Hopf-Frobenius algebras and a simpler Drinfeld double

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The ZX-calculus and related theories are based on so-called interacting Frobenius algebras, where a pair of †-special commutative Frobenius algebras jointly form a pair of Hopf algebras. In this setting we introduce a generalisation of this structure, *Hopf-Frobenius algebras*, starting from a single Hopf algebra which is not necessarily commutative or cocommutative. We provide the necessary and sufficient condition for a Hopf algebra to be a Hopf-Frobenius algebra, and show that every Hopf algebra in $\mathbf{FVect_k}$ is a Hopf-Frobenius algebra. Hopf-Frobenius algebras provide a notion of duality, and give us a "dual" Hopf algebra that is isomorphic to the usual dual Hopf algebra in a compact closed category. We use this isomorphism to construct a Hopf algebra isomorphic to the Drinfeld double that is defined on $H \otimes H$ rather than $H \otimes H^*$.

1 Introduction

In the monoidal categories approach to quantum theory [1, 13] Hopf algebras [31] have a central role in the formulation of complementary observables [12]. In this setting, a quantum observable is represented as special commutative †-Frobenius algebra; a pair of such observables are called *strongly complementary* if the algebra part of the first and the coalgebra part of the second jointly form a Hopf algebra. In abstract form, this combination of structures has been studied under the name "interacting Frobenius algebras" [16] where it is shown that relatively weak commutation rules between the two Frobenius algebras produce the Hopf algebra structure. From a different starting point Bonchi et al [7] showed that a distributive law between two Hopf algebras yields a pair of Frobenius structures, an approach which has been generalised to provide a model of Petri nets [6]. Given the similarity of the two structures it is appropriate to consider both as exemplars of a common family of *Hopf-Frobenius algebras*.

In the above settings, the algebras considered are both commutative and cocommutative. However more general Hopf algebras, perhaps not even symmetric, are a ubiquitous structure in mathematical physics, finding applications in gauge theory [27], topological quantum field theory [3] and topological quantum computing [8]. In this paper we take the first steps towards generalising the concept of Hopf-Frobenius algebra to the non-commutative case, and opening the door to applications of categorical quantum theory in other areas of physics.

Loosely speaking, a Hopf-Frobenius algebra consists of two monoids and two comonoids such that one way of pairing a monoid with a comonoid gives two Frobenius algebras, and the other pairing yields two Hopf algebras, with the additional condition that antipodes are constructed from the Frobenius forms. This schema is illustrated in Figure 1. In Section 3 we give the precise

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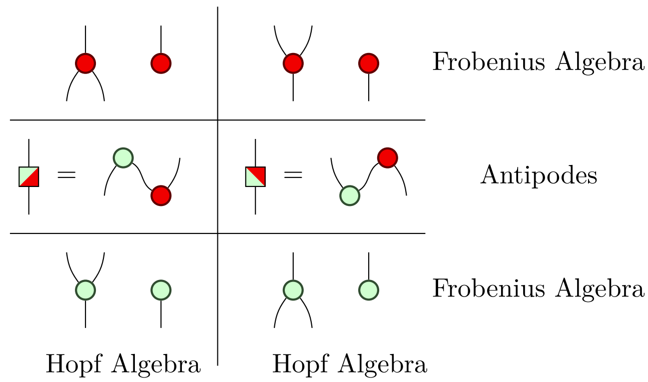


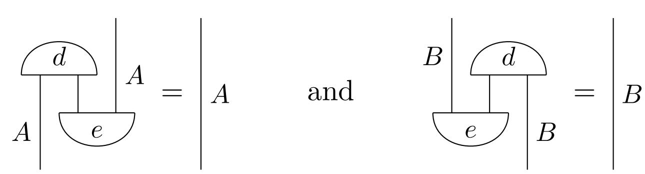
Figure 1: The elements of a Hopf-Frobenius algebra

definition of Hopf-Frobenius algebras and state the necessary and sufficient conditions to extend a Hopf algebra to a Hopf-Frobenius algebra in an arbitrary symmetric monoidal category. It was previously known that in $\mathbf{FVect_k}$, the category of finite dimensional vector spaces, every Hopf algebra carries a Frobenius algebra on both its monoid [26] and its comonoid [14, 24]; in fact every Hopf algebra in $\mathbf{FVect_k}$ is Hopf-Frobenius. In Section 4 we briefly present some examples which are not the usual abelian group algebras. In Section 5 we show the structure of a Hopf-Frobenius algebra can be used to give a simpler version of the Drinfeld double construction.

2 Preliminaries

We assume that the reader is familiar with strict symmetric monoidal categories and their diagrammatic notation; see Selinger [29] for a thorough treatment. We make the convention that diagrams are read from top to bottom. When we work with the dual of an object, we will opt to omit the object names from the wires except where doing so would create ambiguity. Instead, we will assign an orientation to the wires: downwards for the original object, upwards for its dual.

Definition 2.1. In a monoidal category $\mathcal{C}$ with objects A and B, B is *left dual* to A if there exist morphisms $d : I \to A \otimes B$ and $e : B \otimes A \to I$ such that

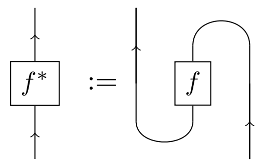


In this circumstance A is *right dual* to B. Note that if $\mathcal{C}$ is symmetric then left duals and right duals coincide.

The morphisms d and e are usually called the unit and counit; for reasons which will become obvious shortly we avoid that terminology and refer to them as the *cap* and the *cup*. Note that if an object has a dual it is unique up to isomorphism (see Lemma C.1).

Definition 2.2. A *compact closed category* [22] is a symmetric monoidal category where every object A has an assigned dual (A^*, d_A, e_A). In the graphical notation we depict the cup and cap in the obvious way:

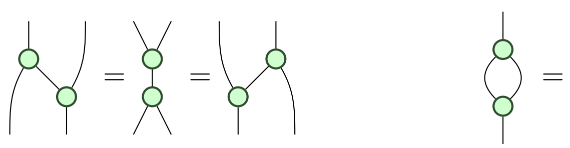
Proposition 2.3 ([22]). *Let $\mathcal{C}$ be a compact closed category. By defining $f^* : B^* \to A^*$ as*



the assignment of duals $A \mapsto A^*$ extends uniquely to a strong monoidal functor $(\cdot)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, with natural isomorphisms $(A \otimes B)^* \cong B^* \otimes A^*$, $A^{**} \cong A$, and $I^* \cong I$ and, further, d and e are natural transformations.

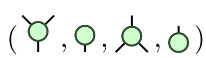
The main foci of this work – Frobenius and Hopf algebras – combine the structure of a monoid and a comonoid on the same object. See Appendix C.2 for basic definitions.

Definition 2.4. A *Frobenius algebra* in a symmetric monoidal category $\mathcal{C}$ consists of a monoid and a comonoid on the same object, obeying the Frobenius law, shown below on the left:



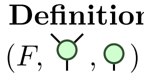
A Frobenius algebra is called *special* or *separable* when it obeys the equation above right, and *quasi-special* when it obeys the special equation up to an invertible scalar factor. A Frobenius algebra is commutative when its monoid is, and cocommutative when its comonoid is.

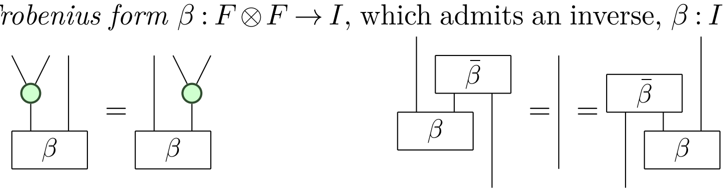
Lemma 2.5. *Every Frobenius algebra induces a cup and a cap which make the object self-dual.*

Proof. Given the Frobenius algebra  define the cup and cap as shown below.

From here the snake equation follows directly. □

Definition 2.4, due to Carboni and Walters [9], has a pleasing symmetry between the monoid and comonoid parts. However, an older equivalent definition will be useful in later sections[1].

Definition 2.6. A *Frobenius algebra* in a symmetric monoidal category $\mathcal{C}$ consists of a monoid  and a *Frobenius form* $\beta : F \otimes F \to I$, which admits an inverse, $\bar{\beta} : I \to F \otimes F$, satisfying:



[1]See Fauser's survey [17] for several equivalent definitions.

