Title: Non-linear stability of vortex formation in swarms of interacting particles

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Abstract: Vortex-like pattern formation is observed in a wide range of biological systems. In previous work we used artificial potential fields to model weak long-range attraction and strong short-range repulsion in a swarm of interacting particles. A dissipation function was defined which minimised the total effective energy of the swarm by aligning neighbouring velocity vectors. Here we extend this work to demonstrate that such vortex-like patterns are stable. The effective energy of the swarm is used as a Lyapunov function to demonstrate that such a swarm of interacting particles will always collapse into a vortex-like state.

Main Text: Swarming patterns have been observed and reported for various species in nature [1]. The coherent flock and the single-mill states are among the most common observed in biological swarms [2–3]. An example of a double-mill pattern, which is occasionally observed, has also been introduced [4]. Emerging vortex patterns among individuals that interact through pair-wise artificial potential fields have been discussed by various authors [5–9]. In particular, we have been shown that the total linear and angular momentum of the swarm are conserved with a pair-wise dissipation function [9]. The vortex pattern was then shown to be a constrained minimum of the total effective energy of the swarm. While it was shown that the vortex pattern was an extremum of the total effective energy, stability was not addressed. In the work reported here we use a Lyapunov function to demonstrate that the swarm will always relax into a vortex-like state. While this is an interesting contribution to the statistical physics of interacting particle systems, it has wider application to the construction of provable behaviours in swarms of interacting robotic agents.

We consider a swarm that consists of N identical particles of equal mass m with position and velocity $(\mathbf{x}_i, \mathbf{v}_i)$ defining the state of the ith particle. Attraction amongst the particles in the swarm is defined through a weak long-range attractive potential $U^a_{ij} = -C_a \exp(-|\mathbf{x}_{ij}|/l_a)$, while collisions between particles are prevented through a strong short-range repulsive potential $U^r_{ij} = C_r \exp(-|\mathbf{x}_{ij}|/l_r)$ [5–7]. The strengths of the attraction and repulsion potentials are denoted by C_a and C_r with ranges l_a and l_r respectively. The particles attempt to align their motion with neighbours through a velocity dependent orientation force $\mathbf{\Lambda}_i$, which is defined as $\mathbf{\Lambda}_i = \sum_j C_o (\mathbf{v}_{ij}.\hat{\mathbf{x}}_{ij}) \exp(-|\mathbf{x}_{ij}|/l_o) \hat{\mathbf{x}}_{ij}$, where $(\hat{\cdot})$ denotes a unit vector, C_o is the strength of the orientation force and l_o is the range of the orientation force. Parallel orientation of the particle velocity vectors then emerges due to the dissipative nature of the orientation force such that motion towards or away from neighbours

is weakly damped, proportional to the component of relative velocity along the vector connecting neighbouring particles, $\mathbf{v}_{ij}.\hat{\mathbf{x}}_{ij}$. This results in a local alignment of particle velocity vectors, as used extensively in rule-based approaches [10]. The exponential term in the orientation force ensures that the effect is localised between neighbours while the pair-wise interaction along $\hat{\mathbf{x}}_{ij}$ leads to conservation of angular momentum.

The evolution of the swarm of interacting particles is now defined through the interaction potential and orientation force such that

$$\dot{\mathbf{x}}_i = \mathbf{v}_i \tag{1a}$$

$$m\dot{\mathbf{v}}_i = -\nabla U^a_i - \nabla U^r_i - \mathbf{\Lambda}_i \tag{1b}$$

where $U_i = \sum_j U_{ij}$ and $\nabla(\cdot) = \partial(\cdot)/\partial \mathbf{x}_i$. The three terms in Eq. (1b) are defined such that $l_r < l_o < l_a$. This arrangement is equivalent to the zone of repulsion, zone of orientation and zone of attraction which has been used successfully in both rule-based simulation [10] and laboratory experimentation with biological swarms [11]. The use of artificial potential fields to mediate interactions between particles provides a continuous representation of these rule-based methods which, unlike rule-based heuristics, is amenable to analytic investigation and formal proof.

