Riordan graphs I: Structural properties*

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Abstract

In this paper, we use the theory of Riordan matrices to introduce the notion of a Riordan graph. The Riordan graphs are a far-reaching generalization of the well known and well studied Pascal graphs and Toeplitz graphs, and also some other families of graphs. The Riordan graphs are proved to have a number of interesting (fractal) properties, which can be useful in creating computer networks with certain desirable features, or in obtaining useful information when designing algorithms to compute values of graph invariants. The main focus in this paper is the study of structural properties of families of Riordan graphs obtained from infinite Riordan graphs, which includes a fundamental decomposition theorem and certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle. We will study spectral properties of the Riordan graphs in a follow up paper.

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1 Introduction

Pascal's triangle is a classical combinatorial object, and its roots can be traced back to the 2nd century BC. In 1991, Shapiro, Getu, Woan and Woodson [31] have introduced the notion of a *Riordan array*, also known as a *Riordan matrix*, in order to define a class of infinite lower triangular matrices with properties analogous to those of the Pascal triangle

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(matrices) [14]:

$$P = \left[\binom{i}{j}\right]_{i,j\geq 0} = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & O & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1}$$

The idea here is to express each column of the Pascal matrix in terms of generating functions. Since the jth column generating function is

$$\sum_{i=0}^{\infty} \binom{i}{j} t^i = \frac{t^j}{(1-t)^{j+1}} = \frac{1}{1-t}\left(\frac{t}{1-t}\right)^j, \ j \geq 0,$$

we move from one column to the next by multiplying by $\frac{t}{1-t}$. It is convenient to start our numbering at 0 so that the leftmost column is the 0th column. In general, we start with $g(t) = g_0 + g_1 t + g_2 t^2 + \cdots$ as the 0th column generating function and then multiply by $f(t) = f_1 t + f_2 t^2 + \cdots$ so that the next column has the generating function $g(t)f(t) = g_0 f_1 t + (g_0 f_2 + g_1 f_1)t^2 + \cdots$. Continuing, we multiply by $f(t)$ again to obtain the generating function $g(t)f(t)^2$ for the 2nd column, and so on. We usually abbreviate $g(t)$ by g, $f(t)$ by f, etc, and the resulting matrix is called the Riordan matrix (g, f). Thus, the generating function for the jth column is gf^j for $j = 0, 1, 2, \ldots$. In the Pascal matrix we have $g = \frac{1}{1-t}$ and $f = \frac{t}{1-t}$.

Since their introduction, Riordan matrices became an active subject of research; see [5, 6, 20, 21, 25, 28, 30, 34] and references there in, for examples of results in this direction. In particular, Riordan matrices found applications in the context of the computation of combinatorial sums [34]. Also, see [10] for a recent paper about Lie theory on the Riordan group, which is the set of invertible Riordan matrices.

The notion of Pascal's triangle was also influential in graph theory and its applications to computer networks. Indeed, in 1983, Deo and Quinn [11] have introduced the *Pascal graphs* in their searching for a class of optimal graphs for computer networks with certain desirable properties, such as

- the design is to be simple and recursive;
- there must be a universal vertex, i.e. a vertex adjacent to all others;
- there must exist several paths between each pair of vertices.

The adjacency matrix of the Pascal graph with n vertices is denoted by PG_n and is defined to be an $n \times n$ symmetric $(0, 1)$-matrix whose main diagonal entries are all 0s and the lower triangular part of the matrix consists of the first $n - 1$ rows of Pascal's triangle modulo 2. These graphs have attracted much attention in the literature (see [3, 9, 13] and references there in). For example, the Pascal graph with 6 vertices is given by

$$\mathcal{A}(PG_6) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

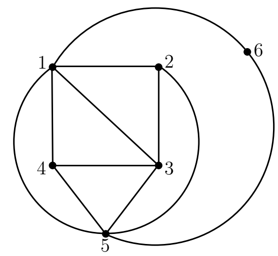


Fig. 1: The Pascal graph PG_6 with 6 vertices

Another important object of interest to us is the well studied notion of a *Toeplitz graph* with n vertices, which is denoted by TG_n (see [15, 16, 17, 18, 19, 24, 29] and references there in for results on Toeplitz graphs). The adjacency matrix $\mathcal{A}(TG_n) = [t_{ij}]$ is an $n \times n$ symmetric $(0,1)$-Toeplitz matrix defined by

$$t_{ij} = \begin{cases} 0 & \text{if } i = j \\ a_{|i-j|} \in \{0,1\} & \text{if } i \neq j. \end{cases}$$

Using Riordan language, $\mathcal{A}(TG_n)$ can be expressed as a Riordan matrix of the Appell type, (g,t) modulo 2 where $g = \sum_{n\geq 0} a_n t^n \in \mathbb{Z}_2[[t]]$. See Section 2.5 for more information about Toeplitz graphs. In this context, if $g = t^{t_1-1} + t^{t_2-1} + \cdots + t^{t_k-1}$, where $1 \leq t_1 < t_2 < \cdots < t_k < n$, then TG_n is denoted by $T_n\langle t_1, t_2, \ldots, t_k\rangle$. For example, the Toeplitz graph $T_6\langle 1,3,5\rangle$ is the bipartite graph given by

$$\mathcal{A}(TG_6) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

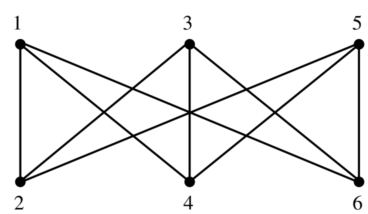


Fig. 2: The Toeplitz graph TG_6 with 6 vertices

In this paper we introduce the notion of a *Riordan graph*. This notion not only provides a far-reaching generalizaiton of the notions of a Pascal graph and a Toeplitz graph, but also extends the theory of Riordan matrices to the domain of graph theory, similarly to the introduction of the Pascal graphs based on Pascal's triangles. The Riordan graphs are proved to have a number of interesting (fractal) properties, which can be useful in creating computer networks with certain desired features, or in obtaining useful information when designing algorithms to compute values of graph invariants. Indeed, the Pascal graphs are just one instance of a family of Riordan graphs having a universal vertex and a simple recursive structure. Thus, other members of that family could be used instead of the Pascal graphs in designing computer networks, and depending on the context, these could be a better choice than the Pascal graphs.

Our basic idea here is in building the infinite adjacency matrix $\mathcal{A}$ based on an infinite Riordan matrix over $\mathbb{Z}$ modulo 2, and considering the leading principal submatrices giving (finite) Riordan graphs. See Section 2.3 for the precise definitions. We introduce various families of Riordan graphs based on the choice of the generating functions defining these graphs via Riordan matrices. For example, Riordan graphs of the *Appell type* are precisely the class of Toeplitz graphs including the complete graphs K_n and bipartite graphs, while Riordan graphs of the *Bell type* include Pascal graphs, *Catalan graphs* and *Motzkin graphs*.

One of the basic questions in our context is whether or not a given labeled or unlabeled graph is a Riordan graph. It turns out that all unlabeled graphs on at most four vertices are Riordan graphs (see Theorem 2.12), while non-Riordan unlabeled graphs always exist for larger graphs on any number of vertices. However, the main focus in this paper is the study of *labeled* Riordan graphs, and we give structural properties of certain families of graphs obtained from infinite Riordan graphs.

Throughout this paper, we normally label graphs on n vertices by the elements of the n-set $[n] := \{1, 2, \ldots, n\}$. However, we also meet graphs labeled by odd numbers, or even numbers, or consecutive subintervals in $[n]$. For two isomorphic graphs, G and H, we write $G \cong H$. For a graph G, V (resp., E) denotes the set of vertices (resp., edges) in G. Also, for a subset S of vertices V, we let $\langle S \rangle$ denote the graph induced by the vertices in $S \subset V$. Finally, we let $\mathbb{N}_0 = \{0, 1, \ldots\}$.

This paper is organized as follows. In Section 2 we review the notion of a Riordan matrix and use it to introduce the notion of an (infinite) Riordan graph in the *labeled* and *unlabeled* cases. However, the main focus in this paper is the *labeled* case, so unless we use the word "unlabeled" explicitely, our Riordan graphs are *labeled*. A number of basic results on Riordan graphs are established in Section 2.3, and this includes the number of Riordan graphs on n vertices (see Proposition 2.8), and the necessary conditions on Riordan graphs (see Theorem 2.11). In Section 2.4 we define the *R-product* $\otimes_R$ of two Riordan graphs and then give its combinatorial interpretation in terms of directed walks in certain graphs (see Theorem 2.20). We also discuss the *ring sum* $\oplus$ of two graphs in Section 2.4 that can be used to define certain classes of Riordan graphs in Section 2.5. These include, but are not limited to Riordan graphs of the *Appell type*, *Bell type*, *Lagrange type*, *checkerboard type*, *derivative type*, and *hitting time type*.

In Section 3 we give structural results applicable to any Riordan graphs. In particular, in Section 3.2 we show that every Riordan graph is a fractal (see Theorem 3.11). Also, the *reverse relabelling* of proper Riordan graphs is defined and studied in Section 3.3. Further, in Section 3.1 we prove the *Riordan Graph Decompostion* theorem (see Theorem 3.4). In addition, in Section 3.4 we give certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle.

In Section 4.1, we consider *io-decomposable* and *ie-decomposable* proper Riordan graphs, and provide a characterization result for these graphs (see Theorem 4.2). One of the main focuses in this paper is the study of Riordan graphs of the *Bell type* conducted in Section 4.2. In particular, we provide two characterization results for io-decomposable Riordan graphs of the Bell type (see Lemma 4.4 and Theorem 4.6) and use the results to show that the *Pascal graphs* and *Catalan graphs* are io-decomposable, while the *Motzkin graphs* are not io-decomposable. Also, in Section 4.2 we study basic graph invariants of io-decomposable Riordan graphs of the Bell type: number of edges, number of universal vertices, clique number, chromatic number, diameter, and others. In Section 4.3 we provide two characterization results (Lemma 4.25 and Theorem 4.26) for ie-decomposable Riordan graphs of the *derivative type*. Finally, in Section 5 we provide concluding remarks and state directions for further research.

We study spectral properties of the Riordan graphs in the follow up paper [8].

2 From Riordan matrices to Riordan graphs

After briefly reviewing the notion of a *Riordan matrix* in the Riordan group, we will introduce the notion of a *Riordan graph*. Then, in Section 2.5 we introduce various families of Riordan graphs.

2.1 A brief introduction to Riordan group

Consider $\kappa[[t]]$ the ring of formal power series $g = \sum_{n\geq 0} g_n t^n$ in the variable t over an integral domain κ. An important operation on formal power series is coefficient extraction, $[t^n]g = g_n$.

Let $L = [\ell_{ij}]_{i,j\geq 0}$ be an infinite matrix over κ. If there exists a pair of generating functions $(g, f) \in \kappa[[t]] \times \kappa[[t]]$, $f(0) = 0$ such that

$$gf^j = \sum_{i\geq 0} \ell_{ij} t^i,\ j \geq 0 \ \text{ or equivalently } \ \ell_{ij} = [t^i]gf^j$$

then the matrix L is called a *Riordan matrix* (or, a *Riordan array*) over κ generated by g and f. Usually, we write $L = (g(t), f(t))$ or (g, f). Since $f(0) = 0$, every Riordan matrix (g, f) is an infinite lower triangular matrix. If a Riordan matrix is invertible, it is called *proper*. Note that (g, f) is invertible if and only if $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$, where f' denotes the first derivative of f.

It is well known [27] that an infinite lower triangular matrix $L = [\ell_{i,j}]_{i,j\geq 0}$ with $\ell_{0,0} \neq 0$ is a proper Riordan matrix given by $L = (g, f)$ if and only if there exists a unique pair of sequences $(a_n)_{n\geq 0}$ with $a_0 \neq 0$ and $(z_n)_{n\geq 0}$ such that for all $i \geq j \geq 0$,

$$\begin{aligned} \ell_{i+1,j+1} &= a_0\ell_{i,j} + a_1\ell_{i,j+1} + \cdots + a_{i-j}\ell_{i,i}; \\ \ell_{i+1,0} &= z_0\ell_{i,0} + z_1\ell_{i,1} + \cdots + z_{i-j}\ell_{i,i}. \end{aligned} \tag{2}$$

Equivalently, there exists a unique pair of generating functions $A = \sum_{n\geq 0} a_n t^n$ and $Z = \sum_{n\geq 0} z_n t^n$ such that

$$g = \frac{1}{1 - tZ(f)}, \quad \text{and} \quad f = tA(f) \text{ or } A = t/\bar{f}.$$

The sequences $(a_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are called the A-sequence and the Z-sequence of the Riordan matrix L, respectively. For example, the Pascal matrix $P = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ in (1) has the A-sequence $(1, 1, 0, \ldots)$ and the Z-sequence $(1, 0, \ldots)$.

Theorem 2.1 [31] *Consider a matrix equation $Bx = c$ where $x = (x_0, x_1, \ldots)^T$. If B is a Riordan matrix (g, f), then the resulting column vector $c = (c_0, c_1, \ldots)^T$ has the generating function*

$$c(t) = \sum_{n\geq 0} c_n t^n = g(t)(x \circ f)(t) = gx(f)$$

where $x(t) = x_0 + x_1 t + x_2 t^2 + \cdots$ and $\circ$ is the composition operator.

This property is called the *fundamental theorem for Riordan matrices* (FTRM) and we write it simply as $(g, f)x = gx(f)$ in terms of generating functions. This leads to the multiplication of Riordan matrices as

$$(g, f) * (h, \ell) = (gh(f), \ell(f)).$$

The set of all proper Riordan matrices under the above *Riordan multiplication* in terms of generating functions forms a group called the *Riordan group* and denoted $(\mathcal{R}, *)$ where $*$ is the usual matrix multiplication. The identity of the group is $(1, t)$, the usual identity matrix and $(g, f)^{-1} = (1/g(\overline{f}), \overline{f})$ where $\overline{f}$ is the *compositional inverse* of f, i.e. $\overline{f}(f(t)) = f(\overline{f}(t)) = t$. Let

$$\begin{aligned}\mathcal{F}_0 &= \{g \in \kappa[[t]] | g(0) \neq 0\},\\ \mathcal{F}_1 &= \{f \in \kappa[[t]] | f(0) = 0, f'(0) \neq 0\} = t\mathcal{F}_0.\end{aligned}$$

Then $(\mathcal{F}_0, \cdot)$ and $(\mathcal{F}_1, \circ)$ form groups under the convolution $\cdot$ and the composition $\circ$ of two formal power series, respectively. Moreover, it can be easily shown that the Riordan group $\mathcal{R}$ is isomorphic to the semidirect product $\rtimes$ of $\mathcal{F}_0$ and $\mathcal{F}_1$:

$$\mathcal{R} \cong \mathcal{F}_0 \rtimes \mathcal{F}_1.$$

It is also well known [20, 25, 30] that there are several important subgroups of the Riordan group. These subgroups will be used for classifying the types of Riordan graphs in Section 2.5.

Appell subgroup. A Riordan matrix of the form

$$(g, t) = \left(\begin{array}{ccccc} g_0 & & & & \\ g_1 & g_0 & & O & \\ g_2 & g_1 & g_0 & & \\ g_3 & g_2 & g_1 & g_0 & \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{array}\right)$$

is called an *Appell matrix* which is a Toeplitz matrix. The set of all invertible Appell matrices forms a normal subgroup of the Riordan group which is called the Appell subgroup:

$$\mathcal{A} := \{(g, t) \in \mathcal{R} \mid g \in \mathcal{F}_0\} \cong \mathcal{F}_0.$$

Lagrange subgroup. A Riordan matrix of the form $(1, f)$ is called a *Lagrange matrix*. The set of all invertible Lagrange matrices forms a subgroup of the Riordan group which is called the Lagrange subgroup:

$$\mathcal{L} := \{(1, f) \in \mathcal{R} \mid f \in \mathcal{F}_1\} \cong \mathcal{F}_1.$$

Lagrange matrices are sometimes called the *associated matrices*.

Using Riordan multiplication we can see that every Riordan matrix (g, f) can be expressed as the product of the Appell matrix (g, t) and the Lagrange matrix $(1, f)$. Since $\mathcal{A} \cap \mathcal{L} = \{(1, t)\}$ it proves that the Riordan group is the semidirect product of $\mathcal{A}$ and $\mathcal{L}$:

$$\mathcal{R} = \{(g, f) = (g, t)(1, f) \in \mathcal{AL}\} = \mathcal{A} \rtimes \mathcal{L}.$$

Bell subgroup. A Riordan matrix of the form (g, tg) or $(f/t, f)$ is called a *Bell matrix*. The set of all invertible Bell matrices forms a subgroup of the Riordan group which is called the Bell subgroup:

$$\mathcal{B} := \{(g, tg) \in \mathcal{R} \mid g \in \mathcal{F}_0\}.$$

Bell matrices are frequently arising in combinatorics. For example,

- Pascal matrix $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ (see (1) for the matrix expression);
- Catalan matrix $(C, tC) = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & O & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $C = \frac{1-\sqrt{1-4t}}{2t}$;
- Motzkin matrix $(M, tM) = \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & O & \\ 2 & 2 & 1 & & & \\ 4 & 5 & 3 & 1 & & \\ 9 & 12 & 9 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $M = \frac{1-t-\sqrt{1-2t-3t^2}}{2t^2}$.

