# Use of the Padé Approximant in Solution to a Model of Vortex Shedding

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## Introduction

The diffusive form of the Van der Pol Oscillator equation

$$\frac{\partial^2\varphi}{\partial t^2}+\varepsilon\Omega(z)(\varphi^2-1)\frac{\partial\varphi}{\partial t}+\Omega^2(z)\varphi-\nu\frac{\partial^3\varphi}{\partial t\partial z^2}=0$$

may be used to model near-wake dynamics of slender bluff bodies. This model is achieved by a continuous distribution of the equation across the spanwise direction. Here, the Padé Approximant is used to obtain an algebraic approximation for this model. This result is then compared to the equivalent Taylor Series Expansion, and the numerical solution returned by the Classical $4^{th}$ Order Runge-Kutta Method.

## Method

### The Van der Pol (VDP) Oscillator

When considered at a single point along the spanwise direction, the diffusive form of the Van der Pol Oscillator reduces to the Classical Van der Pol Oscillator. Arbitrary values are imposed for $0^{th}$, $1^{st}$, and $2^{nd}$ derivatives.

$$\frac{d^2\varphi}{dt^2}+\varepsilon(\varphi^2-1)\frac{d\varphi}{dt}+\varphi=0$$

For the results presented, $\varepsilon=1$.

### Classical Solution to the VDP Porblem

The time variable, $t$, is replaced by the strained coordinate $\tau=\omega t$ where $\omega$ is an asymptotic expansion of the frequency correction. Furthermore, the function $\varphi(t,\varepsilon)$ is replaced by an asymptotic expansion in the parameter $\varepsilon$, $\varphi(t,\varepsilon)=\phi_0(\tau)+\varepsilon\phi_1(\tau)+\varepsilon^2\phi_2(\tau)+\cdots$.

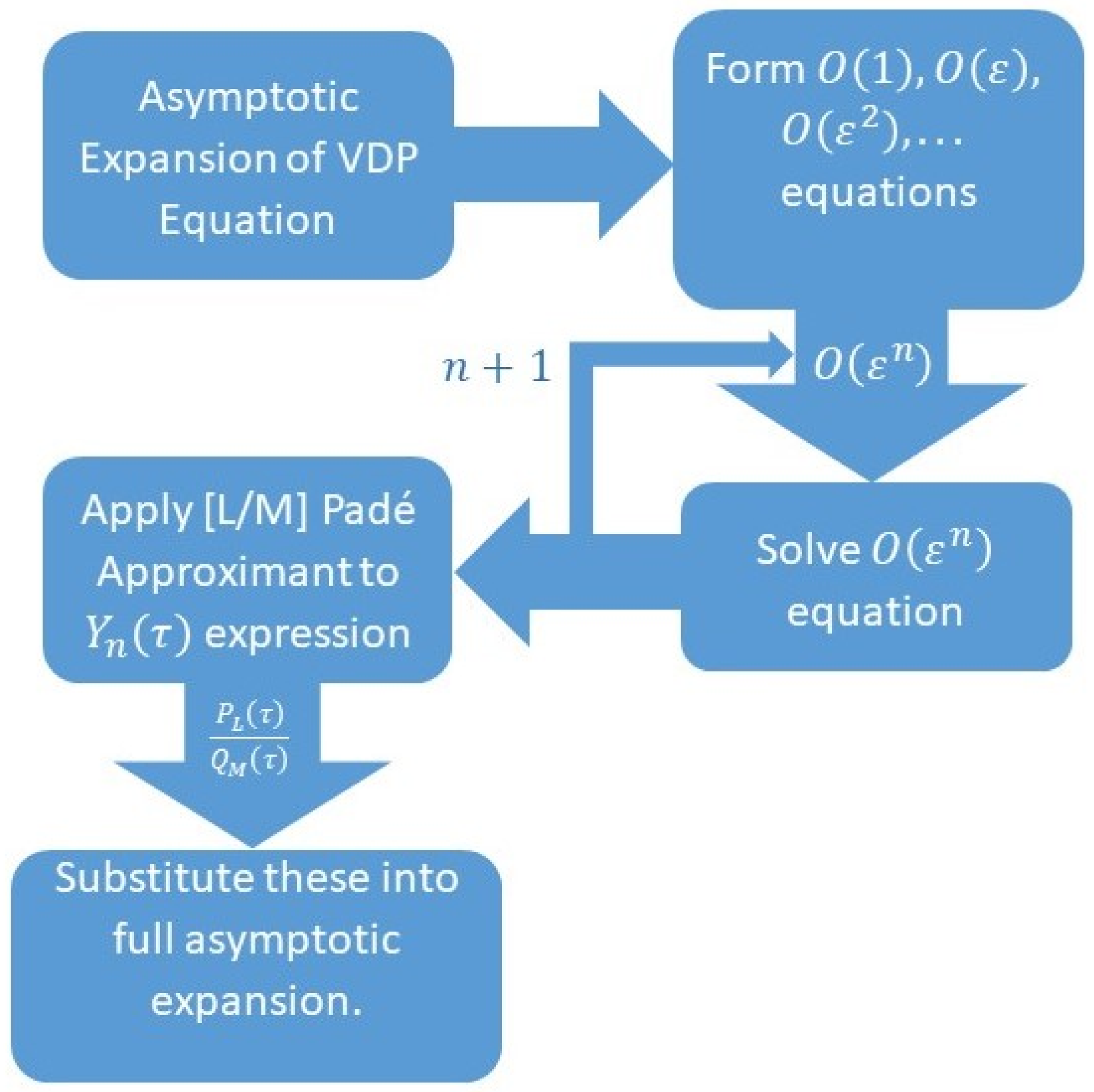
Substituting these values into the original function, the $O(\varepsilon^n)$ equation is solved to give rise to the solution of the $O(\varepsilon^{n+1})$. This gives expressions for $\phi_n(\tau)$ and $\phi_{n+1}(\tau)$ respectively.

