The truncated Euler-Maruyama method for stochastic differential equations with Hölder diffusion coefficients

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Abstract

In stochastic financial and biological models, the diffusion coefficients often involve the term $\sqrt{x}$, or more general $|x|^r$ for $r \in (0, 1)$, which is non-Lipschitz. In this paper, we study the strong convergence of the truncated Euler-Maruyama (EM) approximation first proposed by Mao [24] for one-dimensional stochastic differential equations (SDEs) with superlinearly growing drifts and the Hölder continuous diffusion coefficients.

Keywords. Strong convergence; Truncated EM; Hölder diffusion coefficients

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1 Introduction and motivation

It is well known that under the Lipschitz condition and linear growth condition, an SDE has a unique strong solution and the EM approximation has a strong convergence rate of order 1/2 (see Kloeden and Platen [16]). Without the linear growth condition, the explicit Euler scheme may not converge to the exact solution of an SDE in the strong mean square sense. Even worse, [12] showed that the moments of the standard EM approximate solution at a finite time may diverge to infinity even if the true solution are finite. Many implicit methods have therefore been proposed to study the numerical solutions of SDEs with nonlinear coefficients. For example, Higham, Mao and Stuart [9] proved that the implicit EM numerical solutions converge strongly to the exact solutions of SDEs with the globally one-side Lipschitz continuous drift term and globally Lipschitz continuous diffusion term.

On the other hand, some explicit methods have also been proposed for nonlinear SDEs. For example, [13] proposed the tamed EM schemes to approximate SDEs with the global Lipschitz diffusion coefficient and one-sided Lipschitz drift coefficient, whose numerical solutions converge strongly to the exact solution with order 1/2. Moreover, the tamed Milstein [33] and the stopped EM method [21] as well as their variants have also been proposed to deal with the strong converge problem for nonlinear SDEs. Recently, Mao [24, 25] proposed a new explicit method called the truncated EM method for nonlinear SDEs and established the strong convergence and obtained the convergence rate under the local Lipschitz condition and the Khasminskii-type condition. The authors of [10] showed that the truncated EM method may enable to use a larger stepsize than the tamed Euler method in [28] to achieve the same error. By the truncated technique, this paper aims to study the numerical solutions for SDEs with the superlinear drift coefficient and the Hölder continuous diffusion coefficient since the solutions can be considered in a compact domain under the truncation.

Many interest rate models has the “Aït-Sahalia-type” form

$$dX(t) = (a_1X(t) - a_2X^r(t))dt + \sigma X^\rho(t)dB(t),$$

where $r \geq 1$ and $\rho \geq 1/2$ (for example, [1]). In [32], the authors considered a strongly nonlinear Aït-Sahalia-type interest rate model with $r > 1$ and $\rho \geq 1$ and discussed the backward EM numerical solutions. [4] considered a stochastic SIS epidemic model as follows

$$dI(t) = [\beta I(t)(N - I(t)) - (\mu+\gamma)I(t)]dt + \sqrt{\beta I(t)(N - I(t)) + (\mu+\gamma)I(t)}dB(t),$$

where $\beta,\ \mu,\ \gamma,\ N$ are positive constants, and proved that if $I(0) \in (0, N)$, then $I(t) \in [0, N + (\mu+\gamma)/\beta]$ for all $t > 0$ a.e. If we extend the domain of SIS model above into the whole domain, i.e. $\lambda(x),\ \sigma(x) : \mathbb{R} \to \mathbb{R}$ by considering the following equation:

$$\lambda(x) = \begin{cases} 0, & x < 0, \\ \beta x(N-x) - (\mu+\gamma)x, & 0 \le x \le N + \dfrac{\mu+\gamma}{\beta}, \\ -2(\mu+\gamma)\Big(N + \dfrac{\mu+\gamma}{\beta}\Big), & x > N + \dfrac{\mu+\gamma}{\beta}, \end{cases}$$

and

$$\sigma(x) = \begin{cases} 0, & x < 0, \\ \sqrt{\beta x(N-x) + (\mu+\gamma)x}, & 0 \le x \le N + \dfrac{\mu+\gamma}{\beta}, \\ 0, & x > N + \dfrac{\mu+\gamma}{\beta}, \end{cases}$$

then $\sigma(x)$ is Hölder continuous with exponent $1/2$. In this paper, we are interested in the truncated EM numerical solutions for more general equations than the special Aït-Sahalia-type model and the extened stochastic SIS epidemic model.

When the diffusion coefficients of SDEs are Hölder continuous, [7] investigated the convergence rate of the EM method, but their results are not applicable to the SDEs with superlinearly growing drift coefficients. Moreover, to estimate high order moments, the proof depends on a crucial inequality (see Lemma 3.2 [7]), which is very technical. The authors in [27] studied a tamed Euler-Maruyama scheme for SDEs with Hölder continuous diffusion coefficient and locally Lipschitz continuous drift coefficient. In this paper, we consider the truncated EM method for SDEs with the superlinear drift coefficients and give a direct and more understandable proof of the estimation of high order moments (see Theorem 4.4 below).

This paper is arranged as follows. The next section will give some notation and preparations. Section 3 investigates the convergence of the moment of the truncated EM method at any given time T. The last section gives its strong convergence.

2 Preliminaries, notation and assumptions

Unless otherwise specified, we use the following notation. Let $T \in (0, \infty)$ be a fixed. $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$ with the corresponding inner product $(\cdot,\cdot)$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions.

Consider a scalar SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{2.1}$$

with the initial value $x(0) = x_0 \in \mathbb{R}$, where $f,\ g : \mathbb{R} \to \mathbb{R}$ are Borel measurable. We impose the following standing hypotheses in this paper.

Assumption 2.1. *Assume that the coefficient f satisfies the local Lipschitz condition: for any $R > 0$, there is $K_R > 0$ such that*

$$|f(x) - f(y)| \le K_R|x - y| \tag{2.2}$$

for all $x,\ y \in \mathbb{R}$ with $|x| \vee |y| \le R$.

Assumption 2.2. *Assume that f satisfies the one-side Lipschitz condition and g satisfies the Hölder continuity condition: there are constants $0 \le \alpha < 1/2$, $L_1 > 0$ and $L_2 > 0$ such that*

$$(x - y)(f(x) - f(y)) \le L_1|x - y|^2, \qquad |g(x) - g(y)| \le L_2|x - y|^{\frac{1}{2}+\alpha} \tag{2.3}$$

for all $x,\ y \in \mathbb{R}$.

Assumption 2.3. *Assume that there are $\gamma > 0$ and $H > 0$ such that*

$$|f(x) - f(y)|^2 \le H(1 + |x|^\gamma + |y|^\gamma)|x - y|^2 \tag{2.4}$$

for all $x,\ y \in \mathbb{R}$.

Remark 2.1. Under Assumption 2.2, for all $p > 2$, there is a constant $K = K(p) > 0$, such that

$$xf(x) + \frac{p-1}{2}|g(x)|^2 \le K(1 + |x|^2) \tag{2.5}$$

for all $x \in \mathbb{R}$.

Remark 2.2. Although assumption 2.3 implies assumption 2.1, when we consider the existence of the solution, pth moments of the numerical and exact solutions, we use assumption 2.1. But when the error bound of the numerical solution is considered, assumption 2.3 is enough.