Checkerboard subgroup. A Riordan matrix of the form (g, f) for an even function g and an odd function f is called a *checkerboard matrix*. The set of all invertible checkerboard matrices forms a subgroup of the Riordan group which is called the checkerboard subgroup. The centralizer of the element $(1, -t)$ of the Riordan group is the checkerboard subgroup.

Derivative subgroup. A Riordan matrix of the form (f', f) is called a *derivative matrix*. The set of all invertible derivative matrices forms a subgroup of the Riordan group which is called the derivative subgroup.

Hitting time subgroup. A Riordan matrix of the form $(tf'/f, f)$ is called a *hitting time matrix*. The set of all invertible hitting time matrices forms a subgroup of the Riordan group which is called the hitting time subgroup.

2.2 (0,1)-Riordan matrices

In this section, we introduce the notion of a $(0,1)$-Riordan matrix of order n, which will play an important role in defining Riordan graphs with n vertices in the next section.

First note [10] that the Riordan group $\mathcal{R}$ over $\kappa = \mathbb{R}$ or $\mathbb{C}$ can be described as an inverse limit of an inverse sequence of groups $(\mathcal{R}_n)_{n\in\mathbb{N}}$ of *finite* matrices. So $\mathcal{R}_n$ is a subgroup of the classical Lie group $GL(n,\kappa)$, and elements in $\mathcal{R}_n$ are $n\times n$ Riordan matrices denoted by $(g,f)_n$ which are obtained from (infinite) Riordan matrices (g,f) by taking their leading principal submatrix of order n.

Now consider Riordan matrices over the finite field $\kappa = \mathbb{Z}_2$. We call a Riordan matrix over $\mathbb{Z}_2$ a *binary Riordan matrix* and denote it as $\mathcal{B}(g,f) = [b_{ij}]$. So, $\mathcal{B}(g,f)$ is the $(0,1)$-matrix obtained from a Riordan matrix (g,f) over $\mathbb{Z}$, where $g, f \in \mathbb{Z}_2[[t]]$ or $\mathbb{Z}[[t]]$ with $f(0) = 1$, by taking the coefficients modulo 2, i.e.

$$b_{ij} \equiv [t^i]gf^j \pmod 2 \ \text{ or equivalently } \ \mathcal{B}(g,f) \equiv (g,f) \pmod 2.$$

From now on, we simply write $a \equiv b$ for $a \equiv b \pmod 2$. Also, for $f, g \in \mathbb{Z}[[t]]$, $f \equiv g$ means $[t^n]f \equiv [t^n]g \pmod 2$ for all $n \geq 0$. Since many properties for Riordan matrices are valid for any integral domain κ, we state, without proof, some properties for Riordan matrices over $\mathbb{Z}_2$, which will be useful in this paper.

It is obvious that $\mathcal{B}(g,f)$ is invertible if and only if $g(0) = 1$, $f(0) = 0$ and $f'(0) = 1$. By the properties of the modulo operation, it follows from (2) that an infinite $(0,1)$-lower triangular matrix $B = [b_{i,j}]_{i,j\geq 0}$ with $b_{0,0} = 1$ is an invertible Riordan matrix given by $B = \mathcal{B}(g,f)$ if and only if there exists a unique pair of $(0,1)$-sequences called the *binary A-sequence* $(1, \tilde{a}_1, \tilde{a}_2, \ldots)$ and the *binary Z-sequence* $(\tilde{z}_0, \tilde{z}_1, \ldots)$ such that for all $i \geq j \geq 0$,

$$\begin{aligned} b_{i+1,j+1} &\equiv b_{i,j} + \tilde{a}_1 b_{i,j+1} + \cdots + \tilde{a}_{i-j} b_{i,i}; \\ b_{i+1,0} &\equiv \tilde{z}_0 b_{i,0} + \tilde{z}_1 b_{i,1} + \cdots + \tilde{z}_{i-j} b_{i,i}. \end{aligned}$$

Thus we have the following theorem.

Theorem 2.2 *Let (g,f) be a proper Riordan matrix over $\mathbb{Z}$ with the A-sequence $(a_n)_{n\geq 0}$, $a_0 = 1$ and the Z-sequence $(z_n)_{n\geq 0}$ where $g(0) = 1$. Then, $(g,f) \equiv \mathcal{B}(g,f)$ if and only if $a_n \equiv \tilde{a}_n$ and $z_n \equiv \tilde{z}_n$ for all $n \geq 0$.*

Applying the fundamental theorem for Riordan matrices over $\mathbb{Z}_2$ yields

$$\mathcal{B}(g,f)h \equiv gh(f) \tag{3}$$

where $h \in \mathbb{Z}_2[[t]]$. By this property it can be shown easily that the set of all invertible binary Riordan matrices forms a group under the binary operation

$$\mathcal{B}(g,f)\mathcal{B}(h,\ell) \equiv \mathcal{B}(gh(f), \ell(f)).$$

Now, let $\mathcal{B}(g,f)_n$ be the $n\times n$ $(0,1)$-Riordan matrix obtained from the leading principal submatrix of order n in $\mathcal{B}(g,f)$. The group of $n\times n$ invertible $(0,1)$-Riordan matrices will be denoted by $\mathcal{R}_n(\mathbb{Z}_2)$. If $\kappa=\mathbb{R}$ or $\mathbb{C}$ then the Riordan group $\mathcal{R}_n$ over κ has infinite order. However, the Riordan group $\mathcal{R}_n(\mathbb{Z}_2)$ is a finite group.

Example 2.3 Consider all 3×3 Riordan matrices over $\mathbb{Z}_2$: if $g=g_0+g_1t+g_2t^2$ and $f=f_1t+f_2t^2\in\mathbb{Z}_2[[t]]$ then

$$\mathcal{B}(g,f)_3=\begin{pmatrix} g_0 & 0 & 0\\ g_1 & g_0f_1 & 0\\ g_2 & g_0f_2+g_1f_1 & g_0f_1^2\end{pmatrix}.$$

It can be easily shown that there are 22 binary Riordan matrices of order 3 and exactly eight of them are invertible. Note that $g_0=f_1=1$ in this case:

$$\mathcal{B}(1,t)_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \mathcal{B}(1,t+t^2)_3=\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix},\ \mathcal{B}(1+t^2,t)_3=\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix},$$

$$\mathcal{B}(1+t^2,t+t^2)_3=\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix},\ \mathcal{B}(1+t,t)_3=\begin{pmatrix}1&0&0\\1&1&0\\0&1&1\end{pmatrix},\ \mathcal{B}(1+t,t+t^2)_3=\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix},$$

$$\mathcal{B}(1+t+t^2,t)_3=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix},\ \mathcal{B}(1+t+t^2,t+t^2)_3=\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}.$$

Thus, $\mathcal{R}_3(\mathbb{Z}_2)$ is an eight-element group, and it is isomorphic to the Dihedral group D_4 (also see [5]).

2.3 Riordan graphs

The following definition gives the notion of a Riordan graph in both *labeled* and *unlabeled* cases. For an $n\times n$ matrix A, let $A(i|j)$ denote the $(n-1)\times(n-1)$ matrix obtained from A by deleting ith row and jth column.

Definition 2.4 A simple *labeled* graph G with n vertices is a *Riordan graph* of order n if its adjacency matrix $\mathcal{A}(G)$ is an $n\times n$ symmetric $(0,1)$-matrix expressed by $\mathcal{A}(G)=B+B^T$ for some binary Riordan matrix $B=\mathcal{B}(tg,f)_n$, i.e. B is an $n\times n$ $(0,1)$-lower triangular matrix whose main diagonal entries are all zeros such that

$$B(1|n)=\mathcal{B}(g,f)_{n-1}.$$

We call the graph G a Riordan graph corresponding to the Riordan matrix (g,f) and we denote such a Riordan graph G by $G_n(g,f)$, or simply by G_n when the matrix (g,f) is understood from the context, or it is not important.

A simple *unlabeled* graph is a *Riordan graph* if at least one of its labeled copies is a Riordan graph.

Since the Riordan matrix $(g, f) = [r_{ij}]_{i,j\geq 0}$ is indexed from $i = 0$ and $j = 0$, if $\mathcal{A}(G_n) = [a_{ij}]_{1\leq i,j\leq n}$ where $a_{ii} = 0$, then for $i > j \geq 1$,

$$a_{ij} \equiv [t^{i-1}]tgf^{j-1} \equiv [t^{i-2}]gf^{j-1} = r_{i-2,j-1}. \tag{4}$$

Throughout this paper, we assume that every graph is a simple graph with vertex set $V = [n]$ and edge set E.

Example 2.5 (Labeled Riordan graphs)

(i) The Pascal graph PG_6 in Fig. 1 is the Riordan graph with 6 vertices given by the matrix $\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$.

(ii) The Toeplitz graph TG_6 in Fig. 2 is the Riordan graph with 6 vertices given by the matrix $(1 + t^2 + t^4, t)$.

(iii) The Riordan graph with n vertices given by the Catalan matrix (C, tC) in Section 2.1 is called the *Catalan graph* CG_n. See Fig. 3 for CG_6.

$$\mathcal{A}(CG_6) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

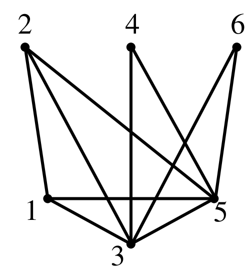


Fig. 3: The Catalan graph CG_6 with 6 vertices

Example 2.6 (Unlabeled graphs)

(i) Let us determine whether the prism graph Pr_6 in Fig. 4 is a Riordan graph.

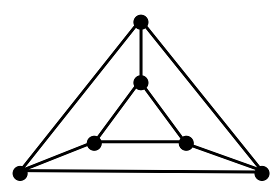


Fig. 4: The prism graph Pr_6 with 6 vertices

There is a vertex labeling whose adjacency matrix is $\mathcal{A}(Pr_6) = B + B^T$ where $B = \mathcal{B}(t^2 + t^3 + t^4, t)_6$ is a Toeplitz matrix of order 6 as shown in Fig. 5. Thus, Pr_6 is a 3-regular Riordan graph given by the Riordan matrix $(t + t^2 + t^3, t)$; see Fig. 5.

$$\mathcal{A}(Pr_6) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix},$$

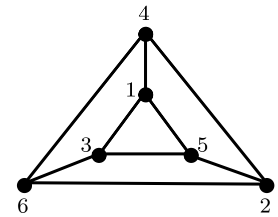


Fig. 5: A labeled prism graph Pr_6 with 6 vertices

(ii) The graph in Fig. 6 with 5 vertices is not a Riordan graph because there is no possible vertex labeling of the graph whose adjacency matrix can be expressed as a binary Riordan matrix (see Proposition 2.13).

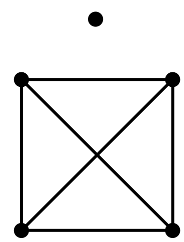


Fig. 6: A non-Riordan graph with 5 vertices

Definition 2.7 A Riordan graph $G_n(g, f)$ is *proper* if the binary Riordan matrix $\mathcal{B}(g, f)_{n-1}$ is proper.

If a Riordan graph $G = G_n(g, f)$ is proper then obviously G is a connected graph, and the Riordan matrix (g, f) is also proper because $g(0) \equiv f'(0) \equiv 1$. The converse to this statement is not true. For instance, $(1, 2t + t^2)$ is a proper Riordan matrix but $G_n(1, 2t + t^2)$ is not a proper Riordan graph.

For any Riordan graph $G_n = G_n(g, f)$, we can think of the sequence of induced subgraphs

$$G_1 = \langle\{1\}\rangle\,,\; G_2 = \langle\{1, 2\}\rangle\,, \ldots, G_{n-1} = \langle\{1, 2, \ldots, n-1\}\rangle\,,$$

each one defined by the same pair of functions, showing the recursive nature of Riordan graphs. From applications point of view, this property implies that when a new node is added to a network, the entire network does not have to be reconfigured.

Proposition 2.8 *The number of Riordan graphs of order* $n \geq 1$ *is* $(4^{n-1} + 2)/3$.

Proof. Let $G = G_n(g, f)$ be a labeled Riordan graph and i be the smallest index such that $g_i = [t^i]g \equiv 1$.

- If $i \geq n - 1$ then G is the null graph N_n.
- If $0 \leq i \leq n - 2$ then we may assume that $g = t^i + g_{i+1}t^{i+1} + \cdots + g_{n-2}t^{n-2} \in \mathbb{Z}_2[t]$ and $f = f_1 t + f_2 t^2 + \cdots + f_{n-i-2}t^{n-i-2} \in \mathbb{Z}_2[t]$.

It follows that the number of possibilities of the coefficients in $\{0,1\}$ to create G is

$$1+\sum_{i=0}^{n-2} 2^{2(n-i-2)} = \frac{4^{n-1}+2}{3}$$

where the 1 corresponds to the null graph. ■

Remark 2.9 The sequence $(a_n)_{n\geq 0}$, $a_n = \frac{4^n+2}{3}$ for the number of Riordan graphs on n vertices has also various combinatorial interpretations as shown in A047849 in the On-Line Encyclopedia of Integer Sequences (OEIS) [33]:

$$1, 2, 6, 22, 86, 342, 1366, \ldots.$$

This sequence has the generating function $(1-3t)/((1-t)(1-4t))$.

Definition 2.10 Any Riordan matrix (g, f) over $\mathbb{Z}$ naturally defines the *infinite* graph

$$G := G(g, f) = \lim_{n\to\infty} G_n(g, f),$$

which we call the *infinite Riordan graph* corresponding to the Riordan matrix (g, f).

We note that even if an *unlabeled* graph is Riordan, its random labeling is likely to result in a non-Riordan graph. The following theorem gives necessary conditions for a graph to be Riordan. These conditions are formulated in terms of the subdiagonal elements in the adjacency matrix of a Riordan graph.

Theorem 2.11 (Necessary conditions for Riordan graphs) *Let $G = G_n(g, f)$ be a Riordan graph with vertex set $V = [n]$ and edge set E. Then, one of the following holds:*

(i) *$i(i + 1) \notin E$ for $1 \leq i \leq n - 1$.*

(ii) *$12 \in E$ and $i(i + 1) \notin E$ for $2 \leq i \leq n - 1$.*

(iii) *$i(i + 1) \in E$ for $1 \leq i \leq n - 1$, i.e. G has the Hamiltonian path $1 \to 2 \to \cdots \to n$.*

Proof. Let $\mathcal{A}(G) = [a_{i,j}]_{1\leq i,j\leq n}$. From the definition of the Riordan matrix (g, f), we have $a_{i+1,i} \equiv g_0 f_1^{i-1}$ for $1 \leq i \leq n-1$ where $g_0 = [t^0]g$ and $f_1 = [t^1]f$. Going through the four possibilities of choosing g_0 and f_1 in $\{0, 1\}$, we obtain the required result. ■

It is known that the number of *unlabeled* graphs on n vertices is given by the sequence $1, 2, 4, 11, 34, 156, \ldots$, which is A000088 in the OEIS [33].

