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Nonlocal diffusion, a Mittag-Leffler function and a two-dimensional Volterra integral equation

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ABSTRACT

In this paper we consider a particular class of two-dimensional singular Volterra integral equations. Firstly we show that these integral equations can indeed arise in practice by considering a diffusion problem with an output flux which is nonlocal in time; this problem is shown to admit an analytic solution in the form of an integral. More crucially, the problem can be re-characterized as an integral equation of this particular class. This example then provides motivation for a more general study: an analytic solution is obtained for the case when the kernel and the forcing function are both unity. This analytic solution, in the form of a series solution, is a variant of the Mittag-Leffler function. As a consequence it is an entire function. A Gronwall lemma is obtained. This then permits a general existence and uniqueness theorem to be proved.

1. Introduction

In this article we shall consider the class of second kind Volterra integral equations of the form

$$y(t) = \int_0^t \int_0^\tau \frac{k(t,\tau,\sigma)y(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau + f(t), \quad 0 \le \alpha,\ \beta < 1, \tag{1.1}$$

where $(t,\tau,\sigma) \in \Omega \doteq \{0 \le \sigma \le \tau \le t \le T\}$ and $y(0) = f(0)$.

In addition, the functions k and f are assumed to be sufficiently smooth and $k(t,t,t) \not\equiv 0$ for all $t \in [0,T]$.

2. A diffusion problem

Consider the nonlocal (in time) diffusion problem

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$$\frac{\partial c}{\partial t}(x,t) = \frac{\partial^2 c}{\partial x^2}(x,t), \quad 0 < x < 1,\ t > 0, \tag{2.1}$$

$$\frac{\partial c}{\partial x}(0,t) = 0, \quad t > 0, \tag{2.2}$$

$$\frac{\partial c}{\partial x}(1,t) = -\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} c(1,\tau)\, d\tau, \quad t > 0, \tag{2.3}$$

subject to initial condition $c(x,0) = c_0$.

Take Laplace Transforms with respect to t:

$$\frac{d^2\bar{c}}{dx^2}(x,s) - s\bar{c}(x,s) = -c_0$$

yielding the solution

$$\bar{c}(x,s) = A(s)\cosh\sqrt{s}x + B(s)\sinh\sqrt{s}x + \frac{c_0}{s},$$

where $\bar{c}(x,s) = \int_0^\infty e^{-st}c(x,t)dt$.

From (2.2) we have

$$\bar{c}(x,s) = A(s)\cosh\sqrt{s}x + \frac{c_0}{s}.$$

Set $x = 1$ and solve for $A(s)$:

$$A(s) = \left(\bar{c}(1,s) - \frac{c_0}{s}\right) / \cosh\sqrt{s}$$

or

$$\bar{c}(x,s) = \frac{(\bar{c}(1,s) - \frac{c_0}{s})}{\cosh\sqrt{s}}\cosh\sqrt{s}x + \frac{c_0}{s}. \tag{2.4}$$

Differentiate with respect to x and employ (2.3):

$$\frac{(\bar{c}(1,s) - \frac{c_0}{s})}{\cosh\sqrt{s}}\sqrt{s}\sinh\sqrt{s} = -L\left[\int_0^t \frac{1}{\sqrt{\pi(t-\tau)}} c(1,\tau)\, d\tau\right] = -\frac{1}{\sqrt{s}}\bar{c}(1,s)$$

by convolution.

Therefore

$$\bar{c}(1,s) = \frac{c_0}{s} - \left(\frac{\coth\sqrt{s}}{\sqrt{s}}\right)\left(\frac{1}{\sqrt{s}}\right)\bar{c}(1,s). \tag{2.5}$$

However [2],

$$L^{-1}\left[\frac{\coth\sqrt{s}}{\sqrt{s}}\right] = 1 + 2\sum_{n=1}^\infty e^{-n^2\pi^2 t} = \frac{1}{\sqrt{\pi t}}\left(1 + 2\sum_{n=1}^\infty e^{-n^2/t}\right). \tag{2.6}$$