We state a result as a lemma for the use of this paper.

Lemma 2.3. *Under Assumption 2.1 and 2.2, the SDE* (2.1) *has a global strong solution $x(t)$ with*

$$\mathbb{E}\Big[\sup_{0\le t\le T} |x(t)|^p\Big] < \infty, \qquad \textit{for all } p > 0. \tag{2.6}$$

Proof. Under Assumption 2.1 and Assumption 2.2, the drift and the diffusion coefficients of SDE (2.1) are continuous, so the existence of the weak solution can easily be obtained from [15, Theorem 2.3, Theorem 2.4, pp.159-164]. Moreover, by (2.3) and (2.5) for each $p > 2$, there is $C = C(p,T) > 0$ such that

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x(t)|^p\Big] \le C\left(1+|x(0)|^p\right),$$

so $x(t)$ exists for all time. For more details see Lemma 3.2 in [9]. By the Lyapunov inequality, for each $p > 0,\ T > 0$, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x(t)|^p\Big] < \infty.$$

By the proof of Theorem 2.1 in [18] which is similar to [35], Assuption 2.1 and Assuption 2.2 guarantee the pathwise uniqueness of SDE (2.1). In view of the Yamada-Watanabe theorem on pathwise uniqueness [35], the SDE (2.1) has a global strong solution. □

Let us now recall the truncated EM method initiated in [24]. To define the truncated EM numerical solutions, we first choose a strictly increasing continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\le u}|f(x)| \le \mu(u), \quad \forall\ u \ge 1.$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(0),\infty)$ to $\mathbb{R}_+$. We also choose a number $\Delta^* \in (0,1]$ and a strictly decreasing function $h : (0,\Delta^*] \to (0,\infty)$ such that

$$h(\Delta^*) \ge \mu(2), \quad \lim_{\Delta\to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{\frac{1}{4}}h(\Delta) \le 1, \quad \forall\ \Delta \in (0,\Delta^*]. \tag{2.7}$$

For a given stepsize $\Delta \in (0,\Delta^*]$, the truncated functions are defined as

$$f_\Delta(x) = f\left((|x| \wedge \mu^{-1}(h(\Delta)))\frac{x}{|x|}\right)$$

for $x \in \mathbb{R}$, where we set $x/|x| = 0$ when $x = 0$. It is easy to see that

$$|f_\Delta(x)| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta), \quad \forall\ x \in \mathbb{R}. \tag{2.8}$$

Remark 2.4. The truncated method used here is to make sure that the moments of the numerical solutions are uniformly bounded. Since g satisfies the linear growth condition, it is not necessary to truncate g.

The discrete-time truncated EM numerical solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are formed by setting $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta + g(X_\Delta(t_k))\Delta B_k$$

for $k = 0,\ 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. According to $\{X_k\}_{k\geq 0}$, let us introduce the following step process

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad t \geq 0$$

on which let us define the following continuous truncated EM solution:

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta(\bar{x}_\Delta(s))ds + \int_0^t g(\bar{x}_\Delta(s))dB(s)$$

for $t \geq 0$. It can be observed easily that $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$ for all $k \geq 0$. Moreover, $x_\Delta(t)$ is an Itô process with the Itô differential

$$dx_\Delta(t) = f_\Delta(\bar{x}_\Delta(t))dt + g(\bar{x}_\Delta(t))dB(t).$$

In this paper, from now on, C stands for a generic positive real constant dependent on $T,\ p,\ K,\ x_0$ etc. but independent of $\Delta,\ R$ and its values may change between occurrences.

The following lemmas are similar to Mao [24] since they do not depend on the local Lipschitz continuity of g.

Lemma 2.5. *For any $\Delta \in (0, \Delta^*]$ and any $\hat{p} > 0$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} \leq C\Delta^{\hat{p}/2}(h(\Delta))^{\hat{p}}, \quad \forall\ 0 \leq t \leq T. \tag{2.9}$$

Consequently

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} = 0, \quad \forall\ 0 \leq t \leq T. \tag{2.10}$$

Proof. Fix any $\Delta \in (0, \Delta^*]$ and $0 \leq t \leq T$, there is a unique integer $k \geq 0$ such that $t_k \leq t \leq t_{k+1}$. By the analysis of Lemma 3.1 in [24], for any $\hat{p} \geq 2$, we can easily derive that

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^{\hat{p}} \leq C\big(\Delta^{\hat{p}-1}\mathbb{E}\int_{t_k}^t |f_\Delta(\bar{x}_\Delta(s))|^{\hat{p}} ds + \Delta^{(\hat{p}-2)/2}\mathbb{E}\int_{t_k}^t |g(\bar{x}_\Delta(s))|^{\hat{p}} ds\big). \tag{2.11}$$

By (2.8), the linear growth condition of g and the Doob martingale inequality

$$\begin{aligned}
\mathbb{E}|x_\Delta(t)|^{\hat{p}} &\leq 3^{\hat{p}-1}\big(|x_0|^{\hat{p}} + \mathbb{E}|\int_0^t f_\Delta(\bar{x}_\Delta(s))ds|^{\hat{p}} + \mathbb{E}|\int_0^t g(\bar{x}_\Delta(s))dB(s)|^{\hat{p}}\big) \\
&\leq 3^{\hat{p}-1}\big(|x_0|^{\hat{p}} + T^{\hat{p}}(h(\Delta))^{\hat{p}} + C_{T,\hat{p}}\mathbb{E}\int_0^t |g(\bar{x}_\Delta(s))|^{\hat{p}} ds\big) \\
&\leq C\big(1 + (h(\Delta))^{\hat{p}} + \int_0^t \mathbb{E}|\bar{x}_\Delta(s)|^{\hat{p}} ds\big),
\end{aligned}$$

from which, we can derive that

$$\sup_{0\le r\le t}\mathbb{E}|\bar{x}_\Delta(r)|^{\hat{p}} \le \sup_{0\le r\le t}\mathbb{E}|x_\Delta(r)|^{\hat{p}} \le C\big(1+(h(\Delta))^{\hat{p}}+\int_0^t \sup_{0\le r\le s}\mathbb{E}|\bar{x}_\Delta(r)|^{\hat{p}}ds\big).$$

By the Gronwall inequality, we can easily get

$$\sup_{0\le t\le T}\mathbb{E}|\bar{x}_\Delta(t)|^{\hat{p}} \le C(1+(h(\Delta))^{\hat{p}}). \tag{2.12}$$

According to (2.8), (2.11), (2.12) and the linear growth condition of g, we immediately have

$$\mathbb{E}|x_\Delta(t)-\bar{x}_\Delta(t)|^{\hat{p}} \le C\big(\Delta^{\hat{p}}(h(\Delta))^{\hat{p}}+\Delta^{\frac{\hat{p}}{2}}+\Delta^{\frac{\hat{p}}{2}}(h(\Delta))^{\hat{p}}\big) \le C\Delta^{\frac{\hat{p}}{2}}(h(\Delta))^{\hat{p}}.$$

For $0<\hat{p}<2$, by the Lyapunov inequality, (2.9) still holds. The proof is completed. □

Lemma 2.6. *Let Assumptions 2.1 and 2.2 hold. Then*

$$\sup_{0<\Delta\le\Delta^*}\sup_{0\le t\le T}\mathbb{E}|x_\Delta(t)|^p \le C, \quad \forall\ T>0. \tag{2.13}$$

The proof of this lemma is the same as Lemma 3.2 in [24]. Since by (2.5) and the linear growth condition of g, we can still easily derive that

$$xf_\Delta(x)+\frac{p-1}{2}|g(x)|^2 \le C(1+|x|^2) \tag{2.14}$$

for all $x\in\mathbb{R}$. The proof of (2.13) only depends on (2.14) and Lemma 2.5.

