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General Decay Anti-synchronization of Multi-weighted Coupled Neural Networks with and without Reaction-Diffusion Terms

Yanli Huang · Jie Hou · Erfu Yang

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Abstract The network models of multi-weighted coupled neural networks (MWCNNs) and multi-weighted coupled reaction-diffusion neural networks (MWCRDNNs) with and without delayed coupling are presented in this paper, respectively. Firstly, on account of the definitions of ψ-type stability and ψ-type function, the concept of decay anti-synchronization is proposed. Then, we investigate the decay anti-synchronization of MWCNNs with and without delayed coupling by designing appropriate nonlinear controllers, and several criteria for ensuring decay anti-synchronization are inferred by means of Lyapunov functional method as well as inequality techniques. Similarly, some conditions for decay anti-synchronization of MWCRDNNs with and without delayed coupling are also respectively derived. Lastly, two numerical examples with simulations are given to validate the correctness of these derived results.

Keywords General decay anti-synchronization · MWCNNs · Nonlinear control · Delayed coupling · Reaction-diffusion terms

Yanli Huang
School of Computer Science and Technology, Tianjin Key Laboratory of Optoelectronic Detection Technology and System, Tianjin Polytechnic University, Tianjin 300387, China
E-mail: huangyanli@tjpu.edu.cn

Jie Hou
School of Computer Science and Technology, Tianjin Polytechnic University, Tianjin 300387, China

Erfu Yang
Department of Design, Manufacture and Engineering Management, Faculty of Engineering, University of Strathclyde, Glasgow G1 1XJ, Scotland, UK

1 Introduction

In recent several decades, increasing attention has been paid to the study of neural networks (NNs) in virtue of their potential applications in various fields, such as optimization, parameter estimation, pattern recognition and so on [4,6,7,15,19,20,26,29,45,56,57]. Accordingly, it has been a hot topic to discuss the dynamic behaviours (e.g., synchronization, passivity and stability) in NNs [17,23,30,38,44,46,48]. In [23], exponential $\mathcal{H}_\infty$ synchronization problem of NNs with discrete time delays was considered. The stability for a class of memristive NNs was concerned and several conditions were established for guaranteeing stability in [30]. Nevertheless, the aforementioned literatures neglect the phenomenon of reaction diffusion in NNs. Strictly speaking, the reaction-diffusion phenomenon in NNs is inevitable when electrons propagate in inhomogeneous electromagnetic fields. Therefore, it is significant to take reaction-diffusion terms into consideration in NNs, and a number of pursuers have studied reaction-diffusion NNs (RDNNs) [11,12,22,32]. The authors discussed the passivity as well as stability of NNs with reaction-diffusion terms in [32].

In particular, when many NNs interconnect with each other, a special kind of complex networks, namely coupled NNs (CNNs), is taken shape. Recently, CNNs have been attracted wide attention because of their extensive applications in various prospects, such as brain science and secure communication. Consequently, a great quantity of meaningful research results on CNNs have been published [10,27,28,42,49]. In [10], the fixed-time synchronization for a kind of CNNs with parameter uncertainties was investigated. The authors discussed the synchronization of memristor-based recurrent CNNs with time-varying delays in [49]. In addition, there are some literatures on the study of coupled reaction-diffusion neural networks (CRDNNs) [31,33,39]. The synchronization and passivity of linearly CRDNNs with adaptive couplings were considered in [33]. In [31], the synchronization and $\mathcal{H}_\infty$ synchronization problems for an array of CRDNNs were considered.

It is well known that synchronization inhere in many practical systems, such as the generation and development of languages, an array of networks consisting of a set of identical delayed NNs and so on. Therefore, there are a large number of investigators devote themselves to studying the synchronization of CNNs and CRDNNs [9,10,28,31,33,39,42,49]. In fact, anti-synchronization is also a fascinating phenomenon in the real world, which widely exists in memristive recurrent NNs, periodic oscillators, and so forth. Moreover, up to now, anti-synchronization has been successfully applied in many fields, for instance, image processing, information science and so on. Hence, it is highly meaningful to study anti-synchronization [16,21,24,40,53]. In [40], the anti-synchronization problem for a class of memristive CNNs was investigated. Nevertheless, only few papers have considered reaction-diffusion phenomena in the study of anti-synchronization [43]. The authors in [43] analyzed anti-synchronization of CRDNNs with time-varying delayed coupling.

As pointed out in [14], it is extremely useful and fascinating topic to estimate the solution's convergence rate of nonlinear systems. Actually, in many practical cases, the convergence time or speed of the system is hard to acquire. Consequently, the conception of decay synchronization is proposed, which is derived from decay stability (ψ-type stability) [1,25,35,36]. In [25], the decay synchronization for a class of bidirectional associative memory NNs was investigated. The authors studied the decay synchronization of NNs with time delays in [1]. It is a pity that the decay anti-synchronization of CNNs and CRDNNs has not been considered in existing research results.

It is worth noting that the network models in the aforementioned literatures are with single weight. In fact, a great many of existing networks are more accurately described by multi-weighted complex dynamical networks (MWCDNs), for instance, transportation networks, social networks, communication networks and so on. Hence, it is of great significance to study MWCDNs [2,3,54,55]. In [2], the global synchronization of MWCDNs was studied and a criterion was obtained for guaranteeing synchronization. The general decay synchronization of delayed MWCDNs was considered by constructing suitable nonlinear feedback controllers in [55]. As a particular sort of MWCDNs, multi-weighted CNNs (MWCNNs) have also been concerned in [8,34]. Wang et al. [34] considered finite-time passivity of MWCNNs and derived several finite-time synchronization conditions. Unfortunately, there is no research results reported on the decay anti-synchronization for MWCNNs as well as multi-weighted CRDNNs (MWCRDNNs) until now.

In view of the foregoing statement, we consider the general decay anti-synchronization of MWCNNs as well as MWCRDNNs, respectively. Firstly, the general decay anti-synchronization is defined by introducing ψ-type function. Additionally, we investigate the decay anti-synchronization for MWCNNs with and without delayed coupling by designing suitable nonlinear controllers and constructing appropriate Lyapunov functional, and some criteria for guaranteeing decay anti-synchronization are obtained. Furthermore, we also discuss the decay anti-synchronization for MWCRDNNs with Dirichlet boundary conditions, and several decay anti-synchronization conditions for MWCRDNNs with and without delayed coupling are put forward.

2 Preliminaries

Definition 1 (see [41]) If the function $\psi(t)$: $\mathbb{R}_+ \to (0, +\infty)$ satisfies the conditions as follows:

1) $\psi(t)$ is nondecreasing and differentiable;
2) $\psi(+\infty) = +\infty$ and $\psi(0) = 1$;
3) $\widehat{\psi}(t) := \frac{\dot{\psi}(t)}{\psi(t)}$ is decreasing;
4) $\forall \alpha, \beta \geqslant 0,\ \psi(\alpha+\beta) \leqslant \psi(\alpha)\psi(\beta)$;

then it is called to be ψ-type function.

Lemma 1 *(see [18]) Let Θ be a cube $|\vartheta_r| < \iota_r (r = 1, 2, \cdots, p)$ and $Z(\vartheta) \in C^1(\Theta)$ be a real-valued function, which satisfies $Z(\vartheta)|_{\partial\Theta} = 0$. Then*

$$\int_\Theta Z^2(\vartheta) d\vartheta \leqslant \iota_r^2 \int_\Theta \left(\frac{\partial Z}{\partial \vartheta_r}\right)^2 d\vartheta,$$

where $\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_p)^T$.

Notations: $\lambda(\cdot)$ signifies the eigenvalue of the corresponding matrix. For any $e(t) = (e_1(t), e_2(t), \cdots, e_M(t))^T \in \mathbb{R}^M$, $\|e(t)\| = \sqrt{\sum_{s=1}^M e_s^2(t)}$. Moreover, if $e(\vartheta, t) = (e_1(\vartheta, t), e_2(\vartheta, t), \cdots, e_M(\vartheta, t))^T \in \mathbb{R}^M$, then $\|e(\cdot, t)\|_2 = \sqrt{\int_\Theta \sum_{s=1}^M e_s^2(\vartheta, t) d\vartheta}$, where $(\vartheta, t) \in \Theta \times \mathbb{R}$ and $\Theta = \{\vartheta = (\vartheta_1, \vartheta_2, \cdots, \vartheta_p)^T |\ |\vartheta_r| < l_r,\ r = 1, 2, \cdots, p\} \subset \mathbb{R}^p$.

