A Stochastic Differential Equation SIS Epidemic Model With Two Independent Brownian Motions

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Abstract. In this paper, we introduce two perturbations in the classical deterministic susceptible-infected-susceptible epidemic model. Greenhalgh and Gray [1] in 2011 use a perturbation on β in SIS model. Based on their previous work, we consider another perturbation on the parameter $\mu + \gamma$ and formulate the original model as a stochastic differential equation (SDE) with two independent Brownian Motions for the number of infected population. We then prove that our Model has a unique and bounded global solution $I(t)$. Also we establish conditions for extinction and persistence of the infected population $I(t)$. Under the conditions of persistence, we show that there is a unique stationary distribution and derive its mean and variance. Computer simulations illustrate our results and provide evidence to back up our theory.

Key words. SIS model, independent Brownian Motion, extinction, persistence, stationary distribution.

1 Introduction

Research on epidemics modelled by introducing deterministic compartmental models makes great contribution to understanding the behaviour of epidemics and helping control of deadly diseases [10, 11]. For example, Capasso[11] introduces the Kermack-Mckendrick model to describe diseases that offer permanent immunity after an individual catching the diseases for a period of time. However, some diseases such as sexually transmitted and bacterial disease do not have permanent immunity. Susceptible individuals will catch the disease at some time to become infected, while after a short period of time infected individuals will become susceptible again. Susceptible-infected-susceptible (SIS) model is a very simple but also commonly used model to describe such epidemic problems [9]. $S(t)$ and $I(t)$ are used to represent the numbers of susceptible and infected populations at time t. The deterministic models is

$$\begin{cases} \frac{\mathrm{d}S(t)}{\mathrm{d}t} = \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t) \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta S(t)I(t) - (\mu+\gamma)I(t) \end{cases} \tag{1.1}$$

with initial values $S_0 + I_0 = N$ and here N is the total size of population. μ is the per capita death rate and γ is the rate at which infected individuals become cured. β is the disease transmission coefficient. With the condition $S + I = N$, we can rewrite the original two ODEs (1.1) into

$$\mathrm{d}I(t) = [\beta(N - I(t))I(t) - (\mu+\gamma)I(t)]\,\mathrm{d}t \tag{1.2}$$

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While deterministic models are not enough to describe problems in real world because parameters are easily influenced by all kinds of circumstances with uncertainty. Thus stochastic models with different environmental noises are more appropriate in epidemic problems. For example, A.Gray *et al.* [1] consider the perturbation on β in deterministic SIS model. They firstly analyse (1.2) in a small time interval $[t, t+\mathrm{d}t)$ with the d notation for small change in any quantity. Hence we have $\mathrm{d}I(t) = I(t+\mathrm{d}t) - I(t)$ in (1.2). Then the disease transmission coefficient β can be regarded as the rate at which each infected individual make contacts with other individuals and the total number of new infections in the small time interval is $\beta I(t)S(t)\,\mathrm{d}t$ and also, a single infected individual makes $\beta\,\mathrm{d}t$ potentially infectious contacts with other individuals in the small time interval. Consequently, when some stochastic environmental factor is introduced on each individual in the population, they replace β by a random variable $\tilde{\beta}$

$$\tilde{\beta}\,\mathrm{d}t = \beta\,\mathrm{d}t + \sigma_1\,\mathrm{d}B_1(t) \tag{1.3}$$

Here $\mathrm{d}B_1(t) = B_1(t+\mathrm{d}t) - B_1(t)$ is the increment of a standard Brownian motion. Hence the potentially infectious contacts made by a single infected individual with another individual in the population in the small time interval $[t, t+\mathrm{d}t)$ are normally distributed with mean $\beta\,\mathrm{d}t$ and variance $\sigma_1^2\,\mathrm{d}t$. Also, Y. Zhao *et al.*[12] use the same perturbation in SIS model with a vaccination and then find the conditions for the disease to become extinct and persist. There are also many other contributions on different epidemic models using multiple environmental noises [5, 6, 7, 13].

Now based on the previous work of A. Gray *et al.* [1], we now consider another perturbation on $(\mu+\gamma)$ with (1.3) existing in traditional SIS model. Within the same small time interval $[t, t+\mathrm{d}t)$, we regard $(\mu+\gamma)I(t)\,\mathrm{d}t$ as the total number of infected individuals becoming cured or pass away in this time interval. In other words, this is the total reduction of infections. Hence each single individual contributes $(\mu+\gamma)\,\mathrm{d}t$ in the reduction of infections in the small time interval $[t, t+\mathrm{d}t)$.
Then we introduce stochasticity on $(\mu+\gamma)$. $(\mu+\gamma)$ is replaced by a random variable $(\tilde{\mu}+\tilde{\gamma})$

$$(\tilde{\mu}+\tilde{\gamma})\,\mathrm{d}t = (\mu+\gamma)\,\mathrm{d}t + \sigma_2\sqrt{N-I(t)}\,\mathrm{d}B_2(t) \tag{1.4}$$

Here we do not simply set $(\tilde{\mu}+\tilde{\gamma})\,\mathrm{d}t = (\mu+\gamma)\,\mathrm{d}t + \sigma_2\,\mathrm{d}B_2(t)$ to be the second perturbation. When susceptible population $S(t) = N - I(t)$ is large, which means there are few infected individuals, the error of estimating μ and γ will be large. Thus we suppose that the variance of estimating $\mu+\gamma$ is proportional to the number of susceptible population. As a result, the reduction of infections caused by medical care and death of a single infected individual in the small time interval $[t, t+\mathrm{d}t)$ is normally distributed with mean $(\mu+\gamma)\,\mathrm{d}t$ and variance ${\sigma_2}^2(N-I(t))\,\mathrm{d}t$. This is also a biologically reasonable model because the variance trends to 0 when $\mathrm{d}t$ goes to 0.

Such a diffusion coefficient in square root form is widely used in financial stochastic differential equations such as Square Root Process. Mao [2] indicates that Square Root Process may be more appropriate if the asset price volatility does not increase dramatically when $S(t)$ increases ($S(t)$ greater than 1), because the variance of error term is proportional to $S(t)$. Meanwhile, in epidemic modelling, Liang and Greenhalgh *et al.* introduce demographic stochasticity [6] in the deterministic SIS model based on Allen's work [14]. The diffusion coefficient of their SDE SIS model is $\sqrt{\beta I(t)(N-I(t)) + (\mu+\gamma)}$ which is very similar to ours. However, to the best of our knowledge, there is not enough work on incorporating white noise with square-root diffusion into the epidemic models. As a result, this paper aims to fill the gap.