Lemma 2.7. *Let Assumptions 2.1 and 2.2 hold. For any real number $R>|x_0|$, define the stopping time*

$$\tau_R = \inf\{t\ge 0: |x(t)|\ge R\},$$

where throughout this paper set $\inf\varnothing=\infty$*. Then*

$$\mathbb{P}(\tau_R\le T) \le \frac{C}{R^p}. \tag{2.15}$$

Note that this lemma is a direct result from (2.6) and recall that $C=C_{T,p,K,x_0}(1+e^{pT})$ standing for a generic positive real constant here is independent of R.

Lemma 2.8. *Let Assumptions 2.1 and 2.2 hold. For any real number $R>|x_0|$ and $\Delta\in(0,\Delta^*]$, define the stopping time*

$$\rho_{\Delta,R} = \inf\{t\ge 0: |x_\Delta(t)|\ge R\},$$

then

$$\mathbb{P}(\rho_{\Delta,R}\le T) \le \frac{C}{R^p}. \tag{2.16}$$

This lemma is a direct result of Lemma 4.1 below and recall that $C=C_{T,p,K,x_0}(1+e^{pT})$ standing for a generic positive real constant here is independent of Δ and R.

3 Convergence at time T

Lemma 3.1. *Let Assumptions 2.2 and 2.3 hold and assume that $R > |x_0|$ is a real number and $\Delta \in (0, \Delta^*]$ is sufficiently small such that $\mu^{-1}(h(\Delta)) \geq R$. Let τ_R and $\rho_{\Delta,R}$ be the same as defined in Lemmas 2.7 and 2.8 respectively. For all $T > 0$, set*

$$\theta_{\Delta,R} = \tau_R \wedge \rho_{\Delta,R} \quad and \quad e_\Delta(t) = x(t) - x_\Delta(t) \quad for \quad 0 < t \leq T.$$

Then

$$\mathbb{E}|e_\Delta(t \wedge \theta_{\Delta,R})| \leq \begin{cases} C\left(\frac{1}{\ln \Delta^{-1}} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\right), & \alpha = 0, \\ C(\Delta^{\frac{1}{2}}h(\Delta))^{2\alpha}, & 0 < \alpha < \frac{1}{2}. \end{cases}$$

Proof. Since $\mu^{-1}(h(\Delta)) \geq R$, if $s \leq \theta_{\Delta,R}$, $|\bar{x}_\Delta(s)| \leq R \leq \mu^{-1}(h(\Delta))$, by the definition of f_Δ, we have $f_\Delta(\bar{x}_\Delta(s)) = f(\bar{x}_\Delta(s))$. By using the method of Yamada and Watanabe [35], we first choose some $\delta > 1$ and $\epsilon > 0$, noting that

$$\int_{\epsilon/\delta}^{\epsilon} \frac{1}{x} = \ln \delta$$

and therefore there is a continuous nonnegative function $\psi_{\delta\epsilon}(x)$, $x \in [0, \infty)$, which is zero outside $[\epsilon/\delta, \epsilon]$, has integral 1 and satisfies (see, e.g., [15, p.168].)

$$\psi_{\delta\epsilon}(x) \leq \frac{2}{x \ln \delta}. \tag{3.1}$$

Define

$$\phi_{\delta\epsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta\epsilon}(z) dz dy, \quad x \in \mathbb{R}.$$

Note that for all $x \in \mathbb{R}$, $|x| \leq \phi_{\delta\epsilon}(x) + \epsilon$ and

$$0 \leq |\phi'_{\delta\epsilon}(x)| \leq 1, \quad \phi''_{\delta\epsilon}(x) = \psi_{\delta\epsilon}(|x|) \leq \frac{2}{|x| \ln \delta} I_{[\varepsilon/\delta \leq |x| \leq \epsilon]}. \tag{3.2}$$

Itô's formula provides

$$\begin{aligned} |e_\Delta(t \wedge \theta_{\Delta,R})| &\leq \epsilon + \phi_{\delta\epsilon}(e_\Delta(t \wedge \theta_{\Delta,R})) \\ &= \epsilon + \int_0^{t \wedge \theta_{\Delta,R}} \phi'_{\delta\epsilon}(e_\Delta(s)) \left[f(x(s)) - f_\Delta(\bar{x}_\Delta(s))\right] ds \\ &\quad + \frac{1}{2} \int_0^{t \wedge \theta_{\Delta,R}} \phi''_{\delta\epsilon}(e_\Delta(s)) \left[g(x(s)) - g(\bar{x}_\Delta(s))\right]^2 ds \\ &\quad + \int_0^{t \wedge \theta_{\Delta,R}} \phi'_{\delta\epsilon}(e_\Delta(s)) \left[g(x(s)) - g(\bar{x}_\Delta(s))\right] dB(s) \\ &= \epsilon + I_1 + I_2 + I_3. \end{aligned} \tag{3.3}$$

By the definition of $\phi_{\delta\epsilon}(x)$, it can be shown that the derivative of $\phi_{\delta\epsilon}(x)$ satisfies

$$\frac{\phi'_{\delta\epsilon}(x)}{x} > 0, \quad x \neq 0.$$

Therefore, since $\sup_{x\in\mathbb{R}} |\phi'_{\delta\epsilon}(x)| \leq 1$, by Assumption 2.2, it holds that for all $x,\, y \in \mathbb{R}$,

$$\begin{aligned}
\phi'_{\delta\epsilon}(x-y)(f(x)-f(y)) &= \begin{cases} \dfrac{\phi'_{\delta\epsilon}(x-y)}{x-y}(x-y)(f(x)-f(y)) & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y, \end{cases} \\
&\leq \begin{cases} L_1 \dfrac{\phi'_{\delta\epsilon}(x-y)}{x-y}|x-y|^2 & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y, \end{cases} \\
&\leq \begin{cases} L_1 \dfrac{|\phi'_{\delta\epsilon}(x-y)|}{|x-y|}|x-y|^2 & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y, \end{cases} \\
&\leq L_1|x-y|. \qquad (3.4)
\end{aligned}$$

