Generalized Ait-Sahalia-type interest rate model with Poisson jumps and convergence of the numerical approximation

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Abstract

In this paper, we consider a generalized Ait-Sahalia interest rate model with Poisson jumps in finance. The analytical properties including positivity, boundedness and pathwise asymptotic estimations of the solution to this model are investigated. Moreover, we prove that the Euler-Maruyama (EM) numerical solution converges to the true solution of the model in probability. Finally, we apply the EM solution to compute some financial quantities. A numerical example is provided to demonstrate the effectiveness of our results.

Keywords: Stochastic interest rate model, Poisson jumps, EM method, Convergence in probability.

1. Introduction

A classical problem in mathematical finance is the pricing of financial assets [1]. One of the asset pricing models reflecting the stochastic fluctuations is described by the following stochastic differential equation (SDE)

$$dy(t) = \kappa(\mu - y(t))dt + by^{\theta}dB(t), \quad y(0) = y_0, \tag{1.1}$$

for any $t > 0$. Here, $B(t)$ is a Brownian motion, $\kappa, \mu \geq 0$, $b, \theta > 0$ and $y_0 > 0$. It is well known that (1.1) contains many famous models such as Merton [2], Vasicek [3], Cox et al. [4], and Brennan [5]. When $\theta = 0.5$, (1.1) is reduced to the mean-reverting square root model, namely, the CIR model. In practice, successful applications of the asset pricing model are heavily dependent on the analytical properties (e.g., pathwise estimation, boundedness) of the true solution to the model. In general, stochastic asset pricing model has no explicit solution. Accordingly, there appear to be a practical need to estimate the true solution of the model via numerical approach. Thus the solution analysis and the convergence of the numerical approximation issues for the asset pricing model have become hot topics of research attracting an ever-increasing interest. Higham

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and Mao [6] investigated the strong convergence properties of EM scheme for CIR model. Wu et al. [7] extended these results to the case of Poisson jumps. Dereich et al. [8] introduced a drift-implicit EM scheme which preserves positivity of solution. Hefter and Herzwurm [9] proposed a Milstein-type scheme and proved the strong convergence results. Analytical properties of mean-reverting γ-model and the convergence result were obtained by Wu et al. [10], they also concentrated on the EM scheme for CIR model with delay [11]. The nonnegativity of solution to the mean-reverting-theta stochastic volatility model was proved by Baduraliya and Mao [12].

In [13] Ait-Sahalia studied several continuous-time models for interest rates. He tested parametric models by examining how closely the parametric model can reproduce features of its nonparametric counterpart. Results of the specification test suggested that a nonlinear stochastic interest model, which is now named as Ait-Sahalia-type model, captures well of the dynamics of the spot rate. Now Ait-Sahalia model has been widely used to volatility and other financial quantities besides interest rate [14, 15]. Furthermore, numerical method for Ait-Sahalia model also attracts researchers' attention. Cheng [16] discussed the analytical properties of this model and showed that the EM solution converges to the true solution in probability. Szpruch et al. [17] presented an implicit numerical method that preserves positivity and boundedness of moments and they proved the strong convergence result for the generalized Ait-Sahalia model. Results on the convergence of numerical solutions and ergodicity of this model can be found in Jiang et al. [18], Jin and Zhang [19].

However, the financial market is sensitive to unforeseen shocks and sudden events. Unexpected events may amplify uncertainty, which in turn exacerbates fear and increases risk perception in the financial system [20]. Numerous studies suggest strong empirical evidence that there exist jumps within financial markets and control systems, see e.g., Ait-Sahalia [21, 22, 23],Ma et al. [24], Li et al. [25]. Jumps risks cannot be ignored in the pricing of financial asserts (see e.g.,[26, 27]). As a more general jump-diffusion process, Lévy process has a wide range of applications in such diverse areas as mathematical finance, financial economics, and stochastic control [28]. For some recent works on Lévy noise, we refer the reader to the literature [28, 29, 30, 31, 32]. In order to characterize the impact of sudden and unforeseeable events on the term structure of interest rate and capture these discontinuous behaviors, jump-diffusion processes have been applied in finance (see e.g., [33, 34, 35, 36, 37]).

Motivated by the above discussion, this paper is concerned with the generalized Ait-Sahalia interest rate model with Poisson jumps of the form

$$\begin{aligned} dy(t) &= (a_{-1}y^{-1}(t) - a_0 + a_1 y(t) - a_2 y^{\gamma}(t))dt + by^{\theta}(t)dB(t) + \delta y(t^-)dN(t), \\ y(0) &= y_0, \end{aligned} \tag{1.2}$$

for $t > 0$. Here, $y_0, a_{-1}, a_0, a_1, a_2, b, \delta > 0$ and $\theta, \gamma > 1$. In addition, $y(t^-) = \lim_{s\to t^-} y(s)$, $B(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process with the compensated Poisson process $\tilde{N}(t) = N(t) - \lambda t$, where λ denotes the jump intensity. Let $B(t)$ and $N(t)$ be independent. We mainly focus on the analytical properties and numerical approximation of the true solution to (1.2). Threefold major contributions are highlighted as follows: (1) a novel generalized Ait-Sahalia model is proposed to account for the phenomenon of Poisson jumps; (2) with the help of Lyapunov functions and stochastic analysis technique, the analytical properties including positivity, boundedeness and pathwise estimation of the solution to the model are obtained for the first time; (3) an appropriate numerical approach is constructed for solving the model and the corresponding convergence result is established. It is worth noticing that our results are not a straightforward generalization of Jiang et al. [18]. Some new techniques are developed to overcome the difficulties due to the appearance of Poisson jumps.

The rest of the paper is arranged as follows. In Section 2, we prove the nonnegativity of the solution to (1.2). In Section 3, we discuss the boundedness of the true solution. Pathwise estimations of the solution to (1.2) are investigated in Section 4. In Section 5, we show the convergence of the EM method applying to the model (1.2). Some applications are illustrated in Section 6. A numerical example together with some ideas on further research is presented in Section 7. Finally, we concludes this paper in Section 8.

2. Positive and global solutions

Throughout this paper, we assume that all the processes are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$ for any $x, y \in \mathbb{R}$. For a set A, $\mathbb{I}_A$ denotes its indicator function. Let $\lfloor u \rfloor$ denote the integer part of u, for any $u \in \mathbb{R}$. Moreover, we set $\inf \emptyset = \infty$. To obtain the desired results, we need the following two lemmas.

Lemma 2.1. *(The Itô formula with jumps [38]) Consider a jump-diffusion process*

$$y(t) = y(0) + \int_0^t f(y(s))ds + \int_0^t g(y(s))dB(s) + \int_0^t h(y(s^-))dN(s),\ t > 0.$$

Let $F(\cdot)$ be a twice continuously differential function. Then

$$\begin{aligned} F(y(t)) = F(y(0)) &+ \int_0^t [F'(y(s))f(y(s)) + \frac{1}{2}F''(y(s))g^2(y(s)) + \lambda(F(y(s) + h(y(s))) - F(y(s)))]ds \\ &+ \int_0^t F'(y(s))g(y(s))dB(s) + \int_0^t [F(y(s^-) + h(y(s^-))) - F(y(s^-))]d\widetilde{N}(t). \end{aligned}$$

Lemma 2.2. *Let $\mathbb{E}\int_0^T |h(s)|^2 ds < \infty$. Then for any $T > 0$*

$$\begin{aligned} &\mathbb{E}\Big|\int_0^T h(s^-)dN(s)\Big|^2 \leq 2\lambda(1 + \lambda T)\mathbb{E}\int_0^T |h(s)|^2 ds, \\ &\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t h(s^-)d\tilde{N}(s)\Big|^2\Big] \leq 4\lambda\mathbb{E}\int_0^T |h(s)|^2 ds, \\ &\mathbb{E}\Big[\sup_{0\leq t\leq T}\Big|\int_0^t h(s^-)dN(s)\Big|^2\Big] \leq (8\lambda + 2\lambda T^2)\mathbb{E}\int_0^T |h(s)|^2 ds. \end{aligned}$$

The proof can be found in Fei et al. [39].

In the context of financial modeling, solution $y(t)$ denotes the asset price or the interest rate. It is essential to show that the solution is positive. The following theorem illustrates this property.

Theorem 2.3. *For any given initial value $y_0 > 0$, there is a unique positive global solution $y(t)$ to (1.2) on $t \geq 0$.*

Proof. Let τ_e be the explosion time. For a sufficiently large positive integer n satisfying $1/n < y(0) < n$, we define the stopping time

$$\tau_n = \inf\{t \in [0, \tau_e) : y(t) \notin [1/n, n]\}.$$

Noting that the coefficients of (1.2) are locally Lipschitz continuous, we can prove that there is a unique local solution $y(t) \in [0, \tau_e)$ for any given initial value $y_0 > 0$ by the classical methods [40]. Let $\tau_\infty = \lim_{n\to\infty} \tau_n$, which implies $\tau_\infty \leq \tau_e$, hence we need to show that $\tau_\infty = \infty$ a.s. , that is $\lim_{n\to\infty} \mathbb{P}\{\tau_n \leq T\} = 0$ for any $T > 0$.