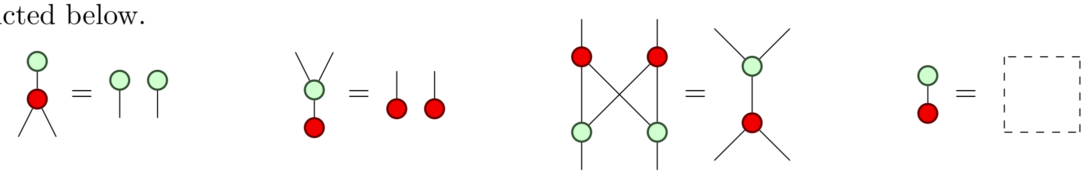
To see that Definition 2.4 implies this definition it suffices to take the cup and cap defined above as β and $\bar{\beta}$. For the converse, we dualise  with β to get a comonoid. For a proof of how this comonoid fulfills the Frobenius law, see Kock [23].

Frobenius forms are far from unique: there is one for each invertible element of the monoid (see AppendixC.3).

Special Frobenius algebras can be understood as arising from a distribution law of comonoids over monoids [25]. In the other direction, distributing monoids over comonoids yields bialgebras.

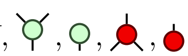
Note. Unlike the preceding section, in our discussion of bialgebras and Hopf algebras, we will use different colours for the monoid and comonoid parts of the structure.

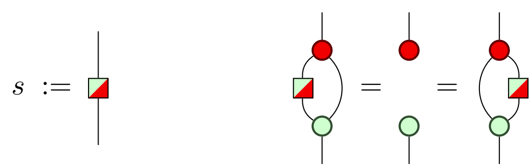
Definition 2.7. A *bialgebra* in symmetric monoidal category $\mathcal{C}$ consists of a monoid and a comonoid on the same object, which jointly obey the *copy*, *cocopy*, *bialgebra*, and *scalar* laws depicted below.



Note that the dashed box above represents an empty diagram. We may equivilantly define a bialgebra as a monoid and a comonoid such that the comonoid is a monoid homomorphism. A *bialgebra morphism* is an morphism of the object which is both a monoid homomorphism and a comonoid homomorphism.

Remark 2.8. Some works, notably on the ZX-calculus [12, 2, 20] and related theories [16], the last axiom is dropped and the other equations modified by a scalar factor, to give a *scaled bialgebra*. Here we use the standard definition: the Frobenius algebras we construct will not be special.

Definition 2.9. A *Hopf algebra* consists of a bialgebra (H, ) and an endomorphism $s : H \to H$ called the *antipode* which satisfies the *Hopf law*:



Where unambiguous, we abuse notation slightly and use H to refer the whole Hopf algebra. Following Street [30], we can define another Hopf algebra H^{op} on the same object, having the same unit and counit, but with the arguments of the multiplication and comultiplication swapped:

Replacing only the comultiplication as above yields a bialgebra H^σ which is not necessarily Hopf. We quote the following basic properties from Street [30].

Proposition 2.10. *For a Hopf algebra H:*

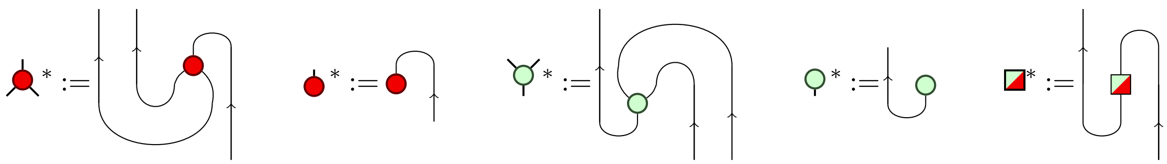
1. *The antipode s is unique.*

2. $s : H^{\mathrm{op}} \to H$ *is a bialgebra homomorphism, i.e.*

3. H^σ *is a Hopf algebra if and only if* s *is invertible, in which case the antipode of* H^σ *is* s^{-1}.

4. *If* H *is commutative or cocommutative then* $s \circ s = \mathrm{id}_H$.

Definition 2.11. Let (H, , , , ,) be a Hopf algebra, and suppose that the object H has a left dual H^*. We define the *dual Hopf algebra* (H^*, *, *, *, *, *) as :



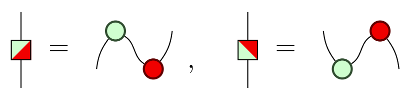
It's straightforward to prove that H^* is indeed a Hopf algebra using the equations of Def 2.1 In later sections it will be helpful to consider duals with respect to different cups and caps, in which case we will vary notation accordingly but the same construction is used in all cases.

3 Hopf-Frobenius Algebras

We now arrive at the main subject of this paper, Hopf-Frobenius algebras in an arbitrary symmetric monoidal category $\mathcal{C}$. These algebras generalise interacting Frobenius algebras [12, 16], and share the same gross structure. It will be helpful to introduce a weaker notion first.

Definition 3.1. A *pre-Hopf-Frobenius algebra* or *pre-HF algebra* consists of an object H bearing a green monoid (,), a green comonoid (,), a red monoid (,), a red comonoid (,) and an endomorphism such that (, , ,) and (, , ,) are Frobenius algebras, and (, , , ,) is a Hopf algebra.

Definition 3.2. A *Hopf-Frobenius algebra*, or *HF algebra*, is a pre-Hopf-Frobenius algebra where satisfies the left equation below,



and with defined as in the right equation above, (, , , ,) is a Hopf algebra.

We refer to the four algebras that make up an HF algebra by the colour[2] of their *multiplication*, so that (, , , ,) is the *green* Hopf algebra, (, , ,) is the *red* Frobenius algebra, etc.

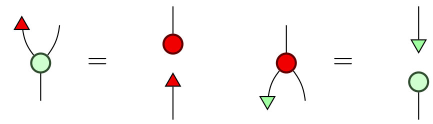
Remark 3.3. Despite their distinct definitions, the two antipodes may coincide; for example if both Frobenius algebras are symmetric, as in a group algebra, then they are equal.

[2]If you are reading this document in monochrome *green* will appear as light grey and *red* as dark grey.

We now move on to the main topic of the section: under what conditions does a Hopf algebra extend to a Hopf-Frobenius algebra? Henceforward, unless otherwise stated, $\mathcal{C}$ will denote a symmetric monoidal category, and H will denote a Hopf algebra $(H, , , , ,)$ in $\mathcal{C}$. Omitted proofs are found in Appendix A.

A key concept is that of an integral. Pareigis [28] proved[3] that in $\mathbf{FPMod_R}$, the category of finitely generated projective modules over a commutative ring, a Hopf algebra has Frobenius structure when its space of integrals is isomorphic to the ring. More generally, Takeuchi[33] and Bespalov et al. [5] gave conditions for the space of integrals in certain braided monoidal categories to be invertible.

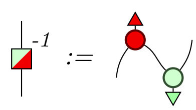
Definition 3.4. A *left (co)integral* on H is a copoint $: H \to I$ (resp. a point $: I \to H$), satisfying the equations:



A *right (co)integral* is defined similarly.

Definition 3.5. An *integral Hopf algebra* $(H, ,)$ is a Hopf algebra H equipped with a choice of right integral, and left cointegral, such that $\circ = \mathrm{id}_I$.

Lemma 3.6. *Let $(H, ,)$ be an integral Hopf algebra. Then the following map is the inverse of the antipode.*



Lemma 3.7. *Let $(H, ,)$ be an integral Hopf algebra, and define*

then β is a Frobenius form for $(H, ,)$ iff γ is a right inverse for β.

Per Definition 2.6, is the counit of this Frobenius algebra and the comultiplication is obtained by dualising with β.

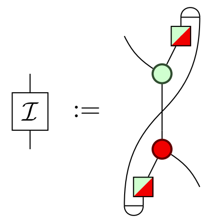
Lemma 3.7 shows an example of a duality structure which produces a Frobenius algebra from an integral Hopf algebra. We wish to treat all such dualities in a uniform way, and also to weaken the requirements on such a duality. For this purpose, we shall define the following concept.

Definition 3.8. Let A and B be objects in a symmetric monoidal category $\mathcal{C}$. A is a *right half dual of* B if there exist morphisms $\cap : I \to A \otimes B$ and $\cup : B \otimes A \to I$ which satisfy the left equation of 2.1. In this circumstance, B is a *left half dual of* A

Half duals are a strict generalisation of duals in the sense of Definition 2.1. Further, any integral Hopf algebra $(H, ,)$ makes H left half dual to itself, via the morphisms β and γ of Lemma 3.7. Unlike true duals, an object may have non-isomorphic half duals. For example, if B is left dual to A, with a section $m : B \hookrightarrow C$ for some retraction $m' : C \twoheadrightarrow B$, then C is a left half dual of A.

[3] This is a generalisation of earlier work by Larson and Sweedler [26] showing that the space of integrals in $\mathbf{FVect_k}$ is always isomorphic to k.