This implies that

$$\begin{aligned}
I_1 &= \int_0^{t\wedge\theta_{\Delta,R}} \phi'_{\delta\epsilon}(e_\Delta(s))\left[f(x(s)) - f_\Delta(\bar{x}_\Delta(s))\right] ds \\
&\leq \int_0^{t\wedge\theta_{\Delta,R}} \phi'_{\delta\epsilon}(e_\Delta(s))\left[f(x(s)) - f(x_\Delta(s))\right] ds \\
&\quad + \int_0^{t\wedge\theta_{\Delta,R}} \left|\phi'_{\delta\epsilon}(e_\Delta(s))\left[f(x_\Delta(s)) - f(\bar{x}_\Delta(s))\right]\right| ds \\
&\leq L_1 \int_0^{t\wedge\theta_{\Delta,R}} |e_\Delta(s)| ds + \int_0^{t\wedge\theta_{\Delta,R}} |f(x_\Delta(s)) - f(\bar{x}_\Delta(s))| ds. \qquad (3.5)
\end{aligned}$$

According to (2.3) and (3.2),

$$\begin{aligned}
I_2 &\leq L_2^2 \int_0^{t\wedge\theta_{\Delta,R}} \frac{1}{|e_\Delta(s)|\ln\delta} I_{[\frac{\epsilon}{\delta}\leq e_\Delta(s)\leq\epsilon]}\big(x(s) - \bar{x}_\Delta(s)\big)^{1+2\alpha} ds \\
&\leq \frac{2^{2\alpha}L_2^2\epsilon^{2\alpha}T}{\ln\delta} + \frac{2^{2\alpha}L_2^2\delta}{\epsilon\ln\delta}\int_0^t (x_\Delta(s) - \bar{x}_\Delta(s))^{1+2\alpha} ds. \qquad (3.6)
\end{aligned}$$

By (3.3), (3.5), (3.6), Assumption 2.3, Lemmas 2.5, 2.6 and the Hölder inequality

$$
\begin{aligned}
& \mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})| \\
\le\ & \epsilon + L_1\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta_{\Delta,R})|ds \\
& +\sqrt{H}\mathbb{E}\int_0^{t\wedge\theta_{\Delta,R}} (1+|x_\Delta(s)|^\gamma+|\bar{x}_\Delta(s)|^\gamma)^{\frac{1}{2}}|x_\Delta(s)-\bar{x}_\Delta(s)|\,ds \\
& +2^{2\alpha}L_2^2\left(\frac{\epsilon^{2\alpha}T}{\ln\delta}+\frac{\delta}{\epsilon\ln\delta}TC_{1+2\alpha}(\Delta^{\frac{1}{2}}h(\Delta))^{1+2\alpha}\right) \\
\le\ & \epsilon + L_1\int_0^t \mathbb{E}|e_\Delta(s\wedge\theta_{\Delta,R})|ds + C\Delta^{\frac{1}{2}}h(\Delta)+C\left(\frac{\epsilon^{2\alpha}}{\ln\delta}+\frac{\delta}{\epsilon\ln\delta}(\Delta^{\frac{1}{2}}h(\Delta))^{1+2\alpha}\right).
\end{aligned}
$$

By the Gronwall inequality, we have

$$\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})|\le C\left(\epsilon+\Delta^{\frac{1}{2}}h(\Delta)+\frac{\epsilon^{2\alpha}}{\ln\delta}+\frac{\delta}{\epsilon\ln\delta}(\Delta^{\frac{1}{2}}h(\Delta))^{1+2\alpha}\right).$$

If $\alpha=0$, taking $\delta=\Delta^{-\frac{1}{8}}$, $\epsilon=-1/\ln\Delta$ yields

$$\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})|\le C\left(\frac{-9}{\ln\Delta}+\Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\left(\Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}+8\Delta^{\frac{1}{8}}(h(\Delta))^{\frac{1}{2}}\right)\right).$$

Since $\Delta^{\frac{1}{8}}(h(\Delta))^{\frac{1}{2}}\le 1$ and $\Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\le 1$, we immediately have

$$\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})|\le C\left(\frac{1}{\ln\Delta^{-1}}+\Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\right).$$

If $0<\alpha<1/2$, take $\epsilon=\Delta^{\frac{1}{2}}h(\Delta)$, $\delta=2$, then

$$\mathbb{E}|e_\Delta(t\wedge\theta_{\Delta,R})|\le C(\Delta^{\frac{1}{2}}h(\Delta))^{2\alpha}.$$

The proof is complete. □

Theorem 3.2. *Let Assumptions 2.2 and 2.3 hold, and $p>1$. If*

$$h(\Delta)\ge\mu\left((\Delta^{\frac{1}{2}}h(\Delta))^{-1/(p-1)}\right),$$

for all sufficiently small $\Delta\in(0,\Delta^*]$, then for every such small Δ,

$$\mathbb{E}|x(T)-x_\Delta(T)|\le\begin{cases} C\left(\frac{1}{\ln\Delta^{-1}}+\Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\right), & \alpha=0,\\ C(\Delta^{\frac{1}{2}}h(\Delta))^{2\alpha}, & 0<\alpha<\frac{1}{2}\end{cases}$$

for all $T>0$.

Proof. For any $\delta > 0$, by the Young inequality, Lemmas 2.3, 2.6, 2.7 and 2.8,

$$\begin{aligned}\mathbb{E}|e_\Delta(T)| &= \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}\le T}| \\ &\le \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + \frac{\delta}{p}\mathbb{E}|e_\Delta(T)|^p + \frac{p-1}{p\delta^{1/(p-1)}}\mathbb{P}(\theta_{\Delta,R}\le T) \\ &\le \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + \frac{C\delta}{p} + \frac{C(p-1)}{p\delta^{1/(p-1)}R^p}.\end{aligned}$$

Since $h(\Delta) \ge \mu\left((\Delta^{\frac{1}{2}}h(\Delta))^{-1/(p-1)}\right)$, thus $\mu^{-1}(h(\Delta)) \ge (\Delta^{\frac{1}{2}}h(\Delta))^{-1/(p-1)}$. We can therefore choose $\delta = \Delta^{\frac{1}{2}}h(\Delta)$ and $R = (\Delta^{\frac{1}{2}}(h(\Delta))^{-1/(p-1)}$ to get

$$\mathbb{E}|e_\Delta(T)| \le \mathbb{E}|e_\Delta(T)I_{\theta_{\Delta,R}>T}| + C\Delta^{\frac{1}{2}}h(\Delta).$$

By Lemma 3.1, we immediately obtain the result of this theorem. □

4 Strong convergence

In this section, we will prove a stronger convergence as follows

$$\lim_{\Delta\to 0}\mathbb{E}\Big[\sup_{0\le t\le T}|x(t)-x_\Delta(t)|^q\Big] = 0.$$

For this purpose, we illustrate a useful lemma here and its proof is very similar to Lemma 4.3 in [24] since it does not depend on the local Lipschitz of g.

Lemma 4.1. *Let Assumptions 2.1, 2.2 hold. Then for each $\bar{p} > 0$,*

$$\sup_{0<\Delta\le\Delta^*}\mathbb{E}\Big[\sup_{0\le t\le T}|x_\Delta(t)|^{\bar{p}}\Big] \le C, \quad \forall\ T > 0.$$

Note that Assumption 2.2 gurantees that g satisfies the linear growth condition, so Assumption 4.1 in [24] is satisfied with $\bar{K} = 2L^2 + 2g^2(0)$ and $r = 2$. The proof of this lemma is same as Lemma 4.3 in [24].

The following lemma gives the moment estimation of $e_\Delta(t)$ before a stopping time.