For any $0 < \alpha < 1$, we define a C^2-function $V : (0, \infty) \to (0, \infty)$ by

$$V(y) = y^\alpha - 1 - \alpha \log y.$$

It is easy to see that $V(y) \to \infty$ as $y \to \infty$ or $y \to 0$. We compute that

$$V'(y) = \alpha(y^{\alpha-1} - y^{-1})$$

and

$$V''(y) = \alpha(\alpha - 1)y^{\alpha-2} + \alpha y^{-2}.$$

Hence,

$$\begin{aligned}
&\mathbb{L}V(y) + \lambda(V(y + \delta y) - V(y))\\
&= \alpha(y^{\alpha-1} - y^{-1})(a_{-1}y^{-1} - a_0 + a_1 y - a_2 y^\gamma) + \frac{b^2}{2}[\alpha(\alpha - 1)y^{\alpha-2} + \alpha y^{-2}]x^{2\theta}\\
&\quad + \lambda[((1 + \delta)y)^\alpha - 1 - \alpha \log((1 + \delta)y) - (y^\alpha - 1 - \log y)]\\
&= a_{-1}\alpha y^{\alpha-2} - a_0\alpha x^{\alpha-1} + a_1\alpha y^\alpha - a_2\alpha y^{\alpha+\gamma-1} - a_{-1}\alpha y^{-2} + a_0\alpha y^{-1}\\
&\quad - a_1\alpha + a_2\alpha y^{\gamma-1} - \frac{b^2\alpha(1 - \alpha)}{2}y^{\alpha+2\theta-2} + \frac{b^2\alpha}{2}y^{2\theta-2}\\
&\quad + \lambda((1 + \delta)^\alpha - 1)y^\alpha - \lambda\alpha \log(1 + \delta),
\end{aligned} \tag{2.1}$$

where $\mathbb{L}V : (0, \infty) \to \mathbb{R}$ is defined by

$$\mathbb{L}V(y) = V'(y)f(y) + \frac{1}{2}V''(y)g^2(y),$$

with $f(y) = a_{-1}y^{-1} - a_0 + a_1 y - a_2 y^\gamma$ and $g(y) = by^\theta$. Recalling that $0 < \alpha < 1, \gamma > 1$ and $\theta > 1$, we can deduce that $\mathbb{L}V(y) + \lambda(V(y + \delta y) - V(y))$ is bounded, say K_1, namely,

$$\mathbb{L}V(y) + \lambda(V(y + \delta y) - V(y)) \leq K_1, \quad y \in (0, \infty). \tag{2.2}$$

By Lemma 2.1, for any $T > 0$ we have

$$\mathbb{E}V(y(T \wedge \tau_n)) \leq V(y_0) + K_1 T. \tag{2.3}$$

Therefore,

$$\mathbb{P}(\tau_n \leq T)[V(1/n) \wedge V(n)] \leq \mathbb{E}V(y(T \wedge \tau_n)) \leq V(y_0) + K_1 T,$$

which means

$$\mathbb{P}(\tau_n \leq T) \leq \frac{V(y_0) + K_1 T}{V(1/n) \wedge V(n)}. \tag{2.4}$$

Thus $\mathbb{P}(\tau_n \leq T) \to 0$, since $V(1/n) \wedge V(n) \to \infty$ as $n \to \infty$. This means $\mathbb{P}(\tau_\infty = \infty) = 1$ as required. □

3. Boundedness

In the modeling of stochastic interest rate, boundedness is a natural requirement. We establish stochastic and moment boundedness for the solution to (1.2) in this section.

3.1. Boundedness of moments

Theorem 3.1. *For any $p \geq 2$, suppose that one of the following two conditions holds:*
(i) $2\theta < \gamma + 1$;
(ii) $2\theta = \gamma + 1$ and $a_2 > (p-1)b^2/2$.
Then there is a constant K_2 such that the solution of (1.2) satisfies

$$\mathbb{E}y^p(t) \leq \frac{y_0^p}{e^t} + K_2, \quad t > 0. \tag{3.1}$$

Proof. For any $p \geq 2$, we define

$$V_1(y,t) = e^t y^p, \quad (y,t) \in (0,+\infty) \times (0,+\infty).$$

Let τ_n be the stopping time defined in Theorem 2.3. We compute

$$\begin{aligned}
&\mathbb{L}V_1(y,t) + \lambda(V_1(y+\delta y,t) - V_1(y,t)) \\
&= e^t[y^p + py^{p-1}(a_{-1}y^{-1} - a_0 + a_1 y - a_2 y^\gamma) + \frac{b^2}{2}p(p-1)y^{p-2+2\theta}] \\
&\quad + \lambda e^t((1+\delta)^p - 1)y^p \\
&= e^t[pa_{-1}y^{p-2} - a_0 p y^{p-1} + (a_1 + 1 + \lambda((1+\delta)^p - 1))y^p \\
&\quad - a_2 p y^{p-1+\gamma} + \frac{b^2}{2}p(p-1)y^{p+2\theta-2}].
\end{aligned}$$

In either condition (i) or condition (ii), we can deduce that there is a constant $K_2 > 0$ such that

$$\mathbb{L}V_1(y,t) + \lambda(V_1(y+\delta y,t) - V_1(y,t)) \leq K_2 e^t.$$

Hence, for any $t \geq 0$,

$$\mathbb{E}\left[e^{t\wedge\tau_n} y^p(t\wedge\tau_n)\right] \leq y_0^p + K_2 e^t.$$

Letting $n \to \infty$ and applying Fatou lemma, we obtain the desired assertion (3.1). □

Theorem 3.2. *For any $p \geq 1$, assume that $2\theta \leq \gamma + 1$ and $\gamma \leq p + 1$. Then there is a constant K_3 such that the solution of (1.2) satisfies*

$$\mathbb{E}y^{-p}(t) \leq \frac{y_0^{-p}}{e^t} + K_3, \quad t > 0. \tag{3.2}$$

Proof. Define

$$V_2(y,t) = e^t y^{-p}, \quad (y,t) \in (0,+\infty) \times (0,+\infty).$$

Let τ_n be the same as before. We compute

$$\begin{aligned}&\mathbb{L}V_2(y,t)+\lambda(V_2(y+\delta y,t)-V_2(y,t))\\&=e^t[y^{-p}-py^{-p-1}(a_{-1}y^{-1}-a_0+a_1y-a_2y^\gamma)+\frac{b^2}{2}p(p+1)y^{-p-2+2\theta}]\\&\quad+\lambda e^t((1+\delta)^p-1)y^{-p}.\end{aligned}$$

Recalling that $2\theta\le\gamma+1$ and $\gamma\le p+1$, we can obtain that there is a constant $K_3>0$ such that

$$\mathbb{L}V_2(y,t)+\lambda(V_2(y+\delta y,t)-V_2(y,t))\le K_3e^t.$$

Hence,

$$\mathbb{E}\left[e^{t\wedge\tau_n}y^{-p}(t\wedge\tau_n)\right]\le y_0^{-p}+K_3e^t.$$

Letting $n\to\infty$ and applying the Fatou lemma, we obtain the desired assertion (3.2). □

By Theorem 3.1 and Theorem 3.2, we obtain the boundedness of $\mathbb{E}y^2(t)$ and $\mathbb{E}y^{-1}(t)$.

Corollary 3.3. *Suppose that one of the following two conditions holds:*
(i) $1<\theta\le(\gamma+1)/2$*;*
(ii) $\theta=(\gamma+1)/2$ *and* $2a_2>b^2$*.*
Then

$$\mathbb{E}y^2(t)<\infty,\quad t>0.$$

Corollary 3.4. *Suppose that* $1<\theta\le(\gamma+1)/2$ *and* $1<\gamma\le 2$*. Then*

$$\mathbb{E}y^{-1}(t)<\infty,\quad t>0.$$

85 The following theorem shows that the average in time of the moments of the solutions $y(t)$ is bounded.

Theorem 3.5. *For any* $\alpha\in(0,1)$*, there is a positive constant* K_α *such that for any initial value* $y_0>0$*, the solution of* (1.2) *satisfies*

$$\limsup_{t\to\infty}\left[\frac{1}{t}\int_0^t\mathbb{E}y^{\alpha+2\theta-2}(s)ds\right]\le K_\alpha.$$

Proof. Noting that $\gamma>1$ and $\theta>1$. Then, there is a positive constant K_α such that

$$\begin{aligned}&\alpha(y^{\alpha-1}-y^{-1})(a_{-1}y^{-1}-a_0+a_1y-a_2y^\gamma)+\frac{b^2\alpha}{2}\Big(-\frac{1}{2}(1-\alpha)x^{\alpha+2(\theta-1)}+x^{2(\theta-1)}\Big)\\&+\lambda((1+\delta)^\alpha-1)y^\alpha-\lambda\alpha\log(1+\delta)\le K_\alpha.\end{aligned}$$

Then we deduce from (2.1) that for any $t\in[0,T]$

$$\frac{b^2}{4}\alpha(1-\alpha)\mathbb{E}\int_0^{t\wedge\tau_n}y^{\alpha+2\theta-2}(s)ds+\mathbb{E}V(y(t\wedge\tau_n))\le V(y_0)+K_\alpha t.$$