Using (2.6) and applying convolution twice we observe that (2.5) transforms to

$$c(1,t) = c_0 - \int_0^t \int_0^\tau \frac{1}{\sqrt{\pi(t-\tau)}}\left(1 + 2\sum_{n=1}^\infty e^{-n^2/(t-\tau)}\right)\frac{1}{\sqrt{\tau-\sigma}} c(1,\sigma)\, d\sigma\, d\tau$$

or

$$c(1,t) = c_0 - \frac{1}{\sqrt{\pi}} \int_0^t \int_0^\tau \frac{1}{(t-\tau)^{1/2}(\tau-\sigma)^{1/2}} \left(1 + 2\sum_{n=1}^{\infty} e^{-n^2/(t-\tau)}\right) c(1,\sigma)\, d\sigma\, d\tau. \tag{2.7}$$

We note that this integral equation is an example of the class of integrals equations given by (1.1).

3. Analytic solution

Consider (2.5) and solve for $\bar{c}(1,s)$:

$$\bar{c}(1,s) = \frac{c_0}{s + \coth\sqrt{s}} = \frac{c_0 \sinh\sqrt{s}}{s\sinh\sqrt{s} + \cosh\sqrt{s}}. \tag{3.1}$$

Using the residue theorem (for details, see Appendix A) we obtain

$$c(1,t) = \frac{c_0}{\pi} \int_0^\infty \frac{e^{-xt}\sin\sqrt{x}\cos\sqrt{x}}{x^2\sin^2\sqrt{x} + \cos^2\sqrt{x}}\, dx \tag{3.2}$$

or, alternatively,

$$c(1,t) = \frac{c_0}{\pi} \int_0^\infty \frac{e^{-y^2 t} y \sin 2y}{y^4 \sin^2 y + \cos^2 y}\, dy \tag{3.3}$$

writing $y = \sqrt{x}$.

Returning to (2.4) and noting that [1]

$$\mathfrak{L}^{-1}\left[\frac{\cosh\sqrt{s}x}{\cosh\sqrt{s}}\right] = \pi \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)\cos\left[\frac{2n-1}{2}\pi x\right] e^{-(2n-1)^2\pi^2 t/4}$$

the solution $c(x,t)$ may be written down as a convolution integral

$$c(x,t) = c_0 + \int_0^t \left\{ \pi \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)\cos\left[\frac{2n-1}{2}\pi x\right] e^{-(2n-1)^2\pi^2(t-u)/4} * \frac{c_0}{\pi}\int_0^\infty \frac{e^{-y^2 u} y\sin 2y}{y^4\sin^2 y + \cos^2 y}\, dy - c_0 \right\} du. \tag{3.4}$$

Upon interchanging the integrals and integrating with respect to u we obtain

$$c(x,t) = c_0 \left\{ 1 + \pi \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)\cos\frac{(2n-1)\pi x}{2} \left[\frac{4}{(2n-1)^2\pi^2}\left(e^{-((2n-1)\pi)^2 t/4} - 1\right) + \frac{1}{\pi}\int_0^\infty \frac{e^{-y^2 t} - e^{-((2n-1)\pi)^2 t/4}}{\left(\frac{((2n-1)\pi)^2}{4} - y^2\right)(y^4\sin^2 y + \cos^2 y)} y \sin 2y\, dy \right] \right\}.$$

However,

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left(\frac{2n-1}{2}\pi x\right)$$

is readily identified as the Fourier series of unity, so further simplification is possible, giving

$$c(x,t) = c_0\left\{\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)}\cos\frac{(2n-1)\pi x}{2}e^{-((2n-1)\pi)^2t/4} + \sum_{n=1}^{\infty}(-1)^{n-1}(2n-1)\cos\frac{(2n-1)\pi x}{2}\int_0^{\infty}\frac{(e^{-y^2t}-e^{-((2n-1)\pi)^2t/4})y\sin 2y\,dy}{(\frac{((2n-1)\pi)^2}{4}-y^2)(y^4\sin^2 y+\cos^2 y)}\right\}. \tag{3.5}$$

