Reliability modeling and preventive maintenance of load-sharing systems with degrading components

Abstract

This paper presents certain new approaches to reliability modeling of systems subject to shared loads. It is assumed that components in the system degrade continuously through additive impact under load. Reliability assessment of such systems is often complicated by the fact that both the arriving load and the failure of components influence the degradation of the surviving components in a complex manner. The proposed approaches seek to ease this problem by first deriving the time to prior failures and the arrival of random loads and then determining the number of failed components. Two separate models capable of analyzing system reliability as well as arriving at system maintenance and design decisions are proposed. The first considers constant load and the other cumulative load. A numerical example is presented to illustrate the effectiveness of the proposed models.

Keywords: load sharing system, continuous degradation, random load, reliability model, preventive maintenance

Notation

$CR(\tau,P)$	Maintenance cost rate
$F_i(t)$	cdf of the *i*th failure time
$F_T(t)$	cdf of the system failure time
$f_i(t)$	pdf of the *i*th failure time
H	Failure threshold of a component
L_k	Magnitude of the *k*th arrived load
$L(t)$	Total load magnitude on the system at time t
$l(t)$	Load magnitude of a component at time t
$M(t)$	Number of loads that have arrived by time t

$N(t)$	Number of failed components by time *t*
N_r	Number of repetitions for Monte Carlo simulation
n	Number of components in the system
P	Degradation threshold for preventive maintenance
$R(t)$	Reliability of the system at time *t*
$r(t)$	Reliability of a component at time *t*
S	Length of a renewal cycle
T	Time to system failure
T_i	Time to the *i*th failure
T_P	Time to reach preventive maintenance threshold
t_j	Arrival time of the *j*th load
$X(t)$	Degradation amount of a component by time *t*
β	Degradation rate
λ	Arrival rate of random loads
μ	Initial degradation amount
$\Phi(\cdot)$	cdf of standard normal distribution
τ	Periodical inspection interval

1. Introduction

It is often assumed in reliability engineering that failures are independent, i.e., failure of a component has little effect on failures of other components. However, many systems are load sharing (all the components work together and share system load) in practice. The assumption of independence is not valid in such a system. If a component fails the same workload has to be shared by the surviving components, so each surviving component experiences an increased load (Singh and Gupta, 2012). The increased load would probably induce a higher failure rate.

Load-sharing systems have been applied widely in industrial practice. For example, in a distributed computer system, servers work together to finish the workload imposed on the system (Levitin and Dai, 2007); in a gear system, the workload is shared by each mesh gear pair with certain load sharing rule (Yu *et al*., 2013); in a power grid, the total electricity demand is distributed across several links in the net (Basu *et al.*, 2012). More examples can be found in (Kvam and Pena, 2005).

There exist numerous works in literature concerning load-sharing systems. However, most of the studies have focused on statistical inference and parameter estimation of lifetime distributions of load-sharing systems (Deshpande *et al*., 2010; Kim and Kvam, 2004; Singh *et al*., 2008; Balakrishnan *et al.*, 2011; Park, 2010; Park, 2013). In contrast, studies on the reliability analysis of load-sharing systems are relatively rare. A reason is that the mechanism of how the load is shared among components is too complex to permit a thorough analysis (Amari *et al*., 2008). Another issue complicating the analysis is that load history affects the reliability of the system within its lifetime.

In most of the literature concerning reliability analysis, it has been assumed for the sake of simplicity that the lifetime of a component follows an exponential distribution (Shao and Lamberson, 1991; Yun *et al*., 2012; Qi *et al*., 2014). However, the assumption of exponential lifetime distribution is questionable in many practical contexts. The restrictive assumption of exponential distribution has been relaxed in some studies (Singh and Gupta, 2012; Durham *et al*., 1997; Liu, 1998; Ibnabdeljalil and Curtin, 1997; Amari and Bergman, 2008). Amari and Bergman (2008) analyzed load-sharing systems with general lifetime distribution with Tampered Failure Rate (TFR) model and Cumulative Exposure (CE) model. It was emphasized that an appropriate model must be carefully chosen to account for the influence of load history, so as to extend the results of the case of exponential distribution to the case of general distribution. Liu (1998) investigated the reliability of *k-out-of-n* systems, where the lifetime distribution of the components is arbitrary. Nonetheless, there is no closed-form solution and numerical methods have to be used to compute the system reliability. Singh and Gupta (2012) analyzed system reliability assuming Lindley lifetime distribution of components and estimated the parameters with Markov Chain Monte Carlo methods. Durham *et al.* (1997) assumed that components follow a Weibull lifetime distribution and studied the failure mechanism of fibers under localized load-sharing rule. Ibnabdeljalil and Curtin (1997) studied the reliability and strength of fiber-reinforced composites under local load-sharing condition. The study demonstrated that the ultimate strength decreases with the composite size and failure occurs by local accumulation of a critical amount of damage.

An assumption implicit in the previous studies is that the components are static; the condition of component is invariant and component failure is sudden and catastrophic. However, in many real world applications, the working environment is usually dynamic and a change in environment may lead to a change in the physics of failures. Many systems and components go through a period of degradation and cease functioning when the degradation amount reaches a critical threshold level. This type of failure is said to be 'soft failure' (Ye *et al.*, 2012). Singpurwalla (1995) pointed out that modelling failure using a stochastic-process approach provides flexibility with respect to describing the failure-generating mechanisms. One advantage of using degradation model is that the degradation level can be detected by inspection/monitoring equipment and therefore the relationship between load and system failure can be more accurately characterized. By taking advantage of the degradation model, we conduct reliability analysis and maintenance policy for load-sharing systems in a continuous degradation context.

In traditional approaches for reliability analysis, proportional hazards models are established to account for the effect of loads on component lifetime distribution (Liu, 1992; Amari and Bergman, 2008). However, for a load-sharing system with continuously degrading components, both the internal degradation process and the external loads have effects on system reliability. The proportional hazards method is limited to a two-state system and cannot model the system behavior with continuous degradation. A novel reliability model is required to address load-sharing system with degrading components. Actually, in the work of Peng *et al.* (2010), the authors developed a reliability model for system with multiple dependent competing failure processes, where a cumulative shock model was adopted to characterize the joint effect of inner degradation and external shocks. In our study for load-sharing system, the cumulative shock model is also used for the service of reliability modeling. While Peng *et al.* (2010) was focused on reliability analysis for a single-component system, we proceed to integrate the cumulative shock model into load-sharing context.

Preventive maintenance strategies for multi-component systems have been studied extensively in the literature (Moghaddam and Usher, 2011; Liu *et al*., 2014; Wang and Pham, 2011; Wang *et al.*, 2014). Most maintenance policies currently in vogue for multi-component systems focus on the economic dependence among components. To the best of our knowledge, few studies have addressed the issue of preventive maintenance for load-sharing systems, although this is a very common situation when a system does not fail completely after a component failure but the loads on others go up.