Letting $t\to\infty$ and using the Fatou lemma, we obatin

$$\frac{b^2}{4}\alpha(1-\alpha)\mathbb{E}\int_0^t y^{\alpha+2\theta-2}(s^-)ds\le V(y_0)+K_\alpha t,$$

which implies the required assertion. □

Theorem 3.6. *If $\theta > 1.5$, then there is a constant K_4 such that the solution of* (1.2) *satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}(y^{-2}(s) + y^2(s))ds \le K_4. \tag{3.3}$$

Proof. If $\theta > 1.5$, we choose $\alpha \in (0.5, 1)$, such that $\alpha + 2(\theta - 1) > 2$. Hence, there is a constant K_4 such that

$$\frac{1}{4}a_{-1}(y^{-2} + y^2) + \mathbb{L}V(y) + \lambda(V(y + \delta y) - V(y)) \le \frac{1}{4}a_{-1}K_4, \quad y \in (0, \infty),$$

where $\mathbb{L}V$ was defined in Theorem 2.3. By (2.1), we have

$$\begin{aligned}\frac{1}{4}\mathbb{E}\int_0^{t\wedge\tau_n}(y^{-2}(s) + y^2(s))ds &\le \frac{1}{4}\mathbb{E}\int_0^{t\wedge\tau_n}(y^{-2}(s) + y^2(s))ds + \mathbb{E}V(y(t\wedge\tau_n))\\ &\le V(y_0) + \frac{1}{4}K_4 t.\end{aligned}$$

Hence, we get

$$\frac{1}{t}\mathbb{E}\int_0^{t\wedge\tau_n}(y^{-2}(s) + y^2(s))ds \le \frac{4V(y_0)}{t} + K_4.$$

Letting $n \to \infty$ gives the desired assertion (3.3). □

3.2. Stochastic boundedness

In this subsection, we show that the solution of (1.2) stay within the interval $(1/n_1, n_1)$ with a large probability for a positive constant n_1.

Theorem 3.7. *For any $\varepsilon \in (0, 1)$ and $y_0 > 0$, there is a constant $n_1 = n_1(y(0), \varepsilon) > 1$ such that the solution of* (1.2) *satisfies*

$$\mathbb{P}(1/n_1 < y(t) < n_1) \ge 1 - \varepsilon. \quad t \ge 0.$$

Proof. Define

$$V_2(y, t) = e^t V(y), \quad (y, t) \in (0, +\infty) \times (0, +\infty),$$

where $V(y) = y^\alpha - 1 - \log y$. Let τ_n be the same as before. For any $t \in [0, T]$, Lemma 2.1 gives

$$\begin{aligned}\mathbb{E}V_2(y(t\wedge\tau_n), t\wedge\tau_n)) &= V(y_0) + \mathbb{E}\int_0^{t\wedge\tau_n} e^s[V(y(s)) + \mathbb{L}V(y(s))]ds\\ &\quad + \lambda\mathbb{E}\int_0^{t\wedge\tau_n} e^s[V(y(s^-) + \delta y(s^-)) - V(y(s^-))]ds.\end{aligned} \tag{3.4}$$

By (2.1), there is a constant K_5 such that

$$V(y) + \mathbb{L}V(y) + \lambda[V(y + \delta y) - V(y] \le K_5, \quad y \in (0, \infty).$$

Hence,

$$\mathbb{E}V_2(y(t\wedge\tau_n), t\wedge\tau_n)) \le V(y_0) + K_5 e^t.$$

Letting $n \to \infty$, we have

$$e^t \mathbb{E}V(y(t)) \leq V(y_0) + K_5 e^t. \tag{3.5}$$

That is

$$\mathbb{E}V(y(t)) \leq \frac{V(y_0)}{e^t} + K_5 \leq V(y_0) + K_5. \tag{3.6}$$

For any sufficiently large integer $n > 1$, (3.6) gives

$$\mathbb{P}(y(t) \leq 1/n) \leq \mathbb{E}\left[\mathbb{I}_{\{y(t)\leq 1/n\}} \frac{V(y(t))}{V(1/n)}\right] \leq \frac{V(y_0) + K_5}{(1/n)^\alpha - 1 - \alpha \log(1/n)} \leq \frac{V(y_0) + K_5}{\alpha \log n - 1}. \tag{3.7}$$

Similarly,

$$\mathbb{P}(y(t) \geq n) \leq \mathbb{E}\left[\mathbb{I}_{\{y(t)\geq n\}} \frac{V(y(t))}{V(n)}\right] \leq \frac{V(y_0) + K_5}{n^\alpha - 1 - \alpha \log n}. \tag{3.8}$$

By (3.7) and (3.8), we have

$$\mathbb{P}(1/n < y(t) < n) > 1 - (V(y_0) + K_5)\Big(\frac{1}{\alpha \log n - 1} + \frac{1}{n^\alpha - 1 - \alpha \log n}\Big).$$

Letting $n \to \infty$, we obtain the desired assertion. □

4. Pathwise asymptotic estimations

We now begin to discuss the pathwise asymptotic properties of the true solution to (1.2).

Theorem 4.1. *Suppose that $1 < \theta \leq 1.5$ and $1 < \gamma \leq 2$. Then for any initial value $y_0 > 0$, the solution of (1.2) satisfies*

$$\liminf_{t\to\infty} \frac{\log y(t)}{\log t} \geq -1 \quad a.s. \tag{4.1}$$

Proof. Define $z(t) = y^{-1}(t)$. By Lemma 2.1, we have

$$\begin{aligned} dz(t) = &[-a_{-1}z^3(t) + a_0 z^2(t) - a_1 z(t) + a_2 z^{2-\gamma}(t) + b^2 z^{3-2\theta}(t)]dt \\ &- bz^{2-\theta}(t)dB(t) - \frac{\delta}{1+\delta} z(t^-)dN(t). \end{aligned} \tag{4.2}$$

Therefore, for $t > 0$, we have

$$\begin{aligned} &\mathbb{E}z(t+1) + \frac{1}{2}a_{-1}\mathbb{E}\int_t^{t+1} z^3(s)ds \\ &= \mathbb{E}z(t) + \mathbb{E}\int_t^{t+1} [-\frac{a_{-1}}{2} z^3(s) + a_0 z^2(s) - (a_1 + \frac{\lambda\delta}{1+\delta})z(s) + a_2 z^{2-\gamma}(s) + b^2 z^{3-2\theta}(s)]ds. \end{aligned} \tag{4.3}$$

Recalling that $1 < \theta \leq 1.5$ and $1 < \gamma \leq 2$, we deduce that there are constants K_6 and K_7 such that

$$-\frac{a_{-1}}{2} z^3 + a_0 z^2 - (a_1 + \frac{\lambda\delta}{1+\delta})z + a_2 z^{2-\gamma} + b^2 z^{3-2\theta} \leq K_6, \quad z \in (0, +\infty)$$

and

$$-a_{-1}z^3 + a_0 z^2 - a_1 z + a_2 z^{2-r} + b^2 z^{3-2\theta} \le K_7, \quad z \in (0, +\infty).$$

Hence, by (4.3), we have

$$\frac{1}{2}a_{-1}\mathbb{E}\int_t^{t+1} z^3(s)ds \le \mathbb{E}z(t) + K_6.$$

For $u \in [t, t+1]$, (4.2) gives

$$z(u) \le z(t) + K_7 - b\int_t^u z^{2-\theta}(s)dB(s) - \frac{\delta}{1+\delta}\int_t^u z(s^-)dN(s). \tag{4.4}$$

Applying the Lyapunov inequality, that is,

$$\mathbb{E}|X|^r \le (\mathbb{E}|X|^s)^{r/s}, \quad 0 < r < s < \infty, \quad X \in L^s(\Omega),$$

we have that

$$\mathbb{E}\left[\int_t^{t+1} z^2(s)ds\right]^{1/2} \le \left(\mathbb{E}\int_t^{t+1} z^2(s)ds\right)^{1/2} \le \left(\left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{2/3}\right)^{1/2} = \left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{1/3},$$

and

$$\mathbb{E}\int_t^{t+1} z(s)ds \le \left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{1/3}.$$

Therefore, by the stochastic inequality, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u z(s^-)dN(s)\right] &= \lambda\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u z(s)ds\right] + \mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u z(s^-)d\tilde{N}(s)\right] \\
&\le \lambda\mathbb{E}\int_t^{t+1} z(s)ds + C_\lambda\mathbb{E}\left[\int_t^{t+1} z^2(s)ds\right]^{1/2} \\
&\le (\lambda + C_\lambda)\left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{1/3},
\end{aligned} \tag{4.5}$$

where C_λ is a constant. Similarly, using the Burkholder-Davis-Gundy inequality and the Lyapunov inequality gives

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u z^{2-\theta}(s)dB(s)\right] &\le 3\mathbb{E}\left[\int_t^{t+1} z^{2(2-\theta)}(s)ds\right]^{1/2} \\
&\le 3\left(\mathbb{E}\int_t^{t+1} z^{2(2-\theta)}(s)ds\right)^{1/2} \\
&\le 3\left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{(2-\theta)/3}.
\end{aligned} \tag{4.6}$$