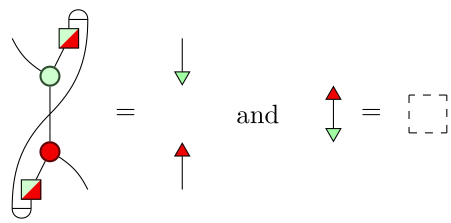
Definition 3.9. Let the object H have a right half dual H^*. The *integral morphisim* $\mathcal{I} : H \to H$ is defined as shown below.



Note that this definition does *not* depend on the choice of half dual – see Lemma A.1

In $\mathbf{FPMod_R}$, $\mathcal{I}$ may be seen as a map from H to the space of left integrals. In fact, it is the retraction of the natural injection from the space of left integrals into H. As such, it acts trivially on integrals, and for every element $v \in H$, $\mathcal{I}(v)$ is a left integral (which may be 0). In Lemma A.3 we show that this holds in the general case.

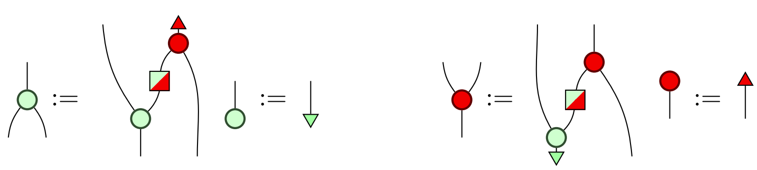
Definition 3.10. We say that a Hopf algebra satisfies the *Frobenius condition* if there exists maps  and  such that



Theorem 3.11. *If H satisfies the Frobenius condition, then H is a pre-HF algebra with the Frobenius forms and their inverses as shown below.*

Further, $(H,$$,$$)$ *is an integral Hopf algebra.*

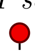
The explicit definitions of the green comonoid and red monoid structures are shown below.



We can rephrase Pareigis' condition as the existence of an integral  in H such that each other integral is equal to a scalar multiple of  .

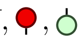
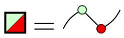
In a monoidal category, an object A is said to have *enough points* if, for all morphisms $f, g : A \to B$, we have

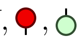
$$(\forall x : I \to A, \quad fx = gx) \Rightarrow f = g \,.$$

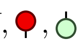
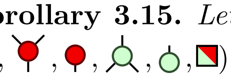
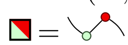
Lemma 3.12. *If H satisfies the Frobenius condition then every left cointegral (right integral) is a scalar multiple of*  *(resp.*  *); further if H has enough points then the converse holds.*

Since $\mathbf{FPMod_R}$ (and $\mathbf{FVect_k}$) are categories where every object has enough points, Lemma 3.12 implies Pareigis' condition is exactly the Frobenius condition.

We now consider the question of when a pre-HF algebra is Hopf-Frobenius. There are two issues to address: whether the red Hopf algebra exists, and whether the two antipodes have the required form.

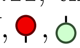
Lemma 3.13. *Let H be a pre-HF algebra;* $(H,$$)$ *is an integral Hopf algebra if and only if* .

Lemma 3.14. *Let H be a pre-HF algebra such that* $(H,$$)$ *is an integral Hopf algebra, and let $(\cdot)^{\circ}$ be the duality defined by the green Frobenius algebra (cf. Lemma 2.5). Then:*

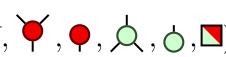
Corollary 3.15. *Let H be a pre-HF algebra such that* $(H,$$)$ *is an integral Hopf algebra.* $(H,$$)$ *forms a Hopf algebra, where* .

Putting all of the above together, we obtain the main result of this section.

Theorem 3.16. *Let H be a Hopf algebra such that the object H has some weak right dual H^*. Then H is a Hopf-Frobenius algebra if and only if H fulfills the Frobenius condition.*

Proof. By Theorem 3.11, the Frobenius condition implies the existence of the two Frobenius algebras, and that $(H,$$)$ is an integral Hopf algebra. By Lemma 3.13,  has the required form. Corollary 3.15 then completes the proof. For the converse direction, it is straightforward to show that every Hopf-Frobenius algebra satisfies the Frobenius condition. See Appendix A. □

One might wonder whether there is any gap between a pre-HF algebra and an HF algebra. The requirement on the form of the antipodes is decisive here. Suppose that we have an HF algebra H. Formally exchanging the colours yields a pre-HF algebra, with two Hopf algebras. However, unless the Frobenius algebras are symmetric, this transformation does not preserve the form of the antipodes hence the resulting structure is not an HF algebra. However the gap is a narrow one, as the following result shows.

Theorem 3.17. *Let H be a pre-HF algebra such that, for some* , $(H,$$)$ *forms a Hopf algebra; then H extends to a Hopf-Frobenius algebra.*

4 Examples

Combined with the results of Larson and Sweedler [26], Pareigis [28], and Lemma 3.12, Theorem 3.16 implies that any Hopf algebra in $\mathbf{FVect_k}$ is Hopf-Frobenius. This allows the direct extension of [16] to non-abelian group algebras, but there are plenty of other examples. We briefly mention some examples which are neither commutative nor cocommutative.

Example 4.1. Let k be a field with a primitive n^{th} root of unity z. The *Taft Hopf algebras* [32] are a family of Hopf algebras in $\mathbf{FVect_k}$ whose antipodes have order $2n$. Generically, the algebra $(H,\mu,1,\Delta,\epsilon,s)$ is generated by elements x and g, such that $x^n = 0$, $g^n = 1$, and $gx = zxg$. The coalgebra is defined $\Delta(x) = 1\otimes x + x\otimes g$, and $\Delta(g) = g\otimes g$, with $\epsilon(x) = 0$ and $\epsilon(g) = 1$. The antipode is $s(x) = -xg^{-1}$, $s(g) = g$, and the rest of the structure follows from the Hopf algebra axioms. We may see that H has the basis $x^{\alpha}g^{\beta}$, where $0 \leq \alpha, \beta, \leq n-1$, so this will imply that H is n^2 dimensional. One can calculate that the left integral of H is

$$\sum_{i=1}^{n} z^{-i} g^i x^{n-1}$$

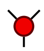
and the right cointegral is the functional that takes x^{n-1} to 1 and every other basis element to 0. We explicitly construct the HF algebra of the Taft Hopf algebra when $n = 2$ in the appendix.

Example 4.2. Hopf algebras which arise as the quantum enveloping algebra of Lie algebras are a type of quantum group. Since these are infinite dimensional, they cannot be Hopf-Frobenius algebras. However their finite dimensional quotients will be Hopf-Frobenius. See Kassel [21] for an example.

Moving away from $\mathbf{FVect_k}$, we consider **Rel**, the category of sets and relations.

Example 4.3. Let G be an infinite group. Following Hasegawa [19] we can construct its group algebra in **Rel**. The integral is $\{(\star, g) \mid g \in G\}$ and the cointegral is the singleton $(1, \star)$. The construction detailed in Theorem 3.11 recovers the expected multiplication and comultiplication relations:

 $:= a \mapsto (b,c)$ such that $a = bc$

 $:= (a,b) \mapsto \begin{cases} a & \text{if } a = b \\ \emptyset & \text{otherwise} \end{cases}$

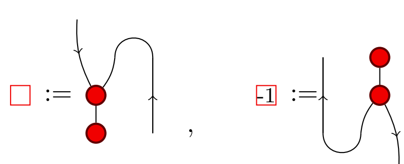
We look forward to discovering more exotic examples.

5 A simpler Drinfeld double

Braided categories of modules over a Hopf algebra are widely used in physics, where they give solutions to the Yang-Baxter equation and in low dimenional topology, where they are used to find invariants. However the category of modules over a Hopf algebra is braided if and only if the Hopf algebra is *quasi-triangular*. The *Drinfeld double* [15] is a construction that takes a Hopf algebra H in $\mathbf{FVect_k}$, and produces a quasi-triangular Hopf algebra $D(H)$ on the object $H \otimes H^*$. In this section we use the self-duality of a Hopf-Frobenius algebra to construct the canonical isomorphism $H \cong H^*$ and thus define a simpler version of the Drinfeld double on $H \otimes H$.

We will assume that $\mathcal{C}$ is a compact closed category. We denote the green and red Hopf algebras of H as $H_{\circ}$ and $H_{\bullet}$ respectively. We use the generalisation of Drinfeld's original construction to symmetric monoidal categories, due to Chen [10].