Lemma 4.2. *Let Assumptions 2.2 and 2.3 hold and assume that $R > |x_0|$ is a real number and $\Delta \in (0, \Delta^*]$ is sufficiently small such that $\mu^{-1}(h(\Delta)) \ge R$. Define $\bar{e}_\Delta(t) = x_\Delta(t) - \bar{x}_\Delta(t)$ and $e^*_\Delta(t) := \sup_{0\le s\le t}|x(s\wedge\theta_{\Delta,R}) - x_\Delta(s\wedge\theta_{\Delta,R})|$, then for all $0 \le t \le T$,*

$$\mathbb{E}|e^*_\Delta(t)| \le \begin{cases} C\left(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\right)^{\frac{1}{2}}, & \alpha = 0, \\ C(\Delta^{\frac{1}{2}}h(\Delta))^{4\alpha^2}, & 0 < \alpha < \frac{1}{2}. \end{cases}$$

Proof. Since $\mu^{-1}(h(\Delta)) \geq R$, if $s \leq \theta_{\Delta,R}$, $|\bar{x}_\Delta(s)| \leq R \leq \mu^{-1}(h(\Delta))$, by the definition of f_Δ, we have $f_\Delta(\bar{x}_\Delta(s)) = f(\bar{x}_\Delta(s))$.

$$\begin{aligned} |e_\Delta(t \wedge \theta_{\Delta,R})| &\leq \epsilon + \phi_{\delta\epsilon}(e_\Delta(t \wedge \theta_{\Delta,R})) \\ &= \epsilon + \int_0^{t\wedge\theta_{\Delta,R}} U_{\delta,\epsilon,\Delta}(s)ds + \frac{1}{2}\int_0^{t\wedge\theta_{\Delta,R}} V_{\delta,\epsilon,\Delta}(s)ds + M_{\delta,\epsilon,\Delta}(t), \quad (4.1) \end{aligned}$$

where

$$\begin{aligned} &U_{\delta,\epsilon,\Delta}(s) = \phi'_{\delta\epsilon}(e_\Delta(s))(f(x(s)) - f(\bar{x}_\Delta(s))), \\ &V_{\delta,\epsilon,\Delta}(s) = \phi''_{\delta\epsilon}(e_\Delta(s))\big(g(x(s)) - g(\bar{x}_\Delta(s))\big)^2, \\ &M_{\delta,\epsilon,\Delta}(t) = \int_0^{t\wedge\theta_{\Delta,R}} \phi'_{\delta\epsilon}(e_\Delta(s))\big(g(x(s)) - g(\bar{x}_\Delta(s))\big)dB(s). \end{aligned}$$

By (2.3), (2.4), (3.2) and (3.4)

$$\begin{aligned} U_{\delta,\epsilon,\Delta}(s) &\leq \phi'_{\delta\epsilon}(e_\Delta(s))\big(f(x(s)) - f(x_\Delta(s))\big) + |\phi'_{\delta\epsilon}(e_\Delta(s))\big(f(x_\Delta(s)) - f(\bar{x}_\Delta(s))\big)| \\ &\leq L_1|e_\Delta(s)| + |f(x_\Delta(s)) - f(\bar{x}_\Delta(s))| \\ &\leq L_1|e_\Delta(s)| + \sqrt{H}(1 + |x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)^{\frac{1}{2}}|\bar{e}_\Delta(s)| \quad (4.2) \end{aligned}$$

and

$$V_{\delta,\epsilon,\Delta}(s) \leq \frac{2^{2\alpha+1}L_2^2\epsilon^{2\alpha}}{\ln\delta} + \frac{2^{2\alpha+1}L_2^2\delta}{\epsilon\ln\delta}|\bar{e}_\Delta(s)|^{1+2\alpha}. \quad (4.3)$$

Substituting (4.2) and (4.3) into (4.1) yields

$$\begin{aligned} e_\Delta(t \wedge \theta_{\Delta,R}) \leq\ & L_1\int_0^{t\wedge\theta_{\Delta,R}} |e_\Delta(s)|ds + \sqrt{H}\int_0^{t\wedge\theta_{\Delta,R}} (1 + |x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)^{\frac{1}{2}}|\bar{e}_\Delta(s)|ds \\ &+ R_{\delta,\epsilon,\Delta} + M_{\delta,\epsilon,\Delta}(t) \quad (4.4) \end{aligned}$$

with

$$R_{\delta,\epsilon,\Delta} := \epsilon + \frac{2^{2\alpha}L_2^2\epsilon^{2\alpha}t}{\ln\delta} + \frac{2^{2\alpha}L_2^2\delta}{\epsilon\ln\delta}\int_0^{t\wedge\theta_{\Delta,R}} |\bar{e}_\Delta(s)|^{1+2\alpha}ds.$$

Due to $|\phi'_{\delta,\epsilon}| \leq 1$, we have

$$d\langle M_{\delta,\epsilon,\Delta}(t)\rangle \leq 2^{2\alpha}L_2^2(|e_\Delta(t)|^{1+2\alpha} + |\bar{e}_\Delta(t)|^{1+2\alpha})I_{t\leq\theta_{\Delta,R}}dt.$$

If $\alpha = 0$, by choosing $\delta = \Delta^{-\frac{1}{8}}$, $\epsilon = -1/\ln\Delta$, we immediately have

$$R_{\delta,\epsilon,\Delta} \leq C\left(\frac{-9}{\ln\Delta} + 8\Delta^{-\frac{1}{8}}\int_0^{t\wedge\theta_{\Delta,R}} |\bar{e}_\Delta(s)|ds\right). \quad (4.5)$$

If $0 < \alpha < 1/2$, take $\epsilon = \Delta^{\frac{1}{2}} h(\Delta)$, $\delta = 2$, and then we get

$$R_{\delta,\epsilon,\Delta} \leq C \left((\Delta^{\frac{1}{2}} h(\Delta))^{2\alpha} + \frac{1}{\Delta^{\frac{1}{2}} h(\Delta)} \int_0^{t\wedge\theta_{\Delta,R}} |\bar{e}_\Delta(s)|^{1+2\alpha} ds \right). \tag{4.6}$$

By (4.4),

$$\begin{aligned} e^*_\Delta(t) &\leq L_1 \int_0^t |e^*_\Delta(s)| ds + \sqrt{H} \int_0^{t\wedge\theta_{\Delta,R}} (1 + |x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)^{\frac{1}{2}} |\bar{e}_\Delta(s)| ds \\ &\quad + R_{\delta,\epsilon,\Delta} + \sup_{0\leq s\leq t} |M_{\delta,\epsilon,\Delta}(s)|. \end{aligned} \tag{4.7}$$

By the Burkholder-Davis-Gundy inequality we have

$$\mathbb{E}\Big[\sup_{0\leq s\leq t} |M_{\delta,\epsilon,\Delta}(s)| \Big] \leq 3\mathbb{E}\langle M_{\delta,\epsilon,\Delta}\rangle^{\frac{1}{2}}(t) \leq 3 \cdot 2^\alpha \cdot L_2(\mathbb{E}A_\Delta(t) + \mathbb{E}B_\Delta(t)), \tag{4.8}$$

where

$$A_\Delta(t) = \left(\int_0^{t\wedge\theta_{\Delta,R}} |e_\Delta(s)|^{1+2\alpha} ds \right)^{\frac{1}{2}}, \quad B_\Delta(t) = \left(\int_0^{t\wedge\theta_{\Delta,R}} |\bar{e}_\Delta(s)|^{1+2\alpha} ds \right)^{\frac{1}{2}}.$$