As a result, we assume that two Brownian motions $B_1(t)$ and $B_2(t)$ are independent. We then substitute two perturbations in our SIS ODE (1.2). We have

$$\mathrm{d}I(t) = [\beta(N - I(t))I(t) - (\mu + \gamma)I(t)]\,\mathrm{d}t + \sigma_1 I(t)(N - I(t))\,\mathrm{d}B_1(t) \\ - \sigma_2 I(t)\sqrt{N - I(t)}\,\mathrm{d}B_2(t) \tag{1.5}$$

with initial value $I(0) = I_0 \in (0, N)$. In the following sections we will concentrate on giving some properties of the solution $I(t)$ of this SDE.

2 Existence of unique positive solution

In order for the model to make sense, we need to prove that the solution of our SDE has a unique global solution which remain within $(0, N)$, with the initial value $I_0 \in (0, N)$.
Theorem 2.1. If $\mu + \gamma \geq \frac{1}{2}\sigma_2^2 N$, then for any given initial value $I(0) = I_0 \in (0, N)$, the SDE has a unique global positive solution $I(t) \in (0, N)$ for all $t \geq 0$ with probability one, namely,

$$\mathbb{P}\{I(t) \in (0, N),\ \forall t \geq 0\} = 1$$

Proof.
The coefficients of our SDE are locally Lipschitz continuous and for any given initial value, there is a unique maximal local solution $I(t)$ on $t \in [0, \tau_e)$, where τ_e is the explosion time [2]. Let $k_0 \geq 0$ be sufficient large to satisfy $\frac{1}{k_0} < I_0 < N - \frac{1}{k_0}$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e)\ : I(t) \notin (1/k, N - 1/k)\}$$

In this paper we set $\inf\emptyset = \infty$. Obviously, τ_k is increasing when $k \to \infty$. And we set $\tau_\infty = \lim_{k\to\infty} \tau_k$. It is clear that $\tau_\infty \leq \tau_e$ almost sure. So if we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $I(t) \in (0, N)$ a.s. for all $t \geq 0$.
Here we assume $\tau_\infty = \infty$ a.s. is not true. Then we can find a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon$$

So we can find an integer $k_1 \geq k_0$ large enough, such that

$$\mathbb{P}\{\tau_k \leq T\} \geq \epsilon \quad \forall k \geq k_1 \tag{2.1}$$

Define a function $V : (0, N) \to \mathbb{R}_+$ by

$$V(x) = -\log x - \log{(N - x)} + \log\frac{N^2}{4}$$

and

$$V_x = -\frac{1}{x} + \frac{1}{N - x},\ V_{xx} = \frac{1}{x^2} + \frac{1}{(N - x)^2}$$

Let $f(t) = \beta(N - I(t))I(t) - (\mu + \gamma)I(t)$, $g(t) = (\sigma_1 I(t)(N - I(t)), -\sigma_2\sqrt{N - I(t)}I(t))$ and $\mathrm{d}B(t) = (\,\mathrm{d}B_1(t),\,\mathrm{d}B_2(t))$.
By Ito formula [2], we have, for any $t \in [0, T]$ and $k \geq k_1$

$$\mathbb{E}V(I(t \wedge \tau_k)) = V(I_0) + \mathbb{E}\int_0^{t\wedge\tau_k} LV(I(s))\,\mathrm{d}s + \mathbb{E}\int_0^{t\wedge\tau_k} V_x g(s)\,\mathrm{d}B(s) \tag{2.2}$$

$\mathbb{E}\int_0^{t\wedge\tau_k} V_x g(s)\,\mathrm{d}B(s) = 0$. Also it is easy to show that

$$
\begin{aligned}
LV(x) &= -\beta(N-x) + (\mu+\gamma) + \beta x - (\mu+\gamma)\frac{x}{N-x} \\
&\quad + \frac{1}{2}(\sigma_1^2(N-x)^2 + \sigma_1^2 x^2 + \sigma_2^2(N-x) + \sigma_2^2\frac{x^2}{N-x}) \\
&\leq -\beta(N-x) + (\mu+\gamma) + \beta x + \frac{1}{2}[\sigma_1^2(N-x)^2 + \sigma_1^2 x^2 + \sigma_2^2(N-x)] \\
&\leq C
\end{aligned}
\tag{2.3}
$$

C is a constant when $\mu+\gamma \geq \frac{1}{2}\sigma_2^2 N$ and $x \in (0, N)$.
Then we have

$$
\begin{aligned}
\mathbb{E}V(I(t\wedge\tau_k)) &\leq V(I_0) + \mathbb{E}\int_0^{t\wedge\tau_k} C\,\mathrm{d}s \\
&\leq V(I_0) + Ct
\end{aligned}
\tag{2.4}
$$

which yields that

$$
\mathbb{E}V(I(T\wedge\tau_k)) \leq V(I_0) + CT \tag{2.5}
$$

Set $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$ and we have $\mathbb{P}(\Omega_k) \geq \epsilon$. For every $\omega \in \Omega_k$, $I(\tau_k, \omega)$ equals either $1/k$ or $N - 1/k$ and we have

$$
V(I(\tau_k, \omega)) = \log\frac{N^2}{4(N-1/k)1/k}
$$

Hence

$$
\begin{aligned}
\infty > V(I_0) + CT &\geq \mathbb{E}[\mathbb{I}_{\Omega_k}(\omega)V(I(\tau_k,\omega))] \\
&\geq \mathbb{P}(\Omega_k)\log\frac{N^2}{4(N-1/k)1/k} \\
&= \epsilon\log\frac{N^2}{4(N-1/k)1/k}
\end{aligned}
$$

letting $k \to \infty$ will lead to the contradiction

$$
\infty > V(I_0) + CT = \infty
$$

So the assumption is wrong and we must have $\tau_\infty = \infty$ almost sure, whence the proof is now completed. However, the condition for our model to have bounded positive solution $\mu+\gamma \geq \frac{1}{2}\sigma_2^2 N$ might be confusing to readers. There are two different ways to understand this condition. In [1] there is no constraint on σ_1 but after adding second perturbation on $\mu+\gamma$, the square root term will trend to infinity very fast when $I(t) \to N$. So there must be a condition on σ_2 to neutralize it. Also, by the classical Feller test in Mao's book [2] on Mean Reverting Square Root Process, there is a very similar result on constraining the coefficient before square root term in order to make the solution always positive.