3 General decay anti-synchronization of MWCNNs with and without delayed coupling

3.1 General decay anti-synchronization of MWCNNs

Consider the following MWCNNs model:

$$\begin{aligned}\dot{Y}_s(t) = &-AY_s(t) + Bg(Y_s(t)) + c_1\sum_{j=1}^{M} G^1_{sj}\Gamma_1 Y_j(t)\\ &+ c_2\sum_{j=1}^{M} G^2_{sj}\Gamma_2 Y_j(t) + \cdots\\ &+ c_m\sum_{j=1}^{M} G^m_{sj}\Gamma_m Y_j(t), \quad s = 1,2,\cdots,M, \qquad (1)\end{aligned}$$

where $\mathbb{R} \ni M > 0$ represents the total number of nodes in the network, $Y_s(t) = (Y_{s1}(t), Y_{s2}(t), \cdots, Y_{sn}(t))^T \in \mathbb{R}^n$ is the state vector of the sth node; $A = \text{diag}(a_1, a_2, \cdots, a_n) \in \mathbb{R}^{n\times n} > 0, B \in \mathbb{R}^{n\times n}$ symbols a constant matrix; $g(Y_s(t)) = (g_1(Y_{s1}(t)), g_2(Y_{s2}(t)), \cdots, g_n(Y_{sn}(t)))^T \in \mathbb{R}^n$ and $\mathbb{R} \ni c_\kappa > 0$ $(\kappa = 1,2,\cdots,m)$ symbols coupling strength for the κth coupling form; $\Gamma_\kappa \in \mathbb{R}^{n\times n}$ $(\kappa = 1,2,\cdots,m)$ is positive definite matrix, which symbols the inner coupling matrix of the κth coupling form; $G^\kappa = (G^\kappa_{sj})_{M\times M} \in \mathbb{R}^{M\times M}$ $(\kappa = 1,2,\cdots,m)$ expresses coupling weight between nodes in the κth coupling form, where G^κ_{sj} is defined as follows: $G^\kappa_{sj} = G^\kappa_{js} > 0$ if and only if there exists a connection between node s and node j for the κth coupling form; if not, $G^\kappa_{sj} = G^\kappa_{js} = 0$ $(s \neq j)$; and

$$G^\kappa_{ss} = -\sum_{\substack{j=1\\ j\neq s}}^{M} G^\kappa_{sj}, \; s = 1,2,\cdots,M.$$

Remark 1 It is well known that a large scale of networks in our real world should be represented by MWCDNs, in which nodes are coupled in the form of multiple coupling, such as social networks, communication networks, transportation networks and so on. Hence, it is of great significance to study MWCDNs [2,3,54,55]. An et al. [3] handled the problem of public traffic network from a new visual field of CDNs with multi-weights and analyzed the impact of congestion degrees, transfers coefficient and passenger flow density between different bus lines on the complex public traffic network to its synchronous ability. Moreover, MWCNNs is a special class of MWCDNs, which has also been studied recently [8,34]. The authors in [8] studied anti-synchronization and pinning control problem of MWCNNs. Unfortunately, there is no research results reported on the decay anti-synchronization for MWCNNs until now, which motivates our research work in this paper.

Consider the network model (1) as the drive system. Then its corresponding response system is introduced as:

$$\begin{aligned}\dot{W}_s(t) = &-AW_s(t) + Bg(W_s(t)) + c_1\sum_{j=1}^{M} G^1_{sj}\Gamma_1 W_j(t)\\ &+c_2\sum_{j=1}^{M} G^2_{sj}\Gamma_2 W_j(t)+\cdots+c_m\sum_{j=1}^{M} G^m_{sj}\Gamma_m W_j(t)\\ &+u_s(t), \quad s = 1,2,\cdots,N, \qquad (2)\end{aligned}$$

where $W_s(t) = (W_{s1}(t), W_{s2}(t), \cdots, W_{sn}(t))^T \in \mathbb{R}^n$ is the state vector of the sth node; $u_s(t) = (u_{s1}(t), u_{s2}(t), \cdots, u_{sn}(t))^T \in \mathbb{R}^n$ is the suitable controller to achieve a certain control objective; A, B, c_κ, G^κ_{sj}, Γ_κ have the same definitions as in system (1).

Assumption 1 *For any $\beta_1, \beta_2 \in \mathbb{R}$, the function $g_i(\cdot)$ $(i = 1,2,\cdots,n)$ satisfies*

$$|g_i(\beta_1) + g_i(\beta_2)| \leqslant F_i|\beta_1 + \beta_2|,$$

where $0 < F_i \in \mathbb{R}$. Take $F = \text{diag}(F_1^2, F_2^2, \cdots, F_n^2) \in \mathbb{R}^{n\times n}$.

Take $e_s(t) = Y_s(t) + W_s(t)$. By (1) and (2), we can obtain

$$\begin{aligned}\dot{e}_s(t) = &-Ae_s(t) + Bg(Y_s(t)) + \sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G^\kappa_{sj}\Gamma_\kappa e_j(t)\\ &+ Bg(W_s(t)) + u_s(t). \qquad (3)\end{aligned}$$

Definition 2 If there exists $\mathbb{R} \ni \lambda > 0$ satisfies the following inequality:

$$\limsup_{t\to\infty} \frac{\log\|e(t)\|}{\log\psi(t)} \leqslant -\lambda,$$

where $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_M^T(t))^T$, $\psi(t)$ is a ψ-type function as defined in Definition 1, then the network (3) is called to be ψ-type stable and λ is the convergence rate.

Before providing our main results, a lemma needs to be given which plays a key role in our proof. What follows is an assumption for preparing the above mentioned lemma.

Assumption 2 *(see [37]) There exist $\eta(t) \in C(\mathbb{R}, \mathbb{R}^+)$ and $\mathbb{R} \ni \sigma > 0$ such that:*

$$\sup_{t\in[0,\infty)} \int_0^t \psi^\sigma(h)\eta(h)dh < \infty, \quad \widehat{\psi}(t) \leqslant 1,$$

in which $\widehat{\psi}(t)$ and $\psi(t)$ are defined in Definition 1.

Lemma 2 *(see [37]) On the premise of Assumption 2, if there exist a differential function $V(t, e_0(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ and two constants $0 < \alpha_1 \in \mathbb{R}$, $0 < \alpha_2 \in \mathbb{R}$ such that*

$$(\alpha_1 \|e_0(t)\|)^2 \leqslant V(t, e_0(t)),$$
$$\dot{V}(t, e_0(t))|_{(3)} + \sigma V(t, e_0(t)) \leqslant \alpha_2 \eta(t),$$

where $e_0(t)$ is a solution of network (3), σ and $\eta(t)$ are well defined in Assumption 2, then we said the system (3) is ψ-type stable, namely the drive-response systems (1) and (2) reach general decay anti-synchronization. In addition, the convergence rate is $\frac{\sigma}{2}$.

Remark 2 In recent decades, NNs have been widely applied in various fields, e.g., optimization, parameter estimation, pattern recognition and so on [4,7,15,26,29]. As a matter of fact, great majority of these applications depend heavily on the dynamical behaviors of NNs. For example, it is vital that each trajectory of the NNs converges to a unique equilibrium point in order to solve optimization problem by utilizing NNs, that is, the NNs is stable. Hence, many researchers have devoted themselves to studying the stability of NNs [30, 44,46]. In addition, as pointed out in [14], it is extremely interesting subject to estimate the solution's convergence rate of nonlinear systems, which is prior condition for theoretical analysis and design. However, in many practical cases, the convergence time or speed of the system is difficult to obtain. In 2016, Wang et al. [36] firstly proposed the definition of general decay stability based on ψ-type function, also known as ψ-type stability. Indeed, when NNs possess ψ-type stability, it is helpful to solve the optimization problem and implement content-addressable memories [47]. Note that anti-synchronization is another interesting phenomenon in many applications, such as in chaotic NNs, memristive NNs, and even periodic oscillator [16,21,24,40]. In fact, anti-synchronization refers to the disappearance of the sum of the relevant state variables of the nodes in a NN or social network. In above existing literatures [25, 36,37], the ψ-type stability is defined as follows:

$$\limsup_{t\to\infty} \frac{\log\|W(t) - Y(t)\|}{\log\psi(t)} \leqslant -\lambda.$$

In Definition 2, the synchronization error $\|W(t)-Y(t)\|$ is changed to $\|W(t) + Y(t)\|$, which means the state variables of two systems (1) and (2) have the same amplitude but the opposite signs. Therefore, the drive-response systems (1) and (2) can reach decay anti-synchronization if the system (3) is ψ-type stable.

In this section, we design the following nonlinear controller for response system (2):

$$u_s(t) = -q_s e_s(t) - \beta_s \frac{\|e(t)\|^2 e_s(t)}{\|e(t)\|^2 + \eta(t)}, \quad s = 1, 2, \cdots, M, \tag{4}$$

where $\mathbb{R} \ni q_s > 0$, $\mathbb{R} \ni \beta_s > 0$.

For convenience, we denote $\beta = \max_{1\leqslant s\leqslant M}\{\beta_s\}$, $\hat{q} = \mathrm{diag}(q_1, q_2, \cdots, q_M)$, $\hat{\beta} = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_M)$.

Theorem 1 *Under Assumptions 1 and 2, the network (3) is ψ-type stable with the convergence rate $\frac{\sigma}{2}$, or the systems (1) and (2) reach general decay anti-synchronization, if*

$$\Phi_1 = I_M \otimes K_1 - (\hat{q} + \hat{\beta}) \otimes I_n + \sum_{\kappa=1}^{m} c_\kappa G^\kappa \otimes \Gamma_\kappa < 0, \tag{5}$$

where $K_1 = -A + \frac{1}{2}(BB^T + F) + \frac{\sigma}{2} I_n$.