It is obvious that by Lemma 2.5

$$\mathbb{E}B_\Delta(t) \leq \left(\int_0^t \mathbb{E}|\bar{e}_\Delta(s)|^{1+2\alpha} ds \right)^{\frac{1}{2}} \leq C(\Delta^{\frac{1}{4}} (h(\Delta))^{\frac{1}{2}})^{1+2\alpha}. \tag{4.9}$$

If $\alpha = 0$, by the Hölder inequality and Lemma 3.1

$$\mathbb{E}A_\Delta(t) \leq \left(\int_0^t \mathbb{E}|e_\Delta(s \wedge \theta_{\Delta,R})| ds \right)^{\frac{1}{2}} \leq C \left(\Delta^{\frac{1}{4}} (h(\Delta))^{\frac{1}{2}} + \frac{1}{\ln \Delta^{-1}} \right)^{\frac{1}{2}}. \tag{4.10}$$

If $\alpha \in (0, 1/2)$, then by Young's inequality and Lemma 3.1,

$$\begin{aligned} \mathbb{E}A_\Delta(t) &\leq \mathbb{E}\left(|e^*_\Delta(t)|^{\frac{1}{2}} (\int_0^{t\wedge\theta_{\Delta,R}} |e_\Delta(s)|^{2\alpha} ds)^{\frac{1}{2}} \right) \\ &\leq \frac{1}{6 \cdot 2^\alpha L_2} \mathbb{E}|e^*_\Delta(t)| + 3 \cdot 2^{\alpha-1} L_2 \int_0^t \mathbb{E}|e_\Delta(s \wedge \theta_{\Delta,R})|^{2\alpha} ds \\ &\leq \frac{1}{6 \cdot 2^\alpha L_2} \mathbb{E}|e^*_\Delta(t)| + C(\Delta^{\frac{1}{2}} h(\Delta))^{4\alpha^2}. \end{aligned} \tag{4.11}$$

Consequently, if $\alpha = 0$, substituting (4.5), (4.8), (4.9), (4.10) into (4.7), together with Lemma

2.5 and Lemma 2.6 yields

$$\begin{aligned}\mathbb{E}e^*_\Delta(t) &\le L_1\int_0^t \mathbb{E}e^*_\Delta(s)ds + C\Big[\Delta^{\frac12}h(\Delta) + \frac{-9}{\ln\Delta} + 8\Delta^{\frac38}h(\Delta) + \Delta^{\frac14}(h(\Delta))^{\frac12} \\ &\quad + \left(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac14}(h(\Delta))^{\frac12}\right)^{\frac12}\Big] \\ &\le L_1\int_0^t \mathbb{E}e^*_\Delta(s)ds + C\Big[\Delta^{\frac12}h(\Delta) + \frac{1}{\ln\Delta^{-1}} + \Delta^{\frac14}(h(\Delta))^{\frac12}(1+8\Delta^{\frac18}(h(\Delta))^{\frac12}) \\ &\quad + \left(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac14}(h(\Delta))^{\frac12}\right)^{\frac12}\Big] \\ &\le L_1\int_0^t \mathbb{E}e^*_\Delta(s)ds + C\left(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac14}(h(\Delta))^{\frac12}\right)^{\frac12}.\end{aligned}$$

The last inequality comes from

$$\Delta^{\frac14}h(\Delta)\le 1 \quad\text{and}\quad \Delta^{\frac12}h(\Delta)\le\Delta^{\frac14}h(\Delta)^{\frac12}\le\left(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac14}(h(\Delta))^{\frac12}\right)^{\frac12}.$$

If $\alpha\in(0,1/2)$, substituting (4.6), (4.8), (4.9), (4.11) into (4.7) gives

$$\mathbb{E}e^*_\Delta(t)\le 2L_1\int_0^t \mathbb{E}e^*_\Delta(s)ds + C(\Delta^{\frac12}h(\Delta))^{4\alpha^2}$$

since $4\alpha^2<2\alpha<1/2+\alpha<1$. Finally, the Gronwall inequality yields the required assertion of this lemma. □

So far, the rates of strong convergence are dependent on h while the use of the truncated EM depends on both μ and h. In the remaining part of this paper, we provide the reader with a simple procedure to construct μ and h. We will show, the rates of strong convergence of the truncated EM for a specific truncation (i.e., for a specific choice of μ and h described below) are the same as the classical EM applied to the scalar SDE established by [7] (where the drift coefficient satisfies the linear growth condition while the diffusion coefficient is Hölder continuous). This indicates, to a good degree, that our specific choice of μ and h described below gives a nice rate of convergence. However, we still do not know if other choices of μ and h may give better convergence rates.

For arbitrary $\epsilon\in(0,1/2)$, define $\mu(u)=\bar{H}u^{1+\bar\gamma}$ ($\bar\gamma\ge\frac{\gamma}{2}$), $h(\Delta)=\bar{H}\Delta^{-\frac{\epsilon}{2}}$. Letting $\Delta^*\in(0,1]$ be sufficiently small such that for all $\Delta\in(0,\Delta^*]$ (2.7) and $\Delta^{\frac{1-\epsilon}{4}}\le\frac{1}{\ln\Delta^{-1}}$ hold, then we have the following result.

Theorem 4.3. *Let Assumptions 2.2 and 2.3 hold and let $\mu(u)$, $h(\Delta)$, Δ^* be defined as above, then*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x(t)-x_\Delta(t)|\Big] \le \begin{cases} C\left(\frac{1}{\ln\Delta^{-1}}\right)^{\frac{1}{2}}, & \alpha=0, \\ C\Delta^{2(1-\epsilon)\alpha^2}, & 0<\alpha<\frac{1}{2} \end{cases}$$

for all $T>0$.

Proof. Let $p>1$ sufficiently large such that $\frac{\epsilon}{2}>\frac{1+\bar{\gamma}}{2(p-1)}$. By the Young inequality, for any $\Delta\in(0,\Delta^*)$, $\delta>0$, and $R>|x_0|$, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|\Big] \le \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|\Big] + \frac{C\delta}{p} + \frac{C(p-1)}{pR^p\delta^{1/(p-1)}}.$$

If we take $\delta=\Delta^{(1-\epsilon)/2}$, $R=\Delta^{-(1-\epsilon)/2(p-1)}$, we immediately have

$$\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|\Big] \le \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|\Big] + C\Delta^{\frac{1-\epsilon}{2}}$$

for any $\Delta\in(0,\Delta^*)$. On the other hand, we see that $\Delta^{-\epsilon/2}\ge\Delta^{-(1-\epsilon)(1+\bar{\gamma})/2(p-1)}$, from which there holds

$$\mu^{-1}(h(\Delta))\ge\Delta^{-(1-\epsilon)/2(p-1)}=R.$$

By Lemma 4.2, if $\alpha=0$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|\Big] &\le C\left(\left(\frac{1}{\ln\Delta^{-1}}+\Delta^{\frac{1-\epsilon}{4}}\right)^{\frac{1}{2}}+\Delta^{\frac{1-\epsilon}{2}}\right) \\ &\le C\left(\frac{1}{\ln\Delta^{-1}}+\Delta^{\frac{1-\epsilon}{4}}\right)^{\frac{1}{2}} \\ &\le C\left(\frac{1}{\ln\Delta^{-1}}\right)^{\frac{1}{2}}.\end{aligned}$$

If $\alpha\in(0,1/2)$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|\Big] &\le C(\Delta^{2(1-\epsilon)\alpha^2}+\Delta^{\frac{1-\epsilon}{2}}) \\ &\le C\Delta^{2(1-\epsilon)\alpha^2}.\end{aligned}$$

The proof is complete. □

Next, we give the higher-order moment estimation of $e_\Delta(t)$.