4. An analytic solution to (1.1) when $k(t,\tau,\sigma)\equiv 1$ and $f(t)\equiv 1$

Recall the integral equation

$$y(t) = \int_0^t\int_0^\tau \frac{k(t,\tau,\sigma)y(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau + f(t), \quad 0\le\alpha,\beta<1. \tag{4.1}$$

By considering a diffusion problem in Section 2 we were able to demonstrate how such integrals might arise and this has provided the motivation for a more general study of this class of integral equations. We shall first consider (4.1) above with $k(t,\tau,\sigma)\equiv 1$ and $f(t)\equiv 1$. We shall see that it is then possible to write down an analytic solution to this integral equation in terms of a variant of the Mittag-Leffler function [4].

Theorem 1. *The integral equation (4.1) with $k(t,\tau,\sigma)\equiv 1$ and $f(t)\equiv 1$ admits the analytic solution*

$$y(t) = E_{2-\alpha-\beta}\big(\Gamma(1-\alpha)\Gamma(1-\beta)t^{2-\alpha-\beta}\big) \tag{4.2}$$

where

$$E_a(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(ak+1)}, \quad a>0, \tag{4.3}$$

is the Mittag-Leffler function.

Proof. By interchanging the order of integration in (4.1), it is readily seen that

$$y(t) = 1 + B(1-\alpha,1-\beta)\int_0^t(t-\tau)^{1-\alpha-\beta}y(\tau)\,d\tau \tag{4.4}$$

where $B(1-\alpha,1-\beta)$ is the Beta function. Set up the Picard iterates

$$y_{n+1}(t) = 1 + B(1-\alpha,1-\beta)\int_0^t(t-\tau)^{1-\alpha-\beta}y_n(\tau)\,d\tau \tag{4.5}$$

with $y_0(t)=1$.

Suppose that after k iterations we have

$$y_k(t) = \sum_{\nu=0}^{k-1} \frac{(\Gamma(1-\alpha))^\nu (\Gamma(1-\beta))^\nu}{\Gamma((2-\alpha-\beta)\nu+1)} t^{(2-\alpha-\beta)\nu} \tag{4.6}$$

and this is true for all $k = 1, 2, \cdots, n$.

We shall now use induction to prove that (4.6) is true for $k = n+1$, and consequently all n. Consider (4.5):

$$\begin{aligned} y_{n+1}(t) &= 1 + B(1-\alpha, 1-\beta) \int_0^t (t-\tau)^{1-\alpha-\beta} \left\{ \sum_{k=0}^{n-1} \frac{(\Gamma(1-\alpha))^k (\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} t^{(2-\alpha-\beta)k} \right\} d\tau \\ &= 1 + B(1-\alpha, 1-\beta) \sum_{k=0}^{n-1} \frac{(\Gamma(1-\alpha))^k (\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} \int_0^t (t-\tau)^{1-\alpha-\beta} \tau^{(2-\alpha-\beta)k} \, d\tau. \end{aligned} \tag{4.7}$$

Now with the variable transformation $\tau = wt$, we see that

$$\int_0^t (t-\tau)^{1-\alpha-\beta} \tau^{(2-\alpha-\beta)k} \, d\tau = B\big(2-\alpha-\beta, (2-\alpha-\beta)k+1\big) t^{(2-\alpha-\beta)(k+1)}.$$

Thus

$$\begin{aligned} y_{n+1}(t) &= 1 + B(1-\alpha, 1-\beta) \sum_{k=0}^{n-1} \frac{(\Gamma(1-\alpha))^k (\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} \\ &\quad * B\big(2-\alpha-\beta, (2-\alpha-\beta)k+1\big) t^{(2-\alpha-\beta)(k+1)} \\ &= 1 + \sum_{k=0}^{n-1} \frac{(\Gamma(1-\alpha))^{k+1} (\Gamma(1-\beta))^{k+1}}{\Gamma((2-\alpha-\beta)(k+1)+1)} t^{(2-\alpha-\beta)(k+1)} \\ &= \sum_{k=0}^{n} \frac{(\Gamma(1-\alpha))^k (\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} t^{(2-\alpha-\beta)k} \end{aligned}$$

