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ADVANCES IN THE LASALLE-TYPE THEOREMS FOR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. This paper considers stochastic functional differential equations (SFDEs) with infinite delay. The main aim is to establish the LaSalle-type theorems to locate limit sets for this class of SFDEs. In comparison with the existing results, this paper gives more general results under the weaker conditions imposed on the Lyapunov function. These results can be used to discuss the asymptotic stability and asymptotic boundedness for SFDEs with infinite delay. In the end, two examples will be given to illustrate applications of our new results established.

1. **Introduction.** In the past few decades, the theory of stochastic functional differential equations (SFDEs) has attracted a great deal of attention, see, for example, [1, 2]. In particular, many papers have been devoted to the study of the stability of SFDEs since the stability has wide applications in automatic control, mechanical system and so on, see, for example, [3, 4]. In particular, the Lyapunov method has been used to deal with stochastic stability by many authors, see, for example, [5, 6, 4].

One of the most important developments of the Lyapunov method is the LaSalle theorem which locates limit sets of deterministic non-autonomous systems, see, for example, [7, 8]. In [9], Mao firstly established stochastic version of the LaSalle

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theorem (i.e., Theorem 1 of LaSalle [8]). Later, Mao and his coauthors established various stochastic versions of the LaSalle theorem for SFDEs with finite delay, by which many classical attraction and stability results for deterministic systems are extended to stochastic systems, see, for example, [10, 11, 12, 13, 14]. Along this line, Wu and Hu [15] established the stochastic LaSalle theorem for neutral SFDEs with infinite delay under the bounded continuous functions space $BC((-\infty;0];\mathbb{R}^n)$ together with the $L^p((-\infty,0];\mathbb{R}^n)$ which denotes all functions $h:(-\infty,0]\to\mathbb{R}^n$ with $\int_{-\infty}^0|h|^p ds<\infty$. However, in [15], we imposed much restrictive conditions on the Lyapunov function if the non-autonomous SFDEs with infinite delay are considered, for example, we requires the polynomial growth condition and this growth is independent of t (namely the inequality (3.2) in Theorem 3.1 of [15]). In fact, it is very difficult to construct an appropriate Lyapunov function satisfying this inequality when we consider non-autonomous systems.

In this paper, we choose the phase space $\mathcal{C}_r$, for given $r>0$, with the fading memory defined by

$$\mathcal{C}_r=\left\{\phi\in C((-\infty,0];\mathbb{R}^n):\sup_{-\infty<\theta\le 0}e^{r\theta}|\phi(\theta)|<\infty\right\} \tag{1.1}$$

with its norm $\|\phi\|_r=\sup_{-\infty<\theta\le0}e^{r\theta}|\phi(\theta)|$, where $C((-\infty,0];\mathbb{R}^n)$ denotes the family of continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$; see [16] for more detail on this phase space and its properties. By using this space, this paper will establish the more general stochastic LaSalle theorems for SFDEs with infinite delay under the weaker conditions comparing with our previous results [15].

Consider an SFDEs with infinite delay of the form

$$dx(t)=f(x_t,t)dt+g(x_t,t)d\omega(t) \tag{1.2}$$

on $t\ge0$ with the initial data $x_0=\xi\in\mathcal{C}_r$, where $x_t=x_t(\theta)\triangleq\{x(t+\theta),-\infty<\theta\le 0\}$, $f:\mathcal{C}_r\times\mathbb{R}_+\to\mathbb{R}^n$ and $g:\mathcal{C}_r\times\mathbb{R}_+\to\mathbb{R}^{n\times m}$ are Borel measurable, $\omega(t)$ is an m-dimensional Brownian motion. To show the dependence of the solution $x(t)$ on the initial data, we also write $x(t)$ as $x(t,\xi)$. Correspondingly, we also write x_t as $x_t(\xi)$. When $-\infty<\theta\le0$ is considered, $x_t(\xi)$ can be written as $x_t(\theta,\xi)$. If Eq.(1.2) has a solution $x(t,\xi)$ with the initial data ξ, then $x_t(\xi)$ is called the solution map.

The rest of this paper is organized as follows. Section 2 provides necessary notations, definitions for preparation of our study. Then we will establish a LaSalle-type theorem for Eq.(1.2) in section 3. In section 4, we will provide two examples to illustrate our new results.

2. **Preliminaries.** Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual condition, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\omega(t)$ be an m-dimensional Brownian motion defined on this probability space. If $x(t)$ is an $\mathbb{R}^n$-value stochastic process, define $x_t=x_t(\theta)\triangleq\{x(t+\theta):-\infty<\theta\le0\}$ for $t\ge0$. Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and $|\cdot|$ the Euclidean norm. Let $\mathbb{R}_+=[0,\infty)$. If A is a vector or a matrix, its transpose is denoted by A^T. For a matrix A, denote its trace norm by $|A|=\sqrt{\text{trace}(A^TA)}$. For a set A, A^c represents its complementary set. Recall that $\mathcal{C}_r$ was defined in (1.1), which is a Banach space with norm $\|\varphi\|_r=\sup_{-\infty<\theta\le0}e^{r\theta}|\varphi(\theta)|$. Let $\mathbf{1}_G$ denote the indicator function of the set G, and $[b]$ the integer part of a real b.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of nonnegative functions $V(x,t)$ from $\mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}_+$ which are continuously twice differentiable in x and once in t, and define an operator $LV : \mathcal{C}_r \times \mathbb{R}_+ \to \mathbb{R}$ by

$$LV(\phi,t) = V_t(\phi(0),t) + V_x(\phi(0),t) f(\phi,t) + \frac{1}{2}\text{trace}(g^T(\phi,t)V_{xx}(\phi(0),t)g(\phi,t)),$$

where

$$V_t(x,t) = \left(\frac{\partial V(x,t)}{\partial t}\right), \quad V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1}, \cdots, \frac{\partial V(x,t)}{\partial x_n}\right),$$

$$V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

If K is a subset of $\mathbb{R}^n$, denote by $d(x,K)$ the Hausdorff semi-distance between $x \in \mathbb{R}^n$ and the set K, that is, $d(x,K) = \inf_{y\in K}|x-y|$. If $\mathcal{W}$ is a real-valued function on $\mathbb{R}^n$, then its kernel is denoted by $Ker(\mathcal{W})$, namely $Ker(\mathcal{W}) = \{x \in \mathbb{R}^n : \mathcal{W}(x) = 0\}$. We denote by $L^1(\mathbb{R}_+;\mathbb{R}_+)$ the family of all Borel measurable functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \gamma(t)dt < +\infty$. Let $\Psi(\mathbb{R}_+;\mathbb{R}_+)$ denote the family of all continuous functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with the property that $\liminf_{t\to\infty}\int_t^{t+\delta}\psi(s)ds > 0$ for any $\delta > 0$. We denote by M_0 the set of probability measures on $(-\infty, 0]$, namely, for any $\mu \in M_0$, $\int_{-\infty}^0 \mu(d\theta) = 1$. For any $k > 0$, let us further define M_k, the subset of M_0, by

$$M_k \triangleq \left\{\mu \in M_0 : \mu^{(k)} \triangleq \int_{-\infty}^0 e^{-k\theta}\mu(d\theta) < \infty\right\}.$$