By (4.4), (4.5) and (4.6), we get

$$\mathbb{E}\left[\sup_{t\le u\le t+1} z(u)\right] \le \mathbb{E}z(t) + K_7 + b\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u z^{2-\theta}(s)dB(s)\right] + \frac{\delta}{1+\delta}\mathbb{E}\left[\sup_{t\le u\le t+1}\int_t^u z(s^-)dN(s)\right]$$
$$\le \mathbb{E}z(t) + K_7 + 3b\left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{(2-\theta)/3} + \frac{\delta(\lambda + C_\lambda)}{1+\delta}\left(\mathbb{E}\int_t^{t+1} z^3(s)ds\right)^{1/3}. \tag{4.7}$$

By the boundedness of $\mathbb{E}z(t)$ and $\mathbb{E}\int_t^{t+1} z^3(s)ds$, we deduce that there is a constant K_8 such that

$$\mathbb{E}\left[\sup_{t\le u\le t+1} z(u)\right] \le K_8. \tag{4.8}$$

Let $\varepsilon > 0$ be arbitrary. Applying the Chebyshev inequality gives

$$\mathbb{P}\left(\sup_{n\le t\le n+1} z(t) > n^{1+\varepsilon}\right) \le \frac{K_8}{n^{1+\varepsilon}}, \quad n = 1, 2, \cdots.$$

By the Borel-Cantelli lemma, we have that for almost $\omega \in \Omega$, there is a $n_0(\omega)$ such that

$$\sup_{n\le t\le n+1} z(t) \le n^{1+\varepsilon}, \text{ for } n \ge n_0,\ n \le t \le n+1$$

which means

$$\frac{\log z(t)}{\log t} \le \frac{(1+\varepsilon)\log n}{\log n} = 1 + \varepsilon.$$

That is

$$\liminf_{t\to\infty} \frac{\log y(t)}{\log t} \ge -(1+\varepsilon).$$

Letting $\varepsilon \to 0$, we obtain the desired assertion (4.9). Thus, the proof is complete. □

Theorem 4.2. *Suppose that $1 < \theta < (\gamma+1)/2$ and $1 < \gamma \le 2$. Then for any initial value $y(0) > 0$, the solution of (1.2) satisfies*

$$\limsup_{t\to\infty} \frac{\log y(t)}{\log t} \le 1 \quad a.s. \tag{4.9}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned} d[y^2(t)] &= [2y(t)(a_{-1}y^{-1}(t) - a_0 + a_1 y(t) - a_2 y^\gamma(t)) + b^2 y^{2\theta}(t)]dt \\ &\quad + 2by^{\theta+1}(t)dB(t) + (2\delta + \delta^2)y^2(t^-)dN(t) \\ &= [2a_{-1} - 2a_0 y(t) + 2a_1 y^2(t) - 2a_2 y^{\gamma+1}(t) + b^2 y^{2\theta}(t)]dt \\ &\quad + 2by^{\theta+1}(t)dB(t) + (2\delta + \delta^2)y^2(t^-)dN(t). \end{aligned} \tag{4.10}$$

Recalling that $1 < \theta < (\gamma+1)/2$ and $1 < \gamma \le 2$. Let $\eta \in (0, 2a_2)$ be arbitrarily small. Then there is a constant $K_9 > 0$ such that

$$\eta y^{\gamma+1} + [2a_{-1} - 2a_0 y + 2a_1\lambda(2\delta + \delta^2)y^2 - 2a_2 y^{\gamma+1} + b^2 y^{2\theta}] \le K_9, \quad y \in (0, \infty). \tag{4.11}$$

By (4.10), (4.11) and Corollary 3.3, we have

$$\begin{aligned}\eta\mathbb{E}\int_t^{t+1} y^{\gamma+1}(s)ds &\leq \eta\mathbb{E}\int_t^{t+1} y^{\gamma+1}(s)ds + \mathbb{E}y^2(t+1)\\ &\leq \mathbb{E}y^2(t) + K_9 < \infty,\end{aligned} \tag{4.12}$$

for any $t \in [0, T]$. Then for any $u \in [t, t+1]$, we have

$$\begin{aligned}y(u) = y(t) &+ \int_t^u [a_{-1}y^{-1}(s) - a_0 + a_1 y(s) - a_2 y^\gamma(s)]ds\\ &+ \int_t^u by^\theta(s)dB(s) + \int_t^u \delta y(s^-)dN(s).\end{aligned}$$

There is a constant K_{10} such that

$$-a_0 + a_1 y - a_2 y^\gamma \leq K_{10}, \quad y \in (0, +\infty).$$

Hence

$$y(u) \leq y(t) + K_{10} + a_{-1}\int_t^u y^{-1}(s)ds + \int_t^u by^\theta(s)dB(s) + \int_t^u \delta y(s^-)dN(s), \tag{4.13}$$

which implies

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq u\leq t+1} y(u)\right] &\leq \mathbb{E}y(t) + K_{10} + a_{-1}\int_t^{t+1}\mathbb{E}y^{-1}(s)ds\\ &+ b\mathbb{E}\left[\sup_{t\leq u\leq t+1}\int_t^u y^\theta(s)dB(s)\right] + \delta\mathbb{E}\left[\sup_{t\leq u\leq t+1}\int_t^u y(s^-)dN(s)\right].\end{aligned} \tag{4.14}$$

By the Burkholder-Davis-Gundy inequality and the Lyapunov inequality, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq u\leq t+1}\int_t^u y^\theta(s)dB(s)\right] &\leq 3\mathbb{E}\left[\int_t^{t+1} y^{2\theta}(s)ds\right]^{1/2}\\ &\leq 3\left(\mathbb{E}\int_t^{t+1} y^{2\theta}(s)ds\right)^{1/2}\\ &\leq 3\left(\mathbb{E}\int_t^{t+1} y^{\gamma+1}(s)ds\right)^{\theta/(\gamma+1)}.\end{aligned} \tag{4.15}$$

In the same way as (4.5) was obtained, we also have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq u\leq t+1}\int_t^u y(s^-)dN(s)\right] &= \lambda\mathbb{E}\left[\sup_{t\leq u\leq t+1}\int_t^u y(s^-)ds\right] + \mathbb{E}\left[\sup_{t\leq u\leq t+1}\int_t^u y(s^-)d\tilde{N}(s)\right]\\ &\leq \lambda\mathbb{E}\int_t^{t+1} y(s)ds + C_\lambda\mathbb{E}\left[\int_t^{t+1} y^2(s)ds\right]^{1/2}\\ &\leq (\lambda + C_\lambda)\left(\mathbb{E}\int_t^{t+1} y^{\gamma+1}(s)ds\right)^{1/(\gamma+1)}.\end{aligned} \tag{4.16}$$

Inserting (4.15) and (4.16) into (4.14), we get

$$\mathbb{E}\left[\sup_{t\le u\le t+1} y(u)\right] \le \mathbb{E}y(t) + K_{10} + a_{-1}\int_t^{t+1} \mathbb{E}x^{-1}(s)ds + 3b\left(\mathbb{E}\int_t^{t+1} y^{\gamma+1}(s)ds\right)^{\theta/(\gamma+1)}$$
$$+ \delta(\lambda + C_\lambda)\left(\mathbb{E}\int_t^{t+1} y^{\gamma+1}(s)ds\right)^{1/(\gamma+1)}. \tag{4.17}$$

Note that $1 < \theta < (\gamma+1)/2$ and $1 < \gamma \le 2$. Using the boundedness of $\mathbb{E}y(t)$, $\mathbb{E}y^{-1}(t)$ and $\mathbb{E}y^2(t)$, combining with (4.12), we can deduce from (4.17) that there is a constant K_{11} such that

$$\mathbb{E}\left[\sup_{t\le u\le t+1} y(u)\right] \le K_{11}. \tag{4.18}$$

Let $\varepsilon > 0$ be arbitrary. Applying the Chebyshev inequality gives

$$\mathbb{P}\left(\sup_{n\le t\le n+1} y(t) > n^{1+\varepsilon}\right) \le \frac{K_{11}}{n^{1+\varepsilon}}, \quad n = 1, 2, \cdots.$$

By the Borel-Cantelli lemma, we have that for almost $\omega \in \Omega$, there is a $n_0(\omega)$, such that

$$\sup_{n\le t\le n+1} y(t) \le n^{1+\varepsilon}, \quad \text{for} \quad n \ge n_0,$$

which means

$$\log y(t) \le (1+\varepsilon)\log n \le (1+\varepsilon)\log t, \quad \text{for} \quad n \le t \le n+1.$$

Hence,

$$\limsup_{t\to\infty} \frac{\log y(t)}{\log t} \le 1 + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain the desired assertion (4.9). □

5. The convergence analysis of the EM method

In this section, we propose an explicit EM scheme to solve (1.2) and prove a convergence result. We first define a discrete EM scheme to (1.2). For a given fixed time step $\Delta \in (0, 1)$ and $Y(0) = y_0 > 0$, the discrete EM approximate solution is defined as below

$$Y_{n+1} = Y_n + f(Y_n)\Delta + b|Y_n|^\theta \Delta B_n + \delta\Delta Y_n \Delta N_n, \tag{5.1}$$

for $n = 0, 1, 2, \cdots$, where $t_n = n\Delta$, $\Delta B_n = B(t_{n+1}) - B(t_n)$, $\Delta N_n = N(t_{n+1}) - N(t_n)$ and $f(y) = a_{-1}y^{-1} - a_0 + a_1 y - a_2 y^\gamma$. It is convenient to use the continuous approximation

$$Y(t) = Y(0) + \int_0^t f(\bar{Y}(s))ds + b\int_0^t |\bar{Y}(s)|^\theta dB(s) + \delta\int_0^t \bar{Y}(s^-)dN(s), \tag{5.2}$$

for $t \in [0, T]$, where $\bar{Y}(t)$ is the following step function

$$\bar{Y}(s) = Y_n, \text{ for } t \in [t_n, t_{n+1}). \tag{5.3}$$

It is easy to see that $\bar{Y}(s) = Y(t_n) = Y_n$, for $t \in [t_n, t_{n+1})$, that is, the discrete and continuous EM solutions coincide at the grid points. The following result generalises Theorem 5.1 in [16] and Theorem 5.4 in [18].