Theorem 2.12 *All unlabeled graphs on at most four vertices are Riordan graphs.*

Proof. As we observed above, the total number of unlabeled graphs on at most four vertices is 18. The Fig. 7 justifies that such graphs are all Riordan graphs. Using the matrices in Example 2.3, we provide the proper labeling and the corresponding Riordan matrices (g, f) in the figure. ■

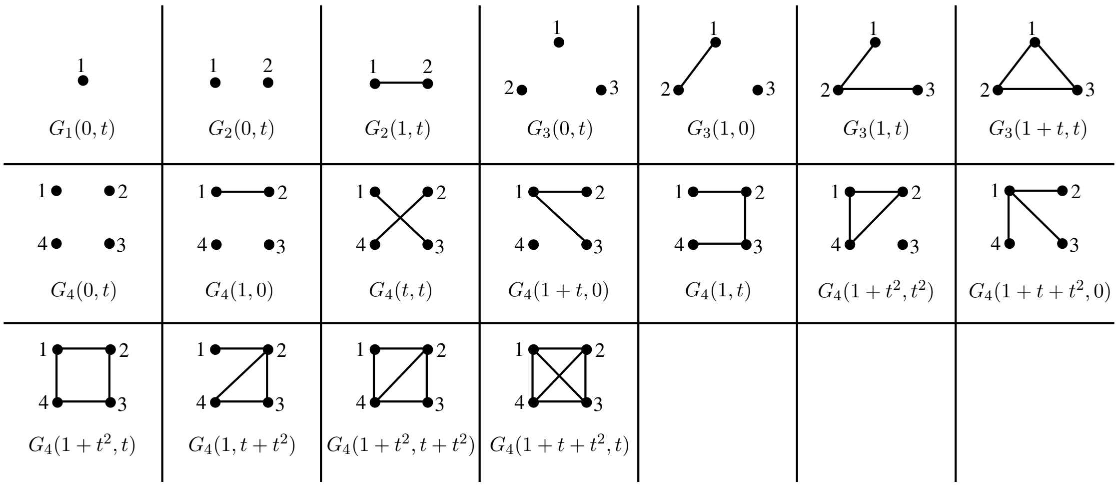


Fig. 7: All unlabeled graphs on at most 4 vertices which are Riordan graphs

We note that the Riordan matrices $\mathcal{B}(1,t+t^2)_3$ and $\mathcal{B}(1+t,t+t^2)_3$ correspond to the same unlabeled graph, and $\mathcal{B}(1+t^2,t+t^2)_3$, $\mathcal{B}(1+t,t)_3$ and $\mathcal{B}(1+t+t^2,t+t^2)_3$ correspond to the same unlabeled graph. On the other hand, the following proposition shows that not all unlabeled graphs on n vertices are Riordan for $n \geq 5$.

Proposition 2.13 *The unlabelled graph $H_{n+1} \cong K_n \cup K_1$ obtained from a complete graph K_n by adding an isolated vertex is not Riordan for $n \geq 4$.*

Proof. Let $H_{n+1} \cong K_n \cup K_1$. Suppose that there exist g and f such that a labeled copy of H_{n+1} is the Riordan graph $G_n(g,f)$. We consider two cases depending on whether the isolated vertex in H_{n+1} is labeled by 1 or not. Assume that the isolated vertex is labeled by 1. Since there are no edges $1i \in E$ for $i = 2, \ldots, n+1$, we have $g = 0$ so that $G_n(g,f)$ is the null graph N_n. This leads to a contradiction. Now, let $i \neq 1$ be the label of the isolated vertex and $\mathcal{A}(H_{n+1}) = [a_{i,j}]_{1 \leq i,j \leq n+1}$. Since n sub-diagonal entries of $\mathcal{A}(H_{n+1})$ by Theorem 2.11 are of the form

$$(a_{2,1}, \ldots, a_{n+1,n}) = \begin{cases} (\underbrace{1,\ldots,1}_{(i-2)\text{times}}, 0, 0, \underbrace{1,\ldots,1}_{(n-i)\ \text{times}}) & \text{if } i \neq n \\ (\underbrace{1,\ldots,1}_{(n-1)\ \text{times}}, 0) & \text{if } i = n, \end{cases}$$

this also leads to a contradiction. Hence the proof follows. ∎

2.4 Operations on Riordan graphs

There are many graph operations studied in the literature (see [2, 23] and references there in). Basically, graph operations produce new graphs from initial ones. However, in general we cannot guarantee that a particular operation applied to Riordan graphs results in a Riordan graph. In this section, we consider graph operations that preserve Riordan graphs. An example of such an operation is the *ring sum* of two graphs defined as follows.

Definition 2.14 Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ we define the *ring sum* $G_1 \oplus G_2 = (V_1 \cup V_2, (E_1 \cup E_2) - (E_1 \cap E_2))$. Thus, an edge is in $G_1 \oplus G_2$ if and only if it is an edge in G_1, or and edge in G_2, but not both.

The ring sum is well defined on Riordan graphs $G_n(g, f)$ and $G_n(h, f)$ with a fixed f and a fixed vertex set (e.g. $[n]$) due to the fact that $(g, f) + (h, f) = (g + h, f)$, so

$$G_n(g, f) \oplus G_n(h, f) = G_n(g + h, f).$$

Every simple graph G with n vertices has an $n \times n$ adjacency matrix $\mathcal{A}(G) = B + B^T$, where B is an $n \times n$ $(0, 1)$-lower triangular matrix with zero diagonal. In what follows we give a new graph operation defined for any simple graphs, not necessary Riordan graphs.

Definition 2.15 Let G and G' be simple labeled graphs with the same vertex set $V = [n]$ whose adjacency matrices of order n are of the form $\mathcal{A}(G) = B + B^T$ and $\mathcal{A}(G') = B' + B'^T$, respectivelty. We define the *R-product* $\otimes_R$ of G and G' to be the graph $G \otimes_R G'$ with the vertex set V whose adjacency matrix of order n is of the form:

$$\mathcal{A}\left(G \otimes_R G'\right) = B_R + B_R^T$$

where $B_R(1|n) = B(1|n)B'(1|n)$.

In particular, if $G = G_n(g, f)$ and $G' = G_n(h, \ell)$ then the *R-product* $\otimes_R$ of G and G' is the Riordan graph (with the vertex set V) given by

$$G_n(g, f) \otimes_R G_n(h, \ell) = G_n(gh(f), \ell(f)). \tag{5}$$

Example 2.16 Let G be the Fibonacci graph $G_n(1, t + t^2)$ and $G' = G_n(\frac{1}{1-t}, \frac{t}{1-t})$ be the Pascal graph. Then, the R-product of these graphs is

$$G \otimes_R G' = G_n\left(\frac{1}{1 - t - t^2}, \frac{t + t^2}{1 - t - t^2}\right).$$

Using (5) we immediately obtain the following theorem.

Theorem 2.17 *Every Riordan graph $G_n(g, f)$ can be expressed as the R-product of $G_n(g, t)$ and $G_n(1, f)$. Furthermore, the set of all proper Riordan graphs forms a group under the R-product $\otimes_R$ given by (5). The identity of the group is the path graph $G_n(1, t)$.*

To give a combinatorial interpretation for the R-product on Riordan graphs, let $B = [b_{ij}]$ be an $n \times n$ (0,1)-lower triangular matrix with zero diagonal. We associate with B a *digraph* $D_c(B)$ such that there is a colored arc from i to j using color c if and only if $b_{ij} = 1$.

Consider Riordan graphs $G_n(g, f)$ and $G_n(h, \ell)$. Let $B_1 = \mathcal{B}(tg, f)_n$, $B_2 = \mathcal{B}(t, t)_n^T$ and $B_3 = \mathcal{B}(th, \ell)_n$ be binary Riordan matrices. Assume that $D_r(B_1) = (V, E_r)$, $D_g(B_2) = (V, E_g)$, and $D_b(B_3) = (V, E_b)$ are the digraphs with the same vertex set $V = [n]$ and the arc sets respectively E_r, E_g and E_b, where r, g and b denote colors red, green, and blue.

Definition 2.18 Given Riordan graphs $G = G_n(g, f)$ and $H = G_n(h, \ell)$ with the vertex set $V = [n]$, the *RGB-graph* $\mathcal{D}_n = \mathcal{D}(G, H)$ is defined to be the digraph $(V, E_r \cup E_g \cup E_b)$ where $V = [n]$. In particular, a directed walk of length 3 in an RGB-graph is called an *RGB-walk* if its first arc is red, the second arc is green, and the third arc is blue.

Example 2.19 Let $G = G_6\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ and $H = G_6\left(\frac{1}{1-t^2}, t^2\right)$. Then, the RGB-graph $\mathcal{D}_6 = \mathcal{D}(G, H)$ is shown in Fig. 8, where solid (resp., dotted, dashed) arrows represent red (resp., green, blue) arrows.

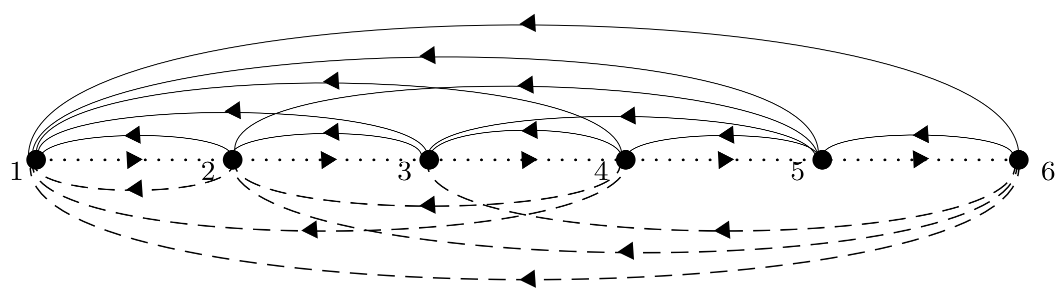


Fig. 8: The RGB-graph $\mathcal{D}_6 = \mathcal{D}(G, H)$

Theorem 2.20 *For Riordan graphs $G = G_n(g, f)$ and $H = G_n(h, \ell)$, let $B_1 = [r_{ij}]$, $B_2 = [g_{ij}]$ and $B_3 = [b_{ij}]$ be the same as in Definition 2.18. Let ω_{ij}, for $i > j$, be the number of RGB-walks from i to j in the RGB-graph $\mathcal{D}(G, H)$. Then,*

$$\omega_{ij} = \sum_{k=j}^{i-1} r_{i,k} g_{k,k+1} b_{k+1,j}.$$

In particular, two vertices i and j in $G \otimes_R H$ are adjacent if and only if $\omega_{ij} \equiv 1$.

Proof. Since $B_1 = \mathcal{B}(tg, f)_n$, $B_2 = \mathcal{B}(t, t)_n^T$ and $B_3 = \mathcal{B}(th, \ell)_n$, B_1 (resp., B_2; B_3) is the adjacency matrix of the directed subgraph of $\mathcal{D}(G, H)$ formed by the red (resp., green; blue) edges. All entries in B_2 are 0 except $g_{k,k+1} = 1$ for $1 \le k \le n-1$. Thus, if $B_1 B_2 B_3 = [d_{ij}]_{1 \le i,j \le n}$ then

$$d_{ij} = \begin{cases} \sum_{k=j}^{i-1} r_{i,k} g_{k,k+1} b_{k+1,j} & \text{if } i > j \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

It implies that if $i > j$ then $d_{i,j}$ counts the number of RGB-walks from i to j in the digraph $\mathcal{D}(G, H)$. Now, let $\mathcal{A} = [a_{ij}]_{1 \le i,j \le n}$ be the adjacency matrix of $G \otimes_R H$. Since

$$\mathcal{B}(gh(f), \ell(f))_{n-1} \equiv \mathcal{B}(g, f)_{n-1} \mathcal{B}(h, \ell)_{n-1},$$

we have

$$a_{ij} \equiv \sum_{k=j}^{i-1} r_{i,k} b_{k+1,j} \ \text{ if } i > j. \tag{7}$$

Since $g_{k,k+1} = 1$ for $1 \leq k \leq n-1$ it follows from (6) and (7) that $d_{i,j} \equiv a_{i,j}$ if $i > j$. It means that, for $i > j$, i and j are adjacent, i.e. $a_{ij} = 1$ if and only if d_{ij} is odd. Hence the proof follows. ■

Example 2.21 Let G and H be the same Riordan graphs as in Example 2.19. Since $B_1 = \mathcal{B}\left(\frac{t}{1-t}, \frac{t}{1-t}\right)_6$, $B_2 = \mathcal{B}(t,t)_6^T$ and $B_3 = \mathcal{B}\left(\frac{t}{1-t^2}, t^2\right)_6$, we obtain

$$D := B_1B_2B_3 = [d_{i,j}] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

By counting RGB-walks in Fig. 8, we see that if $i > j$ then $d_{i,j}$ counts the number of RGB-walks from a vertex i to a vertex j. For instance, $d_{5,1} = 2$ because there are two RGB-walks, 5→1→2→1 and 5→3→4→1 from the vertex 5 to the vertex 1.

In addition, we obtain the adjacency matrix of $G \otimes_R H = G_n\left(1+t, \frac{t^2}{1-t^2}\right)$ as follows:

$$D + D^T \equiv \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

which should be the case by Theorem 2.20.

2.5 Families of Riordan graphs

Recall that every Riordan graph G with n vertices is determined by some Riordan matrix (g, f) over $\mathbb{Z}$ such that $(g, f)_{n-1} \equiv \mathcal{B}(g, f)_{n-1}$. Such a graph G is a Riordan graph corresponding to the Riordan matrix (g, f) and is denoted by $G_n(g, f)$. Thus the families of Riordan graphs are naturally defined by the corresponding classes of Riordan matrices.

In this section, we introduce several classes of Riordan graphs and give examples of graphs in these classes. The names of the classes come from the widely used names of Riordan matrices defining the respective Riordan graphs; such matrices are obtained by imposing various restrictions on the pairs of functions (g, f). Also, in Section 4 we introduce *o-decomposable*, *e-decomposable*, *io-decomposable* and *ie-decomposable* Riordan graphs.

Note that the most general definition of the *null graphs* N_n (also known as the *empty graphs*) in our terms is $G_n(0, f)$ for any f where $f(0) = 0$. Also note that the empty graphs, the star graphs $G_n(\frac{1}{1-t}, 0)$, and the *complete k-ary trees* for $k \geq 2$ defined by $G_n(1 + t + \cdots + t^{k-1}, t^k)$ are examples of non-proper Riordan graphs; other examples of non-proper Riordan graphs can be obtained from (v) in Theorem 3.6, and even more such examples are discussed at the end of this subsection. However, most of Riordan graphs considered in this paper are proper. Inspired by Riordan subgroups given in Section 2.1, we have the following classes of Riordan graphs.

Riordan graphs of the *Appell type*. This class of graphs is defined by an *Appell matrix* (g,t). Let $G = G_n(g,t)$ be a Riordan graph of the Appell type and $g = \sum_{i=0}^{n-2} g_i t^i \in \mathbb{Z}_2[t]$. Then the adjacency matrix $\mathcal{A}(G)$ is given by the symmetric $(0,1)$-Toeplitz matrix

$$\mathcal{A}(G_n) = \begin{pmatrix} 0 & g_0 & g_1 & \cdots & \cdots & g_{n-2} \\ g_0 & 0 & g_0 & \ddots & & \vdots \\ g_1 & g_0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & g_0 & g_1 \\ \vdots & & \ddots & g_0 & 0 & g_0 \\ g_{n-2} & \cdots & \cdots & g_1 & g_0 & 0 \end{pmatrix}.$$

Thus, the class of Riordan graphs of the Appell type is precisely the class of Toeplitz graphs. Examples of graphs in this class are

- the null graphs N_n defined by $G_n(0,t)$;
- the path graphs P_n defined by $G_n(1,t)$;
- the complete graphs K_n defined by $G_n\left(\frac{1}{1-t},t\right)$; and
- the complete bipartite graphs $K_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}$ defined by $G_n(\frac{1}{1-t^2},t)$.

Riordan graphs of the *Bell type*. This class of graphs is defined by a *Bell matrix* (g,tg). Examples of graphs in this class are

- the null graphs N_n defined by $G_n(0,0)$;
- the path graphs P_n defined by $G_n(1,t)$;
- the Pascal graphs PG_n defined by $G_n(\frac{1}{1-t},\frac{t}{1-t})$;
- the *Catalan graphs* CG_n defined by $G_n\left(\frac{1-\sqrt{1-4t}}{2t},\frac{1-\sqrt{1-4t}}{2}\right)$; and
- the *Motzkin graphs* MG_n defined by $G_n\left(\frac{1-t-\sqrt{1-2t-3t^2}}{2t^2},\frac{1-t-\sqrt{1-2t-3t^2}}{2t}\right)$.

Riordan graphs of the *Lagrange type*. This class of graphs is defined by a *Lagrange matrix* $(1,f)$, and it is trivially related to Riordan graphs of the Bell type. Indeed, letting $f = tg$, we see that removing the vertex 1 in $G_n(1,tg)$ gives the graph $G_{n-1}(g,tg)$ of the Bell type. Conversely, given a graph $G_{n-1}(g,tg)$, we can always relabel each vertex i by $i+1$, and add a new vertex labelled by 1 and connected to the vertex 2, to obtain the graph $G_n(1,tg)$ of the Lagrange type. Examples of graphs in this class are

- the path graphs P_n defined by $G_n(1,t)$; and
- the *Fibonacci graph* FG_n defined by $G_n\left(1, t+t^2\right)$.

Riordan graphs of the *checkerboard type.* This class of graphs is defined by a *checkerboard matrix* (g, f) such that g is an even function and f is an odd function. Examples of graphs in this class are

- the path graphs P_n defined by $G_n(1,t)$; and
- the complete bipartite graphs $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ defined by $G_n(\frac{1}{1-t^2}, t)$.