Theorem 4.4. *Under Assumptions 2.2 and 2.3, let $\mu(u)$, $h(\Delta)$, Δ^* defined as above, then*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x(t)-x_\Delta(t)|^q\Big] \le \begin{cases} \frac{C}{\ln\Delta^{-1}}, & \alpha=0, \\ C\Delta^{(1-\epsilon)\alpha}, & 0<\alpha<\frac{1}{2} \end{cases}$$

for all $q\ge2$, $T>0$.

Proof. From above theorem, it is easily observed that for any $q \geq 2$, let $p > q$ sufficiently large such that $\epsilon(p-q) > q(1+\bar{\gamma})$. By the Young inequality, Lemmas 2.3, 2.7, 2.8 and 4.1, for any $\Delta \in (0, \Delta^*)$, $\bar{\delta} > 0$, and $R > |x_0|$, we have

$$\begin{aligned}
\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^q\Big] &= \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^q I_{\theta_{\Delta,R}>T}\Big] + \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^q I_{\theta_{\Delta,R}\le T}\Big] \\
&\le \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^q I_{\theta_{\Delta,R}>T}\Big] + \frac{q\bar{\delta}}{p}\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^p\Big] \\
&\quad + \frac{(p-q)}{p\bar{\delta}^{q/(p-q)}}\mathbb{P}(\theta_{\Delta,R}\le T) \\
&\le \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^q I_{\theta_{\Delta,R}>T}\Big] + \frac{Cq\bar{\delta}}{p} + \frac{C(p-q)}{pR^p\bar{\delta}^{q/(p-q)}}.
\end{aligned}$$

Let $\bar{\delta} = \Delta^{q(1-\epsilon)/2}$, $R = \Delta^{-q(1-\epsilon)/2(p-q)}$, then

$$\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t)|^q\Big] \le \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big] + C\Delta^{\frac{q(1-\epsilon)}{2}}. \tag{4.12}$$

By (4.4), the Jensen inequality, the Burkholder-Davis-Gundy inequality and Assumption 2.2, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big] \\
\le\ & C\{\epsilon^q + \mathbb{E}\left(\int_0^T \sup_{0\le u\le s}|e_\Delta(u\wedge\theta_{\Delta,R})|ds\right)^q + \mathbb{E}\left(\int_0^T (1+|x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)^{\frac{1}{2}}|\bar{e}_\Delta(s)|ds\right)^q \\
&+\frac{\epsilon^{2q\alpha}}{(\ln\delta)^q} + \frac{\delta^q}{\epsilon^q(\ln\delta)^q}\mathbb{E}\left(\int_0^T |\bar{e}_\Delta(s)|^{1+2\alpha}ds\right)^q \\
&+\mathbb{E}\Big[\sup_{0\le t\le T}\Big|\int_0^{t\wedge\theta_{\Delta,R}}\phi'_{\delta,\epsilon}(e_\Delta(s))(g(x(s))-g(\bar{x}_\Delta(s)))dB(s)\Big|^q\Big]\} \\
\le\ & C\{\epsilon^q + \int_0^T \mathbb{E}\Big[\sup_{0\le u\le s}|e_\Delta(u\wedge\theta_{\Delta,R})|^q\Big]ds + \int_0^T \mathbb{E}\left(1+|x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma\right)^{\frac{q}{2}}|\bar{e}_\Delta(s)|^q\,ds \\
&+\frac{\epsilon^{2q\alpha}}{(\ln\delta)^q} + \frac{\delta^q}{\epsilon^q(\ln\delta)^q}\int_0^T \mathbb{E}|\bar{e}_\Delta(s)|^{(1+2\alpha)q}ds + \mathbb{E}\left(\int_0^{T\wedge\theta_{\Delta,R}}|x(s)-\bar{x}_\Delta(s)|^{1+2\alpha}ds\right)^{\frac{q}{2}}\} \\
=:\ & W.
\end{aligned}$$

Applying the Jensen inequality and Lemmas 2.5, 2.6 yields that

$$\begin{aligned} W &\le C\{\epsilon^q + (\Delta^{\frac{1}{2}}h(\Delta))^q + \frac{\epsilon^{2q\alpha}}{(\ln\delta)^q} + \frac{\delta^q}{\epsilon^q(\ln\delta)^q}(\Delta^{\frac{1}{2}}h(\Delta))^{(1+2\alpha)q} \\ &\quad + (\Delta^{\frac{1}{2}}h(\Delta))^{\frac{q(1+2\alpha)}{2}} + \int_0^T \mathbb{E}\Big[\sup_{0\le u\le s}|e_\Delta(u\wedge\theta_{\Delta,R})|^q\Big]du \\ &\quad + \int_0^T \mathbb{E}|e_\Delta(s\wedge\theta_{\Delta,R})|^{\frac{q}{2}(1+2\alpha)}ds\}. \end{aligned}$$

Note that we have also used the estimation

$$|x(s)-\bar{x}_\Delta(s)|^{1+2\alpha} \le C(|e_\Delta(s)|^{1+2\alpha} + |x_\Delta(s)-\bar{x}_\Delta(s)|^{1+2\alpha}).$$

But for any $x>0$, we have $x^{\frac{q}{2}(1+2\alpha)} \le x + x^q$, from which

$$\mathbb{E}|e_\Delta(s\wedge\theta_{\Delta,R})|^{\frac{q}{2}(1+2\alpha)} \le \mathbb{E}\Big[\sup_{0\le u\le s}|e_\Delta(u\wedge\theta_{\Delta,R})|^q\Big] + \mathbb{E}|e_\Delta(s\wedge\theta_{\Delta,R})|.$$

By the Gronwall inequality,

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big] &\le C\Big[\epsilon^q + (\Delta^{\frac{1}{2}}h(\Delta))^q + \frac{\epsilon^{2q\alpha}}{(\ln\delta)^q} + \frac{\delta^q}{\epsilon^q(\ln\delta)^q}(\Delta^{\frac{1}{2}}h(\Delta))^{(1+2\alpha)q} \\ &\quad + (\Delta^{\frac{1}{2}}h(\Delta))^{\frac{q(1+2\alpha)}{2}} + \int_0^T \mathbb{E}|e_\Delta(s\wedge\theta_{\Delta,R})|ds\Big]. \end{aligned}$$