Theorem 5.1. *For any $T > 0$,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|y(t)-Y(t)|^2\right)=0 \quad \textit{in} \quad \textit{probability}. \tag{5.4}$$

Proof. We divide the proof into three steps.
Step 1. For any sufficiently large positive integer n, we define the stopping time

$$\nu_n = \inf\{t \in [0,T] : Y(t) \notin [1/n, n]\}.$$

Let V be the same as before. Then, for $t \in [0,T]$, Lemma 2.1 gives

$$\begin{aligned}\mathbb{E}V(Y(t\wedge\nu_n)) = V(Y_0) + \mathbb{E}\int_0^{t\wedge\nu_n}[V'(Y(s))f(\bar{Y}(s)) + \frac{b^2}{2}V''(Y(s))|\bar{Y}(s)|^{2\theta}]ds\\ + \lambda\mathbb{E}\int_0^{t\wedge\nu_n}[V(Y(s^-)+\delta\bar{Y}(s^-)) - V(Y(s^-))]ds.\end{aligned} \tag{5.5}$$

Recalling (2.1) and (2.2), we have that for $s \in [0, t\wedge\nu_n]$,

$$\begin{aligned}&V'(Y(s))f(\bar{Y}(s)) + \frac{b^2}{2}V''(Y(s))|\bar{Y}(s)|^{2\theta} + \lambda[V(Y(s^-)+\delta\bar{Y}(s^-)) - V(Y(s^-))]\\ &= V'(Y(s))f(Y(s)) + \frac{b^2}{2}V''(Y(s))|Y(s)|^{2\theta} + \lambda[V(Y(s^-)+\delta Y(s^-)) - V(Y(s^-))]\\ &\quad + V'(Y(s))(f(\bar{Y}(s)) - f(Y(s))) + \frac{b^2}{2}V''(Y(s))(|\bar{Y}(s)|^{2\theta} - |Y(s)|^{2\theta})\\ &\quad + \lambda[V(Y(s^-)+\delta\bar{Y}(s^-)) - V(Y(s^-)+\delta Y(s^-))]\\ &\le K_1 + I_1(s) + I_2(s) + I_3(s^-),\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}I_1(s) &= V'(Y(s))(f(\bar{Y}(s)) - f(Y(s))),\\ I_2(s) &= \frac{b^2}{2}V''(Y(s))(|\bar{Y}(s)|^{2\theta} - |Y(s)|^{2\theta}),\\ I_3(s^-) &= \lambda[V(Y(s^-)+\delta\bar{Y}(s^-)) - V(Y(s^-)+\delta Y(s^-))].\end{aligned}$$

If $s \in [0, t\wedge\nu_n]$, then $Y(s^-) \in [1/n, n]$, which implies $\bar{Y}(s) \in [1/n, n]$, we can deduce that there are constants $C_1(n)$ and $C_2(n)$ such that

$$\begin{aligned}|Y^{2\theta}(s) - \bar{Y}^{2\theta}(s)| &\le C_1(n)|Y(s) - \bar{Y}(s)|^{2\theta}\\ &\le C_1(n)|Y(s) - \bar{Y}(s)|(|Y(s)| + |\bar{Y}(s)|)^{2\theta-1}\\ &\le C_1(n)(2n)^{2\theta-1}|Y(s) - \bar{Y}(s)|\end{aligned} \tag{5.7}$$

and

$$|Y^{\gamma}(s) - \bar{Y}^{\gamma}(s)| \le C_2(n)|Y(s) - \bar{Y}(s)|. \tag{5.8}$$

Hence, for any $s \in [0, t \wedge \nu_n]$, we obtain

$$\begin{aligned}
I_1(s) &= V'(Y(s))(f(\bar{Y}(s)) - f(Y(s))) \\
&\le \alpha Y^{-1}(s)(Y^{\alpha}(s) - 1)\Big(a_{-1}\Big|\frac{1}{\bar{Y}(s)} - \frac{1}{Y(s)}\Big| + a_1|\bar{Y}(s) - Y(s)| + a_2|\bar{Y}^r(s) - Y^r(s)|\Big) \\
&\le \alpha n(n^{\alpha} - 1)\Big(a_{-1}\frac{|Y(s) - \bar{Y}(s)|}{|Y(s)||\bar{Y}(s)|} + a_1|Y(s) - \bar{Y}(s)| + a_2 C_2(n)|Y(s) - \bar{Y}(s)|\Big) \\
&\le \alpha n(n^{\alpha} - 1)(a_{-1}n^2 + a_1 + a_2 C_2(n))|Y(s) - \bar{Y}(s)| \\
&=: C_3(n)|Y(s) - \bar{Y}(s)|. \qquad (5.9)
\end{aligned}$$

Similarly, for any $s \in [0, t \wedge \nu_n]$, (5.7) gives

$$\begin{aligned}
I_2(s) &= \frac{b^2}{2}V''(Y(s))(\bar{Y}^{2\theta}(s) - Y^{2\theta}(s)) \\
&\le \frac{b^2}{2}|\alpha(\alpha - 1)Y^{\alpha-2}(s) + \alpha\bar{Y}^{-2}(s)|C_1(n)(2n)^{2\theta-1}|Y(s) - \bar{Y}(s)| \\
&\le \frac{b^2}{2}(2n)^{2\theta-1}C_1(n)(\alpha(\alpha - 1)n^{2-\alpha} + \alpha n^2)|Y(s) - \bar{Y}(s)| \\
&=: C_4(n)|Y(s) - \bar{Y}(s)|. \qquad (5.10)
\end{aligned}$$

Note that the functions y^{α} and $\log y$ are locally Lipschitz continuous for any $y > 0$. Therefore, for $Y(s)$, $\bar{Y}(s) \in [1/n, n]$, we have

$$\begin{aligned}
I_3(s^-) &= \lambda(V(Y(s^-) + \delta\bar{Y}(s^-))) - V(Y(s^-) + \delta Y(s^-)) \\
&\le \lambda\Big(|(Y(s^-) + \delta\bar{Y}(s^-))^{\alpha} - (Y(s^-) + \delta Y(s^-))^{\alpha}| \\
&\quad + \alpha|\log(Y(s^-) + \delta\bar{Y}(s^-))^{\alpha} - \log(Y(s^-) + \delta Y(s^-))^{\alpha}|\Big) \\
&\le \lambda\Big(\alpha n^{1-\alpha}\delta|Y(s^-) - \bar{Y}(s^-)| + n\delta|Y(s^-) - \bar{Y}(s^-)|\Big) \\
&= \lambda\delta(\alpha n^{1-\alpha} + n)|Y(s^-) - \bar{Y}(s^-)| \\
&=: C_5(n)|Y(s^-) - \bar{Y}(s^-)|. \qquad (5.11)
\end{aligned}$$

Inserting (5.9), (5.10) and (5.11) into (5.6), we get from (5.5) that

$$\mathbb{E}V(Y(t \wedge \nu_n)) \le V(y_0)) + K_1 T + (C_3(n) + C_4(n) + C_5(n))\mathbb{E}\int_0^{t\wedge\nu_n} |Y(s) - \bar{Y}(s)|ds. \qquad (5.12)$$

For any $s \in [0, T \wedge \nu_n]$, there is a constant $C_6(n)$ such that $|f(Y_{\lfloor s/\Delta \rfloor})| \le C_6(n)$. By definition (5.1), we have

$$\begin{aligned}
Y(s) - \bar{Y}(s) &= f(Y_{\lfloor s/\Delta \rfloor})(s - \lfloor s/\Delta \rfloor\Delta)) + b|Y_{\lfloor s/\Delta \rfloor}|^{\theta}(B(s) - B(Y_{\lfloor s/\Delta \rfloor})) \\
&\quad + \delta Y_{\lfloor s/\Delta \rfloor}(N(s) - N(Y_{\lfloor s/\Delta \rfloor})) \\
&\le C_6(n)\Delta + bn^{\theta}(B(s) - B(Y_{\lfloor s/\Delta \rfloor})) + \delta n(N(s) - N(Y_{\lfloor s/\Delta \rfloor})). \qquad (5.13)
\end{aligned}$$