3 Extinction

In this section, we will discuss the conditions for the disease to die out in our SDE model (1.5). Here we give the conditions for the solution $I(t)$ of our SDE becoming extinction.

Theorem 3.1. Given that $R_0^S := R_0^D - \frac{\sigma_1^2 N^2 + \sigma_2^2 N}{2(\mu+\gamma)} = \frac{\beta N}{\mu+\gamma} - \frac{\sigma_1^2 N^2 + \sigma_2^2 N}{2(\mu+\gamma)} < 1$*, then for any given initial value* $I(0) = I_0 \in (0, N)$*, the solution of SDE obeys*

$$\limsup_{t\to\infty} \frac{1}{t} \log I(t) < 0 \; a.s. \tag{3.1}$$

if one of the following three conditions is satisfied
• $\sigma_1^2 N + \frac{1}{2}\sigma_2^2 \leq \beta$ *or*
• $\frac{1}{2}\sigma_2^2 \geq \beta$ *or*
• $(\beta - \sigma_1\sqrt{2(\mu+\gamma)}) \vee (\beta - \sigma_1^2 N) < \frac{1}{2}\sigma_2^2 < \beta$
namely, I(t) will trend to zero exponentially a.s. And the disease will die out with probability one.
Proof. Here we use Ito formula

$$\begin{aligned}\frac{\log I(t)}{t} = \frac{\log I_0}{t} + \frac{1}{t}\int_0^t L\tilde{V}(I(s))\,\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_1(N - I(s))\,\mathrm{d}B_1(s) \\ - \frac{1}{t}\int_0^t \sigma_2\sqrt{(N - I(s))}\,\mathrm{d}B_2(s)\end{aligned} \tag{3.2}$$

$L\tilde{V}$ is defined by

$$L\tilde{V}(x) = \beta(N - x) - (\mu+\gamma) - \frac{1}{2}[\sigma_1^2(N-x)^2 + \sigma_2^2(N-x)], x \in (0, N) \tag{3.3}$$

According to the large number theorem for martingales[2], we must have

$$\limsup_{t\to\infty} \frac{1}{t}\{\int_0^t \sigma_1(N - I(s))\,\mathrm{d}B_1(s) - \int_0^t \sigma_2\sqrt{(N - I(s))}\,\mathrm{d}B_2(s)\} = 0 \tag{3.4}$$

So if we can prove $L\tilde{V} \leq \tilde{C} < 0$, then $\limsup_{t\to\infty} \frac{1}{t}\log I(t) < 0$ a.s.($\tilde{C}$ is a constant)
We first examine $L\tilde{V}$ at 0 and N. $L\tilde{V}(N) = -(\mu+\gamma) < 0$ and $L\tilde{V}(0) = \beta N - (\mu+\gamma) - \frac{1}{2}(\sigma_1^2 N^2 + \sigma_2^2 N)$ so we must have firstly

$$L\tilde{V}(0) < 0, \text{which is ensured by} R_0^S < 1 \tag{3.5}$$

$L\tilde{V}(x)$ has the maximal value when

$$x = \hat{x} = \frac{-\beta + \sigma_1^2 N + \frac{1}{2}\sigma_2^2}{\sigma_1^2} = N + \frac{\frac{1}{2}\sigma_2^2 - \beta}{\sigma_1^2} \tag{3.6}$$

and

$$L\tilde{V}(\hat{x}) = \frac{1}{2}\frac{(\beta - \frac{1}{2}\sigma_2^2)^2}{\sigma_1^2} - (\mu+\gamma) \tag{3.7}$$

is the maximal value of $L\tilde{V}$ when $x \in \mathbb{R}$
So we need to discuss with the following three different cases
Case 1. $\hat{x} \leq 0$
With $L\tilde{V} < 0$ at 0 and N, if we have $\hat{x} \leq 0$ Then $L\tilde{V} < 0$ for all $x \in (0, N)$.Consequently,

$$\sigma_1^2 N + \frac{1}{2}\sigma_2^2 \leq \beta \tag{3.8}$$

Case 2. $\hat{x} \geq N$
This is similar with *Case 1.*. $L\tilde{V} < 0$ for all $x \in (0, N)$.So we must have

$$\frac{1}{2}\sigma_2^2 \geq \beta \tag{3.9}$$

Case 3. $\hat{x} \in (0, N)$ In this case we need to make sure the maximal value $L\tilde{V}(\hat{x}) < 0$. So we have

$$L\tilde{V}(\hat{x}) = \frac{1}{2}\frac{(\beta - \frac{1}{2}\sigma_2^2)^2}{\sigma_1^2} - (\mu + \gamma) < 0 \tag{3.10}$$

Also,

$$\frac{1}{2}\sigma_2^2 < \beta \tag{3.11}$$

and

$$\sigma_1^2 N + \frac{1}{2}\sigma_2^2 > \beta \tag{3.12}$$

is required for $\hat{x}$ within $(0, N)$. Rearrange and we therefore have the result for *Case 3*

$$(\beta - \sigma_1\sqrt{2(\mu + \gamma)}) \vee (\beta - \sigma_1^2 N) < \frac{1}{2}\sigma_2^2 < \beta \tag{3.13}$$

Hence when any of the three cases is satisfied, we must have $L\tilde{V} \leq \tilde{C} < 0$ ($\tilde{C}$ is a constant). It then follows that

$$\limsup_{t\to\infty} \frac{\log I(t)}{t} \leq \limsup_{t\to\infty} \frac{\log I_0}{t} + \limsup_{t\to\infty} \frac{1}{t}\tilde{C}t + 0 < 0 \;\; a.s.$$

Therefore we now have obtained the proof of *Theorem 4.1*.