In this paper, we construct reliability models for load-sharing systems subject to degrading components so as to arrive at a preventive maintenance strategy. At first we obtain some preliminary insights by modeling system reliability with constant load. Next we build a reliability model assuming varying loads. Specifically we consider the scenario where the load is random and has a cumulative impact on the system. Finally, we utilize the proposed reliability models to arrive at preventive maintenance decisions.

The remainder of this paper is organized as follows. Section 2 presents the specifications of the system including assumptions, system description and the load sharing rule. In section 3, we construct two reliability models considering constant load and cumulative load separately. A preventive maintenance model with inspection interval and preventive maintenance threshold being optimized simultaneously is developed in Section 4. A numerical example is presented in Section 5 to illustrate the effectiveness of the reliability models at arriving at maintenance policies. Finally, Section 6 concludes the study and makes some suggestions for further work.

2. System specifications

2.1 System description

The following assumptions have been adapted from (Peng *et al*., 2012; Harlow and Phoenix, 1978; Huang and Xu, 2010; Peng *et al*., 2010; Rafiee *et al*., 2014) in formulating the basic reliability model presented in this paper.

1. All the components in the system are identical.
2. Each component is subject to continuous degradation. For a component, denote the degradation amount over time t as $X(t;\mu,\beta)$, where μ is a fixed parameter and β is a random variable.
3. Load is equally distributed on each component.
4. Load imposed on a component has an additive impact on the degradation amount of the component.
5. Each component is deemed to have failed when the degradation amount exceeds a critical threshold *H*.

The system considered here is a parallel system consisting of n identical components. Generally, the degradation process of each component could be in any form of a stochastic process. Linear degradation is assumed (Peng *et al*., 2012), i.e., $X(t)=\mu+\beta t$, where the initial degradation amount μ is a constant and the degradation rate β is a random variable, following a normal distribution, $\beta \sim N(\mu_\beta,\sigma_\beta^2)$. We assume that $\mu_\beta \gg \sigma_\beta$, so that the probability of the degradation level being negative can be neglected. A component is deemed to have failed if its degradation amount $X(t)$ exceeds the threshold, *H*. The assumption of linear degradation is used widely in systems such as micro-electro-mechanical systems (Peng *et al*., 2012; Peng *et al*., 2010) and laser devices (Peng and Tseng, 2009).

Assumption (4) states that the load has an additive impact on the degradation of a component, i.e.,

$$X(t)=\mu+\beta t+l(t) \tag{1}$$

where $l(t)$ is the load imposed on the component. Fig. 1 shows the degradation process of a component with the influence of loads. In the figure, loads arrive at time t_1 and t_2, causing an abrupt change in the degradation amount of the component. It should be noted that the system load can either be a constant or a random variable. We will discuss the reliability model for constant load and cumulative load separately in Section 3.

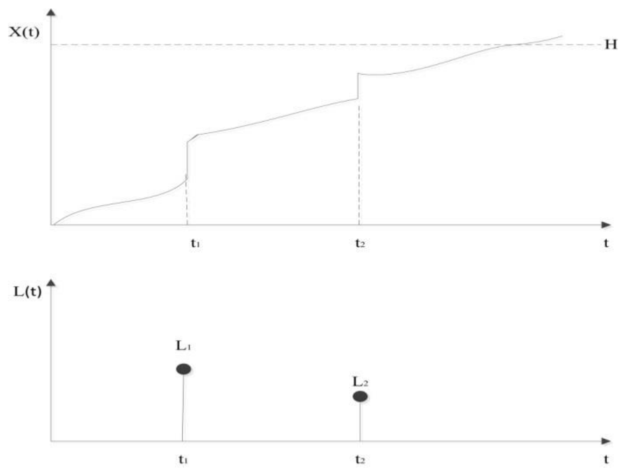

Fig.1: Degradation process of a component subject to loads

2.2 Load-sharing rule

Load-sharing rules determine how system load is distributed among components and how the load on a surviving component changes when some components fail (Harlow and Phoenix, 1978). Typically, load-sharing rules come in three kinds: equal load-sharing rule, local load-sharing rule, and monotone load-sharing rule (Amari *et al*., 2008). An equal load-sharing rule indicates that the total system load is equally distributed among all its components, a local load-sharing rule implies that the load on a failed component is transferred to adjacent components, and a monotone load-sharing rule indicates that the load on the surviving components is non-decreasing when other components fail. In this paper, we focus on the equal load sharing rule, i.e.,

$$l(t) = \frac{L(t)}{n - N(t)} \tag{2}$$

where $L(t)$ is the total system load over time t, $N(t)$ is the number of failed components by time t. The method we use to analyze the reliability of a load-sharing system with equal load-sharing rule can also be applied to system with other load-sharing rules.

If a component fails, the load has to be shared by the remaining components, thus increasing the degradation amount of the surviving components. Fig. 2 shows the degradation process of a component under the influence of failures of other components. In the figure, failures occur at T_1 and T_2, inducing abrupt changes of the degradation amount in the surviving components.

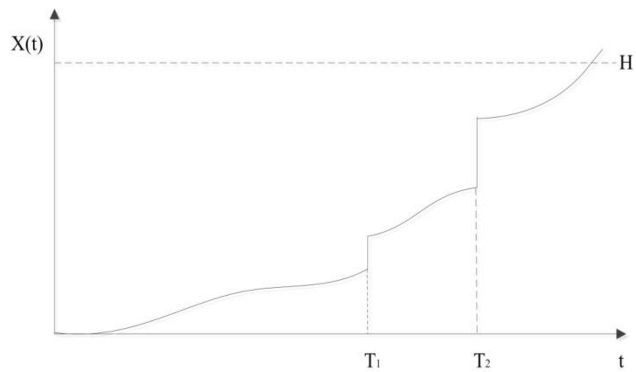

Fig. 2: Degradation process of a component subject to inner failures

3. Reliability models for load-sharing system

As pointed out in the previous section, the degradation amount of a component is influenced by its own degradation process, failure of other components and the imposed system load. The difficulty of conducting a reliability analysis of such a system lies in the dependence between the inner failures of components and the degradation amount of the surviving components, especially when the system load is a random variable (Park, 2013; Huang and Xu, 2010). In this section, we construct separate reliability models for load-sharing systems subject to constant load and cumulative load. More specifically, we formulate a reliability model in Section 3.1 by considering constant load and acquire several preliminary insights. The aim of section 3.1 is to study the effect of inner failures on system reliability. Then we move on to analyze the reliability of a system under cumulative load to jointly analyze the effect of the system load and inner failures on system reliability (see Section 3.2).