Riordan graphs of the *derivative type.* This class of graphs is defined by a *derivative matrix* (f', f). Examples of graphs in this class are

- the null graphs N_n defined by $G_n(0,0)$; and
- the path graphs P_n defined by $G_n(1,t)$.

Riordan graphs of the *hitting time type.* This class of graphs is defined by a *hitting time matrix* $(tf'/f, f)$, $f'(0) = 1$, and it is trivially related to Riordan graphs of the derivative type. Indeed, removing the first row and the first column of the adjacency matrix $\mathcal{A}(G)$ of a graph $G = G_n(tf'/f, f)$ of a hitting time type, we obtain the graph $G_{n-1}(f', f)$ of the derivative type which corresponds to removing the vertex 1 of G. Conversely, given a Riordan graph $G_{n-1}(f', f)$ of the derivative type, one can relabel each vertex i by $i+1$, and add a new vertex, labeled by 1, that is connected to the vertices defined by the coefficients of the function tf'/f to obtain $G_n(tf'/f, f)$.

Thus, Riordan graphs of the Bell type (Section 4.2) and the derivative type (Section 4.3) are of the main interest in Section 4.

As is mentioned above, more classes of Riordan graphs can be introduced using the operations, ring sum $\oplus$ and R-product $\otimes_R$ defined in Section 2.4. Indeed, to illustrate this idea, note that the ring sum $\oplus$ of a Riordan graph $G_n(g, tg)$ of the Bell type and the Riordan graph $G_n((tg)', tg)$ of the derivative type is well defined. Such a sum results in a new class of graphs defined by Riordan matrices of the form (tg', tg). Indeed, since $2g \equiv 0 \pmod 2$, we have

$$G_n(g, tg) \oplus G_n((tg)', tg) = G_n(g + g + tg', tg) = G_n(tg', tg).$$

Note that $G_n(tg', tg)$ is not proper, as is the ring sum of any two proper Riordan graphs.

We end the subsection by noticing that the class of Riordan graphs of the Appell type (i.e. Toeplitz graphs) are closed under the operations $\oplus$ and $\otimes_R$. It is not difficult to show that this class of graphs on n vertices forms a *commutative ring* with the identity element $G_n(1, z)$ and the zero element $G_n(0, z) = N_n$.

2.6 The complement of a Riordan graph

For any Riordan graph $G_n = G_n(g,t)$ of the Appell type, the ring sum $G_n(g,t) \oplus G_n\left(\frac{1}{1-t}, t\right)$ gives the *complement* of G_n, i.e. the graph in which edges of G_n become non-edges, and vice versa.

In general, it is not true that the complement of a Riordan (labeled or unlabeled) graph is Riordan. Indeed, the complement of the star graph $G_n\left(\frac{1}{1-t}, 0\right)$, $n \geq 5$, is a labeled copy of the graph H_n in Proposition 2.13, which is non-Riordan. Thus, Riordan graphs can become non-Riordan, and thus versa, under taking the complement.

Note that the complement of the Riordan graph $G_4(1+t^2, t^2)$ in Fig. 7 is the Riordan graph $G_4(1+t+t^2, t^3)$ showing that the operation of the complement preserves the property of being Riordan for some graphs of non-Appell type.

3 Structural properties of Riordan graphs

We begin with basic properties of a Riordan graph $G = G_n(g,f)$, which can be directly determined in terms of column generating functions of the binary Riordan matrix $\mathcal{B}(g,f) = [b_{ij}]_{i,j\geq 0}$ where $b_{ij} \equiv [t^i]gf^j \in \{0,1\}$. Let

$$\mathcal{B}_n^{(j)} = \mathcal{B}_n^{(j)}(t) := \sum_{i=j}^{n} b_{ij}t^i \in \mathbb{Z}_2[t], \quad j = 0, 1, \ldots$$

be a polynomial in t of degree n which generates the (0,1)-elements b_{ij} in the jth column of $\mathcal{B}(g,f)$. Thus if $\mathcal{A}(G) = B + B^T$ is the $n \times n$ adjacency matrix of G then $B(1|n)$ can be expressed in terms of polynomials in t of degree $n-2$:

$$B(1|n) = \mathcal{B}(g,f)_{n-1} := \left[\mathcal{B}_{n-2}^{(0)}, \mathcal{B}_{n-2}^{(1)}, \ldots, \mathcal{B}_{n-2}^{(n-2)}\right].$$

In this paper, $d_G(i)$ denotes the *degree* of a vertex i in a graph G. If G is understood from the context, we simply write $d(i)$.

Theorem 3.1 (Basic Properties) *Let $G = G_n(g,f)$ be a Riordan graph and let $\mathcal{B}(g,f) = [b_{ij}]_{i,j\geq 0}$. Then,*

(i) *For $i > j \geq 1$, $ij \in E$ if and only if $[t^{i-2}]gf^{j-1} \equiv b_{i-2,j-1} = 1$.*

(ii) *$d(1) = \mathcal{B}_{n-2}^{(0)}(1)$.*

(iii) *$d(n) = \sum_{j=0}^{n-2}[t^{n-2}]\mathcal{B}_{n-2}^{(j)}(t)$.*

(iv) *For $k \notin \{1,n\}$, $d(k) = \mathcal{B}_{n-2}^{(k-1)}(1) + \sum_{j=0}^{k-2}[t^{k-2}]\mathcal{B}_{n-2}^{(j)}(t)$.*

(v) *$|E| = \sum_{j=0}^{n-2}\mathcal{B}_{n-2}^{(j)}(1)$.*

(vi) *If G is proper then it has the Hamiltonian path $1 \to 2 \to \cdots \to n$. In addition, if $b_{n-2,0} = [t^{n-2}]\mathcal{B}_{n-2}^{(0)}(t) = 1$ then G has the Hamiltonian cycle $1 \to 2 \to \cdots \to n \to 1$.*

Proof. The (i)–(iii) are straightforward from (4). The (iv) follows from the fact that the degree of a vertex k is the summation of all entries located in both the $(k-1)$th column and the $(k-2)$th row of $\mathcal{B}(g,f)_{n-1}$. The (v) follows from the fact that the number of edges in G is equal to the number of 1s in $\mathcal{B}(g,f)_{n-1}$. The (vi) follows from the fact that if G is proper then all entries of the subdiagonal in its adjacency matrix are 1s, i.e. i is adjacent to $i+1$ for $i=1,\ldots,n-1$. ∎

The *matching number* $\beta(G)$ is the size of a *maximal matching* in a graph G.

Theorem 3.2 *Let $G=G_n(g,f)$ be a proper Riordan graph. Then $\beta(G)=\lfloor\frac{n}{2}\rfloor$.*

Proof. By (vi) in Theorem 3.1, G has the Hamiltonian path $1\to 2\to\cdots\to n$. Therefore, for even and odd n, respectively, maximal matchings are $\{12,\ 34,\ \ldots,\ (n-1)n\}$ and $\{12,\ 34,\ \ldots,\ (n-2)(n-1)\}$. This completes the proof. ∎

From now on, the following theorems will be useful for studying the structural properties of Riordan graphs in next sections.

Lemma 3.3 *Let $g,f\in\mathbb{Z}_2[[t]]$ where $f(0)=0$. Then, for an integer $s\geq 1$,*

$$(g\circ f)^{2^s}\equiv g\circ f^{2^s}\pmod 2. \tag{8}$$

Proof. We proceed by induction on $s\geq 1$. Let $s=1$ and let $g=g(t):=\sum_{n\geq 0}g_nt^n$. Since $g_n\in\{0,1\}$, $g_n^2=g_n$ for all $n\geq 0$. Thus, we obtain

$$\begin{aligned}(g\circ f)^2=g(f)^2&=\left(\sum_{n\geq 0}g_nf^n\right)^2=\sum_{n\geq 0}\left(\sum_{i=0}^{n}g_ig_{n-i}\right)f^n\\&=g_0^2+2g_0g_1f+(2g_0f_1+g_1^2)f^2+(2g_0g_3+2g_1g_2)f^3+\cdots\\&\equiv\sum_{n\geq 0}g_n^2f^{2n}\equiv\sum_{n\geq 0}g_n(f^2)^n=g(f^2)=g\circ f^2,\end{aligned}$$

which proves (8) when $s=1$. Next, by inductive hypothesis we obtain

$$(g\circ f)^{2^s}\equiv\left(g\circ f^2\right)^{2^{s-1}}\equiv g\circ f^{2^s}=g\left(f^{2^s}\right).$$

Hence the proof is complete. ∎

3.1 Decomposition of Riordan graphs

For any simple graph $G=(V,E)$ of order n with adjacency matrix $\mathcal{A}(G)$ and $V=[n]$, there exists an $n\times n$ permutation matrix P such that

$$P\mathcal{A}(G)P^T=\left(\begin{array}{cc}\mathcal{A}(\langle V_1\rangle)&\tilde{B}\\\tilde{B}^T&\mathcal{A}(\langle V_2\rangle)\end{array}\right) \tag{9}$$

where $\langle V_1\rangle$ and $\langle V_2\rangle$ are respectively induced subgraphs of disjoint vertex sets V_1 and V_2 in which $V_1\bigcup V_2 = V$. Theorem 3.4 gives a description for the adjacency matrix in the case of a Riordan graph where $V_1 = V_o$ and $V_2 = V_e$ are assumed to be the sets of odd and even labeled vertices, respectively.

Theorem 3.4 (Riordan Graph Decomposition) *Let $G = G_n(g, f)$ be a Riordan graph of order n and let $n = n_1 + n_2$ where $n_1 = \lceil n/2\rceil$ and $n_2 = \lfloor n/2\rfloor$. If P is the $n \times n$ permutation matrix of the form $P = (e_1, e_3, \dots, e_{2n_1-1}, e_2, e_4, \dots, 2n_2)^T$ where e_i denotes the unit column vector with 1 in the ith position, then*

$$P\mathcal{A}(G)P^T = \begin{pmatrix} X & \tilde{B} \\ \tilde{B}^T & Y \end{pmatrix} \tag{10}$$

where $X = \mathcal{A}(\langle V_o\rangle)$, $Y = \mathcal{A}(\langle V_e\rangle)$, and

(i) *$\langle V_o\rangle$ is the Riordan graph of order n_1 corresponding to the Riordan matrix $\big(g'(\sqrt{t}), f(t)\big)$;*

(ii) *$\langle V_e\rangle$ is the Riordan graph of order n_2 corresponding to the Riordan matrix $\left(\left(\frac{gf}{t}\right)'(\sqrt{t}), f(t)\right)$;*

(iii) *$\tilde{B} = \mathcal{B}(t(gf)'(\sqrt{t}), f(t))_{n_1\times n_2} + \mathcal{B}((tg)'(\sqrt{t}), f(t))^T_{n_2\times n_1}$.*

Proof. Regarding to the given permutation matrix P, (10) is a direct consequence of (9). If we let $\mathcal{A}(G) = (b_{ij})_{1\le i,j\le n} = B + B^T$ where $B = \mathcal{B}(tg, f)_n$ then

$$P\mathcal{A}(G)P^T = PBP^T + (PBP^T)^T$$

where for $m_1 = 2n_1 - 1$ and $m_2 = 2n_2$,

$$PBP^T = \left(\begin{array}{ccccc|ccccc}
0 & & & & & 0 & & & & \\
b_{31} & 0 & & O & & b_{32} & 0 & & O & \\
b_{51} & b_{53} & 0 & & & b_{52} & b_{54} & 0 & & \\
\vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & \ddots & \\
b_{m_1,1} & b_{m_1,3} & \cdots & b_{m_1,m_1-2} & 0 & b_{m_1,2} & b_{m_1,4} & \cdots & b_{m_1,m_1-1} & 0 \\
\hline
b_{21} & & & & & 0 & & & & \\
b_{41} & b_{43} & & O & & b_{42} & 0 & & O & \\
b_{61} & b_{63} & b_{65} & & & b_{62} & b_{64} & 0 & & \\
\vdots & \vdots & \ddots & \ddots & & \vdots & \vdots & \ddots & \ddots & \\
b_{m_2,1} & b_{m_2,3} & \cdots & b_{m_2,m_2-3} & b_{m_2,m_2-1} & b_{m_2,2} & b_{m_2,4} & \cdots & b_{m_2,m_2-2} & 0
\end{array}\right)$$

$$:= \begin{pmatrix} \hat{X}_{n_1\times n_1} & Z_{n_1\times n_2} \\ W_{n_2\times n_1} & \hat{Y}_{n_2\times n_2} \end{pmatrix}. \tag{11}$$

In particular, if n is odd then PBP^T can be obtained from (11) by deleting the last row and last column.

Now we show that $\hat{X}$, $\hat{Y}$, Z, W are Riordan matrices. First recall that if $(g, f) = (r_{ij})_{i,j\ge0}$ where $r_{ij} = [t^i]gf^j$ then from (4) and $i \ge 2$ and $j \ge 1$, we have

$$b_{ij} \equiv [t^{i-1}]tgf^{j-1} \equiv [t^{i-2}]gf^{j-1} = r_{i-2,j-1}. \tag{12}$$

(i) Let $\hat{X} = (x_{ij})_{1\le i,j\le n_1}$. Then,

$$x_{ij} = \begin{cases} b_{2i-1,2j-1} & \text{if } 1 \le j < i \le n_1 \\ 0 & \text{otherwise.} \end{cases}$$

Using (12) and the property $[t^i]h \equiv [t^{i-1}]h'$ for any $h \in \mathbb{Z}[[t]]$ together with $g(t)^2 \equiv g(t^2)$ from Lemma 3.3, we obtain, for $1 \le j < i \le n_1$,

$$\begin{aligned} x_{ij} &= b_{2i-1,2j-1} \equiv r_{2i-3,2j-2} = [t^{2i-3}]gf^{2j-2} \equiv [t^{2i-4}]\left(gf^{2j-2}\right)' \\ &\equiv [t^{2i-4}]g'f^{2j-2} \equiv [t^{2i-2}]t^2g'(t)f^{2(j-1)}(t) \equiv [t^{2(i-1)}]t^2g'(t)f^{j-1}(t^2) \\ &\equiv [t^{i-1}]tg'(\sqrt{t})f^{j-1}(t). \end{aligned}$$

Thus $\hat{X}$ is an $n_1 \times n_1$ binary Riordan matrix given by

$$\hat{X} = \mathcal{B}\left(tg'(\sqrt{t}), f(t)\right)_{n_1}.$$

Since $\mathcal{A}(\langle V_o\rangle) = \hat{X} + \hat{X}^T$ it follows that $\langle V_o\rangle$ is a Riordan graph on $n_1(=\lceil n/2\rceil)$ vertices of $\left(g'(\sqrt{t}), f(t)\right)$.

(ii) Let $\hat{Y} = (y_{ij})_{1\le i,j\le n_2}$. Then,

$$y_{ij} = \begin{cases} b_{2i,2j} & \text{if } 1 \le j < i \le n_2 \\ 0 & \text{otherwise.} \end{cases}$$

By a similar method as (i), we obtain, for $1 \le j < i \le n_2$,

$$\begin{aligned} y_{ij} &= b_{2i,2j} \equiv r_{2i-2,2j-1} = [t^{2i-2}]gf^{2j-1} = [t^{2i-3}]\frac{gf^{2j-1}}{t} \\ &\equiv [t^{2i-4}]\left(\frac{gf}{t}f^{2j-2}\right)' \equiv [t^{2i-4}]\left(\frac{gf}{t}\right)'f^{2j-2} \equiv [t^{2(i-2)}]\left(\frac{gf}{t}\right)'(t)f^{j-1}(t^2) \\ &\equiv [t^{i-1}]t\left(\frac{gf}{t}\right)'(\sqrt{t})f^{j-1}(t). \end{aligned}$$

Thus, $\hat{Y}$ is an $n_2 \times n_2$ binary Riordan matrix given by

$$\hat{Y} = \mathcal{B}\left(t\left(\frac{gf}{t}\right)'(\sqrt{t}), f(t)\right)_{n_2}.$$

Since $\mathcal{A}(\langle V_e\rangle) = \hat{Y} + \hat{Y}^T$ it follows that $\langle V_e\rangle$ is a Riordan graph on $n_2(=\lfloor n/2\rfloor)$ vertices of $\left(\left(\frac{gf}{t}\right)'(\sqrt{t}), f(t)\right)$.