Hence, for $\Delta \in (0,1)$, we have

$$\mathbb{E}\int_0^{t\wedge\nu_n} |Y(s)-\bar{Y}(s)|ds \le C_6(n)T\Delta + bn^\theta T\Delta^{1/2} + \delta nT\lambda\Delta$$
$$\le T(C_6(n)+bn^\theta+\delta n\lambda)\Delta^{1/2}$$
$$=: C_7(n)\Delta^{1/2}. \tag{5.14}$$

Substituting this into (5.12) yields

$$\mathbb{E}V(Y(t\wedge\nu_n)) \le V(y_0) + K_1T + C(n)\Delta^{1/2}, \tag{5.15}$$

where $C(n) = (C_3(n)+C_4(n)+C_5(n))C_7(n)$. Hence,

$$\mathbb{P}(\nu_n \le T) \le \frac{V(y_0)+K_1T+C(n)\Delta^{1/2}}{V(1/n)\wedge V(n)}. \tag{5.16}$$

Step 2. Let $\sigma_n = \tau_n \wedge \nu_n$. We next show that there exists a constant $D(n)$, which is dependent of n, such that

$$\mathbb{E}\left[\sup_{0\le t\le T\wedge\sigma_n} |y(t)-Y(t)|^2\right] \le D(n)\Delta. \tag{5.17}$$

Note that the function y^θ is locally Lipschitz continuous for $y>0$ and $\theta>1$. Therefore, for $y(s)$, $\bar{Y}(s)\in[1/n,n]$, there is a constant $C_8(n)$ such that

$$|y^\theta(s) - \bar{Y}^\theta(s)|^2 \le C_8(n)|y(s)-\bar{Y}(s)|^2. \tag{5.18}$$

For any $t\in[0,T]$ and $s\in[0,t\wedge\sigma_n]$, which implies $x(s), X(s)\in[1/n,n]$, we have

$$|f(y(s))-f(\bar{Y}(s))| \le a_{-1}\left|\frac{1}{y(s)}-\frac{1}{\bar{Y}(s)}\right| + a_1|y(s)-\bar{Y}(s)| + a_2|y^2(s)-\bar{Y}^2(s)|$$
$$\le a_{-1}\frac{|y(s)-\bar{Y}(s)|}{|y(s)||\bar{Y}(s)|} + a_1|y(s)-\bar{Y}(s)| + a_2(|y(s)|+|\bar{Y}(s)|)|y(s)-\bar{Y}(s)|$$
$$\le (a_{-1}n^2+a_1+2a_2n)|y(s)-\bar{Y}(s)|. \tag{5.19}$$

By (1.2) and (5.2), we get

$$y(t_1\wedge\sigma_n) - Y(t_1\wedge\sigma_n) = \int_0^{t_1\wedge\sigma_n}(f(y(s))-f(\bar{Y}(s)))ds + b\int_0^{t_1\wedge\sigma_n}(y^\theta(s)-|\bar{Y}(s)|^\theta)dB(s)$$
$$+\delta\int_0^{t_1\wedge\sigma_n}(y(s^-)-\bar{Y}(s^-))dN(s), \tag{5.20}$$

for any $t_1 \in [0,T]$. Hence, for any $T \in [0,T]$, by the Hölder and the Doob martingale inequality as well as Lemma 2.2, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t_1\le t}|y(t_1\wedge\sigma_n)-Y(t_1\wedge\sigma_n)|^2\right]\\
&\le 3t\mathbb{E}\int_0^{t\wedge\sigma_n}(f(y(s))-f(\bar{Y}(s)))^2ds+3b^2\mathbb{E}\left[\sup_{0\le t_1\le t}\Big(\int_0^{t_1\wedge\sigma_n}(y^\theta(s)-|\bar{Y}(s)|^\theta)dB(s)\Big)^2\right]\\
&\quad+3\delta^2\mathbb{E}\left[\sup_{0\le t_1\le t}\Big(\int_0^{t_1\wedge\sigma_n}(y(s^-)-\bar{Y}(s^-))dN(s)\Big)^2\right]\\
&\le 3t(a_{-1}n^2+a_1+2a_2n)^2\mathbb{E}\left[\int_0^{t\wedge\sigma_n}|y(s)-\bar{Y}(s)|^2ds\right]+12b^2\mathbb{E}\left[\int_0^{t\wedge\sigma_n}(y^\theta(s)-|\bar{Y}(s)|^\theta)^2ds\right]\\
&\quad+3\delta^2\lambda^2\mathbb{E}\left[\int_0^{t\wedge\sigma_n}(y(s)-\bar{Y}(s))ds\right]^2+K\mathbb{E}\left[\int_0^{t\wedge\sigma_n}(y(s)-\bar{Y}(s))^2ds\right]\\
&\le[3t(a_{-1}n^2+a_1+2a_2n)^2+12b^2C_8(n)+2\delta^3\lambda^2t+K]\mathbb{E}\left[\int_0^{t\wedge\sigma_n}(y(s)-\bar{Y}(s))^2ds\right]\\
&=:C_9(n)\mathbb{E}\left[\int_0^{t\wedge\sigma_n}(y(s)-\bar{Y}(s))^2ds\right]\\
&\le 2C_9(n)\int_0^t\mathbb{E}(y(s\wedge\sigma_n)-Y(s\wedge\sigma_n))^2ds++2C_9(n)\mathbb{E}\left[\int_0^{t\wedge\sigma_n}(Y(s)-\bar{Y}(s))^2ds\right],
\end{aligned}
\tag{5.21}
$$

where K is a constant. In the same way as the computation of (5.14), we can see that there exists a constant $C_{10}(n)$ such that

$$\mathbb{E}\int_0^{t\wedge\sigma_n}(Y(s)-\bar{Y}(s))^2ds\le C_{10}(n)\Delta.$$

Inserting this into (5.21) gives

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t_1\le t}|y(t_1\wedge\sigma_n)-Y(t_1\wedge\sigma_n)|^2\right]\\
&\le 2C_9(n)\int_0^t\mathbb{E}(y(s\wedge\sigma_n)-Y(s\wedge\sigma_n))^2ds+2C_9(n)C_{10}(n)\Delta.
\end{aligned}
$$

Applying the Gronwall inequality yields (5.17).

Step 3. Let $\varepsilon_1,\varepsilon_2\in(0,1)$ be arbitrarily small. Set

$$\bar{\Omega}=\left\{\omega:\sup_{0\le t\le T}|y(t)-Y(t)|^2\ge\varepsilon_1\right\}.$$

Using (5.17), we have

$$
\begin{aligned}
\varepsilon_1\mathbb{P}(\bar{\Omega}\cap\{\sigma_n\ge T\})&\le\mathbb{E}\left[\mathbb{I}_{\{\sigma_n\ge T\}}\mathbb{I}_{\bar{\Omega}}\right]\\
&\le\mathbb{E}\left[\mathbb{I}_{\{\sigma_n\ge T\}}\sup_{0\le t\le T\wedge\sigma_n}|y(t)-Y(t)|^2\right]\\
&\le\mathbb{E}\left[\sup_{0\le t\le T\wedge\sigma_n}|y(t)-Y(t)|^2\right]\\
&\le D(n)\Delta.
\end{aligned}
\tag{5.22}
$$

By (2.4), (5.16) and (5.22), we have

$$\begin{aligned}\mathbb{P}(\bar{\Omega}) &\leq \mathbb{P}(\bar{\Omega} \cap \{\sigma_n \geq T\}) + \mathbb{P}(\sigma_n \leq T) \\ &\leq \mathbb{P}(\bar{\Omega} \cap \{\sigma_n \geq T\}) + \mathbb{P}(\tau_n \leq T) + \mathbb{P}(\nu_n \leq T) \\ &\leq \frac{D(n)}{\varepsilon_1}\Delta + \frac{2V(y_0) + 2K_1 T + c(n)\Delta^{1/2}}{V(1/n) \wedge V(n)}. \end{aligned} \tag{5.23}$$

Recalling that $V(1/n) \wedge V(n) \to \infty$, as $n \to \infty$, we can choose n sufficiently large for

$$\frac{2V(y_0) + 2K_1 T}{V(1/n) \wedge V(n)} \leq \frac{\varepsilon_2}{2}$$

and then choose Δ sufficiently small for

$$\frac{D(n)}{\varepsilon_1}\Delta + \frac{c(n)\Delta^{1/2}}{V(1/n) \wedge V(n)} \leq \frac{\varepsilon_2}{2}$$

to obtain

$$\mathbb{P}(\bar{\Omega}) = \mathbb{P}\left(\sup_{0\leq t\leq T} |y(t) - Y(t)|^2 \geq \varepsilon_1\right) \leq \varepsilon_2, \tag{5.24}$$

which is the desired assertion (5.4). □

Lemma 5.2. *For any $T > 0$,*

$$\lim_{\Delta\to 0}\left(\sup_{0\leq t\leq T} |Y(t) - \bar{Y}(t)|^2\right) = 0 \quad in \quad probability. \tag{5.25}$$