Simulation. In this paper we assume that the unit of time is one day and the population size is measured in units of 1 million. Consequently, our parameters are given by the following values in this section.

$$N = 100, \; \beta = 0.4, \; \mu + \gamma = 0.45, \; \sigma_1 = 0.03$$

In order to find the value of σ_2, we initially need the model to make sense, so we have

$$\sigma_2 \leq 2(\mu + \gamma)/N = 0.94868 \tag{3.14}$$

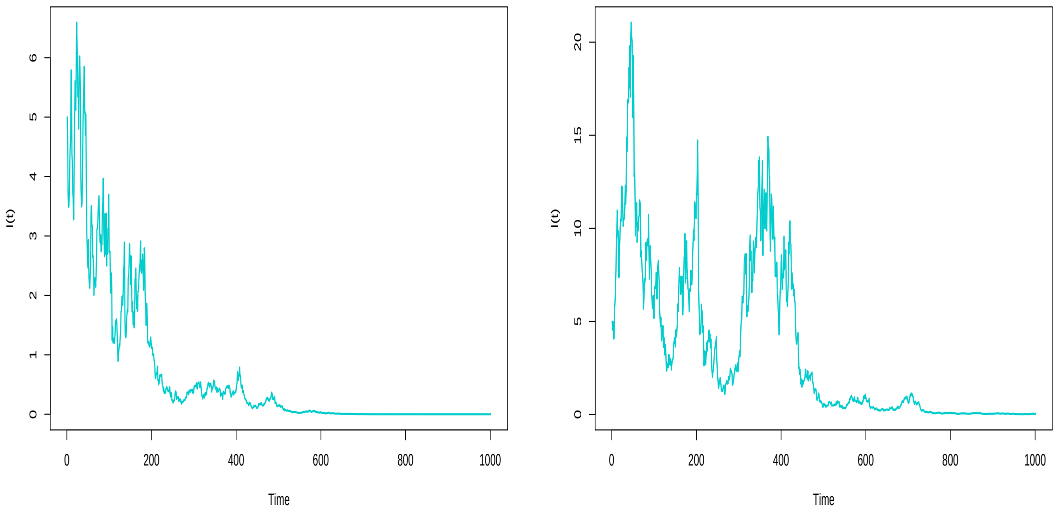
and also if there is extinction in our model, we need

$$R_0^S < 1, \text{which results in } \sigma_2 \geq 0 \tag{3.15}$$

Using these parameters in the other three conditions, we have the corresponding σ_2 to satisfy the three conditions in extinction.

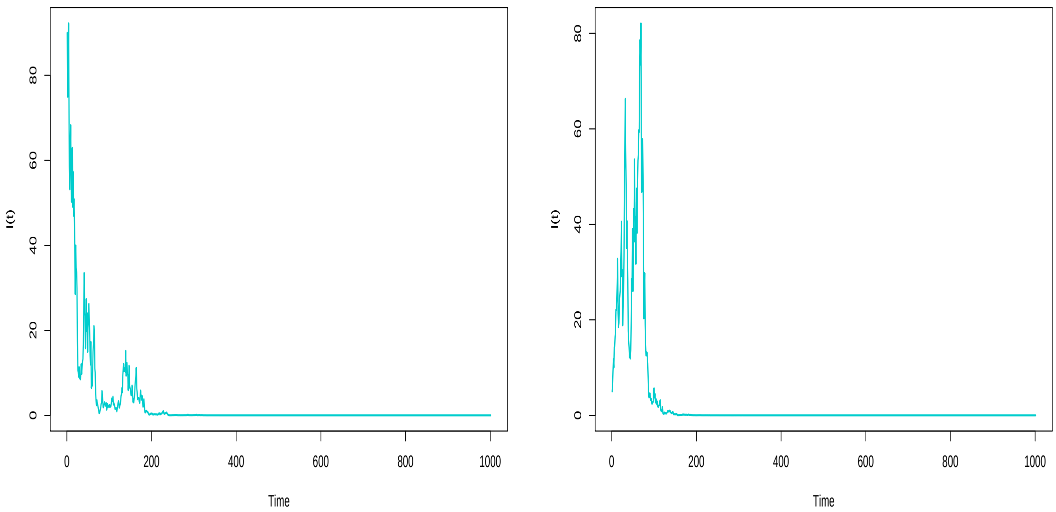
- condition 1:$\sigma_2 \leq 0.78740078$ or,
- condition 2:$\sigma_2 \geq 0.8944271$ or,
- condition 3: $0.78740078 \leq \sigma_2 \leq 0.8944271$

Here we choose 0.3, 0.9 and 0.82 respectively and plot our model by using Euler-Maruyama (EM) Method[2, 3] in R, with step size $\Delta = 0.001$ and both large and small initial values. The computer simulations are presented in Figure 1, 2 and 3. Clearly, our results in this section are illustrated and supported by the simulations. With the values of parameters, the disease will die out.