Proof We construct the following Lyapunov functional for network (3):

$$V_1(t) = \frac{1}{2} \sum_{s=1}^{M} e_s^T(t) e_s(t). \tag{6}$$

It is easy to get $(\frac{1}{\sqrt{2}}\|e(t)\|)^2 \leqslant V_1(t)$. And one has

$$\begin{aligned}\dot{V}_1(t) =& \sum_{s=1}^{M} e_s^T(t)\Big(- Ae_s(t) + Bg(Y_s(t)) + Bg(W_s(t)) \\ &+ \sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa \Gamma_\kappa e_j(t) - q_s e_s(t) \\ &- \beta_s \frac{\|e(t)\|^2 e_s(t)}{\|e(t)\|^2 + \eta(t)}\Big).\end{aligned} \tag{7}$$

Obviously,

$$\begin{aligned}& e_s^T(t) B(g(Y_s(t)) + g(W_s(t))) \\ \leqslant& \frac{1}{2} e_s^T(t)(BB^T + F)e_s(t).\end{aligned} \tag{8}$$

From (7) and (8), we have

$$\begin{aligned}\dot{V}_1(t) =& -\sum_{s=1}^{M} \beta_s e_s^T(t) e_s(t) + \sum_{s=1}^{M} e_s^T(t)(-A + \frac{1}{2}(BB^T + F) \\ &- q_s I_n) e_s(t) - \frac{\|e(t)\|^2}{\|e(t)\|^2 + \eta(t)} \sum_{s=1}^{M} \beta_s e_s^T(t) e_s(t) \\ &+ \sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa e_s^T(t) \Gamma_\kappa e_j(t) \\ &+ \sum_{s=1}^{M} \beta_s e_s^T(t) e_s(t)\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{s=1}^{M}\beta_s e_s^T(t)e_s(t) + \sum_{s=1}^{M} e_s^T(t)(-A+\frac{1}{2}(BB^T+F)\\
&\quad - q_sI_n)e_s(t) + \frac{\eta(t)}{\|e(t)\|^2+\eta(t)}\sum_{s=1}^{M}\beta_s e_s^T(t)e_s(t)\\
&\quad + \sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa e_s^T(t)\varGamma_\kappa e_j(t)\\
&\leqslant \frac{\beta\|e(t)\|^2\eta(t)}{\|e(t)\|^2+\eta(t)} + \sum_{s=1}^{M} e_s^T(t)(-A+\frac{1}{2}(BB^T+F)-q_sI_n\\
&\quad -\beta_sI_n)e_s(t) + \sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa e_s^T(t)\varGamma_\kappa e_j(t). \quad (9)
\end{aligned}
$$

Combine (6) with (9), one has

$$
\begin{aligned}
&\dot{V}_1(t)+\sigma V_1(t)\\
\leqslant&\frac{\beta\|e(t)\|^2\eta(t)}{\|e(t)\|^2+\eta(t)} + \sum_{s=1}^{M} e_s^T(t)(-A+\frac{1}{2}(BB^T+F)\\
&-q_sI_n-\beta_sI_n)e_s(t) + \frac{\sigma}{2}\sum_{s=1}^{M} e_s^T(t)e_s(t)\\
&+\sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa e_s^T(t)\varGamma_\kappa e_j(t)\\
\leqslant&\sum_{s=1}^{M} e_s^T(t)(-A+\frac{1}{2}(BB^T+F)-q_sI_n-\beta_sI_n\\
&+\frac{\sigma}{2}I_n)e_s(t) + \sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa e_s^T(t)\varGamma_\kappa e_j(t)\\
&+\beta\eta(t)\\
=&\beta\eta(t)+e^T(t)\Big[I_M\otimes\big(-A+\frac{1}{2}(BB^T+F)+\frac{\sigma}{2}I_n\big)\\
&-(\hat{q}+\hat{\beta})\otimes I_n + \sum_{\kappa=1}^{m} c_\kappa G^\kappa\otimes\varGamma_\kappa\Big]e(t), \quad (10)
\end{aligned}
$$

where $e(t) = (e_1^T(t), e_2^T(t), \cdots, e_M^T(t))^T$. According to (5), one has

$$
\dot{V}_1(t)+\sigma V_1(t) \leqslant \beta\eta(t).
$$

Take $\alpha_1 = \frac{1}{\sqrt{2}}$, $\alpha_2 = \beta$, then it is easy to obtain the network (3) is ψ-type stable with convergence rate $\frac{\sigma}{2}$. In other words, the drive system (1) and response system (2) reach general decay anti-synchronization.

3.2 General decay anti-synchronization of MWCNNs with delayed coupling

In this section, a MWCNN model with delayed coupling is considered which can be represented as follows:

$$
\begin{aligned}
\dot{Y}_s(t) =& -AY_s(t)+Bg(Y_s(t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa\varGamma_\kappa Y_j(t\\
&-\tau_\kappa(t)),\ s=1,2,\cdots,M, \quad (11)
\end{aligned}
$$

where $Y_s(t)$, $g(\cdot)$, A, B, c_κ, G_{sj}^κ, $\varGamma_\kappa$ are defined similarly as those in Section 3.1, and the time-varying delay $\tau_\kappa(t)$ $(\kappa = 1,2,\cdots,m)$ satisfies $0 \leqslant \tau_\kappa(t) \leqslant \tau$.

Consider the network model (11) as the drive system. Then its corresponding response system is introduced as:

$$
\begin{aligned}
\dot{W}_s(t) =& -AW_s(t)+Bg(W_s(t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa\varGamma_\kappa W_j(t\\
&-\tau_\kappa(t))+u_s(t), \quad (12)
\end{aligned}
$$

where $s = 1,2,\cdots,M$; $W_s(t)$, A, B, c_κ, G_{sj}^κ, $\varGamma_\kappa$, $u_s(t)$ have the same definitions as in system (2); $\tau_\kappa(t)(\kappa = 1,2,\cdots,m)$ has the same definition as in system (11).

Take $e_s(t) = Y_s(t)+W_s(t)$. By (11) and (12), we can obtain

$$
\begin{aligned}
\dot{e}_s(t) =& -Ae_s(t)+Bg(W_s(t))+Bg(Y_s(t))+u_s(t)\\
&+\sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa\varGamma_\kappa e_j(t-\tau_\kappa(t)). \quad (13)
\end{aligned}
$$

We construct the same nonlinear controller (4) for response system (12) in this section.

Theorem 2 *Suppose* $\dot{\tau}_\kappa(t) \leqslant \gamma_\kappa < 1$. *Under Assumptions 1 and 2, the network* (13) *is* ψ*-type stable with the convergence rate* $\frac{\sigma}{2}$, *or the systems* (11) *and* (12) *reach general decay anti-synchronization, if*

$$
\begin{aligned}
\varPhi_2 =& I_M\otimes K_2+\frac{1}{2}\sum_{\kappa=1}^{m} c_\kappa(G^\kappa)^2\otimes(\varGamma_\kappa)^2-(\hat{q}\\
&+\hat{\beta})\otimes I_n < 0, \quad (14)\\
\varLambda_2 =& \tau\sigma+\frac{\sigma}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa} < 1, \quad (15)
\end{aligned}
$$

where $K_2 = -A+\frac{1}{2}(BB^T+F)+(\frac{\sigma}{2}+\tau+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa})I_n$.

Proof Define the following Lyapunov functional for network (13):

$$
V_2(t) = \frac{1}{2}\sum_{s=1}^{M} e_s^T(t)e_s(t)+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{t-\tau_\kappa(t)}^{t} e^T(h)e(h)dh
$$

$$+\int_{-\tau}^{0}\int_{t+\rho}^{t}e^T(h)e(h)dhd\rho. \quad (16)$$

Obviously, $(\frac{1}{\sqrt{2}}\|e(t)\|)^2 \leqslant V_2(t)$ and we can deduce from (16) as follows:

$$\begin{aligned}V_2(t) \leqslant& \frac{1}{2}\sum_{s=1}^{M}e_s^T(t)e_s(t) + \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{t-\tau}^{t}e^T(h)e(h)dh \\ &+ \tau\int_{t-\tau}^{t}e^T(h)e(h)dh \\ =& \left(\tau + \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\right)\int_{t-\tau}^{t}e^T(h)e(h)dh \\ &+ \frac{1}{2}\sum_{s=1}^{M}e_s^T(t)e_s(t). \quad (17)\end{aligned}$$

By calculating the derivative of (16), one has

$$\begin{aligned}\dot{V}_2(t) \leqslant& \sum_{s=1}^{M}e_s^T(t)\Big(-Ae_s(t)+\sum_{\kappa=1}^{m}\sum_{j=1}^{M}c_\kappa G_{sj}^\kappa \Gamma_\kappa e_j(t-\tau_\kappa(t)) \\ &+ Bg(Y_s(t)) + Bg(W_s(t)) - \beta_s\frac{\|e(t)\|^2 e_s(t)}{\|e(t)\|^2+\eta(t)} \\ &-q_s e_s(t)\Big) + \tau e^T(t)e(t) - \int_{t-\tau}^{t}e^T(h)e(h)dh \\ &+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}e^T(t)e(t) - \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t \\ &- \tau_\kappa(t))e(t-\tau_\kappa(t)),\end{aligned} \quad (18)$$

where $e(t-\tau_\kappa(t)) = (e_1^T(t-\tau_\kappa(t)), e_2^T(t-\tau_\kappa(t)), \cdots, e_M^T(t-\tau_\kappa(t)))^T$. Obviously,

$$\begin{aligned}&\sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M}c_\kappa G_{sj}^\kappa e_s^T(t)\Gamma_\kappa e_j(t-\tau_\kappa(t)) \\ \leqslant& \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(t) \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t-\tau_\kappa(t))e(t-\tau_\kappa(t)). \quad (19)\end{aligned}$$