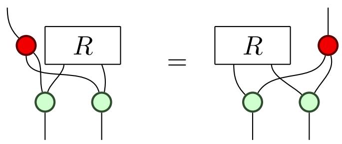
Definition 5.1. Let H be a HF algebra on $\mathcal{C}$. By Proposition C.1, we may define an isomorphism $\square : H \to H^*$, with inverse $\boxed{-1} : H^* \to H$ as

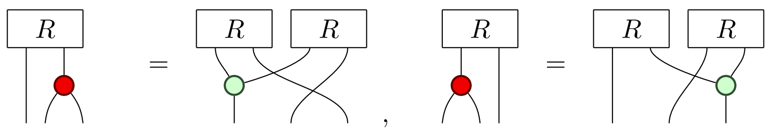


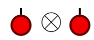
Lemma 5.2. *The morphism* $\square$ *is a Hopf algebra homomorphism between* $H_{\bullet}\sigma$ *and* $H_{\circ}^*$.

Remark 5.3. □ is a natural isomorphism. Due to the limits of space, we are not able to develop this point precisely, but the essential idea is as follows: Given a compact closed category $\mathcal{C}$, we consider the category of HF algebras on $\mathcal{C}$. We have two dual structures - the red dual from the Hopf-Frobeius structure, and the dual from the compact closed structure of $\mathcal{C}$. We find that □ is a natural isomorphism between the functors induced from these dual structures.

Definition 5.4. A Hopf algebra H is *quasi-triangular* if there exists a *universal R-matrix* $R : I \to H \otimes H$ such that

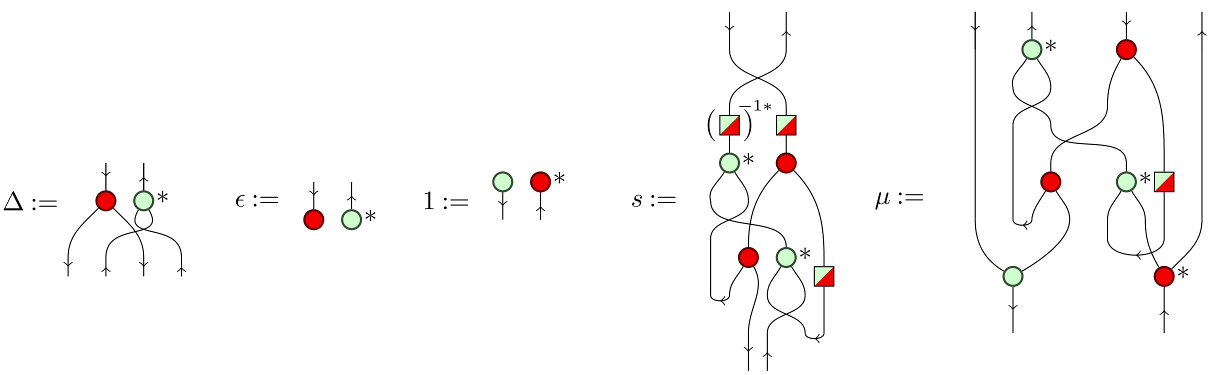
- R is invertible with respect to 
- 

- 

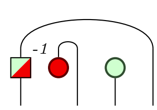
All cocommutative Hopf algebras are quasi-triangular, with  as the universal R-matrix. This definition is motivated by the following theorem [21].

Theorem 5.5. *The category of modules over a Hopf algebra is braided if and only if the Hopf algebra is quasi-triangular*

Definition 5.6. Let H be a Hopf algebra in $\mathcal{C}$ with an invertible antipode. The *Drinfeld double* of H, denoted $D(H) = (H \otimes H^*, \mu, 1, \Delta, \epsilon, s)$, is a Hopf algebra defined in the following manner:



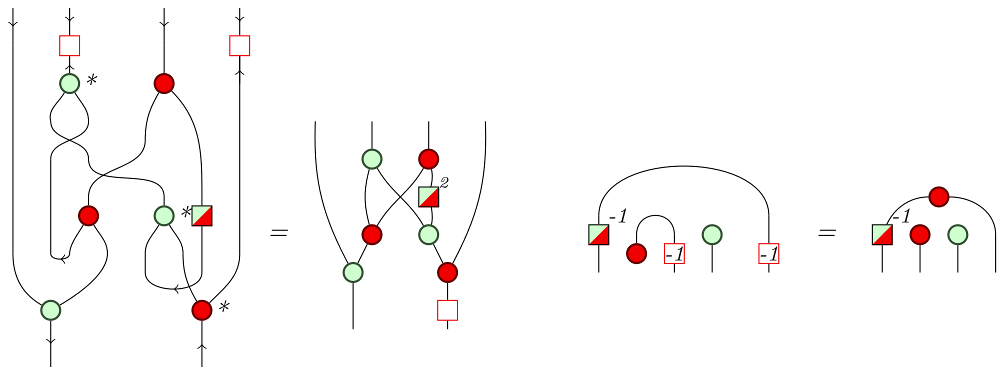
Theorem 5.7 (Drinfeld[15, 10])**.** *$D(H)$ is quasi-triangular, with the universal R-matrix shown below.*



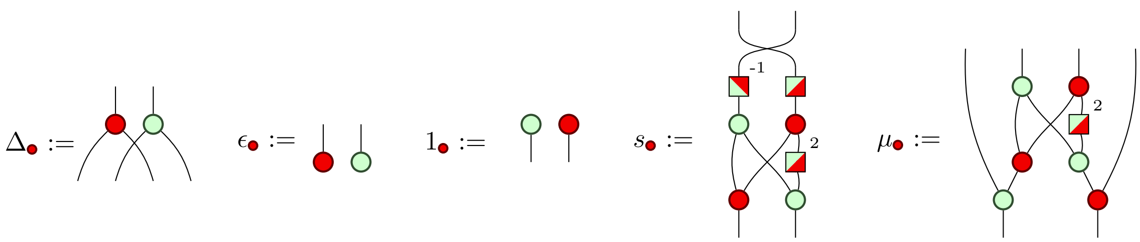
Our goal is to use the Hopf-Frobenius structure to get a Hopf algebra that is isomorphic to the Drinfeld double, but is easier to do diagramatic reasoning with.

We will now use the Hopf-Frobenius structure to derive a Hopf algebra isomorphic to the Drinfeld double. Consider the composite of the map $1 \otimes$□ with the multiplication of the Drinfeld double:

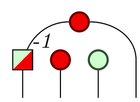
Lemma 5.8.



Definition 5.9. Let H be a HF algebra. The *red Drinfeld double*, denoted $D_{\bullet}(H) = (H \otimes H, \mu_{\bullet}, 1_{\bullet}, \Delta_{\bullet}, \epsilon_{\bullet}, s_{\bullet})$, is a Hopf algebra on the object $H \otimes H$ with structure maps



Corollary 5.10. *$D_{\bullet}(H)$ is a quasi-triangular Hopf algebra isomorphic to the Drinfeld double, with universal R-matrix*



6 Conclusion and further work

We have generalised the notions of interacting Frobenius algebras [12, 16] and interacting Hopf algebras [7] to the non-commutative case, and in the process shown that they are rather common structures. This work could be viewed as an extension of classical results showing that concrete Hopf algebras over finite dimensional vector spaces are also Frobenius algebras [26]. Another perspective is that we make precise how much ambient symmetry is required to obtain a Hopf-Frobenius algebra. The original setting of interacting Frobenius algebras [12] was a †-compact category, which provides a lot of duality on top of the commutative algebras themselves. We show that none of this structure is necessary: all that is required is one-sided *half-dual* for the carrier object. The major question that remains is to pin down exactly when the Frobenius condition holds; as Lemma 3.12 shows, this is tightly related to the existence of integrals. Compact closure does not suffice to guarantee this: in $\mathbf{FPMod_R}$ there are Hopf algebras which are not Frobenius.

While we have established that Hopf algebras are frequently Hopf-Frobenius, the resulting Frobenius algebras need not be well behaved (commutative, dagger, special) as in the original quantum setting [11]. It remains to investigate what Frobenius structures arise from "interesting" Hopf algebras, and whether they have any application in the categorical quantum mechanics programme, or conversely, how HF algebras may be applied in the study of quantum groups.

Weaker structures such as the ill-named coFrobenius algebras or the stateful resource calculus of Bonchi et al [6] perhaps offer an alternative to the nonstandard approach [18] to study infinite dimensional systems. Beyond this, natural generalisations to the braided or planar cases suggest themselves, although this will push diagrammatic reasoning to its limits.

Our new Drinfeld double construction suggests that HF algebras could find applications in topological quantum computation, particularly for error correcting codes, an area where the ZX-calculus is already used [4]. The smallest non-abelian group is S_3, whose group algebra fits in 3 qubits with room to spare.