3. **The LaSalle-type Theorem.** In this section, we will establish the LaSalle-type theorem which includes existence and uniqueness of the global solution for Eq.(1.2) as well as attraction of this solution. we will then give three useful corollaries to show the boundedness and stability. For convenience, we introduce one more notation,

$$\Omega_\alpha \triangleq \left\{\mathcal{W} \in C(\mathbb{R}^n \times \mathbb{R};\mathbb{R}_+) : \sup_{-\infty<t\le 0}\mathcal{W}(x,t) \le L(1+|x|^\alpha)\right.$$
$$\left.\text{for some } L > 0 \text{ and all } x \in \mathbb{R}^n\right\}$$

for $\alpha > 0$, where $C(\mathbb{R}^n \times \mathbb{R};\mathbb{R}_+)$ denotes the family of all continuous nonnegative functions $\mathcal{W}$ on $\mathbb{R}^n \times \mathbb{R}$.

Remark 3.1. If $\mathcal{W} \in \Omega_\alpha$, we have $\sup_{-\infty<s\le 0} e^{\alpha rs}\mathcal{W}(\phi(s),s) < +\infty$ for any $\phi \in \mathcal{C}_r$. In fact,

$$\begin{aligned}\sup_{-\infty<s\le 0} e^{\alpha rs}\mathcal{W}(\phi(s),s) &\le \sup_{-\infty<s\le 0} e^{\alpha rs}L\left(1+|\phi(s)|^\alpha\right)\\ &\le L\sup_{-\infty<s\le 0} e^{\alpha rs} + \sup_{-\infty<s\le 0}\left(e^{rs}|\phi(s)|\right)^\alpha\\ &\le L + \|\phi\|_r^\alpha < +\infty.\end{aligned}$$

To examine existence and uniqueness of the solution, we will impose following assumptions on the coefficients f and g.

(**H1**) Both f and g satisfy the local Lipschitz condition, that is, for any $k > 0$, there exists a constant $b_k > 0$ such that

$$|f(\phi,t) - f(\varphi,t)| \vee |g(\phi,t) - g(\varphi,t)| \leqslant b_k\|\phi - \varphi\|_r,$$

on $t \geq 0$ for those $\phi,\ \varphi \in \mathcal{C}_r$ with $\|\phi\|_r \vee \|\varphi\|_r \leq k$. Moreover,

$$\sup_{0\leq t<\infty} \Big\{ |f(0,t)| \vee |g(0,t)| \Big\} < \infty. \tag{3.1}$$

Theorem 3.2. *Let assumption (**H1**) hold. Assume that there exists a constant $\alpha > 0$ and functions $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $\mathcal{W}_1 \in C(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\mathcal{W}_2 \in \Omega_\alpha$, $a \in \Psi(\mathbb{R}_+; \mathbb{R}_+)$, $\mathcal{W} \in C(\mathbb{R}^n, \mathbb{R}_+)$ and a probability measure $\mu \in M_{\alpha r}$ such that for any $x \in \mathbb{R}^n, \phi \in \mathcal{C}_r, t \geq 0$*

$$\lim_{|x|\to\infty} \Big[\inf_{0\leq t<\infty} V(x,t) \Big] = \infty, \tag{3.2}$$

$$LV(\phi,t) \leq \gamma(t) - \mathcal{W}_1(\phi(0),t) + \int_{-\infty}^0 \mathcal{W}_2(\phi(\theta), t+\theta)\mu(d\theta) \tag{3.3}$$

and

$$\mathcal{W}_1(x,t) - \mathcal{W}_2(x,t) \geq a(t)\mathcal{W}(x). \tag{3.4}$$

Then, for any initial data $\xi \in \mathcal{C}_r$, the following two assertions hold:

(i) *Eq.(1.2) admits a unique global solution. Moreover, for any $\varepsilon \in (0,1)$, there exists a positive $H = H(\xi,\varepsilon)$ such that*

$$\mathbb{P}\{\|x_t(\xi)\|_r \leq H, \forall t \geq 0\} > 1 - \varepsilon. \tag{3.5}$$

(ii) *The $Ker(\mathcal{W}) \neq \varnothing$ and $\lim_{t\to\infty} d(x(t,\xi), Ker(\mathcal{W})) = 0$ a.s..*

Remark 3.3. According to the definition of Ω_α, the bigger α implies that $\mathcal{W}_2$ admits higher nonlinear, which further implies that the coefficients f and g may be higher nonlinear functionals from (3.3). However, recalling definitions of Ω_α and M_k, we have $\Omega_{\alpha_1} \subset \Omega_{\alpha_2}$ provided $\alpha_2 > \alpha_1 > 0$, but $M_{\alpha_1} \supset M_{\alpha_2}$. These shows that when the bigger α is chosen, the set $M_{\alpha r}$ will have to be smaller.

The proof of Theorem 3.2. We divide this proof into two steps.