Combine (18) with (19), one has

$$\begin{aligned}\dot{V}_2(t) \leqslant& \sum_{s=1}^{M}e_s^T(t)\Big(-Ae_s(t)+Bg(Y_s(t)) - \beta_s\frac{\|e(t)\|^2 e_s(t)}{\|e(t)\|^2+\eta(t)} \\ &+ Bg(W_s(t)) - q_s e_s(t)\Big) + \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}e^T(t)e(t) \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(t) \\ &+ \tau e^T(t)e(t) - \int_{t-\tau}^{t}e^T(h)e(h)dh \\ \leqslant& \sum_{s=1}^{M}\beta_s e_s^T(t)e_s(t) + \tau e^T(t)e(t) + \sum_{s=1}^{M}e_s^T(t)(-A+\frac{1}{2}(BB^T \\ &+F) - q_s I_n)e_s(t) + \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(t) \\ &- \frac{\|e(t)\|^2}{\|e(t)\|^2+\eta(t)}\sum_{s=1}^{M}\beta_s e_s^T(t)e_s(t) - \sum_{s=1}^{M}\beta_s e_s^T(t)e_s(t) \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}e^T(t)e(t) - \int_{t-\tau}^{t}e^T(h)e(h)dh \\ \leqslant& \frac{\beta\|e(t)\|^2\eta(t)}{\|e(t)\|^2+\eta(t)} + \sum_{s=1}^{M}e_s^T(t)(-A+\frac{1}{2}(BB^T+F) - q_s I_n \\ &- \beta_s I_n)e_s(t) + \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(t) \\ &+ \tau e^T(t)e(t) - \int_{t-\tau}^{t}e^T(h)e(h)dh \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}e^T(t)e(t). \quad (20)\end{aligned}$$

Combine (17) with (20), we have

$$\begin{aligned}\dot{V}_2(t)+\sigma V_2(t) \leqslant& \frac{\beta\|e(t)\|^2\eta(t)}{\|e(t)\|^2+\eta(t)} + \sum_{s=1}^{M}e_s^T(t)(-A+\frac{1}{2}(BB^T \\ &+F) - q_s I_n - \beta_s I_n)e_s(t) + \tau e^T(t)e(t) \\ &+ \frac{\sigma}{2}\sum_{s=1}^{M}e_s^T(t)e_s(t) - \int_{t-\tau}^{t}e^T(h)e(h)dh \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(t) \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}e^T(t)e(t) + \sigma\Big(\tau \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\Big)\int_{t-\tau}^{t}e^T(h)e(h)dh \\ \leqslant& \sum_{s=1}^{M}e_s^T(t)(-A+\frac{1}{2}(BB^T+F) - q_s I_n - \beta_s I_n \\ &+ \frac{\sigma}{2}I_n)e_s(t) + \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}e^T(t)e(t) \\ &+ \frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(t) \\ &+ \beta\eta(t) + \tau e^T(t)e(t) + \Big(\frac{\sigma}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa} \\ &+ \tau\sigma - 1\Big)\int_{t-\tau}^{t}e^T(h)e(h)dh \\ =& \beta\eta(t) + e^T(t)\Big[I_M\otimes\Big(-A+\frac{1}{2}(BB^T+F)\end{aligned}$$

$$+(\frac{\sigma}{2}+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}+\tau)I_n\Big)-(\hat{q}+\hat{\beta})\otimes I_n$$
$$+\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa(G^\kappa)^2\otimes(\Gamma_\kappa)^2\Big]e(t)+\Big(\tau\sigma-1$$
$$+\frac{\sigma}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\Big)\int_{t-\tau}^{t}e^T(h)e(h)dh.$$

According to (14) and (15), we can obtain

$$\dot{V}_2(t)+\sigma V_2(t)\leqslant\beta\eta(t).$$

Take $\alpha_1=\frac{1}{\sqrt{2}}$, $\alpha_2=\beta$, it is easy to obtain that network (13) is ψ-type stable with convergence rate $\frac{\sigma}{2}$. Then, the drive system (11) and response system (12) reach general decay anti-synchronization.

Remark 3 Recently, some scholars investigated the synchronization and finite-time synchronization of NNs by using integrating inequality technique, instead of conventional methods, such as linear matrix inequality method, matrix measure strategy and so on [50–52]. In [50], Zhang et al. established some new sufficient conditions on global asymptotic synchronization for delayed inertial NNs by constructing different Lyapunov functions and using the constructed integral inequality. These results on the synchronization are novel and interesting. It is noteworthy that the network models in the above mentioned literatures are single-weighted. Actually, numerous existing networks can be represented more precisely by complex networks with multiple weights [2, 3,54,55]. In this section, we derive two sufficient conditions for reaching decay anti-synchronization of the considered MWCNNs by designing appropriate nonlinear controller and using Lyapunov functional method as well as commonly used inequality techniques. To the best of our knowledge, this is the first paper toward to studying the decay anti-synchronization of MWCNNs. In our future work, it would be very interesting to study the decay anti-synchronization of MWCNNs by using some novel inequality techniques, e.g., integral inequality constructed in Lemma 2.1 of [50].

4 General decay anti-synchronization of MWCRDNNs with and without delayed coupling

4.1 General decay anti-synchronization of MWCRDNNs

In this section, the model of MWCNNs with reaction-diffusion terms is considered which can be characterized by the partial differential equations as follows:

$$\frac{\partial Y_s(\vartheta,t)}{\partial t}=-AY_s(\vartheta,t)+D\triangle Y_s(\vartheta,t)+Bg(Y_s(\vartheta,t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{M}c_\kappa G_{sj}^\kappa\Gamma_\kappa Y_j(\vartheta,t),\quad(21)$$

where $s=1,2,\cdots,M$, $Y_s(\vartheta,t)=(Y_{s1}(\vartheta,t),Y_{s2}(\vartheta,t),\cdots,Y_{sn}(\vartheta,t))^T\in\mathbb{R}^n$ is the state vector of the sth neuron at time t and in space ϑ; $\vartheta=(\vartheta_1,\vartheta_2,\cdots,\vartheta_p)^T\in\Theta\subset\mathbb{R}^p$; $D=\mathrm{diag}(d_1,d_2,\cdots,d_n)$, $d_s>0$ is the transmission diffusion coefficient; $g(Y_s(\vartheta,t))=(g_1(Y_{s1}(\vartheta,t)),g_2(Y_{s2}(\vartheta,t)),\cdots,g_n(Y_{sn}(\vartheta,t)))^T\in\mathbb{R}^n$; $\Delta=\sum_{r=1}^{p}\frac{\partial^2}{\partial\vartheta_r^2}$ symbols the Laplace diffusion operator on Θ; A, B, c_κ, Γ_κ, G_{sj}^κ have the same meanings as in Section 3.1.

For the network (21),

$$Y_s(\vartheta,0)=\phi_s(\vartheta)\in\mathbb{R}^n,\quad\vartheta\in\Theta,$$
$$Y_s(\vartheta,t)=0,\quad(\vartheta,t)\in\partial\Theta\times[0,+\infty).$$

Consider the network model (21) as the drive system. Then its corresponding response system is introduced as:

$$\frac{\partial W_s(\vartheta,t)}{\partial t}=-AW_s(\vartheta,t)+D\triangle W_s(\vartheta,t)+Bg(W_s(\vartheta,t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{M}c_\kappa G_{sj}^\kappa\Gamma_\kappa W_j(\vartheta,t)+u_s(\vartheta,t),\quad(22)$$

where $s=1,2,\cdots,M$, $W_s(\vartheta,t)=(W_{s1}(\vartheta,t),W_{s2}(\vartheta,t),\cdots,W_{sn}(\vartheta,t))^T\in\mathbb{R}^n$ is the state vector of the sth neuron at time t and in space ϑ; $u_s(\vartheta,t)=(u_{s1}(\vartheta,t),u_{s2}(\vartheta,t),\cdots,u_{sn}(\vartheta,t))^T\in\mathbb{R}^n$ is the suitable controller to achieve a certain control objective; A, B, D, c_κ, G_{sj}^κ, Γ_κ have the same definitions as in system (21).

For the network (22),

$$W_s(\vartheta,0)=\varphi_s(\vartheta)\in\mathbb{R}^n,\quad\vartheta\in\Theta,$$
$$W_s(\vartheta,t)=0,\quad(\vartheta,t)\in\partial\Theta\times[0,+\infty).$$

Take $e_s(\vartheta,t)=Y_s(\vartheta,t)+W_s(\vartheta,t)$. By (21) and (22), it is easy to get

$$\frac{\partial e_s(\vartheta,t)}{\partial t}=-Ae_s(\vartheta,t)+D\triangle e_s(\vartheta,t)+Bg(Y_s(\vartheta,t))+Bg(W_s(\vartheta,t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{M}c_\kappa G_{sj}^\kappa\Gamma_\kappa e_j(\vartheta,t)+u_s(\vartheta,t).\quad(23)$$

Definition 3 If there exists a constant $\mathbb{R}\ni\lambda>0$ such that

$$\limsup_{t\to\infty}\frac{\log\|e(\cdot,t)\|}{\log\psi(t)}\leqslant-\lambda,$$

where $e(\vartheta,t)=(e_1^T(\vartheta,t),e_2^T(\vartheta,t),\cdots,e_M^T(\vartheta,t))^T$, $\psi(t)$ is a ψ-type function as defined in Definition 1, then the network (23) is called to be ψ-type stable, where λ is the convergence rate when $e(\vartheta,t)\to 0$.

For convenience, we denote

$$\begin{aligned}
Y(\vartheta,t)=&(Y_1^T(\vartheta,t),Y_2^T(\vartheta,t),\cdots,Y_M^T(\vartheta,t))^T,\\
W(\vartheta,t)=&(W_1^T(\vartheta,t),W_2^T(\vartheta,t),\cdots,W_M^T(\vartheta,t))^T,\\
\hat{g}(Y(\vartheta,t))=&(g^T(Y_1(\vartheta,t)),g^T(Y_2(\vartheta,t)),\cdots,g^T(Y_M(\vartheta,t)))^T,\\
\hat{g}(W(\vartheta,t))=&(g^T(W_1(\vartheta,t)),g^T(W_2(\vartheta,t)),\cdots,g^T(W_M(\vartheta,t)))^T.
\end{aligned}$$

For response system (22), the following nonlinear controller is designed :

$$u_s(\vartheta,t)=-q_se_s(\vartheta,t)-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\vartheta,t)}{\|e(\cdot,t)\|^2+\eta(t)}, \tag{24}$$

where $s=1,2,\cdots,M$, $\mathbb{R}\ni q_s>0$, $\mathbb{R}\ni\beta_s>0$.