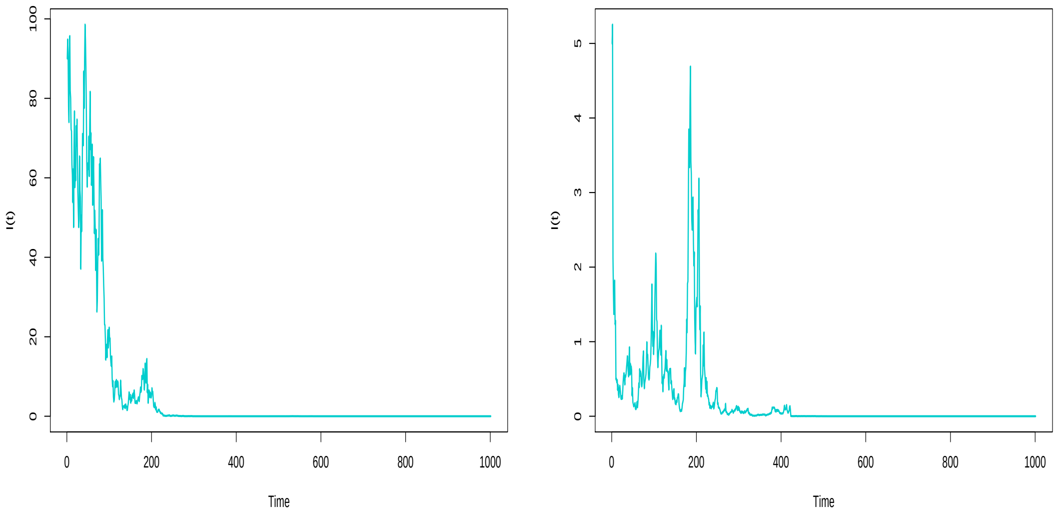
$I(0) = 90$ $I(0) = 5$

Figure 1: Extinction with condition 1



$I(0) = 90$ $I(0) = 5$

Figure 2: Extinction with condition 2



$I(0) = 90$ $\qquad\qquad$ $I(0) = 5$

Figure 3: Extinction with condition 3

4 Persistence

In this section we want to discuss the conditions for the disease to persist in our model. However, there are many definitions of persistence in stochastic dynamic problems[1, 2, 3, 5, 4, 7, 15]. For example, in Mao's book [2] he gives a very general definition of persistence, which only needs the disease to never become extinction with probability 1, such that

$$\liminf_{t\to\infty} I(t) > 0$$

While Greenhalgh and Gray [1] define persistence of their model as oscillations around a positive level. This is a very strong result in epidemic problem. As our works is an extension of [1], we give the theorem 4.1 as following

Theorem 4.1. If $R_0^S = R_0^D - \frac{\sigma_1^2 N^2 + \sigma_2^2 N}{2(\mu+\gamma)} = \frac{\beta N}{\mu+\gamma} - \frac{\sigma_1^2 N^2 + \sigma_2^2 N}{2(\mu+\gamma)} > 1$, *then for any given initial value* $I(0) = I_0 \in (0, N)$, *the solution of (1.5) follows*

$$\limsup_{t\to\infty} I(t) \geq \xi \text{ and } \liminf_{t\to\infty} I(t) \leq \xi \;\; a.s. \tag{4.1}$$

where

$$\xi = \frac{-\beta + \sigma_1^2 N + \frac{1}{2}\sigma_2^2 + \sqrt{\beta^2 - \sigma_2^2\beta - 2\sigma_1^2(\mu+\gamma) + \frac{1}{4}\sigma_2^4}}{\sigma_1^2} \tag{4.2}$$

which is the only positive root of $L\tilde{V} = 0$ *in* $(0, N)$*. I(t) will be above or below the level* ξ *infinitely often with probability one.*

Proof. When $R_0^S > 1$, recall that

$$L\tilde{V}(x) = \beta(N - x) - (\mu+\gamma) - \frac{1}{2}[\sigma_1^2(N-x)^2 + \sigma_2^2(N-x)], x \in (0, N)$$

and we have $L\tilde{V}(0) > 0$, $L\tilde{V}(N) = -(\mu+\gamma) < 0$ and $\xi > \hat{x} = \frac{-\beta+\sigma_1^2 N+\frac{1}{2}\sigma_2^2}{\sigma_1^2}$. So $L\tilde{V}(x)$ is strictly increasing in $(0, 0\vee\hat{x})$ and strictly decreasing in $(0\vee\hat{x}, N)$.
Here we recall (3.2)

$$\begin{aligned}\frac{\log I(t)}{t} = \frac{\log I_0}{t} + \frac{1}{t}\int_0^t L\tilde{V}(I(s))\,\mathrm{d}s + &\frac{1}{t}\int_0^t \sigma_1(N-I(s))\,\mathrm{d}B_1(s)\\ &-\frac{1}{t}\int_0^t \sigma_2\sqrt{(N-I(s))}\,\mathrm{d}B_2(s)\end{aligned}$$

By the large number theorem for martingales[2], there is an $\Omega_2 \subset \Omega$ with $\mathbb{P}\{\Omega_2\} = 1$ such that for every $\omega \in \Omega_2$

$$\lim_{t\to\infty}\frac{1}{t}\{\int_0^t \sigma_1(N-I(s))\,\mathrm{d}B_1(s) - \int_0^t \sigma_2\sqrt{(N-I(s))}\,\mathrm{d}B_2(s)\} = 0 \tag{4.3}$$

Now we assume that $\limsup_{t\to\infty} I(t) \geq \xi\ a.s.$ is not true. Then there must be a small $\epsilon \in (0,1)$ such that

$$\mathbb{P}\{\limsup_{t\to\infty} I(t) \leq \xi - 2\epsilon\} > \epsilon \tag{4.4}$$

Let $\Omega_1 = \{\limsup_{t\to\infty} I(t) \leq \xi - 2\epsilon\}$, then for every $\omega \in \Omega_1$, there exist $T = T(\omega)$ large enough, such that

$$I(t,\omega) \leq \xi - 2\epsilon + \epsilon = \xi - \epsilon, \text{ when } t \geq T(\omega) \tag{4.5}$$

which means when $t \geq T(\omega)$, $L\tilde{V}(I(t,\omega)) \geq L\tilde{V}(\xi-\epsilon)$. So we have for any fixed $\omega \in \Omega_1 \cap \Omega_2$ and $t \geq T(\omega)$

$$\begin{aligned}\liminf_{t\to\infty}\frac{1}{t}\log I(t,\omega) &\geq 0 + \lim_{t\to\infty}\frac{1}{t}\int_0^{T(\omega)} L\tilde{V}(I(s,\omega))\,\mathrm{d}s + \lim_{t\to\infty}\frac{1}{t}L\tilde{V}(\xi-\epsilon)(t-T(\omega))\\ &\geq L\tilde{V}(\xi-\epsilon) > 0\end{aligned}$$

which yields

$$\lim_{t\to\infty} I(t,\omega) = \infty \tag{4.6}$$

and this contradicts with the assumption (4.4). So we must have $\limsup_{t\to\infty} I(t) \geq \xi$ almost sure.