Step 1 (The proof of assertion (i)): In a similar way to what in [17] or [18], we can show that under assumption (**H1**), Eq.(1.2) has a unique maximal local strong solution $x(t,\xi)$ for any initial data $\xi \in \mathcal{C}_r$. To show this solution is global, we only need to prove that $\rho_e = \infty$ a.s., where ρ_e is the explosion time. Define a stopping time $\rho_k = \inf\{t \in [0,\rho_e) : |x(t,\xi)| > k\}$ with the traditional setting $\inf \varnothing = \infty$, where $\varnothing$ denotes empty set. Clearly, ρ_k is increasing as $k \to \infty$ and $\rho_k \to \rho_\infty \leq \rho_e$ a.s.. If we can show $\rho_\infty = \infty$ a.s., then $\rho_e = \infty$ a.s., which implies that $x(t,\xi)$ is actually global. This is equivalent to proving that, for any $t > 0$, $\mathbb{P}(\rho_k \leq t) \to 0$ as $k \to \infty$. Applying the Itô formula to $V(x(t,\xi),t)$ yields

$$\mathbb{E}V(x(t\wedge\rho_k,\xi), t\wedge\rho_k) = V(\xi(0),0) + \mathbb{E}\int_0^{t\wedge\rho_k} LV(x_s(\xi),s)ds.$$

By (3.3), we obtain

$$\begin{aligned}\mathbb{E}V(x(t\wedge\rho_k,\xi), t\wedge\rho_k) \leq & V(\xi(0),0) + \mathbb{E}\int_0^{t\wedge\rho_k} \gamma(s)ds - \mathbb{E}\int_0^{t\wedge\rho_k} \mathcal{W}_1(x(s,\xi),s)ds \\ & + \mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\infty}^0 \mathcal{W}_2(x(s+\theta,\xi), s+\theta)\mu(d\theta)ds.\end{aligned}$$

Note that $\xi \in \mathcal{C}_r, \mu \in M_{\alpha r}$ and $\mathcal{W}_2 \in \Omega_\alpha$. By the Fubini theorem and a substitution technique, we obtain

$$\begin{aligned}
&\int_0^{t\wedge\rho_k}\int_{-\infty}^0 \mathcal{W}_2(x(s+\theta,\xi),s+\theta)\mu(d\theta)ds\\
&=\int_0^{t\wedge\rho_k}\int_{-\infty}^{-s} \mathcal{W}_2(x(s+\theta,\xi),s+\theta)\mu(d\theta)ds\\
&\quad+\int_0^{t\wedge\rho_k}\int_{-s}^{0} \mathcal{W}_2(x(s+\theta,\xi),s+\theta)\mu(d\theta)ds\\
&\le \sup_{-\infty<\theta\le 0} e^{\alpha r\theta}\mathcal{W}_2\left(\xi(\theta),\theta\right)\int_0^{t\wedge\rho_k}\int_{-\infty}^{-s} e^{-\alpha r(s+\theta)}\mu(d\theta)ds\\
&\quad+\int_{-(t\wedge\rho_k)}^{0}\int_{-\theta}^{t\wedge\rho_k} \mathcal{W}_2(x(s+\theta,\xi),s+\theta)ds\mu(d\theta)\\
&\le\frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi)+\int_{-\infty}^0\int_0^{t\wedge\rho_k}\mathcal{W}_2(x(s,\xi),s)ds\mu(d\theta)\\
&=\frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi)+\int_0^{t\wedge\rho_k}\mathcal{W}_2(x(s,\xi),s)ds,
\end{aligned}\tag{3.6}$$

where $A(\xi)$ denotes $\sup_{-\infty<\theta\le 0} e^{\alpha r\theta}\mathcal{W}_2(\xi(\theta),\theta)$, which implies that

$$\begin{aligned}
\mathbb{E}V(x(t\wedge\rho_k,\xi),t\wedge\rho_k) \le& V(\xi(0),0)+\frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi)+\int_0^t\gamma(s)ds\\
&-\mathbb{E}\int_0^{t\wedge\rho_k}\mathcal{W}_1(x(s,\xi),s)-\mathcal{W}_2(x(s,\xi),s)ds.
\end{aligned}$$

Noting that $a(t)$ and $\mathcal{W}(x,t)$ are nonnegative, by (3.4) we obtain

$$\mathbb{E}V(x(t\wedge\rho_k,\xi),t\wedge\rho_k)\le V(\xi(0),0)+\frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi)+\int_0^t\gamma(s)ds$$

Define $h(k)=\inf\{V(x,t):|x|\ge k, t\ge 0\}$. Obviously, $h(k)$ is increasing as $k\to\infty$ and $h(k)\to\infty$.

$$\begin{aligned}
h(k)\mathbb{P}(\rho_k\le t)&\le \mathbb{E}V(x(\rho_k,\xi),\rho_k)\mathbf{1}_{\{\rho_k\le t\}}\\
&=\mathbb{E}V(x(t\wedge\rho_k,\xi),t\wedge\rho_k)\mathbf{1}_{\{\rho_k\le t\}}\\
&\le \mathbb{E}V(x(t\wedge\rho_k,\xi),t\wedge\rho_k)\\
&\le V(\xi(0),0)+\frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi)+\int_0^t\gamma(s)ds,
\end{aligned}\tag{3.7}$$

which implies $\mathbb{P}(\rho_k\le t)\to 0$ as $k\to\infty$. Hence, Eq.(1.2) has a unique global solution for any initial data $x_0=\xi\in\mathcal{C}_r$.

Now, we move on to show (3.5). For any $\xi\in\mathcal{C}_r$, define a stopping time

$$\sigma_k=\inf\{s\ge 0:\|x_s(\xi)\|_r>k\}.$$

Let $H=H(\xi,\varepsilon)>\|\xi\|_r$ satisfy

$$h(H)\ge\frac{1}{\varepsilon}\left\{V(\xi(0))+\frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi)+\int_0^\infty\gamma(s)ds\right\}.\tag{3.8}$$

Employing (3.7) and (3.8) yields

$$\mathbb{P}(\rho_H\le t)\le\varepsilon.$$

It is easy to verify that $\rho_H = \sigma_H$ provided $H > \|\xi\|_r$, see [17]. And then we have

$$\mathbb{P}(\sigma_H \le t) \le \varepsilon,$$

which implies $\mathbb{P}\{\|x_s(\xi)\|_r \le H, \forall s \in [0,t]\} > 1-\varepsilon$. Since t is arbitrary, we have (3.5).