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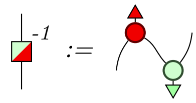
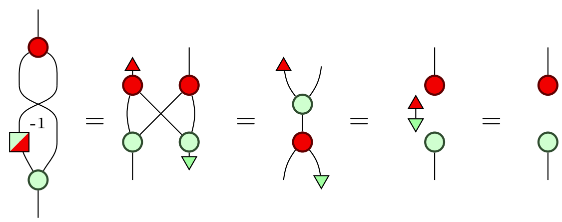
A Proofs omitted from the main body of the paper

Lemma 3.6. *Let* $(H, ,)$ *be an integral Hopf algebra. Then the following map is the inverse of the antipode.*



Proof. From the definition of $^{-1}$, we see that

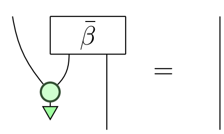


This implies that H^σ is a Hopf algebra. Recall that by Proposition 2.10, when H^σ is a Hopf algebra, then the antipode of H^σ is the inverse of the antipode of H. Thus, $^{-1} \circ = 1$ □

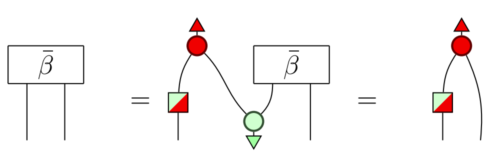
Lemma 3.7. *Let* $(H, ,)$ *be an integral Hopf algebra, and define*

then β *is a Frobenius form for* $(H, ,)$ *iff* γ *is a right inverse for* β*.*

Proof. Note that γ is left inverse to β by Lemma 3.6; hence if it is also right inverse then the conditions of Definition 2.6 are satisfied. Conversely, suppose that β is a Frobenius form; then there exists some $\bar{\beta}$ such that



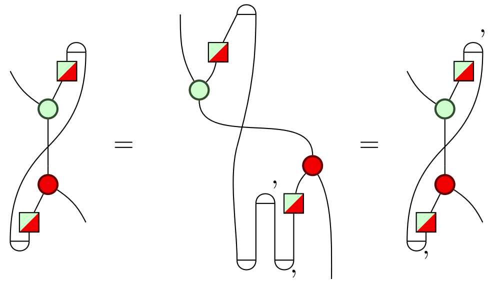
Appealing again to Lemma 3.6 we have



hence, γ is the right inverse of β. □

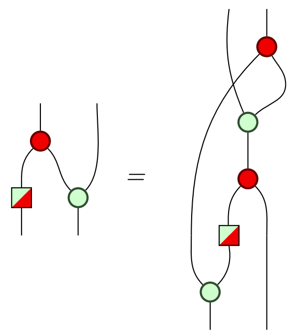
Lemma A.1. *When* H *has two half dual structures,* , *and* $'$, $'$*, then the integral morphisms coincide.*

Proof.

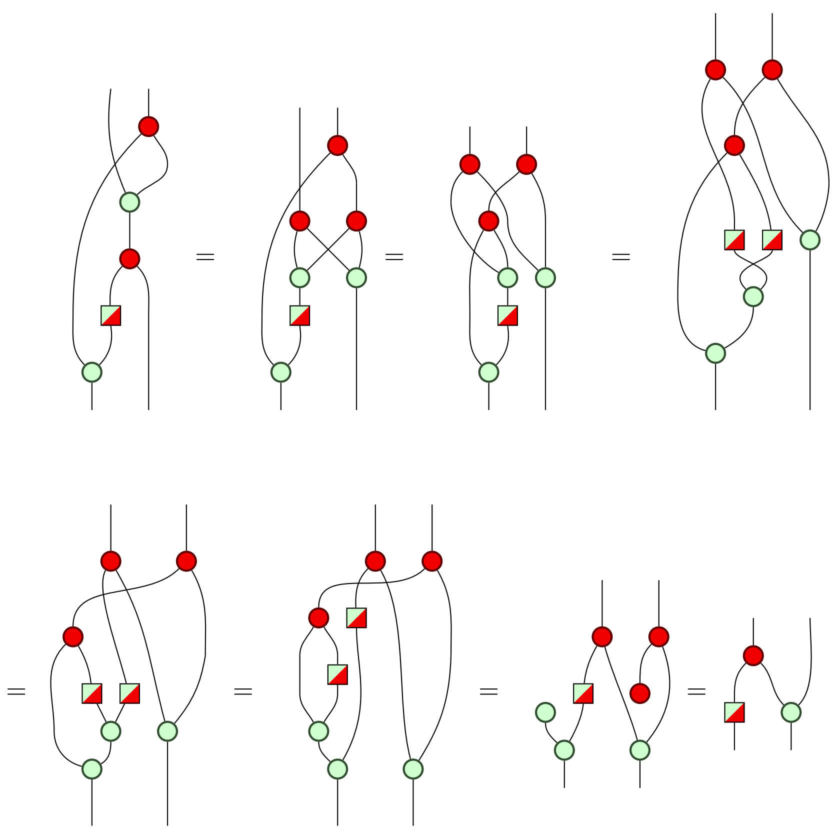


□

Lemma A.2.



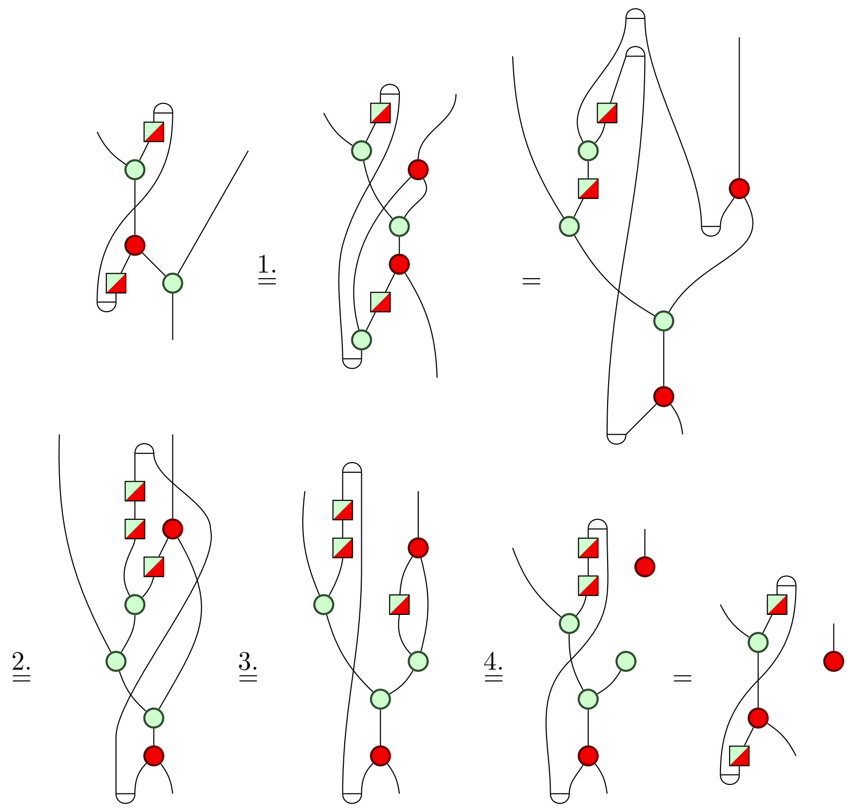
Proof.



□

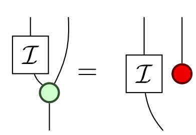
Lemma A.3. *Given a point $p : I \to H$, and copoint $q : H \to I$, $\mathcal{I} \circ p$ is a left cointegral, and p is a left cointegral if and only if $\mathcal{I} \circ p = p$. Similarly, $q \circ \mathcal{I}$ is a right integral, and q is a right integral if and only if $q \circ \mathcal{I} = q$.*

Proof. We begin by composing $\mathcal{I}$ with .

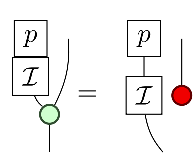


Where (1.) comes from Lemma A.2, (2.) comes from Proposition 2.10, where we use the fact that the antipode is a homomorphism $H^{\sigma} \to H$. (3.) comes from associativity, and (4.) is from the Hopf law.