where use has been made of the standard relationship

$$B(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y). \tag{4.8}$$

The induction step is therefore complete. Thus

$$\begin{aligned} y(t) = \lim_{n\to\infty} y_n(t) &= \sum_{k=0}^{\infty} \frac{(\Gamma(1-\alpha))^k (\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} t^{(2-\alpha-\beta)k} \\ &= E_{2-\alpha-\beta}\big(\Gamma(1-\alpha)\Gamma(1-\beta) t^{2-\alpha-\beta}\big). \end{aligned}$$

Since the Mittag-Leffler function is an entire function, clearly the solution of (4.1) is also an entire function. □

4.1. Relationship with the exponential function

When $\beta = 1-\alpha$, $2-\alpha-\beta = 1$ and so $y(t)$ reduces to

$$y(t) = E_1(z) = E_1\big(\Gamma(1-\alpha)\Gamma(\alpha)t\big) = \exp\big(\Gamma(1-\alpha)\Gamma(\alpha)t\big). \tag{4.9}$$

This is also readily obtainable by considering (4.4) which, in the case of $\beta = 1-\alpha$, reduces to

$$y(t) = B(1-\alpha,\alpha)\int_0^t y(u)du + 1$$

yielding the solution

$$y(t) = \exp\big(B(1-\alpha,\alpha)t\big) = \exp\big(\Gamma(1-\alpha)\Gamma(\alpha)t\big)$$

since $y(0) = 1$.

5. Gronwall inequality

Lemma 1. *Given*

$$y(t) \le \int_0^t \int_0^\tau \frac{k(t,\tau,\sigma)y(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau + f(t), \quad 0<\alpha,\ \beta<1,$$

then

$$y(t) \le F E_{2-\alpha-\beta}\big(K\Gamma(1-\alpha)\Gamma(1-\beta)t^{2-\alpha-\beta}\big) \tag{5.1}$$

where $E_a(z)$ is the Mittag-Leffler function, $|k(t,\tau,\sigma)| < K$ and $|f(t)| < F$.

Proof. Consider the Picard iterates

$$y_{n+1}(t) = F + K\int_0^t \int_0^\tau \frac{y_n(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau, \quad n = 0,1,2,\ldots$$

with $y_0(t) = F$.

Assume that

$$y_k(t) \le F\sum_{k=0}^{\infty} K^k \frac{(\Gamma(1-\alpha))^k(\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} t^{(2-\alpha-\beta)k}$$

is true for all $k = 1,2,\ldots,n$.

The induction proof is essentially the same as in Theorem 4.3 and we obtain

$$y(t) = \lim_{n\to\infty} y_n(t) \le \lim_{n\to\infty} F\sum_{k=0}^{n} K^k \frac{(\Gamma(1-\alpha))^k(\Gamma(1-\beta))^k}{\Gamma((2-\alpha-\beta)k+1)} t^{(2-\alpha-\beta)k}.$$

Again following the ideas of Theorem 4.3 it can be clearly shown that

$$y(t) \le F E_{2-\alpha-\beta}\big(K\Gamma(1-\alpha)\Gamma(1-\beta)t^{2-\alpha-\beta}\big). \qquad \square$$

6. Existence and uniqueness

In this section we shall address the question of existence and uniqueness of a solution to Eq. (1.1). We have the following result.