(iii) Let $Z = (z_{ij})_{1\le i\le n_1, 1\le j\le n_2}$. Then,

$$z_{ij} = \begin{cases} b_{2i-1,2j} & \text{if } 1 \le j < i \le n_1 \\ 0 & \text{otherwise.} \end{cases}$$

By a similar method as (i), we obtain, for $1 \leq j < i \leq n_1$,

$$\begin{aligned} z_{ij} &= b_{2i-1,2j} \equiv r_{2i-3,2j-1} = [t^{2i-3}]gf^{2j-1} \equiv [t^{2i-4}]\left((gf)f^{2j-2}\right)' \\ &\equiv [t^{2i-4}](gf)'f^{2j-2} \equiv [t^{2(i-2)}](gf)'(t)f^{j-1}(t^2) \\ &\equiv [t^{i-1}]t(gf)'(\sqrt{t})f^{j-1}(t). \end{aligned}$$

Thus, Z is an $n_1 \times n_2$ binary Riordan matrix given by

$$Z = \mathcal{B}\left(t(gf)'(\sqrt{t}), f(t)\right)_{n_1 \times n_2}.$$

Let $W = (w_{ij})_{1\leq i\leq n_2, 1\leq j\leq n_1}$. Then,

$$w_{ij} = \begin{cases} b_{2i,2j-1} & \text{if } 1 \leq j \leq i \leq n_2 \\ 0 & \text{otherwise.} \end{cases}$$

By a similar method as above, one can easily show that

$$w_{ij} = [t^{i-1}]\,(tg)'(\sqrt{t})f^{j-1}(t).$$

Thus W is an $n_2 \times n_1$ Riodan matrix given by $\left((tg)'(\sqrt{t}), f(t)\right)$. Since $\tilde{B} = Z + W^T$, the proof is complete. ■

Definition 3.5 Let *o-decomposable Riordan graphs*, standing for *odd decomposable Riordan graphs*, be the class of graphs defined by requiring $Y = O$ in (10), where O is the zero matrix of Y's size. Also, let *e-decomposable Riordan graphs*, standing for *even decomposable Riordan graphs*, be the class of graphs defined by requiring $X = O$ in (10).

The parts (ii) and (iii) in the following theorem justify our choice for the names of o-decomposable and e-decomposable Riordan graphs.

Theorem 3.6 *Let $g \not\equiv 0$ and V_o and V_e be, respectively, the odd and even labelled vertex sets of a Riordan graph $G = G_n(g, f)$. Then,*

(i) *For even n, $\langle V_o\rangle \cong \langle V_e\rangle$ if and only if $[t^{2m-1}]g \equiv [t^{2m}]gf$ for all $m \geq 1$.*

(ii) *The induced subgraph $\langle V_o\rangle$ is a null graph, i.e. G is e-decomposable if and only if $[t^{2m-1}]g \equiv 0$ for all $m \geq 1$, that is, g is an even function modulo 2.*

(iii) *The induced subgraph $\langle V_e\rangle$ is a null graph, i.e. G is o-decomposable if and only if $[t^{2m}]gf \equiv 0$ for all $m \geq 1$, that is, gf is an odd function modulo 2.*

(iv) *G is a bipartite graph with parts V_o and V_e if and only if $[t^{2m+1}]g \equiv [t^{2m}]f \equiv 0$ for all $m \geq 0$, that is, G is of the checkerboard type.*

(v) *There is no edge between a vertex $i \in V_o$ and a vertex $j \in V_e$ if and only if $[t^{2m}]g \equiv [t^{2m}]f \equiv 0$ for all $m \geq 0$, i.e. $G \cong \langle V_o\rangle \bigcup \langle V_e\rangle$.*

Proof. (i) Let n be even. From Theorem 3.4, $\langle V_o\rangle \cong \langle V_e\rangle$ if and only if

$$g' \equiv \left(\frac{gf}{t}\right)'. \tag{13}$$

Since $[t^{2m-1}]h' = 2m[t^{2m}]h \equiv 0$ and $[t^{2m-2}]h' \equiv [t^{2m-1}]h$ for all $h \in \mathbb{Z}[[t]]$, the equation in (13) is equivalent to

$$[t^{2m-1}]g \equiv [t^{2m-1}]\frac{gf}{t} \quad \Leftrightarrow \quad [t^{2m-1}]g \equiv [t^{2m}]gf$$

which proves (i).

(ii) From Theorem 3.4, the induced subgraph $\langle V_o\rangle$ is a null graph if and only if the matrix X in (10) is a zero matrix, i.e.

$$g' \equiv 0 \quad \Leftrightarrow \quad [t^{2m-1}]g \equiv 0 \text{ for all } m \geq 1.$$

Hence the proof follows.

(iii) From Theorem 3.4, the induced subgraph $\langle V_e\rangle$ is a null graph if and only if the matrix Y in (10) is a zero matrix, i.e.

$$\left(\frac{gf}{t}\right)' \equiv 0 \;\; \Leftrightarrow \;\; [t^{2m}]gf \equiv 0 \text{ for all } m \geq 1.$$

Hence the proof follows.

(iv) From Theorem 3.4, G is a bipartite graph with parts V_o and V_e if and only if the matrices X and Y in (10) are zero matrices, i.e.

$$\begin{aligned} g' \equiv 0 \;\text{ and }\; (gf/t)' \equiv 0 \quad &\Leftrightarrow \quad g' \equiv 0 \;\text{ and }\; (f/t)' \equiv 0 \\ &\Leftrightarrow \quad [t^{2m+1}]g \equiv [t^{2m}]f \equiv 0 \text{ for all } m \geq 0. \end{aligned}$$

Hence the proof follows.

(v) Form Theorem 3.4, there is no edge between a vertex $i \in V_o$ and a vertex $j \in V_e$ if and only if the matrix B in (10) is a zero matrix, i.e.

$$\begin{aligned} (gf)'(\sqrt{t}) \equiv 0 \;\text{ and }\; (tg)'(\sqrt{t}) \equiv 0 \quad &\Leftrightarrow \quad (tg)' \equiv 0 \;\textit{ and }\; (f/t)' \equiv 0 \\ &\Leftrightarrow \quad [t^{2m}]g \equiv [t^{2m}]f \equiv 0 \text{ for all } m \geq 0. \end{aligned}$$

Hence the proof follows. ∎

3.2 Fractal properties of Riordan graphs

A *fractal* is an object exhibiting similar patterns at increasingly small scales. Thus, fractals use the idea of a detailed pattern that repeats itself.

In this section, we show that every Riordan graph $G_n(g, f)$ with $f'(0) = 1$ has fractal properties by using the notion of the A-sequence of a Riordan matrix.

Definition 3.7 Let G be a graph. A pair of vertices (u, v) in G is a *cognate pair* of a pair of vertices (i, j) in G if

- $|i-j| = |u-v|$ and
- i is adjacent to $j \Leftrightarrow u$ is adjacent to v.

We denote the cognate pairs as $(i, j) \sim (u, v)$. The set of all cognate pairs of (i, j) is denoted by $\operatorname{cog}(i, j)$. Clearly, the relation $\sim$ on the set $\operatorname{cog}(i, j)$ is an equivalence relation.

Definition 3.8 The A-sequence of the binary Riordan matrix $\mathcal{B}(g, f)$ defining a proper Riordan graph $G_n(g, f)$ with $f'(0) = 1$ is called the *binary A-sequence of the graph.*

Let $G = G_n(g, f)$ be a proper Riordan graph. If G is not a graph of Appell type, i.e. $f \neq t$ then the binary A-sequence of G has of the form

$$A = (\tilde{a}_k)_{k\ge0} = (1, \underbrace{0, \ldots, 0}_{\ell \text{ times}}, 1, \tilde{a}_{\ell+2}, \ldots),\ \ell \ge 0,\ \tilde{a}_k \in \{0, 1\}. \tag{14}$$

The following theorem gives a relationship between cognate pairs and the binary A-sequence of a proper Riordan graph.

Theorem 3.9 *Let $G = G_n(g, f)$ be a proper Riordan graph with a binary A-sequence $A = (\tilde{a}_k)_{k\ge0}$ of the form (14). Assume that $\ell \ge 0$ is the maximum number of consecutive zero terms starting from $\tilde{a}_1$ in the A-sequence. For a given vertex pair (i, j) with $i > j$, if $s \ge 0$ is the least integer satisfying*

$$s \ge \log_2 \frac{i-j}{\ell+1}, \tag{15}$$

then $(i + m2^s, j + m2^s)$ is a cognate pair of (i, j) for each integer m where

$$\frac{1-j}{2^s} \le m \le \frac{n-i}{2^s}. \tag{16}$$

Proof. Let $\mathcal{A}(G) = [b_{ij}]_{1\le i,j\le n} = B + B^T$ where $b_{ij} = b_{ji}$ and $B = (tg, f)_n$. Then we may assume that $g, f \in \mathbb{Z}_2[[t]]$ and $B = [b_{ij}]_{1\le i,j\le n}$ where $b_{ij} = 0$ if $i \le j$. Since the A-sequence of G has the form of (14), its generating function is the polynomial over $\mathbb{Z}_2$:

$$A(t) = 1 + t^{\ell+1} + \tilde{a}_{\ell+2}t^{\ell+2} + \cdots + \tilde{a}_{n-3}t^{n-3}.$$

Using $f \equiv tA(f)$ it follows from Lemma 3.3 that if $k = 2^s$ is an integer for $s \ge 0$ then

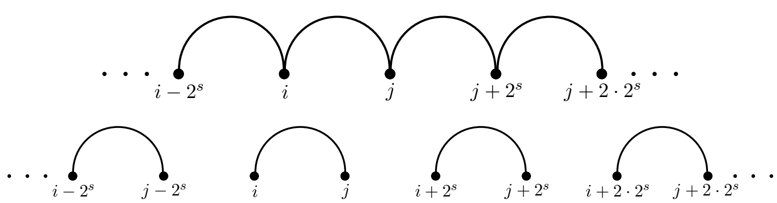
$$\begin{aligned}
b_{i+k,j+k} &\equiv \left[t^{i+k-2}\right] gf^{j+k-1} \equiv \left[t^{i+k-2}\right] gf^{j-1}\,(tA(f))^k \equiv \left[t^{i-2}\right] gf^{j-1}A(f^k) \\
&= \left[t^{i-2}\right] gf^{j-1}\left(1 + f^{k(\ell+1)} + \sum_{m=\ell+2}^{n-3} \tilde{a}_m f^{km}\right) \\
&= b_{ij} + b_{i,j+k(\ell+1)} + \sum_{m=\ell+2}^{n-3} \tilde{a}_m b_{i,j+km}.
\end{aligned}$$

Since $B = [b_{ij}]_{1\le i,j\le n}$ is a lower triangular matrix with zero diagonal, if $j + k(\ell+1) \ge i$ then $b_{i,j+k(\ell+1)} = 0$ so that $b_{i,j+km} = 0$ for all $m \ge \ell+2$. Thus, if s satisfies (15) then

$$b_{i+k,j+k} \equiv b_{i,j},$$

i.e. $(i,j) \sim (i+2^s, j+2^s)$. Moreover, by transitivity of cognate pairs we have $(i,j) \sim (i+m2^s, j+m2^s)$ where m is an integer such that $1 \le j+m2^s < i+m2^s \le n$, i.e. $\frac{1-j}{2^s} \le m \le \frac{n-i}{2^s}$ for the least integer satisfying (15). Hence we complete the proof. ■

In particular, let $\ell = 0$. If i is adjacent to j then the pairs cognate with (i,j) are those connected by edges in the following figures:



for $|i-j| = 2^s$

for $|i-j| < 2^s$

Fig. 9: Cognate pairs in a Riordan graph

Remark 3.10 We note that not every cognate pairs of (i,j) can be expressed as the form $(i+m2^s, j+m2^s)$ for some integers m and $s \ge 0$. It might be interesting to find all cognate pairs of a vertex pair (i,j) of a Riordan graph.

The following theorem shows that every proper Riordan graph $G_n(g,f)$ with $f \ne t$ has a fractal property.

Theorem 3.11 *Let $G = G_n(g,f)$ be the same Riordan graph in Theorem 3.9. Consider a cognate pair $(i+m2^s, j+m2^s)$ of a vertex pair (i,j) with $i > j$. Then, the associated two induced subgraphs are isomorphic as follows:*

$$\langle \{j, j+1, \ldots, i\} \rangle \cong \langle \{j+m2^s, j+1+m2^s, \ldots, i+m2^s\} \rangle .$$

Proof. Let $X = \{j, j+1, \ldots, i\}$ and $Y = \{j+m2^s, j+1+m2^s, \ldots, i+m2^s\}$. Consider a bijection $\sigma : X \to Y$ with $\sigma(k) = k+m2^s$ for $k \in X$. It is enough to show that for any two vertices $a, b \in X$ where $a > b$, $(a,b) \sim (\sigma(a), \sigma(b))$. Consider a vertex pair (a,b) in X. Clearly, $|a-b| = |\sigma(a) - \sigma(b)|$. By Theorem 3.9, if $t \ge 0$ is the least integer satisfying $t \ge \log_2 \frac{a-b}{\ell+1}$ then there exists an integer $\tilde{m}$ such that $a + \tilde{m}2^t$ and $b + \tilde{m}2^t$ where $\tilde{m}$ is an integer that satisfies

$$\frac{1-b}{2^t} \le \tilde{m} \le \frac{n-a}{2^t}. \tag{17}$$

We claim that if we take $\tilde{m} = m2^{s-t}$ then $(a,b) \sim (\sigma(a), \sigma(b))$. Indeed, since $j < b < a < i$ it follows from (16) that

$$\frac{1-b}{2^t} \le \frac{1-j}{2^t} \le m2^{s-t} \le \frac{n-i}{2^t} \le \frac{n-a}{2^t}.$$

Thus $\tilde{m} := m2^{s-t}$ satisfies (17) which implies that $a + \tilde{m}2^t = a + m2^s = \sigma(a)$ and $b + \tilde{m}2^t = b + m2^s = \sigma(b)$. Thus $(a,b) \sim (\sigma(a), \sigma(b))$, which completes the proof. ■

Example 3.12 Consider cognate pairs of $(1,5)$ in the Catalan graph $G = G_n(C, tC)$. Since the binary A-sequence of G is $(1,1,\ldots)$, we have $\ell = 0$ so that $s = 2$. By Theorem 3.9, every vertex pair with the form of $(1+4m, 5+4m)$ is a cognate pair of $(1,5)$ where m is an integer $0 \le m \le \frac{n-5}{4}$. For example, such pairs are $(1,5),(5,9)$ for $n = 9$, and $(1,5),(5,9),(9,13)$ for $n = 15$. Furthermore, if $n = 15$ then by Theorem 3.11 we have

$$\langle\{1,2,3,4,5\}\rangle \cong \langle\{5,6,7,8,9\}\rangle \cong \langle\{9,10,11,12,13\}\rangle\,.$$

Fig. 10 shows a way to draw CG_9 in a "fractal form".

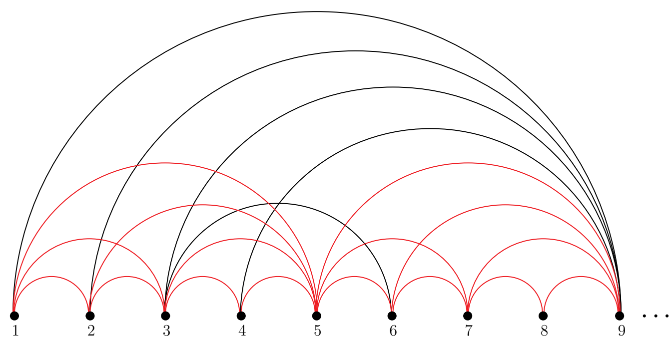


Fig. 10: The Catalan graph $G_9(C, tC)$

3.3 Reverse relabeling of Riordan graphs

Consider a relabeling of a graph G with n vertices labeled by $1,2,\ldots,n$. The relabeling can be done by *reversing* the vertices in $[n]$, that is, by replacing a label i by $n+1-i$ for each $i \in [n]$. Let $\mathcal{A}(G) = B + B^T$ for some $n \times n$ $(0,1)$-lower triangular matrix with zero diagonal. Then, the adjacency matrix of the resulting relabeled graph G' can be obtained from $E\mathcal{A}(G)E = EBE + EB^TE$ where E is the $n \times n$ backward identity matrix corresponding to the reverse relabeling given by

$$E = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{pmatrix}.$$

Since EB^TE is a lower triangular matrix with zero diagonal, the adjacency matrix $\mathcal{A}(G')$ can be expressed by the *flip-transpose* [22] of B defined as $B^F := EB^TE$:

$$\mathcal{A}(G') = B^F + (B^F)^T. \tag{18}$$

Relabeling a Riordan graph does not necessarily result in a Riordan graph. In this section, we show that if $G = G_n(g,f)$ is a proper Riordan graph then the graph resulting in relabeling done by reversing the vertices in $[n]$ is a proper Riordan graph.