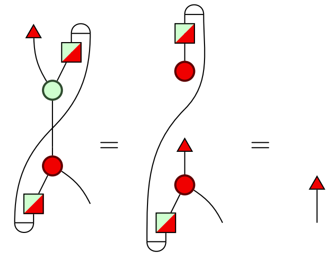
In other words, this tells us that



Hence we may prove that for every point p, $\mathcal{I} \circ p$ is a left cointegral



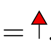
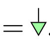
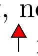
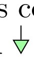
This also clearly imples that if $\mathcal{I} \circ p = p$, then p is a left cointegral. To see the implication in the other direction, let  be a left cointegral. Then

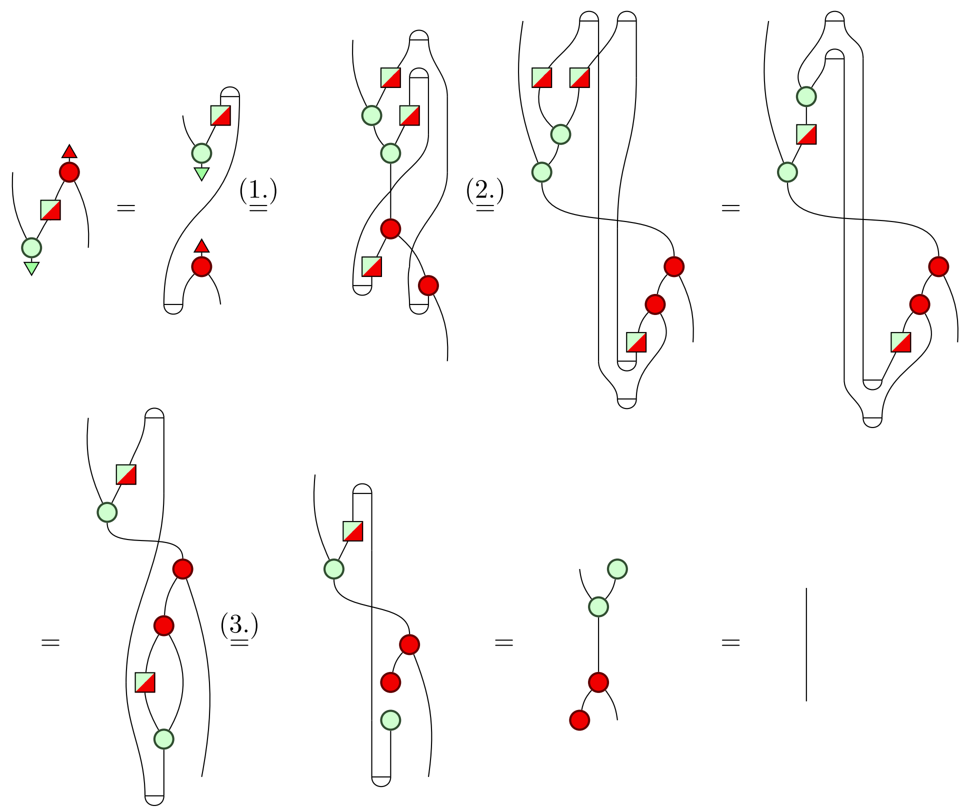


The proof for integrals is similar. □

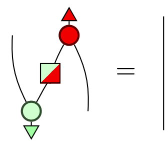
Theorem 3.11. *If H satisfies the Frobenius condition, then H is a pre-HF algebra with the Frobenius forms and their inverses as shown below.*

Further, $(H,$,$)$ *is an integral Hopf algebra.*

Proof. First off, note that the Frobenius condition implies that $\mathcal{I} \circ$ $=$ , and $\circ \mathcal{I} =$ . Hence, by Lemma A.3,  is a left cointegral and  is a right integral. Thus, $(H,$,$)$ is an integral Hopf algebra. We may therefore show that the following maps cancel out:



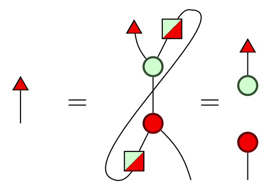
where (1.) is due to the Frobenius condition, (2.) comes from associativity, and (3.) is due to the Hopf law. We then get the following identity



Thus, by Lemma 3.7 we have our result. □

Lemma 3.12. *(forward direction) Let H be a Hopf algebra that fulfills the Frobenius condition. Every left cointegral (right integral) is a scalar multiple of (resp.).*

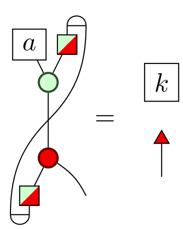
Proof. Let be a left cointegral on H. Then by Lemma A.1, as $= \circ \mathcal{I}$



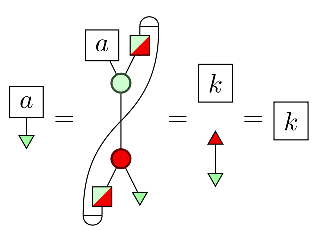
The proof for right integrals is similar. □

Lemma 3.12. *(backward direction) Let H have enough points, and $(H, ,)$ be an integral Hopf algebra. Suppose that every left cointegral (right integral) is a scalar multiple of (resp.). Then H fulfills the Frobenius condition.*

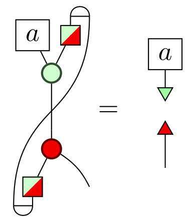
Proof. By Lemma A.3, for all points a (copoint b), $\mathcal{I} \circ a$ is a cointegral and $b \circ \mathcal{I}$ is an integral. Then, by hypothesis



for some scalar k. Since is an integral, $\circ \mathcal{I} =$, so we get the following



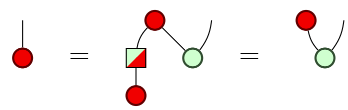
Hence, we see that



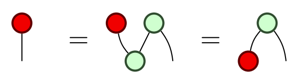
Since H has enough points, we observe that the Frobenius condition is satisfied. ☐

Lemma 3.13. *Let H be a pre-HF algebra; $(H, ,)$ is an integral Hopf algebra if and only if* $=$.

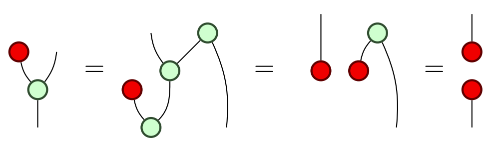
Proof. Note first that $=$ if and only if has an inverse, $^{-1} =$. We will use this second form in the following proof. The implication in one direction follows from Lemma 3.6. For the other direction, suppose that $=$. As we said above, this implies that $^{-1} =$. is a homomorphism, so



which implies simply that

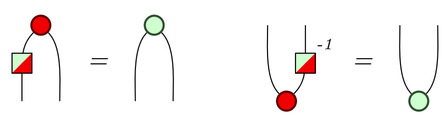


Hence, we get



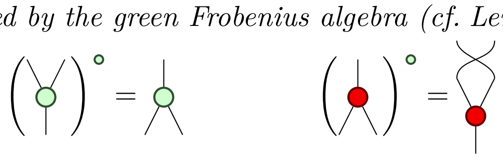
The proof that is a right integral is similar. ☐

Corollary A.4. *Let H be a pre-HF algebra, such that $(H, ,)$ is an integral Hopf algebra. Then*

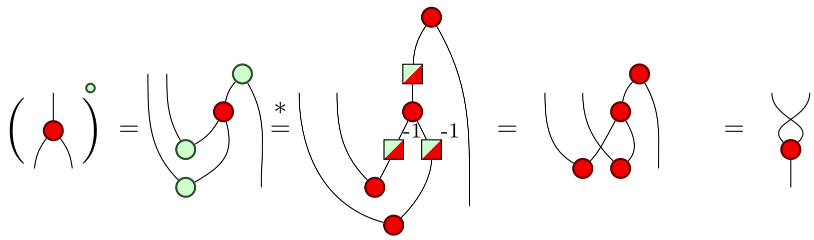


Proof. This comes straight from the fact that $=$ and $^{-1} =$ ☐

Lemma 3.14. *Let H be a pre-HF algebra such that $(H, ,)$ is an integral Hopf algebra, and let $(\cdot)^{\circ}$ be the duality defined by the green Frobenius algebra (cf. Lemma 2.5). Then:*



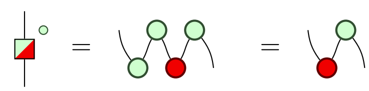
Proof. The first statement is clear from the definition of the green dual and . For the second statement, we see that



where (*) comes from Corollary A.4. □

Corollary 3.15. *Let H be a pre-HF algebra such that* $(H, ,)$ *is an integral Hopf algebra.* $(H, , , , ,)$ *forms a Hopf algebra, where* $=$.

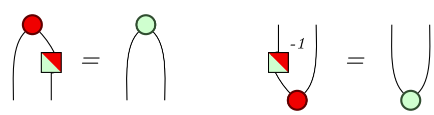
Proof. First off, let $(\cdot)^{\circ}$ be the duality defined by the green Frobenius algebra as in Lemma 2.5. Then



It is clear then, that $= (\ ^{\circ})^{-1}$.