Similarly, if we assume that $\liminf_{t\to\infty} I(t) \leq \xi\ a.s.$ is not true.Then there must be a small $\delta \in (0,1)$ such that

$$\mathbb{P}\{\liminf_{t\to\infty} I(t) \geq \xi + 2\delta\} > \delta \tag{4.7}$$

Let $\Omega_3 = \{\liminf_{t\to\infty} I(t) \geq \xi + 2\delta\}$, then for every $\omega \in \Omega_3$, there exist $T' = T'(\omega)$ large enough, such that

$$I(t,\omega) \geq \xi + 2\delta - \delta = \xi + \delta, \text{ when } t \geq T'(\omega) \tag{4.8}$$

Now we fix any $\omega \in \Omega_3 \cap \Omega_2$ and $t \geq T'(\omega)$ in (3.2) and we have

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log I(t,\omega) &\leq 0 + \lim_{t\to\infty}\frac{1}{t}\int_0^{T'(\omega)} L\tilde{V}(I(s,\omega))\,\mathrm{d}s + \lim_{t\to\infty}\frac{1}{t}L\tilde{V}(\xi+\delta)(t-T'(\omega))\\ &\leq L\tilde{V}(\xi+\delta) < 0\end{aligned}$$

which yields

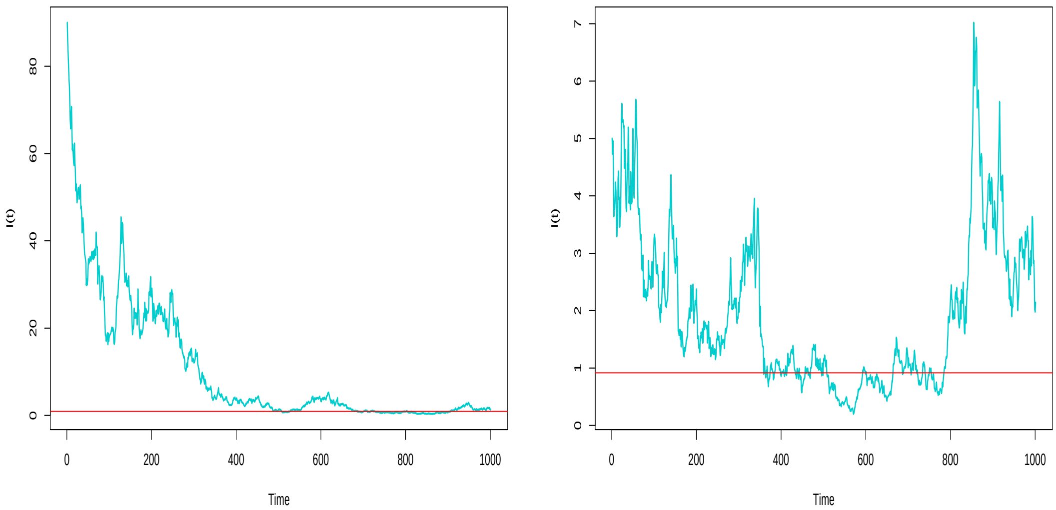
$$\lim_{t\to\infty} I(t,\omega) = 0 \tag{4.9}$$

and this contradicts the assumption (4.7). So we must have $\liminf_{t\to\infty} I(t) \leq \xi$ almost sure.

Simulation. In this section we choose the values of our parameter as following

$$N = 100,\ \beta = 0.5,\ \mu + \gamma = 0.45,\ \sigma_1 = 0.03$$

With $R_0^S > 1$, we have $\sigma_2 < 0.1$. Hence here we choose $\sigma_2 = 0.05$ and the level $\xi = 0.916056$. Similarly, as the level ξ is very closed to zero, we use both large and small initial values and plot the level ξ in the simulation plots to illustrate the results. From Figure 4, it is clear that the number of infectious population will fluctuated around the level ξ. Thus the disease will not die out or explode, which means the disease will persist.



$I(0) = 90$ $\qquad$ $I(0) = 5$

Figure 4: Persistence

5 Stationary Distribution

In this section we will prove that there exists a unique stationary distribution of our SDE model (1.5) when the solution persists between 0 and N. So we give the first theorem this section.

Theorem 5.1 If $R_0^S > 1$, then our SDE model (1.5) has a unique stationary distribution

In order to complete our proof, we need to initially use a well-known result from Khaminskii as a lemma. [8]
Lemma 5 . The SDE model (1.3) has a unique stationary distribution if there is a strictly proper subinterval (a,b) of (0,N) such that $\mathbb{E}(\tau) < \infty$ for all $I_0 \in (0, a] \cup [b, N)$, where

$$\tau = \inf\{t \geq 0 : I(t) \in (a, b)\} \tag{5.1}$$

also,

$$\sup_{I_0 \in [\bar{a}, \bar{b}]} \mathbb{E}(\tau) < \infty \tag{5.2}$$

for every interval $[\bar{a}, \bar{b}] \subset (0, N)$ Now we can prove *Theorem 5.1* using *Lemma 5*.
Proof. Firstly we need to fix any (a, b) such that,

$$0 < a < \xi < b < N \tag{5.3}$$

recall $L\tilde{V}$ in last section, we can see that

$$L\tilde{V}(x) \geq L\tilde{V}(0) \wedge L\tilde{V}(a), \text{if } 0 < x \leq a \tag{5.4}$$

$$L\tilde{V}(x) \leq L\tilde{V}(b), \text{if } b \leq x < N \tag{5.5}$$

also, recall (3.2)

$$\begin{aligned}\log I(t) = \log I_0 + \int_0^t L\tilde{V}(I(s))\,\mathrm{d}s + \int_0^t \sigma_1(N - I(s))\,\mathrm{d}B_1(s) \\ - \int_0^t \sigma_2\sqrt{(N - I(s))}\,\mathrm{d}B_2(s)\end{aligned}$$

and define

$$\tau = \inf\{t \geq 0 : I(t) \in (a, b)\} \tag{5.6}$$

Case 1. For all $t \geq 0$ and any $I_0 \in (0, a)$, from (5.4), we have

$$\begin{aligned}\mathbb{E}\log I(t \wedge \tau) &= \mathbb{E}\log I_0 + \mathbb{E}\int_0^{t\wedge\tau} L\tilde{V}(I(s))\,\mathrm{d}s + 0 \\ &\geq \log I_0 + \mathbb{E}(L\tilde{V}(0) \wedge L\tilde{V}(a))(t \wedge \tau) \qquad (5.7)\end{aligned}$$

From definition of τ, we know that

$$\log a \geq \mathbb{E}\log I(t \wedge \tau) \text{ when } I_0 \in (0, a] \tag{5.8}$$