For convenience, we denote $\beta=\max_{1\leqslant s\leqslant M}\{\beta_s\}$, $\hat{q}=\mathrm{diag}(q_1,q_2,\cdots,q_M)$, $\hat{\beta}=\mathrm{diag}(\beta_1,\beta_2,\cdots,\beta_M)$.

Theorem 3 *Under Assumptions 1 and 2, the network (23) is ψ-type stable with the convergence rate $\frac{\sigma}{2}$, or the systems (21) and (22) achieve general decay anti-synchronization, if*

$$\begin{aligned}
\Phi_3=&I_M\otimes\left(K_1-\sum_{r=1}^{p}\frac{1}{\iota_r^2}D\right)-(\hat{q}+\hat{\beta})\otimes I_n\\
&+\sum_{\kappa=1}^{m}c_\kappa G^\kappa\otimes\Gamma_\kappa<0.
\end{aligned}\tag{25}$$

Proof Construct the following Lyapunov functional for network (23):

$$V_3(t)=\frac{1}{2}\sum_{s=1}^{M}\int_\Theta e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta. \tag{26}$$

Obviously, $(\frac{1}{\sqrt{2}}\|e(\cdot,t)\|)^2\leqslant V_3(t)$. And, one has

$$\begin{aligned}
\dot{V}_3(t)=&\sum_{s=1}^{M}\int_\Theta e_s^T(\vartheta,t)\frac{\partial e_s(\vartheta,t)}{\partial t}d\vartheta\\
=&\sum_{s=1}^{M}\int_\Theta e_s^T(\vartheta,t)\Big(-Ae_s(\vartheta,t)+Bg(Y_s(\vartheta,t))\\
&+D\triangle e_s(\vartheta,t)+Bg(W_s(\vartheta,t))-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\vartheta,t)}{\|e(\cdot,t)\|^2+\eta(t)}\\
&-q_se_s(\vartheta,t)+\sum_{\kappa=1}^{m}\sum_{j=1}^{M}c_\kappa G_{sj}^\kappa\Gamma_\kappa e_j(\vartheta,t)\Big)d\vartheta\\
&+\sum_{s=1}^{M}\beta_s\int_\Theta e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta\\
&-\sum_{s=1}^{M}\beta_s\int_\Theta e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta\\
=&\int_\Theta e^T(\vartheta,t)(I_M\otimes B)(\hat{g}(Y(\vartheta,t))+\hat{g}(W(\vartheta,t)))d\vartheta\\
&+\int_\Theta e^T(\vartheta,t)\Big(\sum_{\kappa=1}^{m}c_\kappa G^\kappa\otimes\Gamma_\kappa-I_M\otimes A-(\hat{\beta}\\
&+\hat{q})\otimes I_n\Big)e(\vartheta,t)d\vartheta+\int_\Theta e^T(\vartheta,t)(I_M\otimes D)\triangle e(\vartheta,t)d\vartheta\\
&+\frac{\eta(t)}{\|e(\cdot,t)\|^2+\eta(t)}\sum_{s=1}^{M}\beta_s\int_\Theta e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta\\
\leqslant&\int_\Theta e^T(\vartheta,t)(I_M\otimes B)(\hat{g}(Y(\vartheta,t))+\hat{g}(W(\vartheta,t)))d\vartheta\\
&+\int_\Theta e^T(\vartheta,t)\Big(\sum_{\kappa=1}^{m}c_\kappa G^\kappa\otimes\Gamma_\kappa-I_M\otimes A-(\hat{\beta}\\
&+\hat{q})\otimes I_n\Big)e(\vartheta,t)d\vartheta+\frac{\beta\|e(\cdot,t)\|^2\eta(t)}{\|e(\cdot,t)\|^2+\eta(t)}\\
&+\int_\Theta e^T(\vartheta,t)(I_M\otimes D)\triangle e(\vartheta,t)d\vartheta,
\end{aligned}\tag{27}$$

where $e(\vartheta,t)=(e_1^T(\vartheta,t),e_2^T(\vartheta,t),\cdots,e_M^T(\vartheta,t))^T$. From the Green's formula, one has

$$\int_\Theta e_{sl}(\vartheta,t)\triangle e_{sj}(\vartheta,t)d\vartheta=-\sum_{r=1}^{p}\int_\Theta\frac{\partial e_{sl}(\vartheta,t)}{\partial\vartheta_r}\frac{\partial e_{sj}(\vartheta,t)}{\partial\vartheta_r}d\vartheta,$$

where $l,j\in\{1,2,\cdots,n\}$, $s=1,2,\cdots,M$. Let $\pi(\vartheta,t)=(I_M\otimes\sqrt{D})e(\vartheta,t)$. From Lemma 1, we can obtain

$$\begin{aligned}
&\int_\Theta e^T(\vartheta,t)(I_M\otimes D)\triangle e(\vartheta,t)d\vartheta\\
=&-\sum_{r=1}^{p}\sum_{s=1}^{M}\sum_{j=1}^{n}\sum_{l=1}^{n}d_l\int_\Theta\frac{\partial e_{sj}(\vartheta,t)}{\partial\vartheta_r}\frac{\partial e_{sl}(\vartheta,t)}{\partial\vartheta_r}d\vartheta\\
=&-\sum_{r=1}^{p}\int_\Theta\left(\frac{\partial e(\vartheta,t)}{\partial\vartheta_r}\right)^T(I_M\otimes D)\frac{\partial e(\vartheta,t)}{\partial\vartheta_r}d\vartheta\\
=&-\sum_{r=1}^{p}\int_\Theta\left(\frac{\partial\pi(\vartheta,t)}{\partial\vartheta_r}\right)^T\frac{\partial\pi(\vartheta,t)}{\partial\vartheta_r}d\vartheta\\
\leqslant&-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_\Theta\pi^T(\vartheta,t)\pi(\vartheta,t)d\vartheta\\
=&-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_\Theta e^T(\vartheta,t)(I_M\otimes D)e(\vartheta,t)d\vartheta.
\end{aligned}\tag{28}$$

Furthermore, one can easily deduce

$$\begin{aligned}
&e^T(\vartheta,t)(I_M\otimes B)(\hat{g}(Y(\vartheta,t))+\hat{g}(W(\vartheta,t))\\
\leqslant&\frac{1}{2}e^T(\vartheta,t)(I_M\otimes(BB^T+F))e(\vartheta,t).
\end{aligned}\tag{29}$$

From (27) to (29), we can obtain

$$\dot{V}_3(t) \leqslant \int_{\Theta} e^T(\vartheta,t)\Big[I_M \otimes \big(-A+\frac{1}{2}(BB^T+F) -\sum_{r=1}^{p}\frac{1}{\iota_r^2}D\big) - (\hat{\beta}+\hat{q}) \otimes I_n + \sum_{\kappa=1}^{m} c_\kappa G^\kappa \otimes \varGamma_\kappa\Big] e(\vartheta,t)d\vartheta + \frac{\beta\|e(\cdot,t)\|^2\eta(t)}{\|e(\cdot,t)\|^2+\eta(t)}. \tag{30}$$

Combine (26) with (30), we have

$$\begin{aligned}\dot{V}_3(t)+\sigma V_3(t) \leqslant& \int_{\Theta} e^T(\vartheta,t)\Big[I_M \otimes \big(-A+\frac{1}{2}(BB^T+F) -\sum_{r=1}^{p}\frac{1}{\iota_r^2}D\big) + \sum_{\kappa=1}^{m} c_\kappa G^\kappa \otimes \varGamma_\kappa - (\hat{\beta} + \hat{q}) \otimes I_n\Big] e(\vartheta,t)d\vartheta + \frac{\beta\|e(\cdot,t)\|^2\eta(t)}{\|e(\cdot,t)\|^2+\eta(t)} \\ &+ \frac{\sigma}{2}\sum_{s=1}^{M}\int_{\Theta} e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta \\ \leqslant& \int_{\Theta} e^T(\vartheta,t)\Big[I_M \otimes \big(-A+\frac{1}{2}(BB^T+F) -\sum_{r=1}^{p}\frac{1}{\iota_r^2}D + \frac{\sigma}{2}I_n\big) - (\hat{\beta}+\hat{q}) \otimes I_n + \sum_{\kappa=1}^{m} c_\kappa G^\kappa \otimes \varGamma_\kappa\Big] e(\vartheta,t)d\vartheta + \beta\eta(t).\end{aligned}$$

According to (25), one has

$$\dot{V}_3(t)+\sigma V_3(t) \leqslant \beta\eta(t).$$

Take $\alpha_1 = \frac{1}{\sqrt{2}}$, $\alpha_2 = \beta$, it is easy to obtain that network (23) is ψ-type stable with convergence rate $\frac{\sigma}{2}$. Then, the drive system (21) and response system (22) reach general decay anti-synchronization.

4.2 General decay anti-synchronization of MWCRDNNs with delayed coupling

In this section, the following MWCRDNNs model with delayed coupling is considered:

$$\frac{\partial Y_s(\vartheta,t)}{\partial t} = -AY_s(\vartheta,t) + D\triangle Y_s(\vartheta,t) + Bg(Y_s(x,t)) + \sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa \varGamma_\kappa Y_j(\vartheta, t-\tau_\kappa(t)), \tag{31}$$

where $s = 1,2,\cdots,M$, $Y_s(\vartheta,t)$, $g(\cdot)$, A, B, D, c_κ, G_{sj}^κ, $\varGamma_\kappa$ are defined similarly as those in Section 4.1, the time-varying delay $\tau_\kappa(t)$ $(\kappa = 1,2,\cdots,m)$ satisfies $0 \leqslant \tau_\kappa(t) \leqslant \tau$.