Proof. By (5.13), for $t \in [0, T \wedge \nu_n]$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\nu_n} |Y(t) - \bar{Y}(t)|^2\right] &\leq C_{11}(n)\Delta^2 + C_{12}(n)\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\nu_n} |B(t) - B(\lfloor t/\Delta\rfloor\Delta)|^2\right] \\ &\quad + C_{13}(n)\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\nu_n} |\tilde{N}(t) - \tilde{N}(\lfloor t/\Delta\rfloor\Delta)|^2\right], \end{aligned} \tag{5.26}$$

where $C_{11}(n) = 3C_6^2(n) + 6\lambda^2\delta^2 n^2$, $C_{12}(n) = 3b^2 n^{2\theta}$ and $C_{13}(n) = 6\delta^2 n^2$. The Doob martingale inequality gives

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T} |B(t) - B(\lfloor t/\Delta\rfloor\Delta)|^4\right] &= \mathbb{E}\left[\max_{0\leq n\leq\lfloor T/\Delta\rfloor-1}\ \sup_{n\Delta\leq t\leq(n+1)\Delta} |B(t) - B(n\Delta)|^4\right] \\ &\leq \sum_{n=0}^{\lfloor T/\Delta\rfloor-1} \mathbb{E}\left[\sup_{n\Delta\leq t\leq(n+1)\Delta} |B(t) - B(n\Delta)|^4\right] \\ &\leq (4/3)^4 \sum_{n=0}^{\lfloor T/\Delta\rfloor-1} \mathbb{E}\left[|B((n+1)\Delta) - B(n\Delta)|^4\right] \\ &\leq \frac{256}{27} \sum_{n=0}^{\lfloor T/\Delta\rfloor-1} \Delta^2 \\ &\leq \frac{256}{27} T\Delta. \end{aligned}$$

By the Hölder inequality, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|B(t)-B(\lfloor t/\Delta\rfloor\Delta)|^2\right] &\le \left(\mathbb{E}\left[\sup_{0\le t\le T}|B(t)-B(\lfloor t/\Delta\rfloor\Delta)|^4\right]\right)^{1/2}\\ &\le \frac{16}{27^{1/2}}T\Delta^{1/2}. \end{aligned}\tag{5.27}$$

Note that $N(T)$ is a Poisson process with intensity λ, which means $\mathbb{P}(N(T)<\infty)=1$, we let these jump points within the intervals $[k_1\Delta,(k_1+1)\Delta)$, $[k_2\Delta,(k_2+1)\Delta)$, $\cdots$, $[k_{N(T)}\Delta,(k_{N(T)}+1)\Delta)$, respectively. Hence, by the Doob martingale inequality, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|\tilde N(t^-)-\tilde N(\lfloor t/\Delta\rfloor\Delta^-)|^2\right] &\le \mathbb{E}\left[\max_{1\le i\le N(T)}\sup_{k_i\Delta\le t<(k_i+1)\Delta}|\tilde N(t^-)-\tilde N(k_i\Delta^-)|^2\right]\\ &\le \mathbb{E}\left[\sum_{i=1}^{N(T)}\sup_{k_i\Delta\le t<(k_i+1)\Delta}|\tilde N(t^-)-\tilde N(k_i\Delta^-)|^2\right]\\ &\le \mathbb{E}N(T)\mathbb{E}\left[\sup_{k_i\Delta\le t<(k_i+1)\Delta}|\tilde N(t^-)-\tilde N(k_i\Delta^-)|^2\right]\\ &\le 4\mathbb{E}N(T)\mathbb{E}\left[|\tilde N((k_i+1)\Delta^-)-\tilde N(k_i\Delta^-)|^2\right]\\ &\le C_{14}\Delta, \end{aligned}\tag{5.28}$$

where C_{14} is a positive constant. Substituting (5.27) and (5.28) into (5.26) gives that there is a constant $C_{15}(n)$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T\wedge\nu_n}|Y(t)-\bar Y(t)|^2\right]\le C_{15}(n)\Delta^{1/2}.\tag{5.29}$$

Let $\varepsilon_1,\varepsilon_2\in(0,1)$ be arbitrary small, we define

$$\tilde\Omega=\left\{\omega:\sup_{0\le t\le T}|Y(t)-\bar Y(t)|^2\ge\varepsilon_1\right\}.$$

Then, we have

$$\begin{aligned}\varepsilon_1\mathbb{P}(\tilde\Omega\cap\{\nu_n\ge T\}) &\le \mathbb{E}\left[\mathbb{I}_{\{\nu_n\ge T\}}\mathbb{I}_{\tilde\Omega}\right]\\ &\le \mathbb{E}\left[\mathbb{I}_{\{\nu_n\ge T\}}\sup_{0\le t\le T\wedge\nu_n}|Y(t)-\bar Y(t)|^2\right]\\ &\le \mathbb{E}\left[\sup_{0\le t\le T\wedge\nu_n}|Y(t)-\bar Y(t)|^2\right]\\ &\le C_{15}(n)\Delta^{1/2}. \end{aligned}\tag{5.30}$$

Thus, by (5.30) and (5.16), we have

$$\begin{aligned}\mathbb{P}(\tilde\Omega) &\le \mathbb{P}(\tilde\Omega\cap\{\nu_n\ge T\})+\mathbb{P}(\nu_n\le T)\\ &\le \frac{C_{15}(n)}{\varepsilon_1}\Delta^{1/2}+\frac{V(y_0)+K_1T+c(n)\Delta^{1/2}}{V(1/n)\wedge V(n)}. \end{aligned}$$

Recalling that $V(1/n) \wedge V(n) \to \infty$, as $n \to \infty$, we can choose n sufficiently large for

$$\frac{V(x_0) + K_1 T}{V(1/n) \wedge V(n)} \leq \frac{\varepsilon_2}{2}$$

and then choose Δ sufficiently small for

$$\frac{C_{15}(n)}{\varepsilon_1}\Delta^{1/2} + \frac{c(n)\Delta^{1/2}}{V(1/n) \wedge V(n)} \leq \frac{\varepsilon_2}{2}$$

to obtain

$$\mathbb{P}(\tilde{\Omega}) = \mathbb{P}\left(\sup_{0\leq t\leq T} |Y(t) - \bar{Y}(t)|^2 \geq \varepsilon_1\right) \leq \varepsilon_2.$$

Thus, we complete the proof. □

Theorem 5.3. *For any* $T > 0$*,*

$$\lim_{\Delta\to 0}\left(\sup_{0\leq t\leq T} |y(t) - \bar{Y}(t)|^2\right) = 0 \quad in \quad probability. \tag{5.31}$$

Proof. For all sufficiently small $\varepsilon \in (0, 1)$,

$$\mathbb{P}\left(\sup_{0\leq t\leq T} |y(t) - \bar{Y}(t)| \geq \varepsilon\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq T} |y(t) - Y(t)| \geq \varepsilon/2\right) + \mathbb{P}\left(\sup_{0\leq t\leq T} |Y(t) - \bar{Y}(t)| \geq \varepsilon/2\right).$$

By Theorem 5.1 and Lemma 5.2, we obtain the desired assertion. □

Remark 5.4. *It should be pointed out that when Poisson process* $N(t)$ *is ignored in the model* (1.2)*, the results on the analytical properties of the solution to the model and convergence of the numerical method reduce to the corresponding ones that were obtained in [10], [16] and [18]. We extend these results to the case of Poisson jumps.*

6. Applications in finance

In this section, we assume that the interest rate or asset price is governed by the model (1.2). Then we use the EM method to approximate some financial quantities.

6.1. Bonds

Let $y(t)$ be the short-term interest rate. Denote by $B(T)$ the price of a bond at the end of period with form

$$B(T) = \mathbb{E}\left[\exp\left(-\int_0^T y(t)dt\right)\right].$$

An approximation to $B(T)$ is given by

$$\bar{B}_\Delta(T) = \mathbb{E}\left[\exp\left(-\int_0^T |\bar{Y}(t)|dt\right)\right],$$

where $\bar{Y}(t)$ is defined in (5.3). Then we have

$$\lim_{\Delta\to 0} |B(T) - \bar{B}_\Delta(T)| = 0.$$

Let $\varepsilon, \xi \in (0,1)$ be arbitrarily small. We need to prove

$$\mathbb{P}\left(\left|\exp\Big(-\int_0^T y(t)dt\Big) - \exp\Big(-\int_0^T |\bar{Y}(t)|dt\Big)\right| \geq \xi\right) < \varepsilon.$$

Note that

$$\begin{aligned}\left|\exp\Big(-\int_0^T y(t)dt\Big) - \exp\Big(-\int_0^T |\bar{Y}(t)|dt\Big)\right| &\leq \left|\int_0^T [y(t) - \bar{Y}(t)]dt\right| \\ &\leq T \sup_{0\leq t\leq T} |y(t) - \bar{Y}(t)|.\end{aligned}$$

Using the Theorem 5.3, we get the desired assertion.

6.2. Barrier options

Let E be the exercise price. Denote by $y(T)$ the asset price at the expiry date T and B the fixed barrier. The expected value of the barrier options at the expiry date, denoted by C, is defined as below

$$C(T) = \mathbb{E}\left[(y(T) - E)^+ \mathbb{I}_{\{0\leq y(t)\leq B, 0\leq t\leq T\}}\right].$$

We define its approximation by

$$\bar{C}_\Delta(T) = \mathbb{E}\left[(\bar{Y}(T) - E)^+ \mathbb{I}_{\{0\leq \bar{Y}(t)\leq B, 0\leq t\leq T\}}\right],$$

where $\bar{Y}(t)$ is the same as before. Then

$$\lim_{\Delta\to 0} |C(T) - \bar{C}_\Delta(T)| = 0.$$

The proof can be found in [10].