Step 2. (The proof of assertion (ii)). We suppose that for almost all $\omega \in \Omega$,

$$\limsup_{t\to\infty} |x(t,\xi)| < \infty \tag{3.9}$$

and

$$\lim_{t\to\infty} \mathcal{W}(x(t,\xi)) = 0. \tag{3.10}$$

Hence there exists a subset $\Omega_0 \subseteq \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that (3.9) and (3.10) hold. For any $\omega \in \Omega_0$, by (3.9), we can find an increasing sequence $\{t_i\}_{i\ge1} \uparrow \infty$ such that $\{x(t_i,\xi)\}_{i\ge1}$ converges to some $y \in \mathbb{R}^n$. Therefore, it follows from (3.10) and continuity of $\mathcal{W}$ that

$$\mathcal{W}(y) = \lim_{i\to\infty} \mathcal{W}(x(t_i,\xi)) = 0,$$

which implies $y \in Ker(\mathcal{W})$ and $Ker(\mathcal{W}) \ne \varnothing$. If there is some $\omega \in \Omega_0$ such that

$$\limsup_{t\to\infty} d(x(t,\xi), Ker(\mathcal{W})) > 0.$$

Then we can find an increasing sequence $\{t_i\}_{i\ge1} \uparrow \infty$ such that

$$\lim_{i\to\infty} d(x(t_i,\xi), Ker(\mathcal{W})) > 0.$$

We can further find a subsequence of $\{t_i\}$, still denoted by $\{t_i\}$, such that $\{x(t_i,\xi)\}_{i\ge1}$ converges to some $y \in \mathbb{R}^n$. The arguments above shows that $y \in Ker(\mathcal{W})$ and whence

$$\lim_{i\to\infty} d(x(t_i,\xi), Ker(\mathcal{W})) = 0,$$

this produces a contradiction. We therefore must have

$$\limsup_{t\to\infty} d(x(t,\xi), Ker(\mathcal{W})) = 0$$

for all $\omega \in \Omega_0$, which implies

$$\lim_{t\to\infty} d(x(t,\xi), Ker(\mathcal{W})) = 0$$

for all $\omega \in \Omega_0$.

Hence, if we can show that (3.9) and (3.10) hold for almost all $\omega \in \Omega$, the desired assertion (ii) will follow. To show (3.9) and (3.10) for almost all $\omega \in \Omega$, applying the Itô formula again to function $V(x(t,\xi),t)$ gives

$$V(x(t,\xi),t) = V(\xi(0),0) + \int_0^t LV(x_s(\xi),s)ds + M(t),$$

where $M(t) = \int_0^t V_x(x(s,\xi),s)g(x_s(\xi),s)d\omega(s)$ is a real-valued continuous local martingale with $M(0) = 0$. By conditions (3.3) and (3.4), using the similar argument as (3.6) we have

$$\begin{aligned} V(x(t,\xi),t) \le & V(\xi(0),0) + \frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi) + \int_0^t \gamma(s)ds \\ & - \int_0^t a(s)\mathcal{W}(x(s,\xi))ds + M(t). \end{aligned} \tag{3.11}$$

Noting that functions a and $\mathcal{W}$ are nonnegative, we have

$$V(x(t,\xi),t) \le V(\xi(0),0) + \frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi) + \int_0^t \gamma(s)ds + M(t).$$

Applying the nonnegative semimartingale convergence theorem we immediately obtain

$$\limsup_{t\to\infty} V(x(t,\xi),t) < \infty \quad a.s.. \tag{3.12}$$

Recalling the definition of $h(k)$, we have $h(|x(t,\xi)|) \le V(x(t,\xi),t)$. Hence we obtain

$$\limsup_{t\to\infty} h(|x(t,\xi)|) \le \limsup_{t\to\infty} V(x(t,\xi),t) < \infty \ \ a.s.,$$

which implies

$$\limsup_{t\to\infty} |x(t,\xi)| < \infty \ \ a.s..$$

We therefore prove (3.9). We now move on to show (3.10) for almost all $\omega \in \Omega$. Taking expectations on both sides of (3.11) yields

$$\mathbb{E}\int_0^t a(s)\mathcal{W}(x(s,\xi))ds \le V(\xi(0),0) + \frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi) + \int_0^t \gamma(s)ds.$$

Letting $t \to \infty$, we obtain

$$\mathbb{E}\int_0^\infty a(s)\mathcal{W}(x(s,\xi))ds \le V(\xi(0),0) + \frac{1}{\alpha r}\mu^{(\alpha r)}A(\xi) + \int_0^\infty \gamma(s)ds < \infty. \tag{3.13}$$

We claim that

$$\liminf_{t\to\infty} \mathcal{W}(x(t,\xi)) = 0 \qquad a.s.. \tag{3.14}$$

If not, there exists a constant $\varepsilon_1 > 0$ such that

$$\mathbb{P}(\Omega_1) \ge \varepsilon_1,$$

where $\Omega_1 = \{\liminf_{t\to\infty} \mathcal{W}(x(t,\xi)) > 2\varepsilon_1\}$. Hence, for any $\omega \in \Omega_1$ there exists a random variable $t_0(\varepsilon_1,\omega) > 0$ such that for all $t > t_0$

$$\mathcal{W}(x(t,\xi)) > \varepsilon_1.$$

Noting that $a \in \Psi(\mathbb{R}_+;\mathbb{R}_+)$, for any $\beta > 0$ there exist two numbers $\rho = \rho(\beta)$ and $t_1 = t_1(\rho,\beta) > 0$ such that for all $t \ge t_1$

$$\int_t^{t+\beta} a(s)ds \ge \rho.$$

Hence, we have

$$\begin{aligned}\mathbb{E}\int_0^\infty a(s)\mathcal{W}(x(s,\xi))ds &\ge \varepsilon_1\mathbb{E}[1_{\Omega_1}\int_{t_0+t_1}^\infty a(s)ds] \\ &= \varepsilon_1\mathbb{E}[1_{\Omega_1}\sum_{k=1}^\infty \int_{t_0+t_1+(k-1)\beta}^{t_0+t_1+k\beta} a(s)ds] \\ &\ge \varepsilon_1\mathbb{E}[1_{\Omega_1}\sum_{k=1}^\infty \rho] = \infty\end{aligned}$$

which contradicts (3.13). Therefore, (3.14) must be true.