Recall that, by Proposition 2.10, the antipode of H^{σ} is the inverse of H. We also see that, in a similar manner to Definition 2.11, $(\cdot)^{\circ}$ will give us another Hopf Algebra, H° : $(H, ^{\circ}, ^{\circ}, ^{\circ}, ^{\circ}, ^{\circ})$. Hence, as H° is a Hopf algebra with antipode $^{\circ}$, and by Lemma 3.14, $(H, , , , ,) = (H^{\circ})^{\sigma}$, so we have our result. □

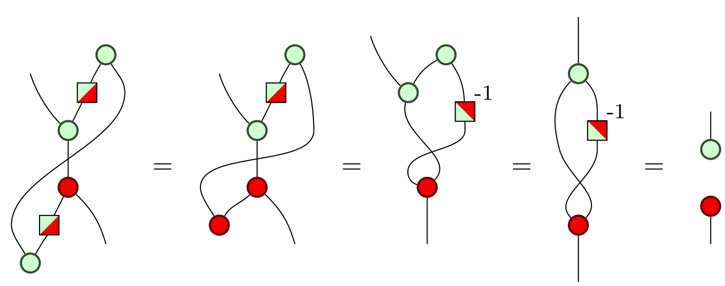
Corollary A.5. *Let H be a pre-HF algebra, such that* $(H, ,)$ *is an integral Hopf algebra. Then*



Theorem 3.16. *Let H be a Hopf algebra such that the object H has some weak right dual H^*. Then H is a Hopf-Frobenius algebra if and only if H fulfills the Frobenius condition.*

Proof. Given a Hopf algebra that fulfills the Frobenius condition, we see that from Theorem 3.11 and Corollary 3.15 that we get a Hopf-Frobenius algebra.

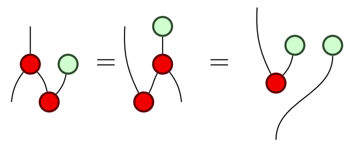
Suppose that H is a Hopf-Frobenius algebra. Then the green Frobenius structure gives H has a half dual structure. Here, we use Corollary A.4 and A.5 to get the first and second equality.



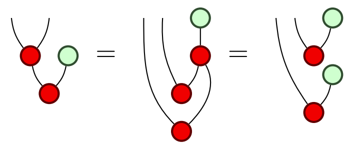
□

Theorem 3.17. *Let H be a pre-HF algebra such that, for some , $(H, , , , ,)$ forms a Hopf algebra; then H extends to a Hopf-Frobenius algebra.*

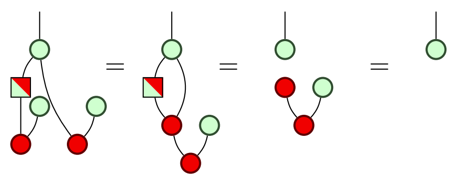
Proof. We begin by showing that is a right integral and is coinvertible. The proof that is a left cointegral and invertible is similar. First off, by the Frobenius algebra structure of



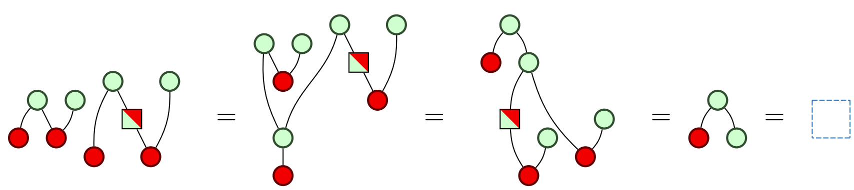
We see that this map gets copied via



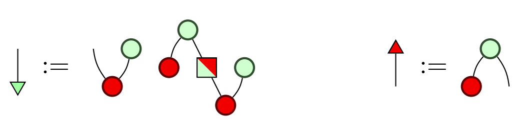
This implies that it is coinvertible as follows



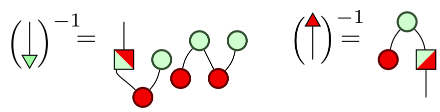
We may use similar proofs to show that is a left cointegral, is copied via , and is invertible. To get an integral Hopf algebra structure, we still need the two integrals to cancel out. Essentially, we need to normalise the integral by multiplying it with the appropriate scalar. We are working in an arbitrary monoidal category, so we need to show that this scalar is invertible.



We define the following copoint and point, and as we have shown above the are an integral and cointegral respectively.



Hence, $(H, ,)$ is an integral Hopf algebra. To extend H to a Hopf-Frobenius algebra, we shall use Corollary 3.15. We will now find two Frobenius algebras such that we may extend H to a pre-HF algebra, with and as the unit and counit. To find these Frobenius algebras, we refer to Proposition C.8, which tells us that given any invertible/ coinvertible element of a Frobenius algebra, we can find a new Frobenius algebra structure. We find that ↑ and ↓ are invertible and coinvertible respectively, with multiplicitive inverses

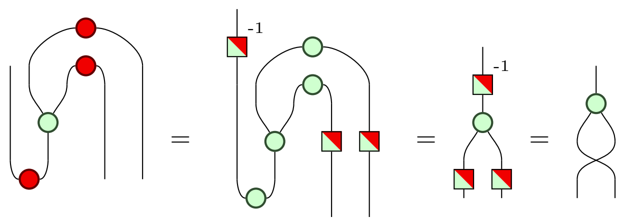


Proposition C.8 tells us that we may construct new Frobenius algebras with ↑ and ↓ as unit and counit of their respective Frobenius algebras. Hence, by Corollary 3.15, we have a Hopf Frobenius algebra.

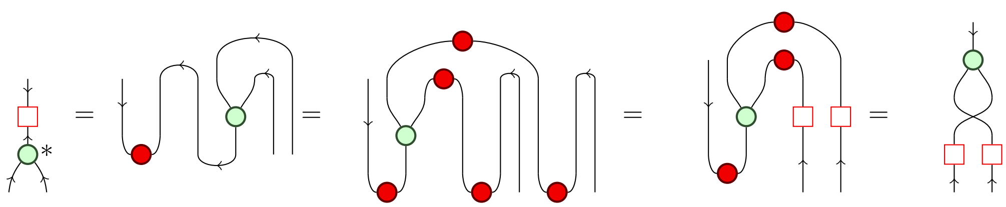
□

Lemma 5.2. *The morphism* □ *is a Hopf algebra homomorphism between* $H_{\bullet}\sigma$ *and* $H^*_{\circ}$

Proof. We will only show that □ is a homomorphism for $\curlywedge^*$, the rest of the structure maps will have similar proofs. We first note that, by Corollary A.4

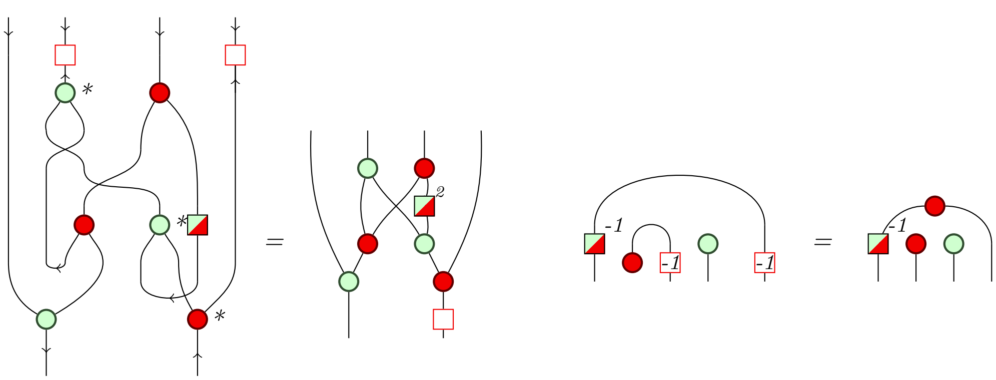


Hence, we see that

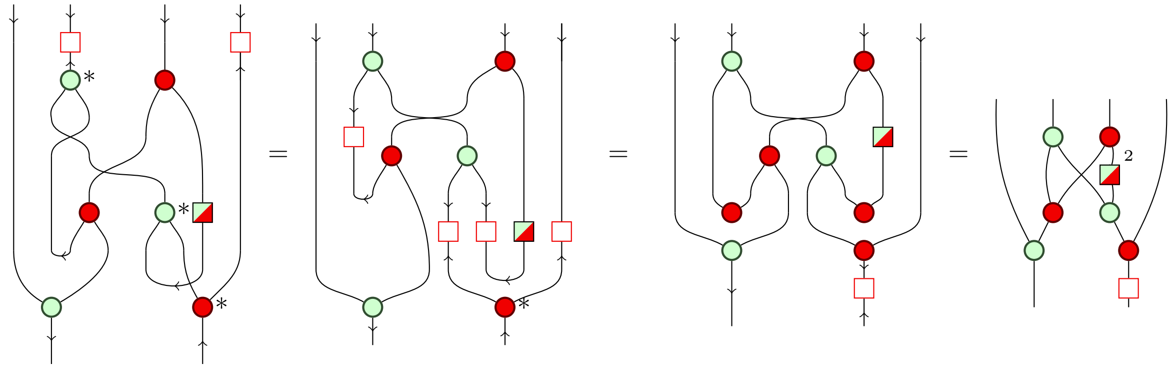


□

Lemma 5.8.