Proposition 3.13 ([6]) *Let $L = (g, f)_n$ be an $n \times n$ leading principal matrix of a proper Riordan matrix (g, f). Then, the flip-transpose $L^F = EL^TE$ is a proper Riordan matrix given by*

$$L^F = (g(\bar{f}) \cdot \bar{f}' \cdot (t/\bar{f})^n, \bar{f})_n$$

Proposition 3.13 implies the following theorem.

Theorem 3.14 *The reverse relabelling of a Riordan graph $G = G_n(g, f)$ with $f'(0) = 1$ is a Riordan graph given by $G' = G_n(g(\bar{f}) \cdot \bar{f}' \cdot (t/\bar{f})^{n-1}, \bar{f})$.*

From (18) we have

$$B^F = \mathcal{B}(tg(\bar{f}) \cdot \bar{f}' \cdot (t/\bar{f})^{n-1}, \bar{f})_n. \tag{19}$$

To give an example for Theorem 3.14 we need the following lemma.

Lemma 3.15 (Lucas Theorem) *Let m and n be nonnegative integers with the base p (a prime) expansions:*

- $m = m_k p^k + m_{k-1}p^{k-1} + \cdots + m_1 p + m_0$ *and*
- $n = n_k p^k + n_{k-1}p^{k-1} + \cdots + n_1 p + n_0$.

Then, the following congruence relation holds:

$$\binom{m}{n} \equiv \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod{p}.$$

Example 3.16 Consider the Catalan graph $G = G_n(C, tC)$ when $n = 2^j - 1$ with $j \geq 2$. Let $g = C = \frac{1-\sqrt{1-4t}}{2t}$ and $f = tC$. By simple computations we obtain $\bar{f} = t + t^2 \equiv t - t^2$ and $g(\bar{f}) = C(\bar{f}) = \frac{1}{1-t}$. Thus,

$$g(\bar{f}) \cdot \bar{f}' \cdot (t/\bar{f})^{2^j-2} \equiv \frac{1}{1-t} \cdot (1-2t) \cdot \left(\frac{1}{1-t}\right)^{2^j-2} \equiv \left(\frac{1}{1-t}\right)^{2^j-1}.$$

Let $\tilde{g} = \sum_{k=0}^{2^j-3}[t^k]1/(1-t)^{2^j-1}$. We claim that $\tilde{g} \equiv 1 - t$. By Newton's binomial theorem we obtain

$$\left(\frac{1}{1-t}\right)^{2^j-1} = \sum_{k=0}^{\infty}\binom{2^j+k-2}{k}t^k \equiv 1 - t + \sum_{k=2}^{\infty}\binom{2^j+k-2}{k}t^k.$$

Now, let $k - 2 = a_i 2^i + \cdots + a_2 2^2 + a_1 2 + a_0 2^0$ where $a_i \in \{0, 1\}$. Since $0 \leq k \leq 2^j - 3$, we have $j \geq i$. Thus, we obtain

$$\begin{aligned} 2^j + k - 2 &= 2^j + a_i 2^i + \cdots + a_2 2^2 + a_1 2 + a_0 2^0 \\ k &= a_i 2^i + \cdots + a_2 2^2 + (a_1 + 1)2 + a_0 2^0. \end{aligned}$$

If $a_1 = 0$ then $\binom{a_1}{a_1+1} = \binom{0}{1} = 0$. Let $a_1 = 1$. Then, we can assume that $a_\ell = 0, a_{\ell-1} = \cdots = a_1 = 1$ where $i \geq \ell - 1$. Then $\binom{a_\ell}{a_\ell+1} = \binom{0}{1} = 0$. By Lucas theorem, for $k \geq 2$,

$$\binom{2^j + k - 2}{k} \equiv 0. \tag{20}$$

Thus, $\tilde{g} = 1 - t$ and by Theorem 3.14, the reverse relabeling of the Catalan graph $G_{2^j-1}(C, tC)$ is a Riordan graph corresponding to the Riordan matrix $(1 - t, t - t^2)$. We note that $(1-t, t-t^2)$ is the inverse matrix of (C, tC). Fig. 12 illustrates, as fractals, the adjacency matrices of the two graphs.

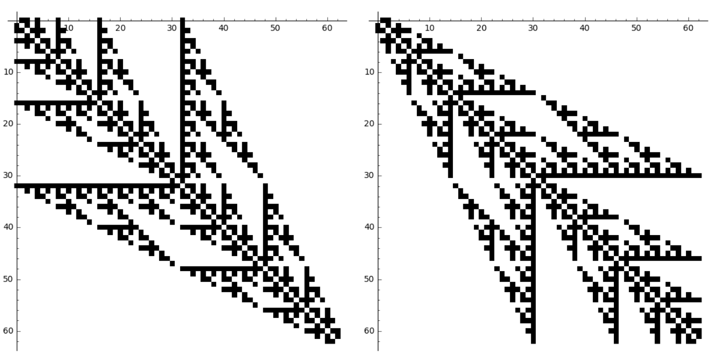


Fig. 12: The fractal natures of a Catalan graph and its reverse labeling

3.4 Eulerian and Hamiltonian Riordan graphs

In this section we give certain conditions on Riordan graphs to have an Eulerian trail/cycle or a Hamiltonian cycle. As the result, we obtain large classes of Riordan graphs which are Eulerian or Hamiltonian.

Theorem 3.17 *Let $G = G_n(g, f)$ be a Riordan graph with $f'(0) = 1$. Then, the degree $d(i)$ of a vertex i is odd (resp., even) if and only if*

$$\phi_{i-1} + \varphi_{n-i} \equiv 1 \ \ (resp.,\ 0)$$

where for $j = 0, \ldots, n-1$

$$\phi_j \equiv [t^j]\frac{tg}{1-f} \quad and \quad \varphi_j \equiv [t^j]\frac{tg(\bar{f}) \cdot (\bar{f})' \cdot (t/\bar{f})^{n-1}}{1-\bar{f}}.$$

Proof. Let $\mathcal{A}(G) = B + B^T$, $\mathbf{d} = (d(1), \ldots, d(n))^T$ and $\mathbf{1} = (1, 1, \ldots, 1)^T \in \mathbb{Z}^n$ where $B = \mathcal{B}(tg, f)_n$. Since $B^T\mathbf{1} = (EB^F E)\mathbf{1} = (EB^F)\mathbf{1}$ where B^F is given by (19), applying the fundamental theorem given by (3) to B and B^F, respectively we obtain

$$\mathbf{d} = \mathcal{A}(G)\mathbf{1} = \left(B + B^T\right)\mathbf{1} = B\mathbf{1} + E(B^F\mathbf{1})$$
$$\equiv \begin{pmatrix} \phi_0 \\ \phi_1 \\ \vdots \\ \phi_{n-1} \end{pmatrix} + \begin{pmatrix} \varphi_{n-1} \\ \varphi_{n-2} \\ \vdots \\ \varphi_0 \end{pmatrix} \equiv \begin{pmatrix} \phi_0 + \varphi_{n-1} \\ \phi_1 + \varphi_{n-2} \\ \vdots \\ \phi_{n-1} + \varphi_0 \end{pmatrix}.$$

Thus, $d(i)$ is odd (resp., even) for $i = 1, \ldots, n$ if and only if $\phi_{i-1} + \varphi_{n-i} \equiv 1$ (resp., 0), as required. ■

Let G be a simple connected graph. Since G has an Eulerian trail if and only if G has no odd vertex or exactly two odd vertices, by Theorem 3.17 we obtain the following corollary.

Corollary 3.18 *Let $G = G_n(g, f)$ be a proper Riordan graph with $f'(0) = 1$. Then, G has an Eulerian trail if and only if ϕ_{i-1} and φ_{n-i} satisfy either for all $i = 1, \ldots, n$,*

$$\phi_{i-1} + \varphi_{n-i} \equiv 0 \tag{21}$$

or for $i_1 \neq i_2$

$$\phi_{i-1} + \varphi_{n-i} \equiv \begin{cases} 1 & \textit{if } i = i_1, i_2 \in \{1, \ldots, n\} \\ 0 & \textit{if } i \in \{1, \ldots, n\} \setminus \{i_1, i_2\} \end{cases}.$$

In particular, if (21) *is satisfied then G is Eulerian.*

Example 3.19 The Pascal graph $PG_n = G_n\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ for $n = 2^k + 1$ $(k \geq 2)$ has an Eulerian trail. We note that PG_n is a proper Riordan graph so that it is a connected graph. Indeed, since $\bar{f} = \frac{t}{1+t}$, simple computations show that

$$\phi_{i-1} \equiv [t^{i-1}]\frac{t}{1-2t} \equiv [t^{i-1}]t,$$
$$\varphi_{n-i} \equiv [t^{n-i}]t(1+t)^{n-1} = \binom{n-1}{i}.$$

By Lucas Theorem, if $n = 2^k + 1$ and $i = 2^k$ then $\binom{n-1}{i} \equiv 1$, and 0 otherwise. Thus,

$$\phi_{i-1} + \varphi_{n-i} \equiv \begin{cases} 1 & \text{if } i = 2 \text{ or } i = 2^k, \\ 0 & \text{if } i \in \{1, \ldots, 2^k + 1\} \setminus \{2, 2^k\}. \end{cases}$$

By Corollary 3.18, PG_{2^k+1} has an Eulerian trail. Similarly, if $n = 2^k$ for $k \geq 3$ then it is shown that PG_n does not have an Eulerian trail.

A polynomial $P = \sum_{k=0}^{n} a_k t^k$ of degree n or a sequence $(a_k)_{k=0}^{n}$ is called *palindromic* if $a_k = a_{n-k}$ for $k = 0, 1, \ldots, n$.

Corollary 3.20 *Let $G = G_n(g,t)$ be a proper Riordan graph of the Appell type with $f'(0) = 1$ and $n \geq 3$ vertices where $g = \sum_{k=0}^{n-2} g_k t^k \in \mathbb{Z}_2[t]$. If g is palindromic with $g(1) \equiv 0$ then G is Eulerian.*

Proof. Since $f = t$, it can be easily shown from Theorem 3.17 that

$$\phi_j = \varphi_j \equiv [t^j]\frac{tg}{1-t} = \begin{cases} 0 & \text{if } j = 0 \\ \sum_{k=0}^{j-1} g_k & \text{if } j = 1, \ldots, n-1. \end{cases}$$

Since $g_0 + g_1 + \cdots + g_{n-2} \equiv 0$ and $g_k \equiv g_{n-2-k}$ for $k = 0, \ldots, n-2$, we obtain $\phi_0 = \phi_{n-1}$ and for $j = 1, \ldots, n-2$,

$$\phi_j \equiv \sum_{k=0}^{j-1} g_k \equiv \sum_{k=j}^{n-2} g_k \equiv \sum_{k=j}^{n-2} g_{n-2-k} \equiv \phi_{n-1-j}.$$

Thus, the sequence $(\phi_k)_{k=0}^{n-1}$ is palindromic. Since $\phi_k = \varphi_k$ it follows from Corollary 3.18 that G is Eulerian. ■

Hamiltonian properties of Toeplitz graphs, which are Riordan graphs of the Appell type, have been studied in [35]. Next, we give more general results. Recall that every proper Riordan graph $G = G_n(g, f)$ has the Hamiltonian path $1 \to 2 \to \cdots \to n$. In particular, if $g = \frac{1}{1-t}$ then G is Hamiltonian, i.e. G has a Hamiltonian cycle.

Theorem 3.21 *Let $G = G_n(g, f)$ be a proper Riordan graph. If one of the following holds then G is Hamiltonian.*

(i) *There exists $i \in \{2, \ldots, n-1\}$ such that $[t^{i-1}]g \equiv 1$ and $[t^{n-2}](gf^{i-1}) \equiv 1$.*
(ii) *$[t]g \equiv 1$ and $[t^2]f \equiv 0$.*

Proof. (i) Let $\mathcal{A}(G) = [a_{ij}]_{1\leq i,j\leq n}$. Since G is proper, we have the path $1 \to 2 \to \cdots \to i$. If $[t^{n-2}](gf^{i-1}) \equiv 1$ for some $i \neq 1$, i.e. $a_{ni} = 1$ then we have the edge $in \in E$. Again, since G_n is proper, we have the path $n \to n-1 \to \cdots \to i+1$. Finally, if $[t^{i-1}]g \equiv 1$ for some $i \neq 1$, i.e. $a_{i+1,1} = 1$ then $(i+1)1 \in E$. Thus we have the following Hamiltonian cycle in G:

$$1 \to 2 \to \cdots \to i \to n \to n-1 \to \cdots \to i+1 \to 1$$

as required.

(ii) Since G is proper and $[t]g \equiv 1$, by Theorem 3.4 we obtain that $\mathcal{A}(\langle V_o \rangle)$ is proper, i.e. we have the path $1 \to 3 \to \cdots \to 2\left\lceil \frac{n}{2} \right\rceil - 1$. Our assumption that $[t^2]f \equiv 0$ implies $[t^2](gf) \equiv 1$. By Theorem 3.4 we obtain that $\mathcal{A}(\langle V_e \rangle)$ is proper, i.e. we have the path $2 \to 4 \to \cdots \to 2\left\lfloor \frac{n}{2} \right\rfloor$. Finally, we obtain the following Hamiltonian cycle in G:

$$1 \to 3 \to \cdots \to 2\left\lceil \frac{n}{2} \right\rceil - 1 \to 2\left\lfloor \frac{n}{2} \right\rfloor \to \cdots \to 4 \to 2 \to 1$$

as required. ■

Example 3.22 The Catalan graph $G = G_n(C, tC)$ when $n = 2^k + 1$ is Hamiltonian. Indeed, it is known [12] that $C_n = [t^n]C \equiv 1$ if and only if $n = 2^k - 1$ for $k \geq 0$:

$$C = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} t^n \equiv 1 + t + t^3 + t^7 + t^{15} + \cdots .$$

Take $i = n - 1$ in (i) of Theorem 3.21. If $n = 2^k + 1$ then we obtain

$$[t^{n-2}]C = [t^{2^k-1}]C \equiv 1 \text{ and } [t^{n-2}]C(tC)^{n-2} = [t^0]C^{n-1} \equiv 1.$$

Thus the Catalan graph $G_{2^k+1}(C, tC)$ is Hamiltonian.

The following result is obtained by Theorem 3.1 in [12] and Proposition 1 in [1].

Lemma 3.23 *The Motzkin number M_n is even if and only if either $n \in S_1$ or $n \in S_2$ where $S_1 = \{4^i(2j-1) - 2 \mid i, j \geq 1\}$ and $S_2 = \{4^i(2j-1) - 1 \mid i, j \geq 1\}$.*

Theorem 3.24 *If $n \neq 4^i(2j-1)$ for $i, j \geq 1$ then the Motzkin graph $G = G_n(M, tM)$ is Hamiltonian for $n \geq 3$.*

Proof. Consider the generating function M of the Motzkin numbers M_n:

$$M = \frac{1 - t - \sqrt{1 - 2t - 3t^2}}{2t^2} = \sum_{n \geq 0} M_n t^n = 1 + t + 2t^2 + 4t^3 + 9t^4 + 21t^5 + 51t^6 + \cdots .$$

Take $i = n - 1$ in (i) of Theorem 3.21. Note that $S_1 \cap S_2 = \emptyset$ in Lemma 3.23. By Lemma 3.23 we obtain

$$[t^{n-2}]M = M_{n-2} \equiv 1 \text{ if and only if } n \notin S_3 \cup S_4$$

where $S_3 := \{4^i(2j-1) \mid i, j \geq 1\}$ and $S_4 := \{4^i(2j-1) + 1 \mid i, j \geq 1\}$. In addition, $[t^{n-2}]M(tM)^{n-2} = [t^0]M^{n-1} \equiv 1$. Thus by Theorem 3.21, if $n \notin S_3 \cup S_4$ then MG_n is Hamiltonian. It remains to prove that if $n \in S_4$ then MG_n is Hamiltonian. To show this, consider the case of $i = 2$ in (i) of Theorem 3.21. Since

$$[t^{n-2}]tM^2(t) \equiv [t^{n-2}]tM(t^2) = [t^{n-2}] \sum_{k \geq 0} M_k t^{2k+1}$$

it follows from $n - 2 = 2k + 1$ that if $n \in S_4$ then

$$k \in \{4^{i-1}(4j-2) - 1 \mid i, j \geq 1\} = \{1, 5, 7, 9, 23, \ldots\}. \tag{22}$$

Since $k \notin S_1 \cup S_2$ for any k in (22), we have $M_k \equiv 1$ by Lemma 3.23 so that $[t^{n-2}]tM^2(t) \equiv 1$. In addition, $[t]M = M_1 \equiv 1$. Thus by Theorem 3.21, if $n \in S_4$ then G is Hamiltonian as required. ■

A complete split graph $CS_{m,n-m}$, $m \leq n$, is a graph on n vertices consisting of a clique K_m on m vertices and a stable set (i.e. independent set) on the remaining $n - m$ vertices, such that any vertex in the clique is adjacent to each vertex in the stable set. The bipartite graph $G(m, n-m)$ obtained from $CS_{m,n-m}$ by deleting all edges in K_m is referred to as the bipartite graph corresponding to $CS_{m,n-m}$.