3.1 Reliability model concerning constant load

In this section, we assume that the load imposed on the system is constant. By using constant load, we mean that only one load with magnitude L is imposed on the system, from the startup of system operation. As the load is equally distributed to the components, the reliability of a surviving component can be obtained as

$$\begin{aligned} r(t) &= \sum_{i=0}^{n-1} P(X(t) < H \mid N(t)=i)\cdot P(N(t)=i) \\ &= \sum_{i=0}^{n-1} P(\mu+\beta t+\frac{L}{n-i} < H \mid N(t)=i)\cdot P(N(t)=i) \end{aligned} \quad (3)$$

where $P(N(t)=i)$ denotes the probability that i components have failed by time t, and L is the constant load imposed on the system.

We assume that the degradation rate β follows a normal distribution, $\beta \sim N(\mu_\beta, \sigma_\beta^2)$, so

$$P(\mu+\beta t+\frac{L}{n-i} < H \mid N(t)=i) = \Phi\left(\frac{H-(\mu+\mu_\beta t+\frac{L}{n-i})}{\sigma_\beta t}\right) \quad (4)$$

Where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of a standard normally distributed variable. To compute $P(N(t)=i)$, we utilize the cdf of the ith failure time and the $(i+1)$th failure time, $F_i(t)$ and $F_{i+1}(t)$ [14]. $F_i(t)$ and $F_{i+1}(t)$ can be obtained by the failure time history (Huang and Xu, 2010). The following theorem indicates how to compute $P(N(t)=i)$ with previous failure times.

Theorem 1: For $i \geq 1$, the probability that the number of failed components by time t is i can be obtained as

$$P(N(t)=i)=\prod_{j=1}^{i-1}(n-j+1)\int_{T_{j-1}}^{T_{j+1}} f_j(T_j)dT_j\cdot(n-i+1)$$
$$\times\left[\int_{T_{i-1}}^{T_{i+1}} f_i(T_i)dT_i\int_{T_i}^{t}(n-i)f_{i+1}(T_{i+1})dT_{i+1}-\int_{T_{i-1}}^{t} f_i(T_i)dT_i\right] \tag{5}$$

where T_i is the time to the ith failure, $T_0=0$, and $f_i(\cdot)$ is the probability density function (pdf) of the ith failure time, defined as

$$f_i(T_i)=\begin{cases}\left.\dfrac{dF_i(t)}{dt}\right|_{t=T_i}, & T_{i-1}<T_i<T_{i+1}\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

Detailed proof is given in the Appendix.

We assume that each component follows a linear degradation process and the degradation rate β follows a normal distribution, so that

$$F_i(t)=P(\mu+\beta t+\frac{L}{n-i}<H\mid N(t)=i-1)=\Phi\left(\frac{H-(\mu+\mu_\beta t+\dfrac{L}{n-i+1})}{\sigma_\beta t}\right) \tag{7}$$

For $T_{i-1}<T_i<T_{i+1}$,

$$f_i(T_i)=\Phi\left(\frac{H-(\mu+\mu_\beta T_i+\dfrac{L}{n-i+1})}{\sigma_\beta T_i}\right)\cdot\left(\frac{\mu-H}{\sigma_\beta T_i^2}\right) \tag{8}$$

Let

$$\Phi_i(t)=\Phi\left(\frac{H-(\mu+\mu_\beta t+\dfrac{L}{n-i+1})}{\sigma_\beta t}\right) \tag{9}$$

Eq. (5) can now be rewritten as

$$P(N(t)=i;i\ge 1)=\prod_{j=1}^{i-1}(n-j+1)\left[\Phi_j\left(T_{j+1}\right)-\Phi_j\left(T_{j-1}\right)\right]\cdot(n-i+1)$$
$$\times\left\{(n-i)\left[\Phi_i\left(T_{i+1}\right)-\Phi_i\left(T_{i-1}\right)\right]\left[\Phi_{i+1}\left(t\right)-\Phi_{i+1}\left(T_i\right)\right]-\left[\Phi_i\left(t\right)-\Phi_i\left(T_{i-1}\right)\right]\right\} \tag{10}$$

A more specific reliability model for a component can be determined based on Eq. (3):

$$
\begin{aligned}
r(t) &= \sum_{i=0}^{n-1} P(X(t) < H \mid N(t) = i) \cdot P(N(t) = i) \\
&= \left(\Phi_1(t)\right)^n + \sum_{i=1}^{n-1} \Phi_{i+1}(t) \cdot \prod_{j=1}^{i-1} (n-j+1)\left[\Phi_j\left(T_{j+1}\right) - \Phi_j\left(T_{j-1}\right)\right] \cdot (n-i+1) \\
&\times \left\{(n-i)\left[\Phi_i\left(T_{i+1}\right) - \Phi_i\left(T_{i-1}\right)\right]\left[\Phi_{i+1}(t) - \Phi_{i+1}(T_i)\right] - \left[\Phi_i(t) - \Phi_i\left(T_{i-1}\right)\right]\right\}
\end{aligned}
\tag{11}
$$

Finally, the reliability of the system can be represented as

$$
\begin{aligned}
R(t) &= P(N(t) < n) = \sum_{i=1}^{n-1} P(N(t) = i) \\
&= \sum_{i=1}^{n-1} \prod_{j=1}^{i-1} (n-j+1)\left[\Phi_j\left(T_{j+1}\right) - \Phi_j\left(T_{j-1}\right)\right] \cdot (n-i+1) \\
&\times \left\{(n-i)\left[\Phi_i\left(T_{i+1}\right) - \Phi_i\left(T_{i-1}\right)\right]\left[\Phi_{i+1}(t) - \Phi_{i+1}(T_i)\right] - \left[\Phi_i(t) - \Phi_i\left(T_{i-1}\right)\right]\right\}
\end{aligned}
\tag{12}
$$

3.2 Reliability model concerning cumulative load

For systems subject to varying loads, the arrival of a load is usually modeled as a renewal process with an exponential, Weibull or Gamma distributed inter-arrival time (Peng *et al*., 2010); the magnitude of a load is modeled as a continuous random variable. In this article, the following specific assumptions are made to model the reliability of the load-sharing system under cumulative load (Rafiee *et al.*, 2014):

1. The load arrives according to a Poisson process with rate λ .
2. The magnitude of each arriving load is an independent and identically distributed (i.i.d.) random variable, following a normal distribution, $L_k \sim N(u_L, \sigma_L^2)$. It is assumed that $\mu_L \gg \sigma_L$, so that the probability of negative loads can be neglected, i.e., $\Phi(-\mu_L / \sigma_L) \approx 0$.