The effective total energy of the swarm ϕ is now defined through a summation to evaluate each pair-wise potential interaction and a summation of the kinetic energy of each particle. Therefore, the total effective energy of the swarm is defined as

$$\phi = \frac{1}{2}\sum_i m\mathbf{v}_i^2 + \sum_i \left(U^a_i + U^r_i\right) \tag{2}$$

Taking the time derivative of Eq. (2) it can be seen that

$$\dot{\phi} = \sum_i \mathbf{v}_i.\left(m\dot{\mathbf{v}}_i + \nabla U^a_i + \nabla U^r_i\right) \tag{3}$$

Then, substituting from Eq. (1b) in Eq. (3), it can further be seen that

$$\dot{\phi} = -\sum_i \mathbf{v}_i . \mathbf{\Lambda}_i \tag{4}$$

and so

$$\dot{\phi} = -\sum_i \mathbf{v}_i . \sum_{j \neq i} C_o \left(\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} \right) \exp \left(- \left| \mathbf{x}_{ij} \right| / l_o \right) \hat{\mathbf{x}}_{ij} \tag{5}$$

We now demonstrate that $\dot{\phi} < 0$ by considering an arbitrary term in the summation as

$$S_{ij} = \left(\mathbf{v}_i . \hat{\mathbf{x}}_{ij} \right) \left(\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} \right) \exp \left(- \left| \mathbf{x}_{ij} \right| / l_o \right) + \left(\mathbf{v}_j . \hat{\mathbf{x}}_{ji} \right) \left(\mathbf{v}_{ji} . \hat{\mathbf{x}}_{ji} \right) \exp \left(- \left| \mathbf{x}_{ji} \right| / l_o \right) \tag{6}$$

However, noting that $\hat{\mathbf{x}}_{ji} = -\hat{\mathbf{x}}_{ij}$ and $\mathbf{v}_{ji} = -\mathbf{v}_{ij}$ it can be seen that

$$S_{ij} = \left(\mathbf{v}_i . \hat{\mathbf{x}}_{ij} \right) \left(\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} \right) \exp \left(- \left| \mathbf{x}_{ij} \right| / l_o \right) - \left(\mathbf{v}_j . \hat{\mathbf{x}}_{ij} \right) \left(\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} \right) \exp \left(- \left| \mathbf{x}_{ij} \right| / l_o \right) \tag{7}$$

and so using the identity $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ it can further be seen that

$$S_{ij} = \left(\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} \right)^2 \exp \left(- \left| \mathbf{x}_{ij} \right| / l_o \right) \tag{8}$$

The rate of change of the total effective energy of the swarm can therefore be written as

$$\dot{\phi} = -\frac{1}{2} \sum_{i=1} \sum_{j \neq i} C_o \left(\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} \right)^2 \exp \left(- \left| \mathbf{x}_{ij} \right| / l_o \right) \tag{9}$$

Since $C_o \geq 0$, the quadratic term in Eq. (9) ensures that $\dot{\phi} < 0$ so that the total effective energy of the swarm is monotonically decreasing.

In previous work we demonstrated that vortex-like patterns could be interpreted as a constrained minimum-energy state [9]. Considering the total effective energy of the swarm

$$E = \left(\frac{1}{2} \sum_i m \mathbf{v}_i{}^2 + \sum_i \left(U^a{}_i + U^r{}_i \right) \right) - \lambda \cdot \left(\sum_i m \mathbf{x}_i \times \mathbf{v}_i - \mathbf{H} \right)$$ and enforcing conservation of total

angular momentum $\mathbf{H}$ through a Lagrange multiplier λ, it was shown that

$$\frac{\partial E}{\partial \mathbf{x}_i} = \left(\nabla U^a{}_i + \nabla U^r{}_i \right) - m\lambda \times \mathbf{v}_i = 0 \tag{10a}$$

$$\frac{\partial E}{\partial \mathbf{v}_i} = m \left(\mathbf{v}_i - \lambda \times \mathbf{x}_i \right) = 0 \tag{10b}$$

so that the constrained minimum-energy state of the swarm corresponds to vortex-like rotation with the velocity vector of each particle normal to its position vector and the vector λ such that $\mathbf{v}_i = \lambda \times \mathbf{x}_i$. The Lagrange multiplier λ was identified as the angular velocity vector of the swarm which is be directed along $\mathbf{H}$. Therefore, it can be seen that in the constrained minimum-energy state $\mathbf{v}_{ij} = \lambda \times \mathbf{x}_{ij}$ and so in Eq. (9) $\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} = \left(\lambda \times \mathbf{x}_{ij} \right) \hat{\mathbf{x}}_{ij}$. However, using the scalar triple product identity $\mathbf{v}_{ij} . \hat{\mathbf{x}}_{ij} = \lambda . \left(\mathbf{x}_{ij} \times \hat{\mathbf{x}}_{ij} \right) = 0$ and so $\dot{\phi} = 0$ in the vortex-like state. It can therefore be concluded that with the orientation force $\mathbf{\Lambda}_i$, a swarm of particles in an initially random state will relax into a spatially coherent vortex-like pattern, as observed in a wide range of biological swarms [1, 3, 12] and in simulation [5, 7, 10, 13]. Again, we note that the use of artificial potential fields to mediate interactions between particles provides a continuous representation of rule-based methods with the length-scales $l_r < l_o < l_a$ equivalent to the zone of repulsion, zone of orientation and zone of attraction used in rule-based simulation [10] and laboratory experimentation [11].