For the network (31),

$$Y_s(\vartheta,t) = \phi_s(\vartheta,t), \quad \vartheta \in \Theta \times [-\tau, 0],$$
$$Y_s(\vartheta,t) = 0, \quad (\vartheta,t) \in \partial\Theta \times [-\tau,+\infty).$$

Consider the network model (31) as the drive system. Then its corresponding response system is introduced as:

$$\frac{\partial W_s(\vartheta,t)}{\partial t} = -AW_s(\vartheta,t) + D\triangle W_s(\vartheta,t) + Bg(W_s(\vartheta,t)) + u_s(\vartheta,t) + \sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa \varGamma_\kappa W_j(\vartheta, t-\tau_\kappa(t)), \quad s = 1,2,\cdots,M, \tag{32}$$

where A, B, D, c_κ, G_{sj}^κ, $\varGamma_\kappa$, $W_s(\vartheta,t)$, $u_s(\vartheta,t)$, $\tau_\kappa(t)$ represent the same meanings as in system (31).

For the network (32),

$$W_s(\vartheta,t) = \varphi_s(\vartheta,t), \quad \vartheta \in \Theta \times [-\tau, 0],$$
$$W_s(\vartheta,t) = 0, \quad (\vartheta,t) \in \partial\Theta \times [-\tau,+\infty).$$

Take $e_s(\vartheta,t) = Y_s(\vartheta,t)+W_s(\vartheta,t)$. By (31) and (32), we can obtain

$$\frac{\partial e_s(\vartheta,t)}{\partial t} = -Ae_s(\vartheta,t) + D\triangle e_s(\vartheta,t) + Bg(Y_s(\vartheta,t)) + \sum_{\kappa=1}^{m}\sum_{j=1}^{M} c_\kappa G_{sj}^\kappa \varGamma_\kappa e_j(\vartheta, t-\tau_\kappa(t)) + Bg(W_s(\vartheta,t)) + u_s(\vartheta,t). \tag{33}$$

We construct the same nonlinear controller (24) for response system (32) in this section.

Theorem 4 *Suppose $\dot{\tau}_\kappa(t) \leqslant \gamma_\kappa < 1$. Under Assumptions 1 and 2, the network (33) is ψ-type stable with the convergence rate $\frac{\sigma}{2}$, or the systems (31)and (32) reach general decay anti-synchronization, if*

$$\varPhi_4 = I_M \otimes (K_2 - \sum_{r=1}^{p}\frac{1}{\iota_r^2}D) + \frac{1}{2}\sum_{\kappa=1}^{m} c_\kappa (G^\kappa)^2 \otimes (\varGamma_\kappa)^2 - (\hat{q}+\hat{\beta}) \otimes I_n < 0, \tag{34}$$

$$\varLambda_4 = \tau\sigma + \frac{\sigma}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa} < 1. \tag{35}$$

Proof Construct the following Lyapunov functional for network (33):

$$V_4(t) = \frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{t-\tau_\kappa(t)}^{t}\int_{\Theta} e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh + \int_{-\tau}^{0}\int_{t+\rho}^{t}\int_{\Theta} e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh d\rho$$

$$+\frac{1}{2}\sum_{s=1}^{M}\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta. \quad (36)$$

Obviously, $(\frac{1}{\sqrt{2}}\|e(\cdot,t)\|)^2 \leqslant V_4(t)$ and we can deduce from (36) as follows:

$$\begin{aligned}V_4(t)\leqslant&\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh\\&+\tau\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh\\&+\frac{1}{2}\sum_{s=1}^{M}\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta\\=&\left(\tau+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\right)\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh\\&+\frac{1}{2}\sum_{s=1}^{M}\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta. \quad (37)\end{aligned}$$

By calculating the derivative of (36), one has

$$\begin{aligned}\dot{V}_4(t)\leqslant&\sum_{s=1}^{M}\int_{\Theta}e_s^T(\vartheta,t)\Big(-Ae_s(\vartheta,t)+\sum_{\kappa=1}^{m}\sum_{j=1}^{M}c_\kappa G_{sj}^{\kappa}\Gamma_\kappa e_j(\vartheta,t-\tau_\kappa(t))\\&+Bg(Y_s(\vartheta,t))+Bg(W_s(\vartheta,t))+D\triangle e_s(\vartheta,t)-q_se_s(\vartheta,t)\\&-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\vartheta,t)}{\|e(\cdot,t)\|^2+\eta(t)}\Big)d\vartheta+\tau\int_{\Theta}e^T(\vartheta,t)e(\vartheta,t)d\vartheta\\&-\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa\int_{\Theta}e^T(\vartheta,t-\tau_\kappa(t))e(\vartheta,t-\tau_\kappa(t))d\vartheta\\&+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{\Theta}e^T(\vartheta,t)e(\vartheta,t)d\vartheta\\&-\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh, \quad (38)\end{aligned}$$

where $e(\vartheta,t-\tau_\kappa(t))=(e_1^T(\vartheta,t-\tau_\kappa(t)),e_2^T(\vartheta,t-\tau_\kappa(t)),\cdots,$ $e_M^T(\vartheta,t-\tau_\kappa(t)))^T$. Obviously,

$$\begin{aligned}&\sum_{\kappa=1}^{m}\sum_{s=1}^{M}\sum_{j=1}^{M}c_\kappa G_{sj}^{\kappa}e_s^T(\vartheta,t)\Gamma_\kappa e_j(\vartheta,t-\tau_\kappa(t))\\\leqslant&\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(\vartheta,t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(\vartheta,t)\\&+\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa e^T(\vartheta,t-\tau_\kappa(t))e(\vartheta,t-\tau_\kappa(t)). \quad (39)\end{aligned}$$

From (38) and (39), we have

$$\begin{aligned}\dot{V}_4(t)\leqslant&\sum_{s=1}^{M}\int_{\Theta}e_s^T(\vartheta,t)\Big(-Ae_s(\vartheta,t)+Bg(Y_s(\vartheta,t))+D\triangle e_s(\vartheta,t)\\&+Bg(W_s(\vartheta,t))-q_se_s(\vartheta,t)-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\vartheta,t)}{\|e(\cdot,t)\|^2+\eta(t)}\Big)d\vartheta\\&+\sum_{s=1}^{M}\beta_s\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta+\tau\int_{\Theta}e^T(\vartheta,t)e(\vartheta,t)d\vartheta\\&+\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa\int_{\Theta}e^T(\vartheta,t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(\vartheta,t)d\vartheta\\&+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{\Theta}e^T(\vartheta,t)e(\vartheta,t)d\vartheta\\&-\sum_{s=1}^{M}\beta_s\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta\\&-\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh\\=&\tau\int_{\Theta}e^T(\vartheta,t)e(\vartheta,t)d\vartheta-\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh\\&+\int_{\Theta}e^T(\vartheta,t)\big(-I_M\otimes A-(\hat{\beta}+\hat{q})\otimes I_n\big)e(\vartheta,t)d\vartheta\\&+\int_{\Theta}e^T(\vartheta,t)(I_M\otimes B)(\hat{g}(Y(\vartheta,t))+\hat{g}(W(\vartheta,t)))d\vartheta\\&+\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa\int_{\Theta}e^T(\vartheta,t)\big((G^\kappa)^2\otimes(\Gamma_\kappa)^2\big)e(\vartheta,t)d\vartheta\\&+\frac{\eta(t)}{\|e(\cdot,t)\|^2+\eta(t)}\sum_{s=1}^{M}\beta_s\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta\\&+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\int_{\Theta}e^T(\vartheta,t)e(\vartheta,t)d\vartheta\\&+\int_{\Theta}e^T(\vartheta,t)(I_M\otimes D)\triangle e(\vartheta,t)d\vartheta\\\leqslant&\int_{\Theta}e^T(\vartheta,t)\Big[I_M\otimes\Big(-A+\frac{1}{2}(BB^T+F)+(\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\\&+\tau)I_n-\sum_{r=1}^{p}\frac{1}{\iota_r^2}D\Big)+\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa(G^\kappa)^2\otimes(\Gamma_\kappa)^2-(\hat{\beta}\\&+\hat{q})\otimes I_n\Big]e(\vartheta,t)d\vartheta+\frac{\beta\|e(\cdot,t)\|^2\eta(t)}{\|e(\cdot,t)\|^2+\eta(t)}\\&-\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh. \quad (40)\end{aligned}$$

Combine (37) with (40), one has

$$\begin{aligned}&\dot{V}_4(t)+\sigma V_4(t)\\\leqslant&\int_{\Theta}e^T(\vartheta,t)\Big[\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa(G^\kappa)^2\otimes(\Gamma_\kappa)^2+I_M\otimes\Big(-A\\&+\frac{1}{2}(BB^T+F)-\sum_{r=1}^{p}\frac{1}{\iota_r^2}D+(\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\\&+\tau)I_n\Big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\vartheta,t)d\vartheta+\sigma\Big(\tau\\&+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\Big)\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh\end{aligned}$$

$$+\frac{\sigma}{2}\sum_{s=1}^{M}\int_{\Theta}e_s^T(\vartheta,t)e_s(\vartheta,t)d\vartheta+\beta\eta(t)$$
$$-\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh$$
$$\leqslant\Big(\tau\sigma-1+\frac{\sigma}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa}\Big)\int_{t-\tau}^{t}\int_{\Theta}e^T(\vartheta,h)e(\vartheta,h)d\vartheta dh$$
$$+\int_{\Theta}e^T(\vartheta,t)\Big[\frac{1}{2}\sum_{\kappa=1}^{m}c_\kappa(G^\kappa)^2\otimes(\Gamma_\kappa)^2+I_M\otimes\Big(-A$$
$$+\frac{1}{2}(BB^T+F)+(\frac{\sigma}{2}+\tau+\frac{1}{2}\sum_{\kappa=1}^{m}\frac{c_\kappa}{1-\gamma_\kappa})I_n$$
$$-\sum_{r=1}^{p}\frac{1}{\iota_r^2}D\Big)-(\hat\beta+\hat q)\otimes I_n\Big]e(\vartheta,t)d\vartheta+\beta\eta(t).$$

According to (34) and (35), we can obtain

$$\dot V_4(t)+\sigma V_4(t)\leqslant\beta\eta(t).$$

Take $\alpha_1=\frac{1}{\sqrt{2}}$, $\alpha_2=\beta$, it is easy to obtain that network (33) is ψ-type stable with convergence rate $\frac{\sigma}{2}$. Then, the drive system (31) and response system (32) reach general decay anti-synchronization.