The magnitude of the cumulative load can be expressed as

$$
L(t) = \begin{cases} \sum_{k=1}^{M(t)} L_k, & if\ M(t) > 0 \\ 0, & if\ M(t) = 0 \end{cases}
\tag{13}
$$

where $M(t)$ is the number of the load arriving at time t . For $M(t) = j$,

$$L(t) \sim N(ju_L, j\sigma_L^2) \tag{14}$$

The reliability of a surviving component can then be expressed as

$$r(t) = \sum_{i=0}^{n-1}\sum_{j=0}^{\infty} P(X(t) < H \mid N(t) = i, M(t) = j) \cdot P(N(t) = i \mid M(t) = j) \cdot P(M(t) = j) \tag{15}$$

The probability that a component survives at time t given the number of failed components and the number of arrived loads, $P(X(t) < H \mid N(t) = i, M(t) = j)$, can then be expressed as

$$\begin{aligned} P(X_L(t) < H \mid N(t) = i, M(t) = j) &= P(\mu + \beta t + \frac{L(t)}{n-i} < H \mid N(t) = i, M(t) = j) \\ &= \Phi\left(\frac{H - (\mu + \mu_\beta t + \frac{j\mu_L}{n-i})}{\sqrt{\sigma_\beta^2 + j\sigma_L^2}}\right) \end{aligned} \tag{16}$$

The number of the load arriving at time t is

$$P(M(t) = j) = \frac{e^{-\lambda t}(\lambda t)^j}{j!} \tag{17}$$

The computation of the probability $P(N(t) = i \mid M(t) = j)$ is relatively complex since both the arrival time of load and the failure time of components influence the surviving components. Two special cases are of particular interest.

Case 1: If the system is subject to no load by time t, it acts as a parallel system with independent components, so that

$$P(N(t) = i \mid M(t) = 0) = \frac{n!}{i!(n-i)!}\left(\Phi\left(\frac{H - (\mu + \mu_\beta t)}{\sigma_\beta t}\right)\right)^{n-i}\left(1 - \Phi\left(\frac{H - (\mu + \mu_\beta t)}{\sigma_\beta t}\right)\right)^{i} \tag{18}$$

Case 2: If no component fails by time t, we obtain

$$P(N(t) = 0 \mid M(t) = j) = \left(\Phi\left(\frac{H - (\mu + \mu_\beta t + \frac{j\mu_L}{n-i+1})}{\sqrt{\sigma_\beta^2 t^2 + j\sigma_L^2}}\right)\right)^{n} \tag{19}$$

In general, when the system is subject to both cumulative load and inner failures, $P(N(t) = i \mid M(t) = j)$ can be determined by the following theorem.

Theorem 2: For $i \geq 1$, $j \geq 1$, the probability that the number of failed components by time *t* is *i*, given the number of arrived loads can be obtained as

$$P(N(t)=i \mid M(t)=j)=\prod_{k=1}^{i-1}\prod_{l=1}^{j-1}\int_{T_{k-1}}^{T_{k+1}}\int_{t_{l-1}}^{t_{l+1}}(n-k+1)g_{k,l}(T_k,t_l)dT_k dt_l$$
$$\times(n-i+1)\left(\begin{aligned}&\int_{T_i}^{t}\int_{t_{j-1}}^{t}(n-i)g_{i+1,j}(T_{i+1},t_j)dT_{i+1}dt_j\cdot\prod_{l=1}^{j-1}\int_{T_{i-1}}^{T_{i+1}}\int_{t_{l-1}}^{t_{l+1}}g_{i,l}(T_i,t_l)dT_i dt_l\\&-\int_{T_{i-1}}^{t}\int_{t_{j-1}}^{t}g_{i,j}(T_i,t_j)dT_i dt_j\end{aligned}\right) \tag{20}$$

where

$$g_{i,j}(T_i,t_j)=\Phi\left(\frac{H-(\mu+\mu_\beta T_i+\frac{(j-1)\mu_L}{n-i+1})}{\sqrt{\sigma_\beta^2 T_i^2+(j-1)\sigma_L^2}}\right)\left(\frac{\mu-H}{\sigma_\beta T_i^2}\right)\lambda e^{-\lambda(t_j-t_{j-1})} \tag{21}$$

The proof is similar to that of Theorem 1.

Combining the above equations, the reliability of a component can be expressed as

$$\begin{aligned}r(t)&=\sum_{i=0}^{n-1}\sum_{j=0}^{\infty}P(X(t)<H \mid N(t)=i, M(t)=j)\cdot P(N(t)=i \mid M(t)=j)\cdot P(M(t)=j)\\
&=\left(\Phi\left(\frac{H-(\mu+\mu_\beta t)}{\sigma_\beta}\right)\right)^{n+1}e^{-\lambda t}+\left(\Phi\left(\frac{H-(\mu+\mu_\beta t+\frac{\mu_L}{n})}{\sqrt{\sigma_\beta^2 t^2+\sigma_L^2}}\right)\right)^{n+1}e^{-\lambda t}\lambda t\\
&+n\left(\Phi\left(\frac{H-(\mu+\mu_\beta t)}{\sigma_\beta}\right)\right)^{n}\left(1-\Phi\left(\frac{H-(\mu+\mu_\beta t)}{\sigma_\beta t}\right)\right)e^{-\lambda t}\\
&+\sum_{i=1}^{n-1}\sum_{j=1}^{\infty}\Phi\left(\frac{H-(\mu+\mu_\beta t+\frac{j\mu_L}{n-i})}{\sqrt{\sigma_\beta^2 t^2+j\sigma_L^2}}\right)\prod_{k=1}^{i-1}\prod_{l=1}^{j-1}\left(\Phi_{k,l}(T_{k+1})-\Phi_{k,l}(T_{k-1})\right)e^{-\lambda(t_{l+1}-t_{l-1})}(n-i+1)\\
&\times\left(\begin{aligned}&(n-i)\left(\Phi_{i+1,j}(t)-\Phi_{i+1,j}(T_i)\right)e^{-\lambda(t-t_{j-1})}\prod_{l=1}^{j-1}\left(\Phi_{i,j}(T_{i+1})-\Phi_{i,j}(T_{i-1})\right)e^{-\lambda(t_{l+1}-t_{l-1})}\\&-\left(\Phi_{i,j}(T_{i+1})-\Phi_{i,j}(T_{i-1})\right)e^{-\lambda(t-t_{j-1})}\end{aligned}\right)\frac{e^{-\lambda t}(\lambda t)^j}{j!}\end{aligned} \tag{22}$$

where

$$\Phi_{i,j}(t)=\Phi\left(\frac{H-(\mu+\mu_\beta t+\frac{(j-1)\mu_L}{n-i+1})}{\sqrt{\sigma_\beta^2 t_i^2+(j-1)\sigma_L^2}}\right)$$