Rearrange and we have

$$\mathbb{E}(t \wedge \tau) \leq \frac{\log\left(\frac{a}{I_0}\right)}{L\tilde{V}(0) \wedge L\tilde{V}(a)}$$

when $t \to \infty$

$$\mathbb{E}(\tau) \leq \frac{\log\left(\frac{a}{I_0}\right)}{L\tilde{V}(0) \wedge L\tilde{V}(a)} < \infty, \forall I_0 \in (0, a] \tag{5.9}$$

Case 2.For all $t \geq 0$ and any $I_0 \in (b, N)$,from (5.5), we have

$$\begin{aligned}\mathbb{E}\log I(t \wedge \tau) &= \mathbb{E}\log I_0 + \mathbb{E}\int_0^{t\wedge\tau} L\tilde{V}(I(s))\,\mathrm{d}s + 0 \\ &\leq \log I_0 + \mathbb{E}(L\tilde{V}(b))(t \wedge \tau) \qquad (5.10)\end{aligned}$$

From definition of τ, we know that

$$\log b \leq \mathbb{E}\log I(t \wedge \tau) \text{ when } I_0 \in (b, N] \tag{5.11}$$

Rearrange and we have

$$\log b \leq \log I_0 + L\tilde{V}(b)\mathbb{E}(t \wedge \tau)$$

$$\mathbb{E}(t \wedge \tau) \leq -\frac{\log\left(\frac{b}{I_0}\right)}{\mid L\tilde{V}(b) \mid}$$

when $t \to \infty$

$$\mathbb{E}(\tau) \leq -\frac{\log\left(\frac{b}{I_0}\right)}{\mid L\tilde{V}(b) \mid} \leq \infty \ \ \forall I_0 \in (b, N] \tag{5.12}$$

Combine the results from both *Case 1* and *Case 2* and we complete the proof of *Theorem 6.1*. Now we need to give the mean and variance of the stationary distribution.

Theorem 5.2 If $R_0^S > 1$ and denote m and v as the mean and variance of the stationary distribution of SDE model (1.5). Then we have

$$m = \frac{2\beta(R_0^S - 1)(\mu + \gamma)}{2\beta^2 - \sigma_1^2(\beta N + \mu + \gamma) - \sigma_2^2\beta} \tag{5.13}$$

and

$$v = \frac{\beta N - \mu - \gamma}{\beta} m - m^2 \tag{5.14}$$

Proof. For any $I_0 \in (0, N)$, we firstly recall (1.5) in the integral form

$$I(t) = I_0 + \int_0^t [\beta(N - I(s))I(s) - (\mu + \gamma)I(s)]\,\mathrm{d}s + \int_0^t \sigma_1 I(s)(N - I(s))\,\mathrm{d}B_1(s) \\ \int_0^t -\sigma_2 I(s)\sqrt{N - I(s)}\,\mathrm{d}B_2(s) \tag{5.15}$$

Dividing both sides by t and when $t \to \infty$, applying the ergodic property of the stationary distribution [8] and also the large number theorem of martingales, we have the result that

$$0 = (\beta N - \mu - \gamma)m - \beta m_2 \tag{5.16}$$

where m, m_2 are the mean and second moment of the stationary distribution. Also,we need to consider (3.2) as well

$$\frac{\log I(t)}{t} = \frac{\log I_0}{t} + \frac{1}{t}\int_0^t L\tilde{V}(I(s))\,\mathrm{d}s + \frac{1}{t}\int_0^t \sigma_1(N - I(s))\,\mathrm{d}B_1(s) \\ - \frac{1}{t}\int_0^t \sigma_2\sqrt{(N - I(s))}\,\mathrm{d}B_2(s) \tag{5.17}$$

when $t \to \infty$. We have

$$\frac{1}{2}\sigma_1^2 m_2 - (\sigma_1^2 N + \frac{1}{2}\sigma_2^2 - \beta)m = \beta N - \mu - \gamma - \frac{1}{2}\sigma_1^2 N^2 - \frac{1}{2}\sigma_2^2 N \tag{5.18}$$

Note that $\beta N - \mu - \gamma - \frac{1}{2}\sigma_1^2 N^2 - \frac{1}{2}\sigma_2^2 N = (R_0^S - 1)(\mu + \gamma)$. Rewrite this

$$\frac{1}{2}\sigma_1^2 m_2 - (\sigma_1^2 N + \frac{1}{2}\sigma_2^2 - \beta)m = (R_0^S - 1)(\mu + \gamma) \tag{5.19}$$

Rearrange and we have

$$m = \frac{2\beta(R_0^S - 1)(\mu + \gamma)}{2\beta^2 - \sigma_1^2(\beta N + \mu + \gamma) - \sigma_2^2\beta} \tag{5.20}$$

Hence

$$v = m_2 - m^2 = \frac{\beta N - \mu - \gamma}{\beta} m - m^2 \tag{5.21}$$

Simulation. In this section we choose the values of our parameter as following

$$N = 100, \ \beta = 0.5, \ \mu + \gamma = 0.45, \ \sigma_1 = 0.02, \ \sigma_2 = 0.05$$