### Applying the Padé Approximant at Each Order

For the Padé Approximant to be applied, a power series expansion must first be found for each $\phi_n(\tau)$. If the $N^{th}$ order expansion is of the form $\phi_n(\tau)\approx\sum_{k=0}^{N}c_k\tau^k$, then the coefficients of the Padé Approximation $\frac{P_L(\tau)}{Q_M(\tau)}$ are found by solving the following equation.

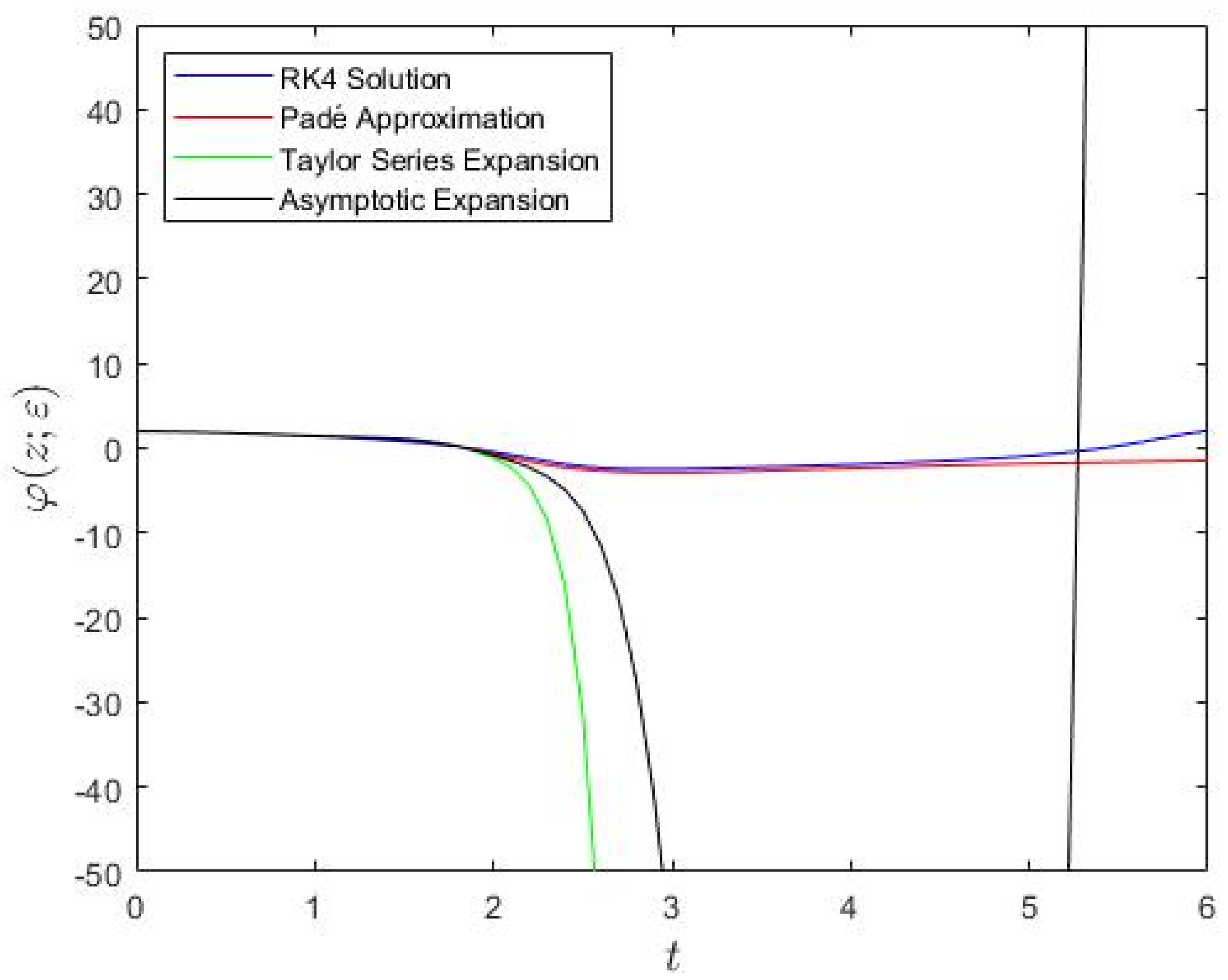
$$-\begin{bmatrix} c_0\\ c_1\\ c_2\\ \vdots\\ c_{L-1}\\ c_L\\ c_{L+1}\\ \vdots\\ c_{L+M}\\ c_{L+M+1}\end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots\cdots & 0\\ c_0 & 0 & 0 & \cdots & \cdots & \ddots & -1 & 0\ \cdots & 0\\ c_1 & c_0 & 0 & \cdots & \cdots & \cdots & \ddots & -1\ \cdots & 0\\ c_2 & c_1 & c_0 & \ddots & \cdots & \cdots\cdots & \cdots & \ddots\ \ddots & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots\cdots & \cdots & \cdots\ \ddots & -1\\ c_{L-1} & c_{L-2} & \cdots\cdots & c_0 & 0 & \cdots & \cdots & \cdots & 0\\ c_L & c_{L-1} & \cdots\cdots & c_1 & 0 & \ddots & \cdots & \cdots & 0\\ c_{L+1} & c_L & \cdots\cdots & c_2 & 0 & \cdots & \ddots & \cdots & 0\\ \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ c_{L+M} & c_{L+M-1} & \cdots\cdots & c_{L-1} & 0 & \cdots & \cdots & \cdots & 0\end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3\\ \vdots\\ q_M\\ p_0\\ p_1\\ p_2\\ \vdots\\ p_L\end{bmatrix}$$

Once the coefficients are found, the approximation for each $\phi$-term may be constructed. These are then inserted into the asymptotic expansion of $\varphi(t,\varepsilon)$ to obtain the approximation of the function as a collection of Padé Approximations arranged in an asymptotic expansion. This process is summarised by the following chart.




Algorithm for applying the Padé Approximant in the VDP Oscillator problem.

## Results




As can be seen, the VDP Oscillator models the formation (and dissipation) of vortices in the near-wake region as sinusoidal (although influenced by secularity). The Padé Approximant - as descibred - is a more accurate representation of the target function (represented by the RK4 solution) than either the corresponding Taylor Series or the Asymptotic Expansion used.

## Conclusions

- As $\varepsilon\to 0$, the patterns associated with vortex shedding become increasingly sinusoidal for a given spanwise position;
- The Padé Approximant can be used to give a more accurate representation than the other techniques presented (excluding the numerical method);
- Although the selection of $\varepsilon=1$ undermines the accuracy of the asymptotic expansion, the resulting Padé Approximant **is not** similarly affected;
- The accuracy is improved as secularity tends towards being insignificant (for all techniques); although,
- The advantages of the Padé Approximant are more clearly demonstrated when secularity is a notable factor.