$\Phi_{i,j}(t)$ can be interpreted as the cdf of failure time under *i* inner failures and *j* external loads. Computation of the reliability in Eq. (22) requires the distribution of previous arrival times of loads and failure time of components, which can be solved by iteration. The reliability of the system can then be assessed as

$$R(t)=P(N(t)<n)=\sum_{i=1}^{n-1}P(N(t)=i)=\sum_{i=1}^{n-1}\sum_{j=1}^{\infty}P(N(t)=i\,|\,M(t)=j)P(M(t)=j)$$

$$=\sum_{i=1}^{n-1}\sum_{j=1}^{\infty}\left\{\begin{array}{l}\prod_{k=1}^{i-1}\prod_{l=1}^{j-1}\int_{T_{k-1}}^{T_{k+1}}\int_{t_{l-1}}^{t_{l+1}}(n-k+1)g_{k,l}(T_k,t_l)dT_k dt_l\times(n-i+1)\\ \left(\begin{array}{l}\int_{T_i}^{t}\int_{t_{j-1}}^{t}(n-i)g_{i+1,j}(T_{i+1},t_j)dT_{i+1}dt_j\cdot\prod_{l=1}^{j-1}\int_{T_{i-1}}^{T_{i+1}}\int_{t_{l-1}}^{t_{l+1}}g_{i,l}(T_i,t_l)dT_i dt_l\\ -\int_{T_{i-1}}^{t}\int_{t_{j-1}}^{t}g_{i,j}(T_i,t_j)dT_i dt_j\end{array}\right)\end{array}\right\}\frac{e^{-\lambda t}(\lambda t)^j}{j!} \tag{23}$$

Although the developed reliability models are applied for parallel systems, they can be easily extended to *k-out-of-n* systems. For a *k-out-of-n*: *G* system, system reliability can be expressed as the probability that less than *k* components fail by time *t*. By using Eq. (12), we can have the system reliability as $R(t)=P(N(t)\le k)=\sum_{i=1}^{k}P(N(t)=i)$.

4. Preventive Maintenance Policy

The system considered in this paper is highly integrated, e.g., a micro-electro-mechanical system where repair or replacement of any individual component is particularly difficult or even impossible (Peng *et al.*, 2010). Thus, maintenance action has to be taken for the whole system. A preventive maintenance model with periodic inspection is developed in this article. To evaluate the performance of the maintenance policy, we adopt a long-run cost rate model where the preventive maintenance threshold *P* and the periodic inspection interval τ are the two decision variables. The maintenance policy works as follows.

At the *i*th inspection time $i\tau$, if

(1) The degradation level of the system exceeds the failure threshold, $X(i\tau) > H$, then the system is replaced. Additional cost may be incurred during the downtime of the system.

(2) The degradation level of the system exceeds the threshold for preventive maintenance but still works, $H > X(i\tau) > P$, then preventive maintenance is undertaken.

(3) The degradation level of the system does not exceed the threshold for preventive maintenance, $X(i\tau) < P$, then the system is left unchanged.

Remark: For a parallel system, the system fails when all the components have failed. As the component in the system fails one by one, failure time of the system is equal to the failure time of the last failed component and the degradation process of the system can be characterized as the degradation process of the last failed component.

Both the preventive maintenance and replacement will bring the system back to the state of "as good as new". A renewal cycle is defined as the time interval between two consecutive maintenance actions (preventive maintenance or replacement). A renewal process is executed during succeeding cycles while further costs are incurred within each cycle. From the basic renewal theory, the long-run cost rate *CR* can be computed by (Grall *et al*., 2002)

$$CR = \lim_{t \to \infty} \frac{C(t)}{t} = \frac{E[TC]}{E[S]} \tag{24}$$

where $C(t)$ is the total maintenance cost incurred until time *t*, *TC* is the total cost within the renewal cycle and *S* is the length of the renewal cycle.

The cost items include the inspection cost, the preventive maintenance cost, the replacement cost, and the penalty cost due to the malfunction of the system (Peng *et al*., 2010). The expected total maintenance cost of a renewal cycle can then be expressed as

$$E[TC] = \begin{cases} C_I E[N_I] + C_F E[\rho] + C_R, & \text{if replacement takes place} \\ C_I E[N_I] + C_P, & \text{if preventive maintenance takes place} \end{cases} \tag{25}$$

Synthesizing the above two maintenance actions at the end of a renewal cycle, Eq. (25) can be rewritten as

$$E[TC] = C_I E[N_I] + (C_F E[\rho] + C_R) E\left[1_{\{\text{replacement}\}}\right] + C_P E\left[1_{\{\text{preventive maintenance}\}}\right] \tag{26}$$

where $1_{\{\bullet\}}$ is the indicator function, C_I is the cost of each inspection, C_F is the cost rate during system downtime, C_R is the cost of replacement, C_P is the cost of preventive maintenance, N_I is number of inspections within a renewal cycle, and ρ is the system downtime, i.e., time interval between system failure to the next inspection time.

The number of inspections within a renewal cycle is related to the system degradation level, in a manner that $X(i\tau) > P > X((i-1)\tau)$. Let T_P denote the time when the system degradation level reaches *P*, so that the expected number of inspections within a renewal cycle, $E[N_I]$ can be expressed as

$$\begin{aligned} E[N_I] &= \sum_{i=1}^{\infty} iP(N_I = i) = \sum_{i=1}^{\infty} iP\left(X(i\tau) > P > X((i-1)\tau)\right) \\ &= \sum_{i=1}^{\infty} i\left(F_{T_P}(i\tau) - F_{T_P}((i-1)\tau)\right) \end{aligned} \tag{27}$$

where $F_{T_P}(\cdot)$ is the cdf of T_P, which can be computed in a way similar to the calculation of $F_T(\cdot)$.

The system downtime is the interval from the system failure time to the next inspection time, that is, $\rho = i\tau - T$. The expected system downtime is given as

$$\begin{aligned} E[\rho] &= \sum_{i=1}^{\infty} E[\rho \mid N_I = i] P(N_I = i) \\ &= \sum_{i=1}^{\infty} \int_{(i-1)\tau}^{i\tau} \left((i\tau - t) dF_T(t)\right)\left(F_{T_P}(i\tau) - F_{T_P}((i-1)\tau)\right) \end{aligned} \tag{28}$$

The expected value of replacement occurring at the end of a renewal cycle can be expressed as

$$E\left[1_{\{\text{replacement}\}}\right]=\sum_{i=1}^{\infty}P\left\{X\left(i\tau\right)>H\mid N_I=i\right\}P\left(N_I=i\right)$$
$$=\sum_{i=1}^{\infty}F_T\left(i\tau\right)\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right) \tag{29}$$

Similarly, the expected value of preventive maintenance occurring at the end of a renewal cycle is given by

$$E\left[1_{\{\text{preventive maintenance}\}}\right]=\sum_{i=1}^{\infty}P\left\{H>X\left(i\tau\right)>P\mid N_I=i\right\}P\left(N_I=i\right)$$
$$=\sum_{i=1}^{\infty}\left(F_T\left(i\tau\right)-F_{T_P}\left(i\tau\right)\right)\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right) \tag{30}$$

The length of a renewal cycle is determined by the number of inspections and the interval between two inspections, that is, $S=N_I\tau$. Its expected value can be computed as

$$E\left[S\right]=\sum_{i=1}^{\infty}P\left\{S\mid N_I=i\right\}P\left(N_I=i\right)$$
$$=\sum_{i=1}^{\infty}i\tau\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right) \tag{31}$$