Finally, in order to illustrate the formation of vortex-like patterns using the mechanism discussed above, a planar swarm of $N = 50$ particles is considered. The particles in the swarm are randomly distributed over a unit disk with a random distribution of initial velocities. The free parameters are selected such that $l_r < l_o < l_a$ so that the swarm experiences weak long-range attraction, strong short range repulsion and local velocity alignment. It can be seen from Fig. 1 that the swarm slowly relaxes into a vortex-like pattern. As the swarm relaxes, the time rate of change of the total effective energy of the swarm vanishes, as shown in Fig. 2.

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Figure 1. Formation of a vortex-like pattern in a swarm of interacting particles (N=50) with C_a=1, C_r=2, C_o=0.1, l_a =1, l_r =0.2, l_o=0.5 for non-dimensional time t=0 until t=7 (top left to bottom right).

Figure 2. Time rate of change of the total effective energy of the swarm.
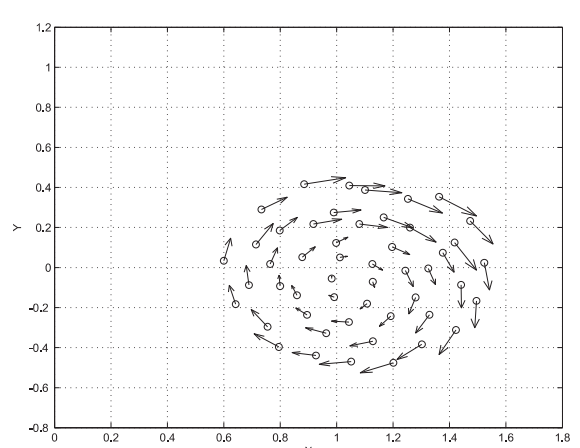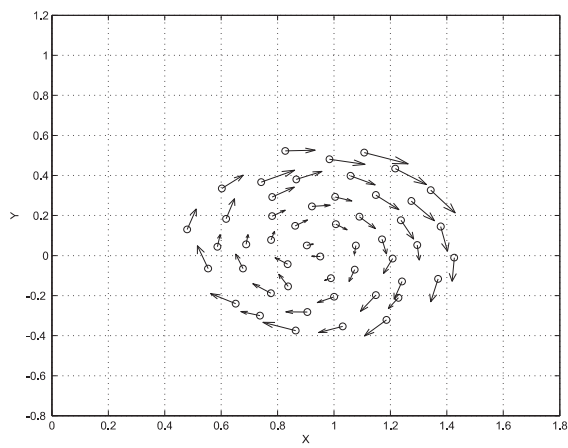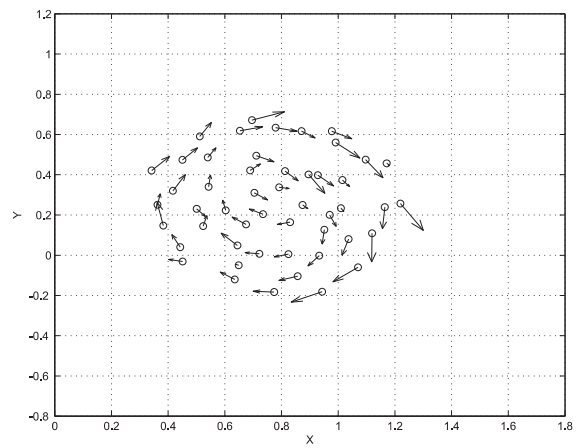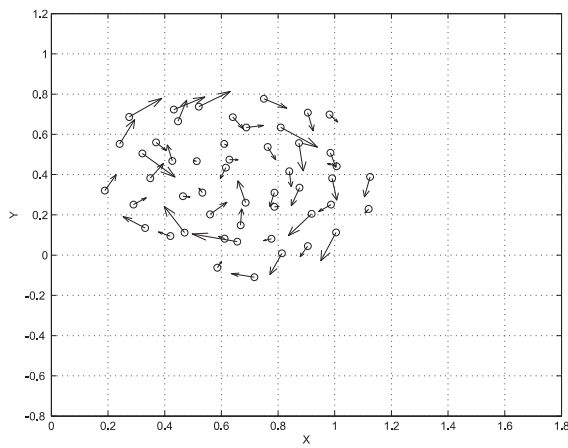
Figure 1

Figure 2