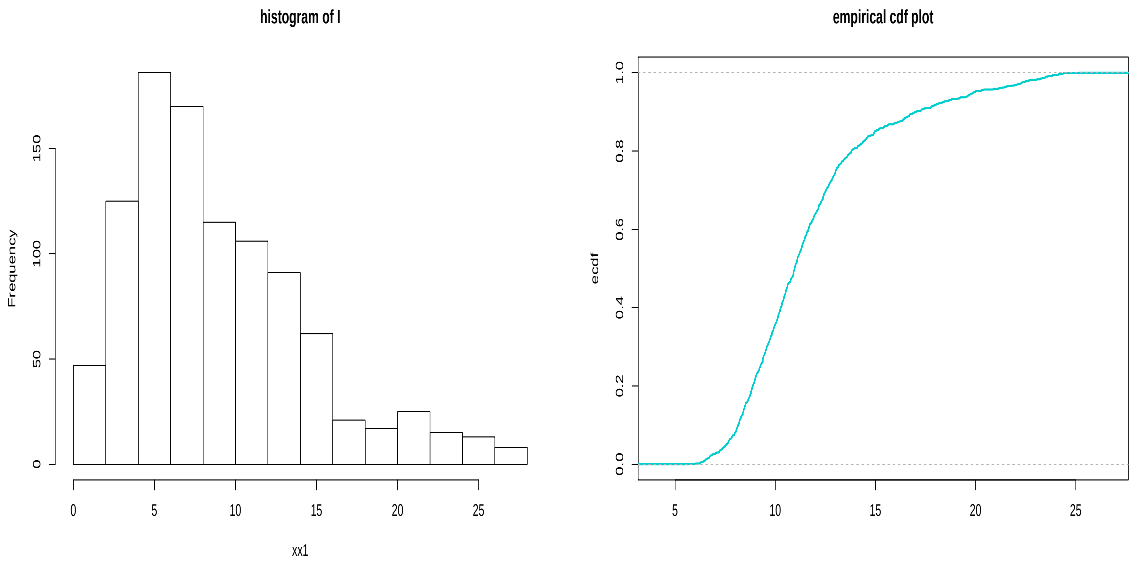
$R_0^S = 1.06389 > 1$ so the disease will persist and there is a stationary distribution of our model. And for these parameters, the mean and variance of the stationary distribution of our model is

$$m = 6.23982, \ v = 23.4628$$

In order to reach the stationary distribution in our simulation, we set a long run of 20000 iterations with step size $\Delta = 0.001$ and then calculate the mean and variance for the last 1000 iterations. The results from simulations show that

$$m = 6.531954, \ v = 23.2428$$

Figure 5 also displays the path of $I(t)$ and the empirical cumulative distribution functions for the last 1000 samples of the simulation.



$I(t) : \sigma_1 = 0.02, \sigma_2 = 0.05$ ecdf

Figure 5: Stationary Distribution

Conclusion

In this paper we introduce another perturbation on $\mu + \gamma$ based on Greenhalgh and Gray's research[1] with a different form. This SIS SDE model with two independent Brownian motion has very similar properties as theirs[1]. We firstly prove that our model has a unique and positive solution which is bounded with $(0, N)$ with probability 1. Then we define the Stochastic Reproduction Number of our model, which needs a weaker condition for the model to be extinction compared to the classical deterministic model and the previous model with one perturbation. When $R_0^S < 1$, we find the further three conditions for the disease to die out. As long as one of these is satisfied, the disease will die out with probability one. When $R_0^S > 1$, we prove that the solution of our model will oscillate around a positive level ξ almost surely. Under this circumstance, we find the unique positive stationary distribution of our SDE model with the expression of mean and variance. Importantly, simulations with different values of parameters are produced to illustrate and support our theoretical results.

Our new perturbation clearly needs σ_2 not too large from *Theorem 2.1* to ensure a unique bounded positive solution of (1.5). However this perturbation extends the requirements for $R_0^S < 1$ compared to the deterministic SIS model and the results in [1]. This means for those parameters that will not cause the disease to die out in the deterministic model as well as Gray's model [1], extinction will become possible if we add the new perturbation. Meanwhile, we find the unique stationary distribution with no extra conditions, which means that adding our new perturbation in Gray's model [1] will have similar results.

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References

[1] A. Gray, D.Greenhalgh, L.Hu, X.Mao and J.Pan, A Stochastic Differential Equation SIS Epidemic Model, *SIAM Journal on Applied Mathematics*, vol.71, no.3, pp. 876-902, 2011.

[2] X.Mao, Stochastic Differential Equations and Applications 2nd ed., *Horwood Chichester*, UK, 2008.

[3] X.Mao, C.Yuan, Stochastic Differential Equations with Markovian Switching, *Imperial College Press*, London, UK, 2006.

[4] A. Gray, D.Greenhalgh, X.Mao and J.Pan, The SIS Epidemic Model With Markovian Switching, *Journal of Mathematical Analysis and Applications*, 394(2), pp.876-902, 2011.

[5] Y.Liang, D.Greenhalgh, X.Mao, A Stochastic Differential Equation Model for the Spread of HIV amongst People Who Inject Drugs, *Hindawi Publishing Corporation, Computational and Mathematical Methods in Medicine*, vol.2016, Article ID 6757928, 14 pages, 2016.

[6] D.Greenhalgh, Y. Liang, X.Mao, Demographic Stochasticity in the SDE SIS Epidemic Model, *Discrete and Continuous Dynamical Systems - Series B*, vol. 20, No. 9, 30.09.2015, p. 2859-2884, 2015.

[7] D.Greenhalgh, Y. Liang, X.Mao, SDE SIS Epidemic Model with Demographic Stochasticity and Varying Population Size, *Journal of Applied Mathematics and Computation*, vol.276, pp.218–238, 2016.

[8] R.Z. Hasminskii, Stochastic Stability of Differential Equations, *Sijthoff & Noordhoff, Alphenaan den Rijn*, The Netherlands, Germantwn, MD, 1980.

[9] H. W. Hethcote and J. A. Yorke, Gonorrhea Transmission Dynamics and Control, *Lecture Notes in Biomath* 56, Springer-Verlag, Berlin,1984.

[10] B. Shugin, L. Stone, Z. Agur, Pulse Vaccination Strategy in the SIR Epidemic Model, *Bulletin of Mathematical Biology*, 60, 1123-1148, 1998

[11] V. Capasso, A generalization of the Kermack-McKendrick deterministic epidemic model, 2005 *Mathematical Biosciences*, vol. 42, Issues 1–2, pp.43-61, 1978

[12] Y.Zhao, D. Jiang and D. O'Regan, The extinction and persistence of the stochastic SIS epidemic model with vaccination, *Physica A: Statistical Mechanics and its Applications*, vol.392,Issue 20, pp.4916-4927, 2013

[13] G.Hu, M.Liu, K. Wang, The asymptotic behaviours of an epidemic model with two correlated stochastic perturbations, *Applied Mathematics and Computation*, vol.218, Issue 21,pp.10520-10532,2012

[14] E. J. Allen, Modelling with Ito Stochastic Differential Equations, *Springer- Verlag*, Dordecht, 2007

[15] Y. Cai, X. Mao, Stochastic prey-predator system with foraging arena scheme, *Applied Mathematical Modelling*, 64. pp. 357-371. ISSN 0307-904X, 2018