Based on Eq. (24) to Eq. (31), the long-run maintenance cost rate, $CR\left(\tau,P\right)$ can be obtained as

$$CR\left(\tau,P\right)=\frac{C_I\sum_{i=1}^{\infty}i\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right)+C_P\sum_{i=1}^{\infty}\left(F_T\left(i\tau\right)-F_{T_P}\left(i\tau\right)\right)\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right)}{\sum_{i=1}^{\infty}i\tau\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right)}$$
$$+\frac{\left(C_F\sum_{i=1}^{\infty}\int_{(i-1)\tau}^{i\tau}\left(\left(i\tau-t\right)dF_T\left(t\right)\right)\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right)+C_R\right)\cdot\sum_{i=1}^{\infty}F_T\left(i\tau\right)\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right)}{\sum_{i=1}^{\infty}i\tau\left(F_{T_P}\left(i\tau\right)-F_{T_P}\left(\left(i-1\right)\tau\right)\right)} \tag{32}$$

The optimal value of τ and P can be obtained by solving

$$\left(\tau*,P*\right)=\arg\min_{\tau,P}CR\left(\tau,P\right) \tag{33}$$

Due to the interaction between τ and P and the complexity of $CR\left(\tau,P\right)$, it is difficult to obtain an analytical result of $\left(\tau*,P*\right)$. Hence, we turn to a numerical method to jointly optimize the inspection interval τ and the threshold for preventive maintenance P.

5. Numerical Example

To investigate the reliability model and maintenance policy, we took a load-sharing redundant microengine system as an example. A microengine often fails due to the visible wear on rubbing surfaces between the gear and the pin joint (Peng *et al*., 2010). The wear is mainly caused by the ageing degradation process. Meanwhile, external load shocks contribute to the debris between the gear and the pin joint. Usually in a system, multiple microengines work together to perform tasks, which can thus be modelled as a load-sharing system. We consider a system consisting of three microengnes in a load-sharing parallel structure. Each microengine goes through a linear degradation process individually, $X(t) = \mu + \beta t$, where the initial degradation amount μ is a constant and the degradation rate β is a random variable, following a normal distribution, $\beta \sim N(\mu_\beta, \sigma_\beta^2)$. In the following section, we consider the case that the system is subject to constant load L and cumulative load $L(t)$ respectively. The cumulative loads arrive according to a homogeneous Poisson process with rate λ. The magnitude of each load arrived is assumed to be an i.i.d random variable, following a normal distribution, $L_k \sim N(u_L, \sigma_L^2)$. A microengine fails when the overall wear amount exceeds the threshold, *H*. Preventive maintenance is conducted according to the observation of the visible wear amount. Table 1 summarizes the associated parameters for reliability analysis and maintenance policy.

Table 1: Summary of the parameters

Parameter	value	source
μ	0	Peng et al. (2010)
μ_β	8.4823×10^{-9} μm^3	Peng et al. (2010)
σ_β	6.0016×10^{-10} μm^3	Peng et al. (2010)
λ	2.5×10^{-5}	Peng et al. (2010)

μ_L	1×10^{-4} μm^3	Peng et al. (2010)
σ_L	2×10^{-5} μm^3	Peng et al. (2010)
H	0.00125μm^3	Peng et al. (2010)
L	0.0002 μm^3	Peng et al. (2010)
C_I	\$500	Assumption
C_P	\$10,000	Assumption
C_R	\$30,000	Assumption
C_F	\$10	Assumption

5.1 Reliability analysis

We first analyzed the reliability of the system under constant load. The system reliability function is given Eq. (12). However, it is very difficult to have the complete form of reliability function due to the complexity of the expression. Hence, we turn to Monte Carlo simulation to get the reliability function.

We compute system reliability with the number of repetitions N_r=10,000. Fig. 3 shows the reliability variation noted within the time period $\left[0.5\times10^5, 1.5\times10^5\right]$. Fig. 4 shows the pdf of failure time under constant load. The pdf was computed numerically using the equation $f_T(t) = \left(F_T(t+\Delta t) - F_T(t)\right)/\Delta t$, where Δt is the time increment. We can observe that the system reliability began to decrease at time $t = 0.98\times10^5$ and reached 0 at time $t = 1.12\times10^5$. We believe that small variation in the degradation process accounts for the rapid decrease. We also investigated the impact of failure threshold on system reliability and undertook a sensitivity analysis of failure threshold (see Fig. 5). Note that the failure threshold had shifted from 0.00115 μm^3 to 0.00135 μm^3, which implies that higher failure threshold increases the reliability of the system.

Also, to examine the effect of number of repetitions, we selected the system reliability at several times and compare the results for various N_r. We selected the system reliability from $t = 1\times10^5$ to $t = 1.1\times10^5$ and compare the results for various N_r (varying from 100 to 10,000). Table

4 showed the system reliability for various number of repetitions N_r. It is indicated that when $N_r \geq 1,000$, the difference of system reliability is within 0.01, which implies the effectiveness of the Monte Carlo simulation method.