Lemma 3.25 ([4]) *In a split graph $CS_{m,n-m}$, if $m < n-m$ then $CS_{m,n-m}$ is not Hamiltonian. If $m = n-m$ then $CS_{m,n-m}$ is Hamiltonian if and only if the corresponding bipartite graph $G(m,\, n-m)$ is Hamiltonian.*

Theorem 3.26 *If $G = G_n(g, f)$ is an improper Riordan graph with $[t]f \equiv 0$ then G_n is not Hamiltonian.*

Proof. If f satisfies $[t]f \equiv 0$ then the Riordan graph $G_n(g, f)$ for any g is a subgraph of $G_n(\frac{1}{1-t}, t^2)$. Thus it is sufficient to prove the claim for $G_n(\frac{1}{1-t}, t^2)$. Let $\mathcal{A}(G) = [a_{i,j}]_{1\le i,j\le n}$ and let $C_j(t)$ be the jth column generating function of $\mathcal{B}(tg, f)$ where $g = \frac{1}{1-t}$ and $f = t^2$. We may assume that $i > j$ and let $n = n_1 + n_2$ where $n_1 = \lceil n/2 \rceil$ and $n_2 = \lfloor n/2 \rfloor$. Then $C_{n_2+1}(t) = g \cdot t^{2n_2}$, i.e. $a_{i,j} = 0$ if $j \ge n_2 + 1$. Thus the subgraph of G induced by $\langle\{n_2 + 1, \ldots, n\}\rangle$ is a null graph. Clearly, G is a subgraph of the complete split graph CS_{n_2,n_1}. If n is odd then by Lemma 3.25 CS_{n_2,n_1} is not Hamiltonian. It follows that G is not Hamiltonian. Otherwise, n is even. Again, by Lemma 3.25, CS_{n_2,n_1} is Hamiltonian if the corresponding bipartite graph $G(n_2, n_1)$ is Hamiltonian. But in this case $C_{n_2}(t) = g\, t^{n-2}$. Now if $g_0 \equiv 1$, then $a_{n/2,(n-2)} \equiv 1$ and otherwise if $g_0 \equiv 0$ then $a_{n/2,(n-2)} \equiv 0$. Thus the vertex $n/2$ has maximum degree 1 in the graph $G(n_2, n_1)$, i.e. $G(n_2, n_1)$ is not Hamiltonian. Again, by Lemma 3.25, $CS_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ has no Hamiltonian cycle so that G_n is not Hamiltonian. ■

Proposition 3.27 *Let $G = G_n(g, f)$ be an e-decomposable/o-decomposable Riordan graph and n be even. Then G is Hamiltonian if and only if the bipartite graph with partitions V_o and V_e is Hamiltonian. Moreover, if G is an e-decomposable Riordan graph and n is odd then G is not Hamiltonian.*

Proof. Let G be an e-decomposable/o-decomposable Riordan graph and n be even. In this case, G contains an independent set on $n/2$ vertices and so G is a subgraph of the complete split graph $CS_{n/2,n/2}$. By Lemma 3.25, $CS_{n/2,n/2}$ is Hamiltonian if and only if the corresponding bipartite graph $G(n/2,\, n/2)$ is Hamiltonian.

If G is an e-decomposable Riordan graph and n is odd, then G is a subgraph of the complete split graph $CS_{(n-1)/2,\,(n+1)/2}$. By Lemma 3.25 we obtain the desired result. ■

We end this section by observing that every Riordan graph $G = G_n(g, f)$ of the checkerboard type of odd order n is not Hamiltonian since G is bipartite of odd order by (iv) in Theorem 3.6.

4 Four families of Riordan graphs

In this section, we define io-decomposable and ie-decomposable Riordan graphs, and also consider Riordan graphs of the Bell type and of the derivative type.

4.1 io-decomposable and ie-decomposable Riordan graphs

Definition 4.1 Let $G = G_n(g,f)$ be a proper Riordan graph with the odd and even vertex sets V_o and V_e, respectively.

- If $\langle V_o\rangle \cong G_{\lceil n/2\rceil}(g,f)$ and $\langle V_e\rangle$ is a null graph then G_n is *io-decomposable.*
- If $\langle V_o\rangle$ is a null graph and $\langle V_e\rangle \cong G_{\lfloor n/2\rfloor}(g,f)$ then G_n is *ie-decomposable.*

"io" and "ie" stand for "isomorphically odd" and "isomorphically even", respectively.

Theorem 4.2 *Let $G = G_n(g,f)$ be a proper Riordan graph.*

(i) *G is io-decomposable if and only if $g' \equiv g^2$ and $gf \equiv t\cdot(f/t)'$.*

(ii) *G is ie-decomposable if and only if $g' \equiv 0$ and $t^2g \equiv tf' + f$.*

Proof. (i) Since $g(t^2) \equiv g^2(t)$ by Lemma 3.3, by the definitions and Theorem 3.4, G is io-decomposable if and only if the matrix X in (10) is given by

$$\mathcal{A}\left(G_{\lceil n/2\rceil}(g'(\sqrt{t}), f(t))\right) = \mathcal{A}\left(G_{\lceil n/2\rceil}(g(t), f(t))\right) \quad \Leftrightarrow \quad g'(t) = g(t^2) \equiv g^2(t)$$

and

$$\begin{aligned} G_{\lfloor n/2\rfloor}\left(\left(\frac{gf}{t}\right)'(\sqrt{t}), f(t)\right) \cong N_{\lfloor n/2\rfloor} \quad &\Leftrightarrow \quad 0 \equiv \left(\frac{gf}{t}\right)' \equiv g^2\frac{f}{t} + g\cdot\left(\frac{f}{t}\right)' \\ &\Leftrightarrow \quad gf \equiv t\cdot(f/t)'. \end{aligned}$$

(ii) Similarly, G is ie-decomposable if and only if the matrix X in (10) is given by

$$G_{\lceil n/2\rceil}(g'(\sqrt{t}), f(t)) \cong N_{\lceil n/2\rceil} \quad \Leftrightarrow \quad g'(t) \equiv 0$$

and

$$\begin{aligned} &\mathcal{A}\left(G_{\lfloor n/2\rfloor}\left(\left(\frac{gf}{t}\right)'(\sqrt{t}), f(t)\right)\right) = \mathcal{A}\left(G_{\lfloor n/2\rfloor}(g(t), f(t))\right) \\ \Leftrightarrow \quad & g^2 \equiv \left(\frac{gf}{t}\right)' \equiv g'\frac{f}{t} + g\cdot\left(\frac{f}{t}\right)' \equiv g\cdot\left(\frac{f}{t}\right)' \\ \Leftrightarrow \quad & t^2g \equiv tf' + f. \end{aligned}$$

■

4.2 Riordan graphs of the Bell type

In this section, we consider some properties of a Riordan graph $G_n(g,tg)$ of Bell type with odd and even vertex sets V_o and V_e, respectively.

Theorem 4.3 *Any Riordan graph $G = G_n(g, tg)$ of the Bell type is o-decomposable in which its adjacency matrix is permutational equivalent to*

$$\begin{pmatrix} X & \tilde{B} \\ \tilde{B}^T & O \end{pmatrix}$$

where X is the adjacency matrix of $\langle V_o \rangle = G_{\lceil n/2 \rceil}(g'(\sqrt{t}), tg(t))$ and

$$\tilde{B} = \mathcal{B}(tg, tg)_{\lceil n/2 \rceil, \lfloor n/2 \rfloor} + \mathcal{B}((tg)'(\sqrt{t}), tg))^T_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}. \tag{23}$$

.

Proof. Since for $m \geq 1$

$$[t^{2m}]t \cdot g^2(t) = [t^{2m-1}]g^2(t) \equiv [t^{2m-1}]g(t^2) \equiv 0,$$

by (iii) of Theorem 3.6 G_n is o-decomposable. Its adjacency matrix immediately follows from by Theorem 3.4. ■

Now, we consider io-decomposable Riordan graphs of the Bell type. We first derive conditions on the A-sequences of such graphs.

Lemma 4.4 *A Riordan graph $G = G_n(g, tg)$ is io-decomposable if and only if*

$$g^2 \equiv g', \text{ i.e. } [t^j]g \equiv [t^{2j+1}]g.$$

Proof. Since $\langle V_e \rangle$ is a null graph, by Theorem 4.2 G is io-decomposable if and only if $g' \equiv g^2$. ■

Remark 4.5 Let $[t^j]g = g_j$ with $g_0 = 1$. By Lemma 4.4, a Riordan graph $G_n(g, tg)$ is io-decomposable if and only if $g_{2m} \equiv g_{(2m+1)2^k-1}$ for each $m \geq 0$ and $k \geq 1$, i.e.

$$g = \sum_{n\geq 0} g_{2n} \left(\sum_{k\geq 0} t^{(2n+1)2^k-1} \right).$$

Since g depends only on its even coefficients, the number of io-decomposable Riordan graphs of the Bell type is equal to $2^{\lfloor n/2 \rfloor - 1}$.

Theorem 4.6 *A Riordan graph $G = G_n(g, tg)$ is io-decomposable if and only if its binary A-sequence is of the form $(1, 1, a_2, a_2, a_4, a_4, \ldots)$, $a_{2j} \in \{0, 1\}$ whose generating function is*

$$A(t) \equiv (1 + t) + (1 + t) \sum_{j\geq 1} a_{2j} t^{2j}.$$

Proof. Let G be io-decomposable. Since there is a unique binary generating function $A(t) = \sum_{i\geq 0} a_i t^i \in \mathbb{Z}_2[t]$ such that $g \equiv A(tg)$, by applying derivative to both sides, we obtain $g' \equiv (g + tg') \cdot A'(tg)$. Since $g^2 \equiv g'$ by Lemma 4.4, it implies

$$g \equiv (1 + tg) \cdot A'(tg), \text{ i.e. } A(tg) \equiv (1 + tg) \cdot A'(tg). \tag{24}$$

Since $A'(t) \equiv \sum_{i\geq 0} a_{2i+1}t^{2i}$, the equation (24) is equivalent to

$$\begin{aligned}\sum_{j\geq 0} a_j(tg)^j &\equiv (1+tg)\left(\sum_{i\geq 0} a_{2i+1}(tg)^{2i}\right)\\ &= a_1 + a_1tg + a_3(tg)^2 + a_3(tg)^3 + a_5(tg)^4 + a_5(tg)^5 + \cdots.\end{aligned}$$

Thus $a_{2i} \equiv a_{2i+1}$ for $i \geq 0$ as desired. ■

Corollary 4.7 *Let $A(t)$ be a generating function of the binary A-sequence for a proper Riordan graph $G = G_n(g, tg)$. Then we have:*

(i) *If $A(t) \equiv \sum_{j=0}^{2s+1} t^n$ or $A(t) \equiv (1+t)^{2t-1}$ then G is io-decomposable.*

(ii) *If $A(t) \equiv \sum_{j=0}^{2s} t^n$ or $A(t) \equiv (1+t)^{2t}$ then G is not io-decomposable.*

Proof. (i) If $A(t) \equiv \sum_{j=0}^{2s+1} t^n$ then clearly G is io-decomposable by Theorem 4.6. Now let $A(t) = (1+t)^{2t-1}$. Since (g, tg) is of the Bell type, we obtain

$$g \equiv A(tg) = (1+tg)^{2t-1}.$$

Applying derivative to both sides and using $2t-1 \equiv 1$, we obtain

$$g' \equiv (g+tg')(1+tg)^{2t-2} = \frac{g+tg'}{1+tg}(1+tg)^{2t-1} \equiv \frac{g+tg'}{1+tg}g.$$

It implies that

$$g'(1+tg) \equiv (g+tg')g, \text{ i.e. } g' \equiv g^2.$$

By Lemma 4.4, G is io-decomposable.

(ii) If $A \equiv \sum_{j=0}^{2s} t^n$ then clearly G is not io-decomposable by Theorem 4.6. Now let $A = (1+t)^{2t}$. Using $g = A(tg) \equiv (1+tg)^{2t}$ we obtain $g' \equiv 0$. Since $g^2(t) \equiv g(t^2) \not\equiv 0$ we have $g' \not\equiv g^2$. Thus, G is not io-decomposable. ■

Example 4.8 Note that the Pascal graph PG_n, the Catalan graph CG_n and the Mottkin graph MG_n have the generating functions $1+t$, $(1+t)^{-1}$ and $1+t+t^2$ for binary A-sequences, respectively. By Corollary 4.7 we see that PG_n and CG_n are io-decomposable, but MG_n is not io-decomposable.

Lemma 4.9 *If $G = G_n(g, tg)$ is io-decomposable then the number $m(G)$ of edges of G is given by*

$$m(G) = 2m(G_{\lceil n/2\rceil}) + m(H_{\lfloor n/2\rfloor+1})$$

where $H_{\lfloor n/2\rfloor+1} \cong G_{\lfloor n/2\rfloor+1}\left((tg)'(\sqrt{t}), tg\right)$.

Proof. Let $\sigma(A_n)$ denote the number of 1s in the adjacency matrix $A_n = \mathcal{A}(G)$. Since G is io-decomposable, by (23) we obtain

$$\sigma(A_n) = \sigma(A_{\lceil n/2 \rceil}) + 2\sigma(B_1) + 2\sigma(B_2) \tag{25}$$

where $B_1 = \mathcal{B}((tg)'(\sqrt{t}), tg)_{\lfloor n/2 \rfloor \times \lceil n/2 \rceil}$ and $B_2 = \mathcal{B}(tg, tg)_{\lceil n/2 \rceil \times \lfloor n/2 \rfloor}$. We can see that

$$2\sigma(B_1) = \sigma\left(\mathcal{A}\left(H_{\lfloor n/2 \rfloor + 1}\right)\right) \quad \text{and} \quad 2\sigma(B_2) = \sigma(A_{\lceil n/2 \rceil}).$$

Applying this to (25), we obtain

$$\sigma(A_n) = 2\sigma(A_{\lceil n/2 \rceil}) + \sigma\left(\mathcal{A}\left(H_{\lfloor n/2 \rfloor + 1}\right)\right)$$

which implies the desired result. ■

Since $d_{G_n}(n) = m(G_n) - m(G_{n-1})$, by Lemma 4.9 we obtain the following lemma.

Lemma 4.10 *If a Riordan graph $G_n = G_n(g, tg)$ is io-decomposable then*

(i) $d_{G_n}(n) = 2\{m(G_k) - m(G_{k-1})\} = 2\, d_{G_k}(k)$ *if* $n = 2k - 1$

(ii) $m(H_{k+1}) - m(H_k) = d_{H_{k+1}}(k+1)$ *if* $n = 2k$

where $m(G_0) = 0$ *and* $H_j \cong G_j((tg)'(\sqrt{t}), tg)$.

Example 4.11 (a) Consider the Catalan graph $G_n = G_n(C, tC)$ which is io-decomposable. Since $C = 1 + tC^2$ we obtain

$$(tC)' = \frac{d}{dt}\left(t + (tC)^2\right) \equiv 1 + 2(tC)(tC)' \equiv 1.$$

Thus it follows from Lemmas 4.9 and 4.10 that

$$m(G_n) = 2\, m\left(G_{\lceil n/2 \rceil}\right) + m\left(G_{\lfloor n/2 \rfloor}\right) + 1.$$

Equivalently, we obtain

$$m(G_n) = \begin{cases} 2\, m(G_k) + m(G_{k-1}) + 1 & \text{if } n = 2k - 1 \\ 3\, m(G_k) + 1 & \text{if } n = 2k. \end{cases}$$

This recurrence relation gives

$$m(G_{2^k}) = \frac{3^k - 1}{2} \quad \text{and} \quad m\left(G_{2^k+1}\right) = \frac{3^k - 1}{2} + 2^k.$$

(b) Let $G_n = G_n\left(\frac{1}{1-t}, \frac{t}{1-t}\right)$ be the Pascal graph which is io-decomposable. Since $(tg)' = \left(\frac{t}{1-t}\right)' \equiv \frac{1}{1-t^2}$, by Lemma 4.9 we obtain

$$m(PG_n) = 2m(PG_{\lceil n/2 \rceil}) + m(PG_{\lfloor n/2 \rfloor + 1}).$$

This recurrence relation gives

$$m(PG_{2^k+1}) = 3^k \quad \text{and} \quad m(PG_{2^k}) = 3^k - 2^k$$

which also known as in [32].

Definition 4.12 A vertex in a graph G is *universal* (or *apex*, or *dominating vertex*) if it is adjacent to all other vertices in G.

