic, Restricted Three-body Problem'. AIAA Journal, Vol. 7, No. 6, June 1969.

[11] Pyragus, K., 'Continuous control of chaos by self-controlling feedback'. Physics Letters A, Vol. 170, pp. 421-428, 1992.

[12] Chen, G., Yu, X.,'On Time-Delayed Feedback Control of Chaotic Systems'. IEEE Trans. on Circuits and systems-I: Fundamental Theory and applications, Vol. 46, No.6, pp. 767-772, June 1999.