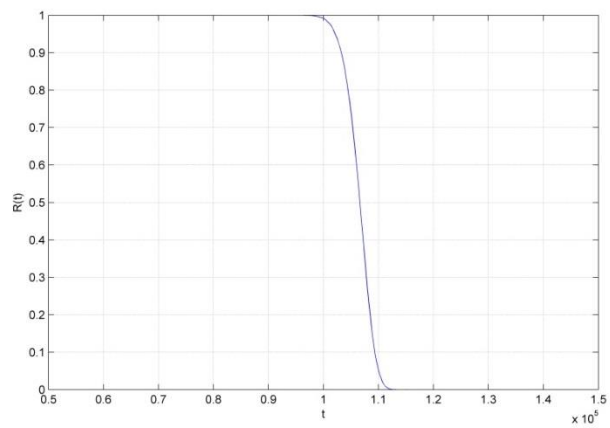

Fig. 3: Reliability of system under constant load

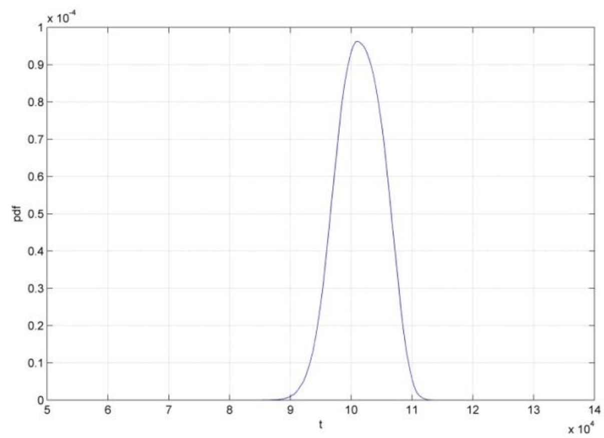

Fig. 4: pdf of system reliability with constant load

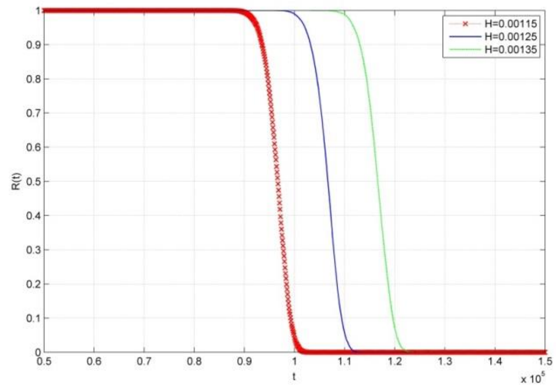

Fig. 5: Sensitivity analysis of failure threshold

Table 2: reliability variation with number of repetitions

N_r \ t	30	32	34	36	38	40	42	44	46	48	50
100	0.990	0.980	0.950	0.940	0.860	0.720	0.60	0.470	0.300	0.150	0.030
500	0.998	0.984	0.97	0.926	0.852	0.748	0.61	0.442	0.268	0.118	0.042
1,000	0.99	0.976	0.953	0.916	0.847	0.738	0.593	0.429	0.264	0.129	0.049
5,000	0.989	0.978	0.961	0.923	0.847	0.739	0.596	0.437	0.272	0.136	0.059
10,000	0.992	0.978	0.951	0.919	0.845	0.740	0.603	0.436	0.270	0.138	0.053

According to Eq. (23), we plotted the reliability of the system subject to cumulative load, as shown in Fig. 6. We can observe that the reliability of the system started to descend at time $t = 0.8 \times 10^5$ and hit 0 at time $t = 1.7 \times 10^5$. Compared with the reliability variation with constant load, the system reliability with cumulative load possessed longer deteriorating duration. This is due to the fact that the randomness of the arriving load adds more uncertainty to the degradation process. For system subject to cumulative load, monitoring/inspection techniques play a more significant role in reducing the uncertainty of the system. We also plotted the pdf of failure time $f_T(t)$ in Fig. 7.

We are interested in the parameter of failure threshold H and load arrival rate λ, and make sensitivity analysis of the two parameters, as shown in Fig. 8 and Fig. 9. Fig. 8 suggests that a larger threshold would lead to a higher reliability performance. In Fig. 9, when λ increases from $\lambda = 1.5\times10^{-5}$ to $\lambda = 3.5\times10^{-5}$, system reliability shifts to the left. This indicates that reliability deteriorates faster when the system is subject to loads with higher arrival rates.

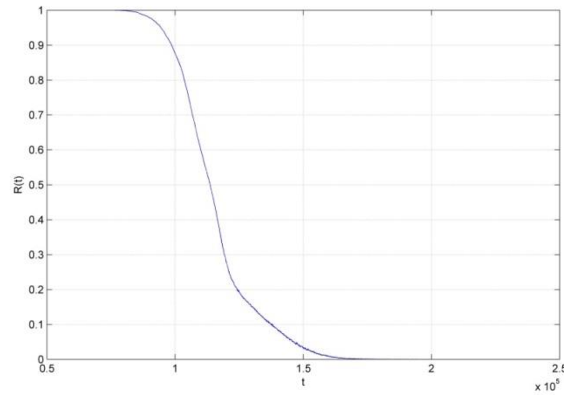

Fig. 6: Reliability of a system under cumulative load

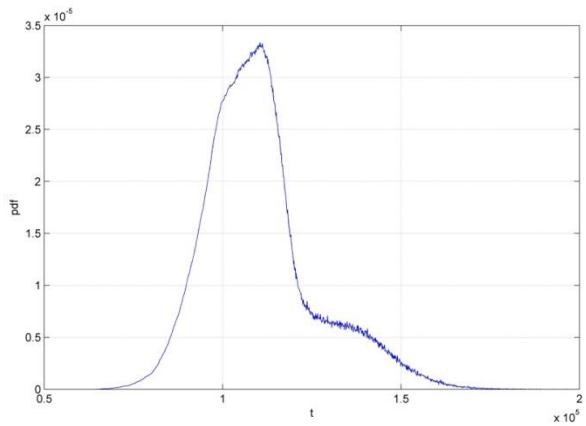

Fig. 7: pdf of system reliability under cumulative load

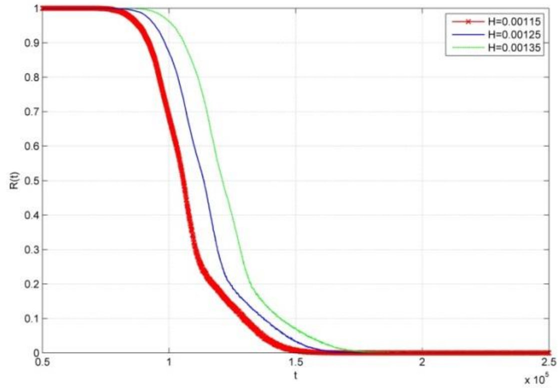

Fig. 8: Sensitivity analysis of failure threshold

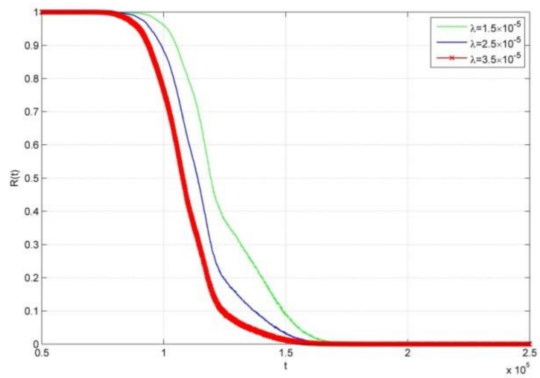

Fig. 9: Sensitivity analysis of λ

5.2 Optimal preventive maintenance policy

The optimal values in Eq. (33) were determined using a numerical method. The cost parameters are listed in Table 1. Table 3 shows the maintenance cost rate results $CR(\tau, P)$ as a function of inspection interval and preventive maintenance threshold for a system subject to constant load. Note that the minimum cost rate was 5.245 at $(\tau*, P*) = (10.5\times10^4, 7.7\times10^{-4})$.

For a system subject to cumulative load, we can obtain the minimum cost rate of 5.437, at $(\tau*, P*) = (11\times10^4, 5.7\times10^{-4})$. Table 4 shows the variation of maintenance cost rate $CR(\tau, P)$ as a function of τ and P.

From Table 3 and Table 4, we can observe that the cost rate $CR(\tau, P)$ decreases with τ before $\tau = \tau*$, and increases afterwards. However, the effect of preventive maintenance threshold P is rather obscure. It can be concluded that the inspection interval τ is the main contributor to the variation of the cost rate.