It is known [11] that the Pascal graphs PG_n which are io-decomposable have two or three universal vertices for $n \geq 2$. The following theorem gives a result in this direction for any io-decomposable Riordan graphs of the Bell type with $2^i + 1$ or $2^i + 2$ vertices.

Theorem 4.13 *Let $G_n = G_n(g, tg)$ be io-decomposable. If $n = 2^i + 1$ for $i \geq 0$, then G_n and G_{n+1} have at least one universal vertex, namely the vertex $2^i + 1$.*

Proof. Let $n = 2^i + 1$. It is enough to show that $d_{G_n}(n) = 2^i$. We prove this by induction on $i \geq 0$. Let $i = 0$. Since $g(0) = 1$, $d_{G_2}(2) = 1$. Thus it holds for $i = 0$. Let $i \geq 1$. Then,

$$\begin{aligned}
d_{G_n}(n) &= 2\{m(G_{2^{i-1}+1}) - m(G_{2^{i-1}})\} \qquad \text{(by Lemma 4.10)} \\
&= 2\{2m(G_{2^{i-2}+1}) + m(H_{2^{i-2}+1}) - 2m(G_{2^{i-2}}) - m(H_{2^{i-2}+1})\} \quad \text{(by Lemma 4.9)} \\
&= 2^2\{m(G_{2^{i-2}+1}) - m(G_{2^{i-2}})\} \\
&= 2\, d_{G_{2^{i-1}+1}}(2^{i-1} + 1) \qquad \text{(by Lemma 4.10)} \\
&= 2^i \qquad \text{(by the induction hypothesis).}
\end{aligned}$$

Thus if $n = 2^i + 1$ then the vertex n is a universal vertex of G_n. In addition, two vertices n and $n+1$ are adjacent in G_{n+1} since G_{n+1} is proper. Thus the vertex n is also a universal vertex of G_{n+1} if $n = 2^i + 1$. ■

Theorem 4.14 *An io-decomposable Riordan graph $G_n(g, tg)$ is $(\lceil \log_2 n \rceil + 1)$-partite.*

Proof. We proceed by induction on $n \geq 2$. Let $n = 2$. Since $G_2(g, tg)$ is clearly bi-partite, the theorem holds for $n = 2$. Let $n \geq 3$. Since $\langle V_o \rangle \cong G_{\lceil n/2 \rceil}(g, tg)$ is io-decomposable, and $\langle V_e \rangle$ is a null graph, by the induction hypothesis, $G_n(g, tg)$ is the $(\lceil \log_2 \lceil n/2 \rceil \rceil + 2)$-partite graph. Now it is enough to show that $\lceil \log_2 \lceil n/2 \rceil \rceil = \lceil \log_2 n \rceil - 1$. For all $k \geq 0$, when $2^k < n \leq 2^{k+1}$ we have

$$\lceil \log_2 \lceil n/2 \rceil \rceil = k = \lceil \log_2 n \rceil - 1.$$

Hence we obtain the desired result. ■

Remark 4.15 We note that if $G_n(g, tg)$ is io-decomposable then it is $(\lceil \log_2 n \rceil + 1)$-partite with partitions $V_1, V_2, \ldots, V_{\lceil \log_2 n \rceil + 1}$ such that

$$V_j = \left\{ 2^{j-1} + 1 + i2^j \mid 0 \leq i \leq \left\lfloor \frac{n - 1 - 2^{j-1}}{2^j} \right\rfloor \right\} \quad \text{if } 1 \leq j \leq \lceil \log_2 n \rceil$$

and $V_{\lceil \log_2 n \rceil + 1} = \{1\}$.

Riordan arrays frequently arising in combinatorics are of Bell type. So it would be interesting to study graph invariants for Riordan graphs of the Bell type.

Definition 4.16 A *clique* is a subset of vertices of a graph G such that its induced subgraph is a complete graph. The *clique number* of G is the number of vertices in a maximum clique in G, and it is denoted by $\omega(G)$.

Theorem 4.17 *For $n \geq 1$, if $G = G_n(g, tg)$ is io-decomposable then*

$$\omega(G) = \lceil \log_2 n \rceil + 1.$$

Proof. It follows from Theorem 4.14 that

$$\omega(G) \leq \lceil \log_2 n \rceil + 1. \tag{26}$$

Let $\tilde{V} = \{1\} \cup \{2^i + 1 \mid 0 \leq i \leq \lceil \log_2 n \rceil - 1\} \subseteq V$. By Theorem 4.13, for every i, $0 \leq i \leq \lceil \log_2 n \rceil$, the vertex $2^i + 1 \in \tilde{V}$ is adjacent to all vertices in $\{1\} \cup \{2^j + 1 \mid 0 \leq j \leq i - 1\}$. Thus the induced subgraph of $\tilde{V}$ is the complete graph $K_{\lceil \log_2 n \rceil + 1}$. By (26) we obtain the desired result. ∎

Since the complete graph K_5 is not planar, from Theorem 4.17 we immediately obtain the following corollary.

Corollary 4.18 *An io-decomposable graph $G_n(g, tg)$ is not planar for all $n \geq 9$.*

It is known [11] that the Pascal graph PG_n is planar for $n \leq 7$ but it is not for $n \geq 8$. Also we may check that the Catalan graph CG_n is planar for $n \leq 8$ but it is not for $n \geq 9$.

Definition 4.19 The *chromatic number* of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color, and it is denoted by $\chi(G)$.

Theorem 4.20 *For $n \geq 1$, if a Riordan graph $G = G_n(g, tg)$ is io-decomposable then*

$$\chi(G) = \lceil \log_2 n \rceil + 1.$$

Proof. Since a k-partite graph is k-colorable and $\chi(G) \geq \omega(G)$, we obtain the desired result by Theorems 4.14 and 4.17. ∎

Definition 4.21 The *distance* between two vertices u, v in a graph G is the number of edges in a shortest path between u and v, and it is denoted by $d(u, v)$. The *diameter* of G is the maximum distance between all pairs of vertices, and it is denoted by $\mathrm{diam}(G)$.

It is obvious that if G has a universal vertex then $\mathrm{diam}(G) = 1$ or 2.

Theorem 4.22 *If a Riordan graph $G = G_n(g, tg)$ is io-decomposable then*

$$\mathrm{diam}(G) \leq \lfloor \log_2 n \rfloor. \tag{27}$$

In particular, if $n = 2^k + 2$ or $2^{k+1} + 1$, for $k \geq 1$, then $\mathrm{diam}(G) = 2$.

Proof. We proceed by induction on $n \geq 2$. Let $n = 2$. Since clearly $\text{diam}(G) = 1 \leq \lfloor \log_2 2 \rfloor$, the statement is true for $n = 2$. Suppose $n \geq 3$. Let $V_1 = \{i \mid 1 \leq i \leq 2^b + 1\}$ and $V_2 = \{i \mid 2^b + 1 \leq i \leq n\}$ where b is an integer such that $2^b < n \leq 2^{b+1}$. By Theorem 4.13, $2^b + 1$ is a universal vertex in the induced subgraph $\langle V_1 \rangle$. Thus, $\text{diam}(\langle V_1 \rangle) \leq 2$. From Theorem 3.11 and Lemma 4.6, we obtain $\langle V_2 \rangle \cong G_{n-2^b}(g, tg)$. By induction hypothesis, we have

$$\text{diam}(\langle V_2 \rangle) \leq \lfloor \log_2 (n - 2^b) \rfloor \leq \lfloor \log_2 n \rfloor - 1. \tag{28}$$

Let $u \in V_1 \backslash \{2^b + 1\}$ and $v \in V_2 \backslash \{2^b + 1\}$. Now it is enough to show that $d(u, v) \leq \lfloor \log_2 n \rfloor$. Since $2^b + 1$ is an universal vertex of $\langle V_1 \rangle$, it follows from (28) that

$$d(u, v) \leq d(u, 2^b + 1) + d(2^b + 1, v) \leq \lfloor \log_2 n \rfloor$$

which proves (27). Now let $n = 2^k + 2$ or $2^{k+1} + 1$ for $k \geq 1$. By Theorem 4.13, every io-decomposable Riordan graph $G = G_n(g, tg)$ has at least one universal vertex. Thus $\text{diam}(G)$ is 1 or 2. Now it is enough to show that G is not a complete graph for $n \geq 4$. Let $\mathcal{A}(G) = [a_{i,j}]_{1 \leq i,j \leq n}$. Since $a_{4,2} \equiv [t^2] t g^2(t) \equiv [t] g(t^2) = 0$, G cannot be K_n for $n \geq 4$. Hence we obtain the desired result. ■

Corollary 4.23 *Let $n \geq 6$ and a Riordan graph $G = G_n(g, tg)$ be io-decomposable. If $2^k + 1 < n < 2^{k+1}$ then*

$$\text{diam}(G) \leq \lfloor \log_2 (n - 2^k) \rfloor + 1.$$

Proof. Let $V_1 = \{i \mid 1 \leq i \leq 2^k + 1\}$ and $V_2 = \{i \mid 2^k + 1 \leq i \leq n\}$. By Theorem 3.11 and Lemma 4.6, we obtain $\langle V_2 \rangle \cong G_{n-2^k}(g, tg)$. Thus, by (27) we have

$$\text{diam}(G_{n-2^k}) \leq \lfloor \log_2 (n - 2^k) \rfloor. \tag{29}$$

Let $u \in V_1 \backslash \{2^k + 1\}$ and $v \in V_2 \backslash \{2^k + 1\}$. Since the vertex $2^k + 1$ is a universal vertex in the induced subgraph $\langle V_1 \rangle$, it follows from (29) that

$$d(u, v) \leq d(u, 2^k + 1) + d(2^k + 1, v) \leq \lfloor \log_2 (n - 2^k) \rfloor + 1,$$

which completes the proof. ■

4.3 Riordan graphs of the derivative type

We now consider some properties of a Riordan graph $G_n(f', f)$ of the derivative type.

Theorem 4.24 *Any Riordan graph $G = G_n(f', f)$ of the derivative type is e-decomposable in which its adjacency matrix is permutational equivalent to*

$$\begin{pmatrix} O & \tilde{B} \\ \tilde{B}^T & Y \end{pmatrix}$$

where Y is the adjacency matrix of $\langle V_e \rangle = G_{\lfloor n/2 \rfloor} \left((f'f/t)' (\sqrt{t}), f(t) \right)$ and

$$\tilde{B} = \mathcal{B}(tf'(t), f(t))_{\lceil n/2 \rceil \times \lfloor n/2 \rfloor} + \mathcal{B}(f'(\sqrt{t}), f(t))^T_{\lfloor n/2 \rfloor \times \lceil n/2 \rceil}.$$

Proof. Let $f = \sum_{i \geq 1} f_i t^i$. Since $[t^{2m-1}]f' = 2mf_{2m} \equiv 0$ for $m \geq 1$, it follows from (ii) in Theorem 3.6 that G_n is o-decomposable. The adjacency matrix of G immediately follows from by Theorem 3.4. ∎

Now, we turn our attention to ie-decomposable Riordan graphs of the derivative type.

Lemma 4.25 *A Riordan graph $G = G_n(f', f)$ is ie-decomposable if and only if*

$$(t + t^2)f' \equiv f, \ \textit{i.e.}\ [t^{2m-1}]f \equiv [t^{2m}]f \ \textit{for all}\ m \geq 1.$$

Proof. Since $\langle V_o \rangle$ in G is a null graph and $f'' \equiv 0$ for all $f \in \mathbb{Z}[[t]]$, by Theorem 4.2 G is ie-decomposable if and only if $(t + t^2)f' \equiv f$. ∎

Theorem 4.26 *A Riordan graph $G = G_n(f', f)$ is ie-decomposable if and only if its binary A-sequence is of the form $(1, 1, a_2, 0, a_4, 0, a_6, 0, \ldots)$ where a_{2i} is 0 or 1 for $i \geq 1$, i.e. $A'(t) \equiv 1$.*

Proof. Since $f \equiv tA(f)$ it follows from Lemma 4.25 that G is ie-decomposable if and only if $(1 + t)f' \equiv f/t \equiv A(f)$. Applying derivative to both sides of $f \equiv tA(f)$ we have

$$f' \equiv A(f) + tf'A'(f) \equiv (1 + t)f' + tf'A'(f).$$

After simplification we obtain $A'(f) \equiv 1$, i.e. $A'(t) \equiv 1$. Since G is proper, it follows that G is ie-decomposable if and only if $A(t) \equiv 1 + t + \sum_{i \geq 1} a_{2i}t^{2i}$ where $a_{2i} \in \{0, 1\}$. ∎

Example 4.27 Consider $G = G_n\left(\frac{1}{1+t^2}, \frac{t}{1+t}\right)$. Then $A(t) = 1 + t$. Since $A'(t) = 1$, it follows from Theorem 4.26 that G is ie-decomposable. For instance, if $n = 9$ then

$$P \left(\begin{array}{cccc|ccccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ \hline
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right) P^T = \left(\begin{array}{ccccc|cccc}
 & & & & & 1 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 0 & 1 \\
 & & O & & & 0 & 1 & 1 & 1 \\
 & & & & & 1 & 1 & 1 & 1 \\
 & & & & & 0 & 0 & 0 & 1 \\ \hline
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right)$$

where $P = [e_1 \mid e_3 \mid \cdots \mid e_9 \mid e_2 \mid e_4 \mid \cdots \mid e_8]^T$.

Theorem 4.28 *Let $G = G_n(f', f)$ be ie-decomposable. Then, G is $(\lfloor \log_2 n \rfloor + 1)$-partite for $n \geq 1$.*

Proof. We proceed by induction on n. Since $G_1(f', f)$ is a single vertex and $G_2(f', f)$ is bi-partite, the theorem holds for $n = 1, 2$. Assume $n \geq 3$. Since $\langle V_e \rangle \cong G_{\lfloor n/2 \rfloor}$ is ie-decomposable and $\langle V_o \rangle$ is a null graph, it follows from induction hypothesis that G_n is $(\lfloor \log_2 \lfloor n/2 \rfloor \rfloor + 2)$-partite. Now it is enough to show that $\lfloor \log_2 \lfloor n/2 \rfloor \rfloor = \lfloor \log_2 n \rfloor - 1$. If $2^k \leq n < 2^{k+1}$ for all $k \geq 1$ then $\lfloor \log_2 \lfloor n/2 \rfloor \rfloor = k - 1 = \lfloor \log_2 n \rfloor - 1$. Thus we obtain the desired result. ∎

5 Concluding remarks and open problems

In this paper, we use the notion of a Riordan matrix to introduce the notion of a Riordan graph, and based on it, to introduce the notion of an *unlabelled* Riordan graph. The studies conducted by us are aimed at *structural properties* of (various classes of) Riordan graphs; *spectral properties* of Riordan graphs are studied by us in the follow up paper [8].

Even though our paper establishes a number of fundamental structural results, many more such results are yet to be discovered. In particular, we would like to extend our results on graph properties for Riordan graphs of the Bell type to other family of Riordan graphs. Other specific problems we would like to be solved are as follows.

Problem 1 Characterize *unlabelled* Riordan graphs.

Problem 2 Enumerate *unlabelled* Riordan graphs.

Problem 3 Characterize Riordan graphs whose complements are Riordan in labelled and unlabelled cases. See Section 2.6 for relevant observations.

Problem 4 What is the complexity of recognizing labelled/unlabelled Riordan graphs?

Problem 5 Characterize Riordan graphs in terms of forbidden subgraphs, or otherwise.

Problem 6 Find graph invariants not considered in this paper for io-decomposable Riordan graphs of the Bell type, e.g. the *independence number*, *Wiener index*, *average path length*, and so on.

Let G_n be an io-decomposable Riordan graph of the Bell type. Then, one can check that $\mathrm{diam}(G_1) = 0$ and $\mathrm{diam}(G_n) = 1$ for $n = 2, 3$. Based on computer experiments, in the ArXiv version of our paper [7], we have published the following conjecture.

Conjecture 1 Let G_n be an io-decomposable Riordan graph of the Bell type. Then,

$$2 = \mathrm{diam}(PG_n) \leq \mathrm{diam}(G_n) \leq \mathrm{diam}(CG_n)$$

for $n \geq 4$. Moreover, PG_n is the only graph in the class of io-decomposable graphs of the Bell type whose diameter is 2 for *all* $n \geq 4$.

However, it was shown in [26] that Conjecture 1 is false, because its statement does not work for certain graphs G_n and values of n. See [26] for the updated version of Conjecture 1. In either case, we conclude our paper with the following conjecture.

Conjecture 2 We have that $diam(CG_{2^k}) = k$ and there are no io-decomposable Riordan graphs $G_{2^k} \ncong CG_{2^k}$ of the Bell type satisfying $diam(G_{2^k}) = k$ for all $k \geq 1$.

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