The above examples show that the EM method can be used to estimate finance quantities.

7. A numerical example

Let us consider the following Ait-Sahalia interest rate model with Poisson jumps

$$\begin{aligned}&dy(t) = (a_{-1}y^{-1}(t) - a_0 + a_1 y(t) - a_2 y^\gamma(t))dt + by^\theta(t)dB(t) + \delta y(t^-)dN(t),\\ &y(0) = y_0.\end{aligned} \tag{7.1}$$

We use a part of the parameters that were estimated from financial data (see [13, 41]), namely,

$$\begin{aligned} a_{-1} &= 1.041\times 10^{-4}, & a_0 &= 5.652\times 10^{-3},\\ a_1 &= 9.648\times 10^{-2}, & a_2 &= 5.349\times 10^{-1},\\ b &= 1.329\times 10^{-2}, & \delta &= 1.041\times 10^{-4},\\ \gamma &= 2, & \theta &= 1.4999,\\ y_0 &= 6.0\%, & \lambda &= 0.20,\end{aligned}$$

where y_0 denotes the initial value for the short-term rate of interest and λ the mean number of jumps per unit time. Let $y(t)$ be the true solution of (7.1) and $Y(t)$ the corresponding EM solution defined in (5.2). In this section, sample average is used to approximate the expected value.

7.1. The moment boundedness

In order to test the moment boundedness of the true solution, we carry out numerical experiments using MATLAB. Fig.3 shows the values of $\mathbb{E}y(t)$, $\mathbb{E}y^2(t)$ and $\mathbb{E}y^{-1}(t)$ versus t for the Ait-Sahalia model (7.1), approximated by EM scheme (5.1) with step size $\Delta = 0.005$ and 500 sample paths, respectively. We see from Fig.3 that sample averages of $Y(t)$, $Y^2(t)$ and $Y^{-1}(t)$ are bounded, confirming the results of Theorem 3.1 and 3.2.

7.2. The pathwise estimations

We perform a simulation result plotted in Fig.2 on the evolution of $\log Y(t)/\log t$ with time t for 50 solution paths of (7.1). The points have been joined by blue straight lines for clarity. The two curves $1+\varepsilon$ and $-(1+\varepsilon)$ are added as red dotted and black dash-dot lines, respectively. It seems that curves representing $\log Y(t)/\log t$ lie between the two straight lines $-(1+\varepsilon)$ and $1+\varepsilon$ for all the samples if time t is sufficiently large, say, $t \geq 20$. On the other hand, by virtue of Theorems 4.1 and 4.2, we obtain that for any $\varepsilon > 0$, there is a random variable T_ε such that the true solution $y(t)$ of (7.1) satisfied

$$-(1+\varepsilon) \leq \log y(t)/\log t \leq 1+\varepsilon, \ \text{for } \forall t \geq T_\varepsilon,$$

with probability one. Clearly, simulation experiments illustrate the validity of the theoretical results.

7.3. Convergence

To show the convergence result, we will focus on the error between the true solution $y(T)$ and EM solution Y_L at the endpoint T, where $T = L\Delta$. Define the strong error e_Δ^{strong} by

$$e_\Delta^{\text{strong}} := \mathbb{E}|y(T) - Y_L|.$$

Since it is difficult to obtain an explicit solution to (7.1), the EM solution with step size $\Delta = 2^{-12}$ is used as a reference solution. Fig.2 plots the strong error at endpoint $T = 10$ with $\Delta = 2^{-i}$($i = 5, 6, 7, 8, 9$) on a log-log scale for 200 sample paths. Table 1 shows the sample mean of $|Y(T) - Y_L|$ with 200 sample points for different step sizes Δ and different endpoints T. From Table 1 and Fig.2, we deduce that there is a constant C such that $e_\Delta^{\text{strong}} \leq C\Delta^{1/2}$. Thanks to the Markov inequality, we get

$$\mathbb{P}(|y(T) - Y_L| \geq \Delta^{1/4}) \leq \frac{\mathbb{E}|y(T) - Y_L|}{\Delta^{1/4}} \leq C\Delta^{1/4},$$

which means that the error at a fixed point in $[0, T]$ is small with probability close to one. Thus, the simulation results illustrate the convergence of EM method.

Remark 7.1. *Compared with some similar works (e.g., [10, 16, 18, 42]), our numerical experiments show the simulation results on the moment boundedness and the pathwise estimations of the solution to the model as well as the convergence of the numerical scheme for the first time. It is interesting to observe from Fig.1 that the EM solution is strongly convergent with order one-half and Fig.2 that the curves denoting $\log y(t)/\log t$ are far below the straight line $1+\varepsilon$, which is the theoretical estimate for the upper bounds of $\log y(t)/\log t$, for all sample paths. These observations indicate that our theoretical results are somewhat conservative. We will improve this estimate bounds and propose a more appropriate numerical method for solving this model which preserves nonnegativity and converges strongly to the true solution in the future.*

Table 1: Sample mean of $|Y(T)-Y_L|$ with 200 sample points for different step sizes Δ and different endpoints T

Δ	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
$T = 10$	0.3207e-006	0.2093e-006	0.1559e-006	0.0976e-006	0.0632e-006
$T = 15$	0.3736e-006	0.2812e-006	0.2007e-006	0.1253e-002	0.0807e-006
$T = 20$	0.4190e-006	0.2921e-006	0.2182e-006	0.1519e-006	0.0970e-006
$T = 25$	0.4880e-006	0.3672e-006	0.2608e-006	0.1775e-006	0.1137e-006

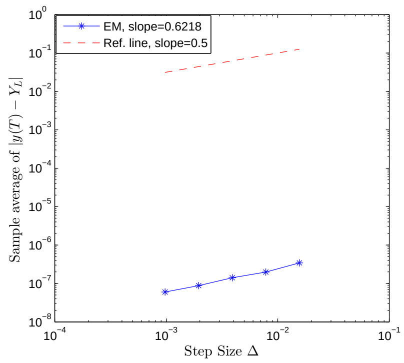

Fig. 1. Strong error by EM at time $T = 10$ with 200 sample paths

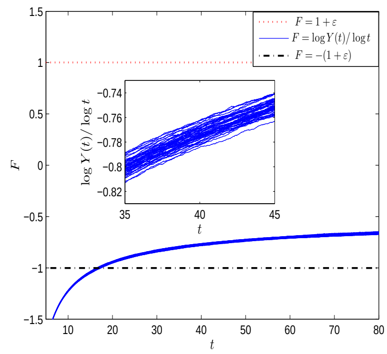

Fig. 2. $\log Y(t)/\log t$ vs. t by EM with $\varepsilon = 0.001$ for 50 sample paths

8. Conclusion

In this paper, we extend the generalized Ait-Sahalia interest rate model to the case of Poisson jumps. By means of the Lyapunov function method and stochastic analysis technique, the analytical properties of the solution to this model are studied. Also, a convergence result for EM method is obtained. Finally, a simulation example is given to illustrate the effectiveness of our results.

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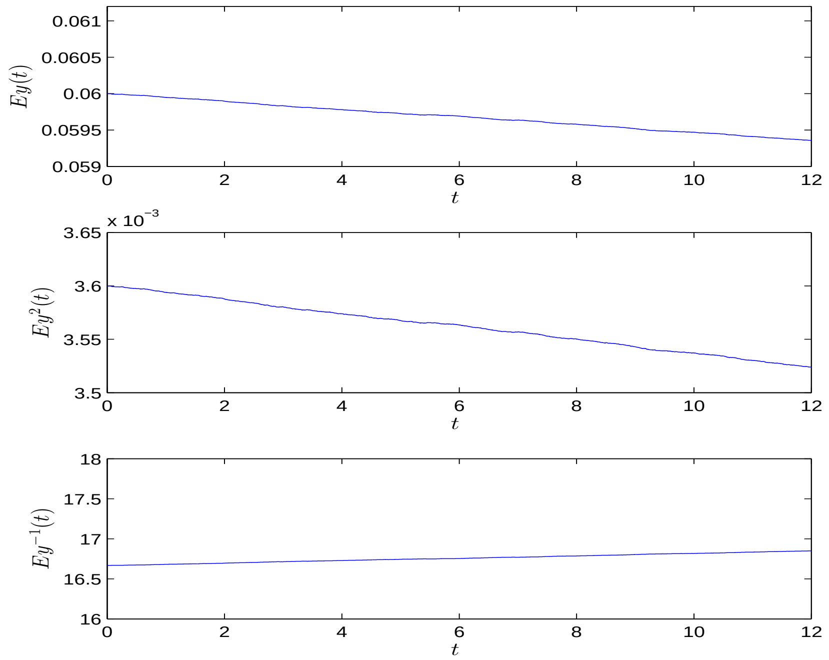

Fig. 3. Sample averages of $Y(t)$, $Y^2(t)$ and $Y^{-1}(t)$ by EM with 500 sample paths

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