Since $\mu^{-1}(h(\Delta)) \ge R$, by Lemma 3.1, if $\alpha=0$, then choose $\delta=\Delta^{-\frac{1}{8}}$, $\epsilon=-1/\ln\Delta$,

$$\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big] \le C\left(\frac{1}{\ln\Delta^{-1}} + \Delta^{\frac{1}{4}}(h(\Delta))^{\frac{1}{2}}\right) \le \frac{C}{\ln\Delta^{-1}}. \quad (4.13)$$

If $\alpha\in(0,1/2)$, then taking $\epsilon=\Delta^{\frac{1}{2}}h(\Delta)$, $\delta=2$ gives

$$\mathbb{E}\Big[\sup_{0\le t\le T}|e_\Delta(t\wedge\theta_{\Delta,R})|^q\Big] \le C(\Delta^{\frac{1}{2}}h(\Delta))^{2\alpha}. \quad (4.14)$$

This, together with (4.12) and (4.13), yields the required assertion of this theorem. □

Example. Consider the following SDE

$$dx(t) = (ax(t)-bx^3(t))dt + c|x(t)|^{\frac{1}{2}+\alpha}dB(t), \quad (4.15)$$

with $x(0)=x_0$, $0<\alpha<\frac{1}{2}$, where $B(t)$ is a scalar Brownian motion and $a,\ b,\ c$ are positive numbers. Clearly, $f(x)=ax-bx^3$ satisfies Assumption 2.1. Also,

$$(x-y)(f(x)-f(y)) \le a(x-y)^2, \qquad |g(x)-g(y)| \le c(x-y)^{\frac{1}{2}+\alpha},$$

which shows that Assumption 2.2 is satisfied. Moreover,

$$|f(x) - f(y)|^2 \leq H(1 + |x|^\gamma + |y|^\gamma)|x - y|^2$$

holds with $\gamma = 4$, $H = 2a^2 + 9b^2$, so Assumption 2.3 is satisfied. To apply Theorem 4.3, noting that

$$\sup_{|x| \leq u} |f(x)| \leq \bar{H} u^3 \qquad u \geq 1,$$

where $\bar{H} = a + b + c$, we can set $\mu(u) = \bar{H}u^3$ and for all $0 < \epsilon < \frac{1}{2}$, $h(\Delta) = \bar{H}\Delta^{-\frac{\epsilon}{2}}$. We can therefore conclude that the truncated EM solutions of the SDE (4.15) satisfy

$$\mathbb{E}\Big[\sup_{0 \leq t \leq T} |x(t) - x_\Delta(t)|\Big] \leq C\Delta^{2(1-\epsilon)\alpha^2} \qquad 0 < \alpha < \frac{1}{2},$$

for all $T > 0$.

Finally, let us give numerical simulations to illustrate our theoretical results of Theorem 4.3. Choose $a = b = c = 1$. Considering the fact that the solution of equation (4.15) is unknown, we get the numerical "exact" solution by very small stepsize , e.g., $\Delta = 1/320000$ and we perform 1000 sample paths of the Truncated EM method. Compared with the results of Corollary 2.3 in [7] (convergence order is $2\alpha^2$ if $0 < \alpha < 1/2$), Figure 1 and Figure 2 show that we can obtain the same convergence order for the nonlinear equations. Fig. 2 also shows that if α tends to 1/2, the convergence order should be 1/2, which is the classical results in the previous papers. The convergence orders are listed in Figure 1 and Figure 2.

Conclusion remark

Our result depends on the Yamada-Watanabe method for a scalar SDE with Hölder continuous diffusion coefficient and this method is not applicable for m-dimensional SDEs. To consider SDEs with m independent perturbations, we should consider the existence of the strong solution for general m-dimensional SDEs with Hölder continuous diffusion coefficients firstly.

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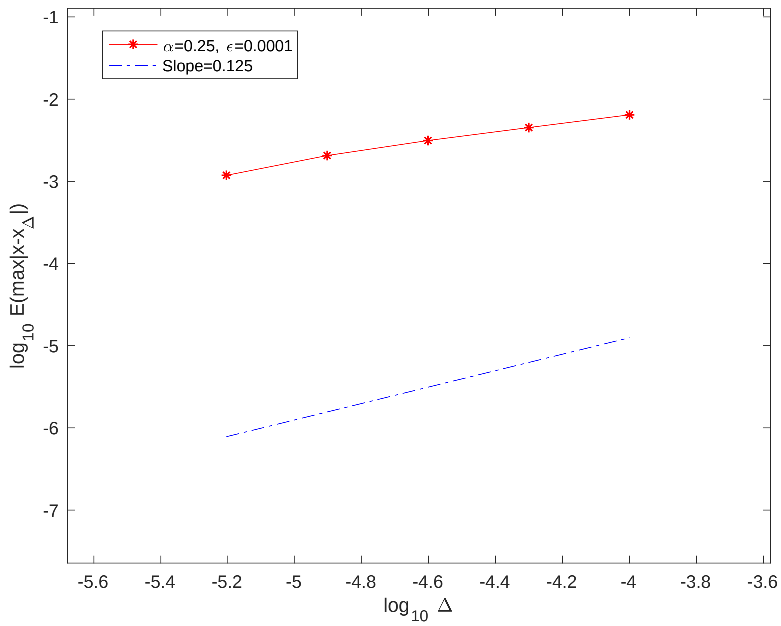

Figure 1: The convergence order with $\alpha = 0.25$ and $\epsilon = 0.0001$.

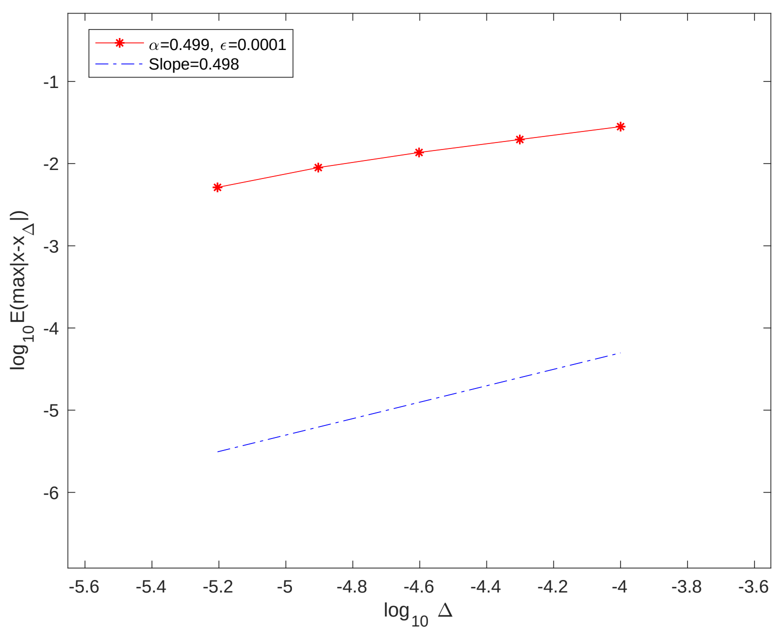

Figure 2: The convergence order with $\alpha = 0.499$ and $\epsilon = 0.0001$.

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