Theorem 2. *Let $k(t,\tau,\sigma)$ and $f(t)$ be sufficiently smooth functions on Ω and $[0,T]$ respectively. Then Eq. (1.1) has a unique solution.*

Proof. From (4.2) we observe that

$$\phi(t) = E_{2-\alpha-\beta}\big(K\Gamma(1-\alpha)\Gamma(1-\beta)t^{2-\alpha-\beta}\big)$$

is the solution to the following problem

$$\phi(t) = 1 + K\int_0^t\int_0^\tau \frac{\phi(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau. \tag{6.1}$$

Also, it follows from the definition of the Mittag-Leffler function that $\phi(t) > 1$, $\forall t \in [0,T]$, so that we can define the norm in $C[0,T]$:

$$\|x\|_\phi = \sup_{t\in[0,T]}\left\{\frac{|x(t)|}{\phi(t)}\right\} \tag{6.2}$$

which is equivalent to the infinite norm in $C[0,T]$, (see [3]). We again set up the Picard iteration for solving (1.1),

$$\begin{aligned} y_0(t) &= f(t) \\ y_{n+1}(t) &= f(t) + \int_0^t\int_0^\tau \frac{k(t,\tau,\sigma)y_n(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau \end{aligned} \tag{6.3}$$

and prove that the operator $H : C[0,T] \mapsto C[0,T]$ defined by

$$Hx(t) = f(t) + \int_0^t\int_0^\tau \frac{k(t,\tau,\sigma)x(\sigma)}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau$$

is well defined in $C[0,T]$ and is a contraction in the Banach space $(C[0,T], \|\cdot\|_\phi)$.

The fact that $Hx \in C[0,T]$ follows easily from f and k being smooth functions. Now

$$\begin{aligned} \big|Hx_1(t) - Hx_2(t)\big| &\le \int_0^t\int_0^\tau \frac{|k(t,\tau,\sigma)|\,|x_1(\sigma)-x_2(\sigma)|}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau \\ &\le K\int_0^t\int_0^\tau \frac{|x_1(\sigma)-x_2(\sigma)|}{(t-\tau)^\alpha(\tau-\sigma)^\beta}\,d\sigma\,d\tau \end{aligned} \tag{6.4}$$

where $K = \max_\Omega\{|k(t,\tau,\sigma)|\}$. Dividing (6.4) through by $\phi(t)$ and using the definition of the norm $\|\cdot\|_\phi$ we can write

$$
\begin{aligned}
\frac{|Hx_1(t) - Hx_2(t)|}{\phi(t)} &\le \frac{K}{\phi(t)} \int_0^t \int_0^\tau \frac{|x_1(\sigma) - x_2(\sigma)|}{\phi(\sigma)} \frac{\phi(\sigma)}{(t-\tau)^\alpha (\tau-\sigma)^\beta} \, d\sigma \, d\tau \\
&\le \frac{K}{\phi(t)} \|x_1 - x_2\|_\phi \int_0^t \int_0^\tau \frac{\phi(\sigma)}{(t-\tau)^\alpha (\tau-\sigma)^\beta} \, d\sigma \, d\tau \\
&= \frac{K}{\phi(t)} \|x_1 - x_2\|_\phi \frac{1}{K} \big(\phi(t) - 1\big) \\
&= \left(1 - \frac{1}{\phi(t)}\right) \|x_1 - x_2\|_\phi = \delta \|x_1 - x_2\|_\phi
\end{aligned}
$$

where $\delta < 1$ since $\phi(t) > 1$, $\forall t \in [0, T]$.

It remains to take the supremum of the left-hand side to give the result

$$\|Hx_1 - Hx_2\|_\phi \le \delta \|x_1 - x_2\|_\phi.$$

This proves that H is a contraction. The Banach fixed point theorem can then be applied to show that the Picard iteration converges to a solution of (1.1).

To prove uniqueness we require the Gronwall lemma of the previous section. If $x_1(t)$ and $x_2(t)$ are both solutions of (1.1) then $z(t) = x_1(t) - x_2(t)$ is also a solution of (1.1) with $f(t) \equiv 0$. Clearly $z(t)$ satisfies the assumption of Lemma 1. This lemma can then be applied to $z(t)$ to give

$$z(t) \le F E_{2-\alpha-\beta}\big(K\Gamma(1-\alpha)\Gamma(1-\beta)t^{2-\alpha-\beta}\big)$$

where F is the maximum of the forcing term f, which in this case is identically zero. Hence $z(t)$ is identically zero and this proves the uniqueness of the solution. □

Remark 1. Note that Theorem 2 remains true even when Eq. (1.1) is nonlinear i.e. $k \equiv k(t, \tau, \sigma, y)$ as long as k is Lipschitz continuous in the y-variable, i.e.

$$\big|k(t, \tau, \sigma, y_1) - k(t, \tau, \sigma, y_2)\big| \le L\|y_1 - y_2\|, \quad \forall (t, \tau, \sigma) \in \Omega$$

where L is the Lipschitz constant.