Suppose that (3.10) is not true. Then there exists a number $\varepsilon_2 > 0$ such that

$$\mathbb{P}(\Omega_2) \ge 3\varepsilon_2, \tag{3.15}$$

where $\Omega_2 = \{\limsup_{t\to\infty} \mathcal{W}(x(t,\xi)) > 2\varepsilon_2\}$. In view of (3.5), we can choose a constant $H_1 = H_1(\xi, \varepsilon_2) > \|\xi\|_r$ such that

$$\mathbb{P}(\Omega_3) > 1 - \varepsilon_2, \tag{3.16}$$

where $\Omega_3 = \{\sup_{0\le t<\infty} \|x_t(\xi)\|_r \le H_1\}$. It is easy to see from (3.15) and (3.16) that

$$\mathbb{P}(\Omega_2 \cap \Omega_3) > 2\varepsilon_2.$$

For any $\delta > 0$ (we will choose appropriate δ below), let us now define a sequence of stopping times

$$\begin{aligned}
\tau_{H_1} &= \inf\{t \ge 0 : |x(t,\xi)| > H_1\},\\
\sigma_1 &= \inf\{t \ge 0 : \mathcal{W}(x(t,\xi)) \ge 2\varepsilon_2\},\\
\sigma_2 &= \inf\{t \ge \sigma_1 : \mathcal{W}(x(t,\xi)) \le \varepsilon_2\},\\
\sigma_{2k+1} &= \inf\{t \ge \sigma_{2k} + \delta : \mathcal{W}(x(t,\xi)) \ge 2\varepsilon_2\}, \qquad k = 1, 2, \cdots,\\
\sigma_{2k+2} &= \inf\{t \ge \sigma_{2k+1} : \mathcal{W}(x(t,\xi)) \le \varepsilon_2\}, \qquad k = 1, 2, \cdots,
\end{aligned}$$

where throughout this paper we set $\inf \varnothing = \infty$.

Noting that (3.14) and the definitions of Ω_2 and Ω_3, we have for any $k \ge 1$

$$\tau_{H_1}(\omega) = \infty \qquad \text{and} \qquad \sigma_k(\omega) < \infty, \qquad \forall \omega \in \Omega_2 \cap \Omega_3 \tag{3.17}$$

and

$$\sigma_{2k} < \infty \qquad \text{provided} \qquad \sigma_{2k-1} < \infty. \tag{3.18}$$

By (3.17)and (3.18), we compute

$$\begin{aligned}
&\mathbb{E}\int_0^\infty a(s)\mathcal{W}(x(t,\xi))ds\\
&\ge \mathbb{E}\left[\sum_{k=1}^\infty 1_{\{\sigma_{2k-1}<\infty, \sigma_{2k}<\infty, \tau_{H_1}=\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k}} a(s)\mathcal{W}(x(s,\xi))ds\right]\\
&\ge \varepsilon_2 \mathbb{E}\left[\sum_{k=1}^\infty 1_{\{\sigma_{2k-1}<\infty, \tau_{H_1}=\infty\}} \int_{\sigma_{2k-1}}^{\sigma_{2k}} a(s)ds\right]\\
&\ge \varepsilon_2 \mathbb{E}\left[\sum_{k=1}^\infty 1_{\Omega_2\cap\Omega_3} \int_{\sigma_{2k-1}}^{\sigma_{2k}} a(s)ds\right]
\end{aligned} \tag{3.19}$$

If therefore we can show that

$$\mathbb{E}\left[\sum_{k=1}^\infty 1_{\Omega_2\cap\Omega_3} \int_{\sigma_{2k-1}}^{\sigma_{2k}} a(s)ds\right] = \infty. \tag{3.20}$$

Then we derive from (3.19) and (3.20) that

$$\mathbb{E}\int_0^\infty a(s)\mathcal{W}(x(t,\xi))ds \ge \varepsilon_2 \mathbb{E}\left[\sum_{k=1}^\infty 1_{\Omega_2\cap\Omega_3} \int_{\sigma_{2k-1}}^{\sigma_{2k}} a(s)ds\right] = \infty,$$

which contradicts (3.13). This implies (3.10). Hence, we now only need to prove that (3.20) holds. By the local Lipschitz condition and (3.1), for any $k > 0$ there is a constant $c_k > 0$ such that

$$|f(\phi, t)| \vee |g(\phi, t)| \le c_k$$

for all $t \geq 0$ and $\phi \in \mathcal{C}_r$ with $\|\phi\|_r \leq k$. Making use of this property as well as the Doob Martingale inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left[1_{\{\tau_{H_1}\wedge\sigma_{2k-1}\}}\sup_{0\leq t\leq T}|x(\tau_{H_1}\wedge(\sigma_{2k-1}+t))-x(\tau_{H_1}\wedge\sigma_{2k-1})|^2\right]\\
&\leq 2\mathbb{E}\left[1_{\{\tau_{H_1}\wedge\sigma_{2k-1}\}}\sup_{0\leq t\leq T}|\int_{\tau_{H_1}\wedge\sigma_{2k-1}}^{\tau_{H_1}\wedge(\sigma_{2k-1}+t)}f(x_s(\xi),s)ds|^2\right]\\
&\quad+2\mathbb{E}\left[1_{\{\tau_{H_1}\wedge\sigma_{2k-1}\}}\sup_{0\leq t\leq T}|\int_{\tau_{H_1}\wedge\sigma_{2k-1}}^{\tau_{H_1}\wedge(\sigma_{2k-1}+t)}g(x_s(\xi),s)d\omega(s)|^2\right]\\
&\leq 2\mathbb{E}\left[1_{\{\tau_{H_1}\wedge\sigma_{2k-1}\}}\int_{\tau_{H_1}\wedge\sigma_{2k-1}}^{\tau_{H_1}\wedge(\sigma_{2k-1}+T)}|f(x_s(\xi),s)|ds\right]^2\\
&\quad+8\mathbb{E}\left[1_{\{\tau_{H_1}\wedge\sigma_{2k-1}\}}\int_{\tau_{H_1}\wedge\sigma_{2k-1}}^{\tau_{H_1}\wedge(\sigma_{2k-1}+T)}|g(x_s(\xi),s)|^2ds\right]\\
&\leq 2T^2c_{H_1}^2+8Tc_{H_1}^2\\
&=2(T+4)Tc_{H_1}^2,
\end{aligned}\tag{3.21}$$

where in the last inequality we use the fact that $\tau_{H_1}=\sigma_{H_1}$. Since $\mathcal{W}(\cdot)$ is continuous in $\mathbb{R}^n$, it is uniformly continuous in the closed ball $\bar{B}_{H_1}=\{x\in\mathbb{R}^n : |x|\leq H_1\}$. Hence, there exists $\kappa=\kappa(\varepsilon_2)>0$ such that for any $x,y\in\bar{B}_{H_1}$