Remark 4 As is well known, linear matrix inequality, matrix measure theory, stability theory method are commonly used methods for studying the synchronization of NNs [5,13,23,24,40]. In [5], Cao and Wan presented several stability and synchronization criteria for inertial BAM NNs by using matrix measure strategies. However, the majority of network models in above-mentioned results are described by ordinary differential equations. Actually, reaction-diffusion effects are very important and unavoidable in NNs once the electrons transport in inhomogeneous magnetic field. Therefore, it is necessary and meaningful to study the CNNs with reaction-diffusion terms [31,33,39,43], which are described by partial differential equations, i.e., the state of each neuron is not only dependent on the time variable, but also intensively dependent on space variable. Unfortunately, there are no results reported on the decay anti-synchronization of CRDNNs with multi-weights. In this section, the definition of ψ-type stability for MWCRDNNs is firstly given, which is equivalent to decay anti-synchronization of the considered drive-response systems of MWCRDNNs. Then, several decay anti-synchronization criteria for MWCRDNNs are established by designing appropriate controller. Actually, the main difficulty for obtaining decay anti-synchronization conditions in our paper comes from the reaction-diffusion terms and multi-weighted terms, which cannot be dealt with by those traditional techniques used in the above literatures. Due to this, by constructing new Lyapunov functionals, and employing Green's formula, Lemmas 1 and 2, and some other inequality techniques, we establish some conditions for achieving decay anti-synchronization of MWCRDNNs in Theorems 3 and 4, which are dependent on the reaction-diffusion terms and multi-weighted terms.

5 Numerical Examples

Example 5.1. Consider the following MWCNN with delayed coupling:

$$\dot Y_s(t)=-AY_s(t)+Bg(Y_s(t))+0.2\sum_{j=1}^{6}G_{sj}^1\Gamma_1Y_j(t-\tau_1(t))$$
$$+0.2\sum_{j=1}^{6}G_{sj}^2\Gamma_2Y_j(t-\tau_2(t))+0.1\sum_{j=1}^{6}G_{sj}^3\Gamma_3Y_j(t$$
$$-\tau_3(t)),\quad s=1,2,\cdots,6, \tag{41}$$

where $g_i(\delta)=\frac{|\delta+1|-|\delta-1|}{8}$ $(i=1,2,3)$, $A=\mathrm{diag}(0.3,0.4,0.6)$, $\Gamma_1=\mathrm{diag}(0.2,0.3,0.1)$, $\Gamma_2=\mathrm{diag}(0.1,0.2,0.1)$, $\Gamma_3=\mathrm{diag}(0.2,0.2,0.3)$, $\tau_1(t)=\frac{1}{20}-\frac{1}{20}e^{-t}$, $\tau_2(t)=\frac{1}{20}-\frac{1}{30}e^{-t}$, $\tau_3(t)=\frac{1}{20}-\frac{1}{40}e^{-t}$, $\tau=\frac{1}{20}$, $\gamma_1=\frac{1}{20}$, $\gamma_2=\frac{1}{30}$, $\gamma_3=\frac{1}{40}$, choose the following matrices as B, G^1, G^2 and G^3 respectively

$$B=\begin{pmatrix}0.3&0&0.1\\0.1&0.2&0\\0.1&0&0.4\end{pmatrix},$$

$$G^1=\begin{pmatrix}-0.6&0.2&0.1&0.1&0&0.2\\0.2&-0.7&0.3&0&0.2&0\\0.1&0.3&-0.6&0&0.1&0.1\\0.1&0&0&-0.5&0.2&0.2\\0&0.2&0.1&0.2&-0.5&0\\0.2&0&0.1&0.2&0&-0.5\end{pmatrix},$$

$$G^2=\begin{pmatrix}-0.5&0&0.3&0.1&0.1&0\\0&-0.4&0.2&0.1&0&0.1\\0.3&0.2&-0.8&0.1&0&0.2\\0.1&0.1&0.1&-0.4&0&0.1\\0.1&0&0&0&-0.3&0.2\\0&0.1&0.2&0.1&0.2&-0.6\end{pmatrix},$$

$$G^3=\begin{pmatrix}-0.6&0.1&0&0.2&0.2&0.1\\0.1&-0.5&0.2&0.1&0.1&0\\0&0.2&-0.4&0&0&0.2\\0.2&0.1&0&-0.6&0.3&0\\0.2&0.1&0&0.3&-0.8&0.2\\0.1&0&0.2&0&0.2&-0.5\end{pmatrix}.$$

Obviously, $g_i(\cdot)$ $(i=1,2,3)$ satisfies Assumption 1 with $F_i=0.25$. Consider (41) as the drive system, then its corresponding response system is shown as follows:

$$\begin{aligned}\dot{W}_s(t) =& - AW_s(t) + Bg(W_s(t)) + 0.2\sum_{j=1}^{6} G^1_{sj}\Gamma_1 W_j(t-\tau_1(t)) \\ &+ 0.2\sum_{j=1}^{6} G^2_{sj}\Gamma_2 W_j(t-\tau_2(t)) + 0.1\sum_{j=1}^{6} G^3_{sj}\Gamma_3 W_j(t \\ &- \tau_3(t)) + u_s(t),\ s = 1, 2, \cdots, 6, \qquad (42)\end{aligned}$$

the parameters in the controller (4) are chosen as follows: $\hat{q} = \mathrm{diag}(0.8, 0.6, 0.4, 0.2, 0.5, 0.2)$, $\hat{\beta} = \mathrm{diag}(0.3, 0.5, 0.4, 0.3, 0.2, 0.5)$ and $\eta(t) = e^{-0.2t}$. Then the nonlinear controller (4) is transferred as follows:

$$\begin{cases} u_1(t) = -0.8e_1(t) - 0.3\dfrac{\|e(t)\|^2 e_1(t)}{\|e(t)\|^2 + e^{-0.2t}}, \\ u_2(t) = -0.6e_2(t) - 0.5\dfrac{\|e(t)\|^2 e_2(t)}{\|e(t)\|^2 + e^{-0.2t}}, \\ u_3(t) = -0.4e_3(t) - 0.4\dfrac{\|e(t)\|^2 e_3(t)}{\|e(t)\|^2 + e^{-0.2t}}, \\ u_4(t) = -0.2e_4(t) - 0.3\dfrac{\|e(t)\|^2 e_4(t)}{\|e(t)\|^2 + e^{-0.2t}}, \\ u_5(t) = -0.5e_5(t) - 0.2\dfrac{\|e(t)\|^2 e_5(t)}{\|e(t)\|^2 + e^{-0.2t}}, \\ u_6(t) = -0.2e_6(t) - 0.5\dfrac{\|e(t)\|^2 e_6(t)}{\|e(t)\|^2 + e^{-0.2t}}. \end{cases} \qquad (43)$$

The other parameters in (42) are defined the same as those in (41). We choose $\sigma = 0.02$, in other words, the convergence rate is $\frac{\sigma}{2} = 0.01$. Through a simple operation based on the above parameters, we have

$$\begin{aligned}\lambda(\Phi_2) =&\{-1.1714, -1.1726, -0.2960, -1.0269, -1.0229, \\ &-0.4269, -0.8963, -0.8943, -0.8722, -0.4963, \\ &-0.4952, -0.7699, -0.7729, -0.7237, -0.5718, \\ &-0.5953, -0.6272, -0.6256\}, \\ \Lambda_2 =& 0.0062 < 1,\end{aligned}$$

which satisfy the conditions (14) and (15).

According to Theorem 2, the systems (41) and (42) reach general decay anti-synchronization under the nonlinear controller (43). The simulation result is shown in Fig. 1.