7. Concluding remarks

This paper has been concerned with a special class of two dimensional Volterra integral equations. A nonlocal diffusion problem was introduced and it was shown that this problem could be re-characterized as a two-dimensional Volterra integral equation of type (1.1). The diffusion problem was shown to have an analytic solution.

With this motivation the integral equation was studied in more detail. We showed that, when the kernel and forcing function are both chosen to have the value of unity, this gave rise to an analytic solution in terms of a variant of the Mittag-Leffler function. A Gronwall inequality was demonstrated and then employed to prove existence and uniqueness.

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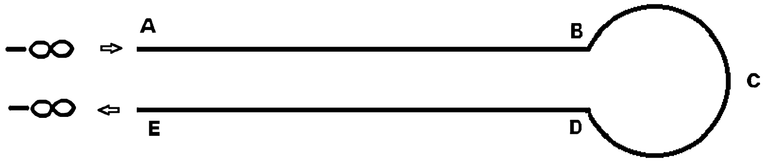

Fig. 1. Hankel contour.

Appendix A

From the Hankel contour (see Fig. 1) we observe that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\bar{c}(1,s)ds = -\sum_{H_k}\frac{1}{2\pi i}\int_{H_k} e^{st}\bar{c}(1,s)ds$$

where the sum is over the segments AB and DE. This is evident since the integrals around the circle of radius R ($R \to \infty$) and around the circle BCD of radius ϵ ($\epsilon \to 0$) are both zero.

Thus

$$\begin{aligned} c(1,t) &= \mathfrak{L}^{-1}\left[\frac{e^{st}c_0}{s+\coth\sqrt{s}}\right] \\ &= -\frac{c_0}{2\pi i}\left\{\int_{AB}\frac{e^{st}}{s+\coth\sqrt{s}}ds + \int_{DE}\frac{e^{st}}{s+\coth\sqrt{s}}ds\right\} \\ &= -\frac{c_0}{2\pi i}(I_1+I_2). \end{aligned}$$

Consider I_1 and write $s = xe^{i\pi} = -x$ and hence $\sqrt{s} = \sqrt{x}e^{i\pi/2} = i\sqrt{x}$.

Thus

$$I_1 = \lim_{\substack{\epsilon\to 0\\ R\to\infty}}\int_{-R}^{-\epsilon}\frac{e^{st}}{s+\coth\sqrt{s}}ds = -\int_0^\infty \frac{e^{-xt}}{(x-i\cot\sqrt{x})}dx.$$

Consider I_2 and write $s = xe^{-i\pi} = -x$, $\sqrt{s} = \sqrt{x}e^{-i\pi/2} = -i\sqrt{x}$.

Thus, similarly

$$I_2 = \int_0^\infty \frac{e^{-xt}}{x+i\cot\sqrt{x}}dx.$$

So

$$\begin{aligned} c(1,t) &= -\frac{c_0}{2\pi i}\left[-\int_0^\infty \frac{e^{-xt}}{(x-i\cot\sqrt{x})}dx + \int_0^\infty \frac{e^{-xt}}{(x+i\cot\sqrt{x})}dx\right] \\ &= \frac{c_0}{\pi}\int_0^\infty \frac{e^{-xt}\cot\sqrt{x}}{x^2+\cot^2\sqrt{x}}dx, \end{aligned}$$

or equivalently,

$$c(1,t) = \frac{c_0}{\pi} \int_0^\infty \frac{e^{-xt} \sin \sqrt{x} \cos \sqrt{x}}{x^2 \sin^2 \sqrt{x} + \cos^2 \sqrt{x}} dx.$$

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