$$|\mathcal{W}(x)-\mathcal{W}(y)|<\varepsilon_2 \tag{3.22}$$

provided $|x-y|<\kappa$. We furthermore choose $T_0=T_0(\varepsilon_2,\kappa,H_1)>0$ such that

$$\frac{2(T_0+4)T_0c_{H_1}^2}{\kappa^2}<\varepsilon_2.$$

By the Chebyshev inequality, it then follows from (3.21) that

$$\mathbb{P}\left(\{\tau_{H_1}\wedge\sigma_{2k-1}\}\cap\Omega_k^1\right)\leq\frac{2(T_0+4)T_0c_{H_1}^2}{\kappa^2}<\varepsilon_2, \tag{3.23}$$

where $\Omega_k^1=\{\sup_{0\leq t\leq T_0}|x(\tau_{H_1}\wedge(\sigma_{2k-1}+t))-x(\tau_{H_1}\wedge\sigma_{2k-1})|\geq\kappa\}$. Consequently, by (3.17) and (3.23), we have

$$\begin{aligned}
\mathbb{P}\left(\Omega_2\cap\Omega_3\cap\Omega_k^1\right)&\leq\mathbb{P}\left(\{\tau_{H_1}=\infty,\sigma_{2k-1}<\infty\}\cap\Omega_k^1\right)\\
&=\mathbb{P}\left(\{\tau_{H_1}\wedge\sigma_{2k-1}<\infty,\tau_{H_1}=\infty\}\cap\Omega_k^1\right)\\
&\leq\mathbb{P}\left(\{\tau_{H_1}\wedge\sigma_{2k-1}<\infty\}\cap\Omega_k^1\right)\\
&<\varepsilon_2.
\end{aligned}$$

Recalling the definition of Ω_k^1, we have

$$\begin{aligned}
&\mathbb{P}\left(\Omega_2\cap\Omega_3\cap\{\sup_{0\leq t\leq T_0}|x(\tau_{H_1}\wedge(\sigma_{2k-1}+t))-x(\tau_{H_1}\wedge\sigma_{2k-1})|<\kappa\}\right)\\
&=\mathbb{P}(\Omega_2\cap\Omega_3)-\mathbb{P}(\Omega_2\cap\Omega_3\cap\Omega_k^1)\\
&>2\varepsilon_2-\varepsilon_2=\varepsilon_2.
\end{aligned}$$

Using (3.22), we have

$$
\begin{aligned}
&\mathbb{P}(\Omega_2\cap\Omega_3\cap\Omega_k^2)\\
&\geq \mathbb{P}\left(\Omega_2\cap\Omega_3\cap\{\sup_{0\leq t\leq T_0}|x(\tau_{H_1}\wedge(\sigma_{2k-1}+t))-x(\tau_{H_1}\wedge\sigma_{2k-1})|<\kappa\}\right)\\
&>\varepsilon_2,
\end{aligned}
$$

where $\Omega_k^2=\{\sup_{0\leq t\leq T_0}|\mathcal{W}(x(\sigma_{2k-1}+t))-\mathcal{W}(x(\sigma_{2k-1}))|\leq\varepsilon_2\}$. Noting that $a\in\Psi(\mathbb{R}_+,\mathbb{R}_+)$, for $T_0>0$ above, there exist two constants $\varepsilon_3=\varepsilon_3(T_0)>0$ and $t_3=t_3(\varepsilon_3,T_0)>0$ such that for all $t\geq t_3$

$$
\int_t^{t+T_0} a(s)ds\geq\varepsilon_3.
$$

Recalling the definition of the stopping time σ_{2k+1}, let $\delta=T_0$. Hence, for any $\omega\in\Omega_2\cap\Omega_3, k\geq[t_3/\delta]+2$ we have $\sigma_{2k-1}\geq t_3$. Hence, for any $\omega\in\Omega_2\cap\Omega_3, k\geq[t_3/\delta]+2$, we have

$$
\int_{\sigma_{2k-1}}^{\sigma_{2k-1}+\delta} a(s)ds\geq\varepsilon_3.
$$

Noting that $\sigma_{2k}-\sigma_{2k-1}\geq\delta=T_0$ provided $\omega\in\Omega_2\cap\Omega_3\cap\Omega_k^2$, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\sum_{k=1}^{\infty}1_{\Omega_2\cap\Omega_3}\int_{\sigma_{2k-1}}^{\sigma_{2k}}a(s)ds\right]&\geq\mathbb{E}\left[\sum_{k=[\frac{t_3}{\delta}]+2}^{\infty}1_{\Omega_2\cap\Omega_3}\int_{\sigma_{2k-1}}^{\sigma_{2k}}a(s)ds\right]\\
&\geq\mathbb{E}\left[\sum_{k=[\frac{t_3}{\delta}]+2}^{\infty}1_{\Omega_2\cap\Omega_3\cap\Omega_k^2}\int_{\sigma_{2k-1}}^{\sigma_{2k}}a(s)ds\right]\\
&\geq\mathbb{E}\left[\sum_{k=[\frac{t_3}{\delta}]+2}^{\infty}1_{\Omega_2\cap\Omega_3\cap\Omega_k^2}\int_{\sigma_{2k-1}}^{\sigma_{2k-1}+T_0}a(s)ds\right]\\
&\geq\varepsilon_3\mathbb{E}\left[\sum_{k=[\frac{t_3}{\delta}]+2}^{\infty}1_{\Omega_2\cap\Omega_3\cap\Omega_k^2}\right]\\
&>\varepsilon_3\sum_{k=[\frac{t_3}{\delta}]+2}^{\infty}\varepsilon_2=\infty,
\end{aligned}
$$

which implies (3.20). Then we prove (3.10). This completes the proof. □

Remark 3.4. Let us make some comparison between our new theorem and that in [15]. Consider a special case of our new theorem where we let $a(t)=1,\gamma(t)=0$ and choose

$$
\begin{aligned}
\mathcal{W}_1(x)&=\sum_{i=1}^{N}\sum_{j=1}^{n}L_{ij}|x_j|^{\alpha_i}+G(x),\\
\mathcal{W}_2(x)&=\sum_{i=1}^{N}\sum_{j=1}^{n}L_{ij}|x_j|^{\alpha_i},
\end{aligned}
$$

in which N is a positive integer, $L_{ij},\alpha_i\geq 0, i=1,2,\cdots,N, j=1,2,\cdots,n$, and $G\in C(\mathbb{R}^n;\mathbb{R}_+)$. If initial data $\xi\in BC((-\infty;0];\mathbb{R}^n)\cap L^\alpha((-\infty;0];\mathbb{R}^n)$, where

$\alpha = \max_{1\le i\le N}\alpha_i$, whence we can let $r = 0$ and this special case becomes [15, Theorem 3.1] if the neutral term there is omitted. Note that $BC((-\infty;0];\mathbb{R}^n)\subset\mathcal{C}_r$. In this paper we use the phase space $\mathcal{C}_r$, which is more general except that we must impose an additional condition on probability measure $\mu\in M_{\alpha r}$ which is smaller than M_0.