Example 5.2. Consider the following MWCRDNN with delayed coupling:

$$\begin{aligned}\frac{\partial Y_s(\vartheta,t)}{\partial t} =& - AY_s(\vartheta,t) + D\triangle Y_s(\vartheta,t) + Bg(Y_s(\vartheta,t)) \\ &+ 0.1\sum_{j=1}^{6} G^1_{sj}\Gamma_1 Y_j(\vartheta, t-\tau_1(t)) \\ &+ 0.2\sum_{j=1}^{6} G^2_{sj}\Gamma_2 Y_j(\vartheta, t-\tau_2(t)) \\ &+ 0.3\sum_{j=1}^{6} G^3_{sj}\Gamma_3 Y_j(\vartheta, t-\tau_3(t)), \qquad (44)\end{aligned}$$

where $s = 1, 2, \cdots, 6$, $g_i(\delta) = \frac{|\delta+1|-|\delta-1|}{4}$ $(i = 1, 2, 3)$, $A = \mathrm{diag}(0.4, 0.6, 0.5)$, $D = \mathrm{diag}(0.3, 0.2, 0.4)$, $\Theta = \{\vartheta|-1 < \vartheta < 1\}$, $\Gamma_1 = \mathrm{diag}(0.1, 0.3, 0.2)$, $\Gamma_2 = \mathrm{diag}(0.4, 0.2, 0.2)$, $\Gamma_3 = \mathrm{diag}(0.1, 0.1, 0.2)$, $\tau_1(t) = \frac{1}{20} - \frac{1}{10}e^{-t}$, $\tau_2(t) = \frac{1}{20} - \frac{1}{20}e^{-t}$, $\tau_3(t) = \frac{1}{20} - \frac{1}{30}e^{-t}$, $\tau = \frac{1}{20}$, $\gamma_1 = \frac{1}{10}$, $\gamma_2 = \frac{1}{20}$, $\gamma_3 = \frac{1}{30}$, choose the following matrices as B, G^1, G^2 and G^3 respectively

$$B = \begin{pmatrix} 0.4 & 0.1 & 0.2 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.3 \end{pmatrix},$$

$$G^1 = \begin{pmatrix} -0.6 & 0.1 & 0.2 & 0.2 & 0.1 & 0 \\ 0.1 & -0.7 & 0 & 0.3 & 0.1 & 0.2 \\ 0.2 & 0 & -0.3 & 0 & 0 & 0.1 \\ 0.2 & 0.3 & 0 & -0.7 & 0.2 & 0 \\ 0.1 & 0.1 & 0 & 0.2 & -0.5 & 0.1 \\ 0 & 0.2 & 0.1 & 0 & 0.1 & -0.4 \end{pmatrix},$$

$$G^2 = \begin{pmatrix} -0.4 & 0.1 & 0 & 0.2 & 0.1 & 0 \\ 0.1 & -0.5 & 0.3 & 0.1 & 0 & 0 \\ 0 & 0.3 & -0.6 & 0.1 & 0 & 0.2 \\ 0.2 & 0.1 & 0.1 & -0.5 & 0 & 0.1 \\ 0.1 & 0 & 0 & 0 & -0.2 & 0.1 \\ 0 & 0 & 0.2 & 0.1 & 0.1 & -0.4 \end{pmatrix},$$

$$G^3 = \begin{pmatrix} -0.6 & 0.3 & 0.1 & 0 & 0.2 & 0 \\ 0.3 & -0.8 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & -0.6 & 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0.1 & -0.4 & 0.2 & 0 \\ 0.2 & 0.1 & 0 & 0.2 & -0.7 & 0.2 \\ 0 & 0.1 & 0.2 & 0 & 0.2 & -0.5 \end{pmatrix}.$$

Obviously, $g_i(\cdot)$ $(i = 1, 2, 3)$ satisfies Assumption 1 with $F_i = 0.5$. Consider (44) as the drive system, then its corresponding response system is shown as follows:

$$\begin{aligned}\frac{\partial W_s(\vartheta,t)}{\partial t} =& - AW_s(\vartheta,t) + D\triangle W_s(\vartheta,t) + Bg(W_s(\vartheta,t)) \\ &+ 0.1\sum_{j=1}^{6} G^1_{sj}\Gamma_1 W_j(\vartheta, t-\tau_1(t)) + u_s(\vartheta,t) \\ &+ 0.2\sum_{j=1}^{6} G^2_{sj}\Gamma_2 W_j(\vartheta, t-\tau_2(t)) \\ &+ 0.3\sum_{j=1}^{6} G^3_{sj}\Gamma_3 W_j(\vartheta, t-\tau_3(t)), \qquad (45)\end{aligned}$$

where $s = 1, 2, \cdots, 6$, the parameters in the controller (24) are chosen as follows: $\hat{q} = \mathrm{diag}(0.7, 0.6, 0.5, 0.6, 0.4, 0.2)$, $\hat{\beta} = \mathrm{diag}(0.2, 0.3, 0.6, 0.4, 0.5, 0.3)$ and $\eta(t) = e^{-0.1t}$. Then the nonlinear controller (24) is transferred as fol-

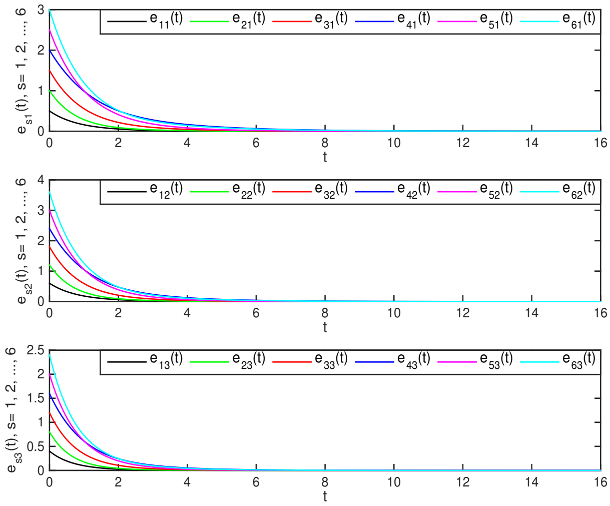

Fig. 1 $e_{sj}(t),\ s=1,2,\cdots,6,\ j=1,2,3.$

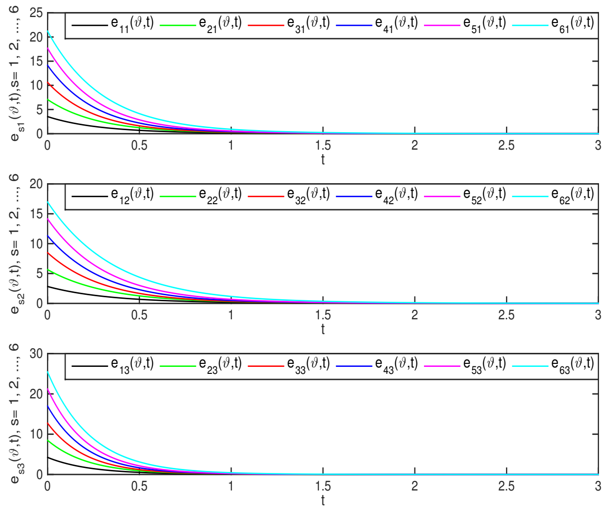

Fig. 2 $e_{sj}(\vartheta,t),\ s=1,2,\cdots,6,\ j=1,2,3.$

lows:

$$\begin{cases} u_1(\vartheta,t) = -0.7e_1(\vartheta,t) - 0.2\dfrac{\|e(\cdot,t)\|^2 e_1(\vartheta,t)}{\|e(\cdot,t)\|^2 + e^{-0.1t}}, \\ u_2(\vartheta,t) = -0.6e_2(\vartheta,t) - 0.3\dfrac{\|e(\cdot,t)\|^2 e_2(\vartheta,t)}{\|e(\cdot,t)\|^2 + e^{-0.1t}}, \\ u_3(\vartheta,t) = -0.5e_3(\vartheta,t) - 0.6\dfrac{\|e(\cdot,t)\|^2 e_3(\vartheta,t)}{\|e(\cdot,t)\|^2 + e^{-0.1t}}, \\ u_4(\vartheta,t) = -0.6e_4(\vartheta,t) - 0.4\dfrac{\|e(\cdot,t)\|^2 e_4(\vartheta,t)}{\|e(\cdot,t)\|^2 + e^{-0.1t}}, \\ u_5(\vartheta,t) = -0.4e_5(\vartheta,t) - 0.5\dfrac{\|e(\cdot,t)\|^2 e_5(\vartheta,t)}{\|e(\cdot,t)\|^2 + e^{-0.1t}}, \\ u_6(\vartheta,t) = -0.2e_6(\vartheta,t) - 0.3\dfrac{\|e(\cdot,t)\|^2 e_6(\vartheta,t)}{\|e(\cdot,t)\|^2 + e^{-0.1t}}. \end{cases} \tag{46}$$

The other parameters in (45) are defined the same as those in (44). We choose $\sigma = 0.2$, in other words, the convergence rate is $\frac{\sigma}{2} = 0.1$. Through a simple operation based on the above parameters, we have

$$\begin{aligned} \lambda(\Phi_4) =& \{-0.3649, -0.5621, -0.6401, -0.7607, -0.7647, \\ & -0.7676, -1.2384, -0.8634, -1.1396, -1.1611, \\ & -1.0600, -1.0414, -1.0343, -1.0380, -0.9584, \\ & -0.9624, -0.9607, -0.9606\}, \\ \Lambda_4 =& 0.0732 < 1, \end{aligned}$$

which satisfy the conditions (34) and (35).

According to Theorem 4, the systems (44) and (45) reach general decay anti-synchronization under the nonlinear controller (46). The simulation result is displayed in Fig. 2.

6 Conclusion

This paper has proposed the concept of general decay anti-synchronization by introducing ψ-type stability and ψ-type function. Then, the decay anti-synchronization for MWCNNs and MWCRDNNs has been investigated, respectively. By utilizing Lyapunov functional approach, some inequality techniques, as well as the construction of nonlinear controllers, several novel criteria for ensuring decay anti-synchronization for these two networks have been obtained. Additionally, we also have considered the decay anti-synchronization for these networks with delayed coupling. Finally, two examples have been given to verify the correctness and validity for our results.

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Conflict of Interest The authors declare that they have no conflict of interest.

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