Proof. This is clear from the definition of , Lemma 5.2 and Corollary A.4. We explicitly spell out the first statement here.



The proof of the second statement follows immediately from the definition of . □

B Taft Hopf algebra for $n = 2$

Here we shall state the Hopf-Frobenius algebra for the 4 dimensional Taft Hopf algebra explicitly. It is generated by g and x, and has the structure

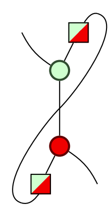
	1	x	g	gx
1	1	x	g	gx
x	x	0	$-gx$	0
g	g	gx	1	x
gx	gx	0	$-x$	0

	
1	$1 \otimes 1$
x	$1 \otimes x + x \otimes g$
g	$g \otimes g$
gx	$g \otimes gx + gx \otimes 1$

	
1	1
x	0
g	1
gx	0

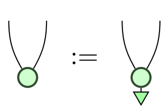
	
1	1
x	gx
g	g
gx	$-x$

FVect$_{\mathbf{k}}$ is a compact closed category, so the integral projection is the map

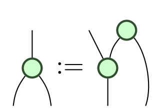


$\mathcal{I}$	
1	0
x	$x - gx$
g	0
gx	0

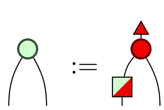
Hence, the element $x - gx$ is a left cointegral, and the right integral is the delta function for x, δ_x. Hence, by Theorem 3.11, these shall be our unit and counit respectively. It is now possible to construct the resulting Hopf-Frobenius algebra, but we shall explicity state the structure maps. The green Frobenius algebra is



	1	x	g	gx
1	0	1	0	0
x	1	0	0	0
g	0	0	0	1
gx	0	0	-1	0

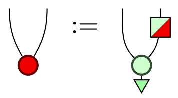


1	$1 \otimes x + gx \otimes g - g \otimes gx + x \otimes 1$
x	$x \otimes x + gx \otimes gx$
g	$g \otimes x + x \otimes g - 1 \otimes gx + gx \otimes 1$
gx	$gx \otimes x + x \otimes gx$

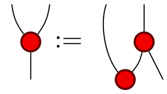


$= 1 \otimes x + gx \otimes g - g \otimes gx + x \otimes 1$

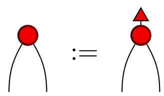
and the red Frobenius algebra



	1	x	g	gx
1	0	0	0	-1
x	1	0	0	0
g	0	1	0	0
gx	0	0	-1	0



	1	x	g	gx
1	0	0	0	-1
x	1	x	0	0
g	0	g	0	0
gx	0	0	$-g$	$-gx$



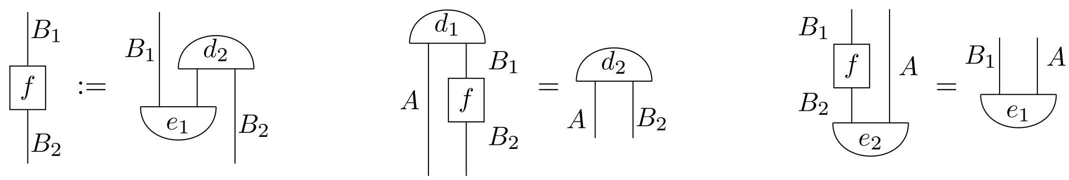
$= 1\otimes x + x\otimes g - g\otimes gx - gx\otimes 1$

C Additional Background Material

In this section we provide additional definitions and basic properties to flesh out the background material of Section 2.

C.1 Categories with duals

Proposition C.1. *In a monoidal category $\mathcal{C}$ suppose that A has two right duals (B_1, d_1, e_1) and (B_2, d_2, e_2); then there exists an isomorphism $f : B_1 \cong B_2$, satisfying the equations shown below.*

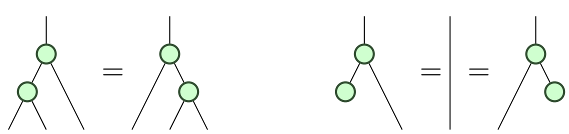


Proof. Define f as shown above; the required equations follow immediately. □

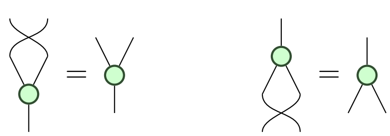
C.2 Monoids and Comonoids

Definition C.2. A *monoid* in a monoidal category $\mathcal{C}$ consists of an object M, a binary multiplication $\mu : M \otimes M \to M$ and a unit morphism $\eta : I \to M$ obeying the familiar associativity and unit laws, shown in diagram form below.

A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$, concretely depicted below.

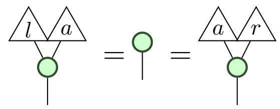


A (co)monoid is called (co)commutative if its (co)multiplication is invariant under the exchange map, as depicted below.



In this paper we will *not* assume commutativity or cocommutativity.

Definition C.3. Given a monoid $(M, \ , \ )$, a point $a : I \to M$ is *left invertible* if there exists a point $l : I \to M$ satisfying the left equation below; it is *right invertible* if there exists $r : I \to M$ satisfying the right equation; it is *invertible* if it is both left and right invertible, in which case the two inverses coincide.



Co-invertibility of co-points $\alpha : M \to I$ with respect to a comonoid is defined dually.

C.3 Frobenius algebras

In the following lemmas we will assume that we have a given Frobenius algebra $(F, \ , \ , \ , \ )$.

Definition C.4. A Frobenius algebra is called *symmetric* if its cap (or equivalently its cup) is invariant under the symmetry.

Proof. See Kock [23] □

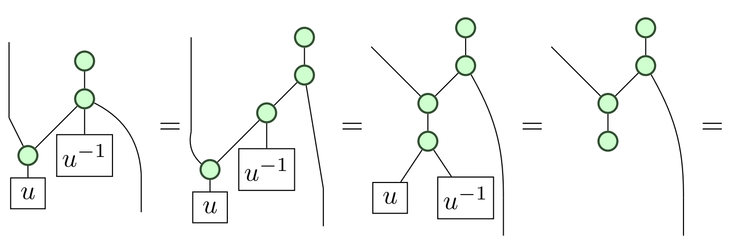
Lemma C.5. *There is a bijective correspondence between invertible points for the monoid and coinvertible copoints for the comonoid.*

Proof. Let $(\cdot)^\circ$ be the duality induced by the cup and cap; then $u : I \to F$ is invertible iff and only if $u^\circ : F \to I$ is coinvertible. □

Lemma C.6. *Let u be a coinvertible element of the comonoid. Define*

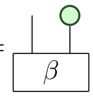
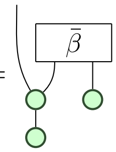
Then $\beta(u)$ is a Frobenius form for the monoid $(F, \ , \ ,)$.

Proof. We must show that the equations of 2.6 hold. The first follows from associativity of the monoid. For the second we have:

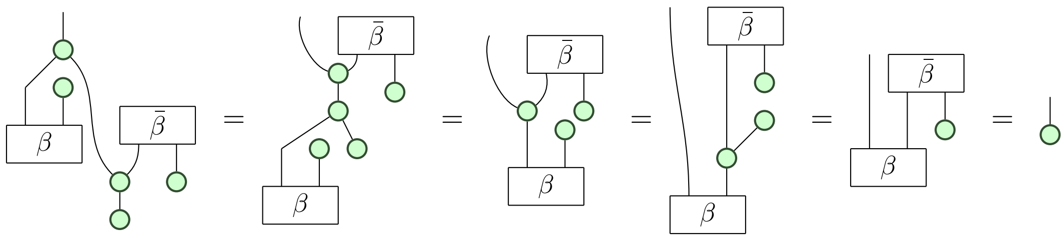


and similarly for the other side. Note that $\beta(u) = \beta(v)$ implies $u = v$ by the uniqueness of inverses. □

Lemma C.7. *Suppose that β is a Frobenius form on* *; then we obtain a coinvertible element $u : F \to I$ as follows:*

$u :=$  $u^{-1} :=$ 

Proof. We need only to show that u^{-1} is the coinverse of u.



□

Combining the three preceding lemmas we obtain:

Proposition C.8. *There is a bijective correspondence between the invertible elements of a monoid and the Frobenius forms definable on it.*