Corollary 3.5. *Let (**H1**) hold. If there exists a number $\alpha>0$ and functions $V\in C^{2,1}(\mathbb{R}^n\times\mathbb{R}_+;\mathbb{R}_+),\gamma\in L^1(\mathbb{R}_+;\mathbb{R}_+),a_1\in\Psi(\mathbb{R}_+;\mathbb{R}_+),a_2\in C(\mathbb{R};\mathbb{R}_+),(a_1-a_2)\in\Psi(\mathbb{R}_+;\mathbb{R}_+),u_1,u_2\in C(\mathbb{R}^n;\mathbb{R}_+),a_2u_2\in\Omega_\alpha$ and a probability measure $\mu\in M_{\alpha r}$ such that*

$$\lim_{|x|\to\infty}\left[\inf_{0\le t<\infty}V(x,t)\right]=\infty,$$

$$LV(\phi,t)\le\gamma(t)-a_1(t)u_1(\phi(0))+\int_{-\infty}^0 a_2(t+\theta)u_2(\phi(\theta))\mu(d\theta)$$

and

$$u_1(x)\ge u_2(x)$$

for any $x\in\mathbb{R}^n,t\in\mathbb{R}_+$ and $\phi\in\mathcal{C}_r$. Then we have $Ker(u_1-u_2)\cap Ker(u_2)\ne\varnothing$ and $\lim_{t\to\infty}d(x(t,\xi),Ker(u_1-u_2)\cap Ker(u_2))=0$ a.s..

Proof. Define $\mathcal{W}_1(x,t)=a_1(t)u_1(x)$ and $\mathcal{W}_2(x,t)=a_2(t)u_2(x)$. Then we have

$$\begin{aligned}\mathcal{W}_1(x,t)-\mathcal{W}_2(x,t)&=a_1(t)u_1(x)-a_2(t)u_2(x)\\&=a_1(t)(u_1(x)-u_2(x))+(a_1(t)-a_2(t))u_2(x).\end{aligned}$$

Note that $a_1\in\Psi(\mathbb{R}_+;\mathbb{R}_+),(a_1-a_2)\in\Psi(\mathbb{R}_+;\mathbb{R}_+)$. It follows from Theorem 3.2 that

$$Ker(u_1(x)-u_2(x))\ne\varnothing,\quad Ker(u_2)\ne\varnothing$$

and

$$\lim_{t\to\infty}d(x(t,\xi),Ker(u_1-u_2))=0,\quad\lim_{t\to\infty}d(x(t,\xi),Ker(u_2))=0\quad a.s..$$

Those imply $Ker(u_1-u_2)\cap Ker(u_2)\ne\varnothing$ and

$$\lim_{t\to\infty}d(x(t,\xi),Ker(u_1-u_2)\cap Ker(u_2))=0\quad a.s..$$

□

Corollary 3.6. *Let the assumptions of Theorem 3.2 hold. Assume also that $\mathcal{W}(x)=0$ if and only if $x=0$. Then for any $\xi\in\mathcal{C}_r$*

$$\lim_{t\to\infty}x(t,\xi)=0\quad a.s..$$

Corollary 3.7. *Let the assumptions of Theorem 3.2 hold. If $Ker(\mathcal{W})$ is bounded, then for any $\xi\in\mathcal{C}_r$*

$$\lim_{t\to\infty}|x(t,\xi)|\le C\quad a.s.,$$

where $C=\sup\{|x|:x\in Ker(\mathcal{W})\}$.

4. **Examples.** In this section, we will give two examples to illustrate the applications of our results established in previous section.

Example 4.1. Consider the following two-dimensional infinite delay stochastic integro-differential equations.

$$d\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{3}{2}x_2(t) \\ -(1-a\sin^2(t))x_1(t)-(1+2a\sin^2(t))x_2(t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \int_{-\infty}^0 b\sin(t+\theta)x_1(t+\theta)+cx_2(t+\theta)\mu(d\theta) \end{pmatrix} d\omega(t) \tag{4.1}$$

with initial data $x_0=\xi\in\mathcal{C}_r$, where $0\le a<1, a+2b^2<1, c^2<1/4, \mu\in M_{2r}$ and $\omega(t)$ is a scalar Brownian motion. It is easy to see that the coefficients of Eq.(4.1) satisfy assumption (**H1**). Let $V(x,t)=x_1^2+x_1x_2+x_2^2$. Obviously, $V(x,t)\in C^{2,1}(\mathbb{R}^2;\mathbb{R}_+)$ and

$$\lim_{|x|\to\infty}\left[\inf_{0\le t<\infty} V(x,t)\right]=\infty.$$

Then we compute

$$\begin{aligned} LV(\phi,t) =&(2\phi_1(0)+\phi_2(0))\frac{3}{2}\phi_2(0) \\ &+(\phi_1(0)+2\phi_2(0))(-(1-a\sin^2(t))\phi_1(0)-(1+2a\sin^2(t))\phi_2(0)) \\ &+\left(\int_{-\infty}^0 b\sin(t+\theta)\phi_1(\theta)+c\phi_2(\theta)\mu(d\theta)\right)^2 \\ \le&-(1-a\sin^2 t)\phi_1^2(0)-(\frac{1}{2}+4a\sin^2(t))\phi_2^2(0) \\ &+\int_{-\infty}^0 [b\sin(t+\theta)\phi_1(\theta)+c\phi_2(\theta)]^2\mu(d\theta) \\ \le&-(1-a\sin^2 t)\phi_1^2(0)-(\frac{1}{2}+4a\sin^2(t))\phi_2^2(0) \\ &+2\int_{-\infty}^0 b^2\sin^2(t+\theta)\phi_1^2(\theta)+c^2\phi_2^2(\theta)\mu(d\theta) \\ =&-\mathcal{W}_1(\phi(0),t)+\int_{-\infty}^0 \mathcal{W}_2(\phi(\theta),t+\theta)\mu(d\theta), \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \mathcal{W}_1(x,t)&=(1-a\sin^2 t)x_1^2+(\frac{1}{2}+4a\sin^2(t))x_2^2 \\ \mathcal{W}_2(x,t)&=2b^2\sin^2(t)x_1^2+2c^2x_2^2. \end{aligned}$$

We compute

$$\begin{aligned} \mathcal{W}_1(x,t)-\mathcal{W}_2(x,t)&=(1-a\sin^2 t)x_1^2+(\frac{1}{2}+4a\sin^2(t))x_2^2-2b^2\sin^2(t)x_1^2-2c^2x_2^2 \\ &\ge\min\left\{1-(a+2b^2)\sin^2 t,\frac{1}{2}-2c^2+4a\sin^2(t)\right\}(x_1^2+x_2^2). \end{aligned}$$

Clearly, the right-hand side of Eq.(4.2) depends on the variant t. We therefore cannot apply the theorem in [15]. Note that $0\le a<1, a+2b^2<1$ and $c^2<1/4$. It is easy to see that $\min\left\{1-(a+2b^2)\sin^2 t,\frac{1}{2}-2c^2+4a\sin^2(t)\right\}\in\Psi(\mathbb{R}_+;\mathbb{R}_+)$ and $Ker(x_1^2+x_2^2)=\{(0,0)^T\}$. By Corollary 3.6, we can conclude that the unique global solution $x(t,\xi)$ for any initial data $\xi \in \mathcal{C}_r$ to the Eq.(4.1) has the property that

$$\lim_{t\to\infty} |x(t,\xi)| = 0 \quad a.s..$$

Example 4.2. Consider the following scalar nonlinear stochastic functional differential equation with infinite delay

$$\begin{aligned} dx(t) =& \Big(e^{-\lambda t} - x^3(t) - \sin^2(t)x^3(t) - x(t)\sin^2(x(t))\Big)dt \\ &+ \left(\int_{-\infty}^0 \frac{\sqrt{3}}{2}x^2(t+\theta) + \sin(t+\theta)x^2(t+\theta)\mu(d\theta)\right)d\omega(t) \end{aligned} \tag{4.3}$$

with initial data $x_0 = \xi \in \mathcal{C}_r$, where $\lambda > 0, \mu \in M_{4r}$ and $\omega(t)$ is a scalar Brownian motion. It is easy to see that its coefficients satisfy the assumption (**H1**). Let $V(x,t) = x^2$. Obviously, $V \in C^{2,1}(\mathbb{R};\mathbb{R}_+)$ and

$$\lim_{|x|\to\infty}\left[\inf_{0\le t<\infty} V(x,t)\right] = \infty.$$

Then the operator $LV : \mathcal{C}_r \times \mathbb{R}_+ \to \mathbb{R}$ has the form

$$\begin{aligned} LV(\phi,t) =& 2\phi(0)[e^{-\lambda t} - \phi^3(0) - \sin^2(t)\phi^3(0) - \phi(0)\sin^2(\phi(0))] \\ &+ \left(\int_{-\infty}^0 \frac{\sqrt{3}}{2}\phi^2(\theta) + \sin(t+\theta)\phi^2(\theta)\mu(d\theta)\right)^2 \\ \le& 2\phi(0)e^{-\lambda t} - 2\phi^4(0) - 2\sin^2(t)\phi^4(0) - 2\phi^2(0)\sin^2(\phi(0)) \\ &+ \int_{-\infty}^0 [\frac{3}{2}\phi^4(\theta) + 2\sin^2(t+\theta)\phi^4(\theta)]\mu(d\theta) \\ \le& \frac{3}{2}e^{-\frac{4\lambda t}{3}} - \left(\frac{3}{2}\phi^4(0) + 2\sin^2(t)\phi^4(0) + 2\phi^2(0)\sin^2(\phi(0))\right) \\ &+ \int_{-\infty}^0 [\frac{3}{2}\phi^4(\theta) + 2\sin^2(t+\theta)\phi^4(\theta)]\mu(d\theta) \\ \le& \gamma(t) - \mathcal{W}_1(\phi(0),t) + \int_{-\infty}^0 \mathcal{W}_2(\phi(\theta), t+\theta)\mu(d\theta), \end{aligned} \tag{4.4}$$

where $\gamma(t) = 3e^{-\frac{4\lambda t}{3}}/2, \mathcal{W}_1(x,t) = 3x^4/2 + 2\sin^2(t)x^4 + 2x^2\sin^2(x)$, and $\mathcal{W}_2(x,t) = 3x^4/2 + 2\sin^2(t)x^4$. Then we have

$$\begin{aligned} \mathcal{W}_1(x,t) - \mathcal{W}_2(x,t) &= \frac{3}{2}x^4 + 2\sin^2(t)x^4 + 2x^2\sin^2(x) - \frac{3}{2}x^4 - 2\sin^2(t)x^4 \\ &= 2x^2\sin^2(x). \end{aligned} \tag{4.5}$$

It is obvious that the right-hand side of the inequality (4.4) depends on t. Hence we cannot apply the theorem in [15]. However, applying (3.12) and Theorem 3.2, we can conclude that the solution of Eq.(4.3) has the properties that

$$\limsup_{t\to\infty} |x(t,\xi)|^2 < \infty \quad a.s. \quad \text{and} \quad \lim_{t\to\infty} d(x(t,\xi), Ker(\mathcal{W})) = 0 \quad a.s.,$$

where $Ker(\mathcal{W}) = \{x \in \mathbb{R} : x^2\sin^2(x) = 0\} = \{k\pi : k = 0, \pm1, \pm2, \cdots\}$. And if $\lim_{t\to\infty} x(t,\xi)$ exists, these imply that

$$\lim_{t\to\infty} x(t,\xi) = \rho\pi \quad a.s.,$$

where ρ is a random variable taking values in $\{0, \pm 1, \pm 2, \cdots\}$ and it may depend on the initial data ξ.

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