Table 3: Maintenance cost rate with constant load *vs* P and τ

τ (10^4) / P(10^{-4})	8	8.5	9	9.5	10	10.5	11	11.5	12	12.5	13
7.5	6.831	6.609	6.165	5.784	5.435	5.261	5.414	5.619	5.797	5.962	6.128
7.7	6.831	6.666	6.213	5.749	5.435	**5.245**	5.417	5.613	5.799	5.960	6.118
7.9	6.847	6.703	6.250	5.808	5.456	5.264	5.408	5.619	5.800	5.962	6.121
8.1	6.837	6.768	6.313	5.858	5.448	5.253	5.420	5.621	5.811	5.962	6.122
8.3	6.842	6.798	6.439	5.900	5.501	5.258	5.412	5.615	5.793	5.967	6.127
8.5	6.843	6.878	6.490	5.977	5.534	5.275	5.418	5.621	5.799	5.962	6.121
8.7	6.853	6.897	6.608	6.043	5.564	5.280	5.417	5.619	5.807	5.970	6.117
8.9	6.843	6.919	6.650	6.152	5.680	5.305	5.427	5.608	5.798	5.966	6.124
9.1	6.842	6.957	6.802	6.223	5.698	5.326	5.444	5.641	5.794	5.970	6.124
9.3	6.847	6.986	6.835	6.353	5.756	5.366	5.460	5.623	5.792	5.964	6.122
9.5	6.846	6.994	6.902	6.449	5.784	5.417	5.483	5.625	5.808	5.973	6.118

Table 4: Maintenance cost rate with cumulative load *vs* P and τ

τ (10^4) / P (10^{-4})	8	8.5	9	9.5	10	10.5	11	11.5	12	12.5	13
5.5	6.511	6.194	5.838	5.602	5.510	5.504	5.475	5.613	5.752	5.874	6.078
5.7	6.509	6.116	5.888	5.587	5.480	5.455	**5.437**	5.628	5.762	5.965	6.008
5.9	6.479	6.154	5.823	5.616	5.538	5.472	5.521	5.548	5.771	5.944	6.088
6.1	6.564	6.144	5.846	5.599	5.518	5.464	5.534	5.623	5.742	5.899	6.062
6.3	6.485	6.167	5.871	5.636	5.517	5.447	5.529	5.644	5.785	5.900	6.025
6.5	6.448	6.154	5.869	5.624	5.488	5.475	5.486	5.638	5.717	5.911	6.086
6.7	6.422	6.178	5.856	5.588	5.494	5.494	5.474	5.662	5.822	5.945	6.046

6.9	6.375	6.132	5.844	5.682	5.495	5.495	5.539	5.610	5.762	5.843	6.119
7.1	6.340	5.881	6.802	5.590	5.507	5.507	5.499	5.634	5.727	5.910	6.012
7.3	6.431	5.814	6.835	5.624	5.496	5.496	5.540	5.642	5.758	5.890	6.119
7.5	6.332	5.836	6.902	5.642	5.523	5.523	5.511	5.585	5.786	5.860	6.076

6. Conclusion

This paper has developed two reliability models for assessing the reliability of load-sharing systems with continuously degrading components. The first has considered the case of constant load and assessed the effect of failures of components on the surviving components. The second has modeled systems subject to cumulative load and examined the influence of random load and the influence of inner failures. Finally, the proposed models have been utilized to formulate preventive maintenance policies for load-sharing system.

Future investigations can aim at relaxing some assumptions in this study. For example, this study considers cumulative load model; future works can consider competing failure modes, where the system may fail due to either soft failures (*e.g.*, degradation) or catastrophic failures (*e.g.*, shocks). In addition, this study conducts reliability analysis and maintenance policy for load-sharing systems in parallel structure. Extension to other complex structures (*e.g.*, parallel-series, series-parallel bridge structure) is also of interest to investigate. Also, compared with equal load-sharing rule used in this study, other load-sharing rules (*e.g.*, local load-sharing rule or monotone load-sharing rule) may be more practical in some real applications. Load-sharing systems with non-identical components can also be investigated.

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Appendix

The probability that i components have failed by time t can be expressed as

$$\begin{aligned} &P(N(t)=i) = P(N(t)\geq i) - P(N(t)\geq i+1) \\ &= P(T_i \leq t) - P(T_{i+1}\leq t) \\ &= F_i(t) - F_{i+1}(t) \end{aligned} \tag{A1}$$

Note that the time to the *i*th failure, T_i is a random variable. The probability that a component fails in an infinitesimal interval dT_i at time T_i is given as

$$P_e(t = T_i \mid T_1, T_2, ..., T_{i-1}) = f_i(T_i)dT_i \tag{A2}$$

Note that there are $n-i+1$ surviving components in the system by time $T_{i-1} < t < T_i$, then the probability that the *i*th failure occurs in an infinitesimal interval dT_i at time T_i is

$$P(t = T_i \mid T_1, T_2, ..., T_{i-1}) = (n-i+1) f_i(T_i)dT_i \tag{A3}$$

Since $F_i(t)$ is related with the previous failure times, we can obtain the joint probability of the *i*th failure time and the previous inner failure times as

$$\begin{aligned} &P(T_i \leq t, T_1, T_2, ..., T_{i-1}) \\ &= P(T_i \leq t \mid T_1, T_2, ..., T_{i-1}) \cdot P(t = T_{i-1} \mid T_1, T_2, ..., T_{i-2}) \cdot ... \cdot P(t = T_2 \mid T_1) \cdot P(t = T_1) \\ &= \int_{T_{i-1}}^{t} (n-i+1) f_i(T_i)dT_i \cdot \prod_{j=1}^{i-1} (n-j+1) f_j(T_j)dT_j \end{aligned} \tag{A4}$$

$F_i(t)$ can be viewed as the marginal distribution of $P(T_i \leq t, T_1, T_2, ..., T_{i-1})$, and can be obtained as

$$\begin{aligned} &F_i(t) = P(T_i \leq t; T_1, T_2, ..., T_{i-1}) \\ &= \int_{T_{i-1}}^{t} (n-i+1) f_i(T_i)dT_i \cdot \prod_{j=1}^{i-1} \int_{T_{j-1}}^{t} (n-j+1) f_j(T_j)dT_j \\ &= \int_{T_{i-1}}^{t} (n-i+1) f_i(T_i)dT_i \cdot \prod_{j=1}^{i-1} \int_{T_{j-1}}^{T_{j+1}} (n-j+1) f_j(T_j)dT_j \end{aligned} \tag{A5}$$

The second item is due to $T_1 < T_2 < ... < T_{i-1} < T_i$, and the third item is due to the definition of $f_i(T_i)$ in Eq.(6).

Then, we can get

$$
\begin{aligned}
&P(N(t)=i)=F_i(t)-F_{i+1}(t)\\
&=\prod_{j=1}^{i-1}(n-j+1)\int_{T_{j-1}}^{T_{j+1}} f_j(T_j)dT_j\cdot(n-i+1)\cdot \\
&\left(\int_{T_{i-1}}^{T_{i+1}} f_i(T_i)dT_i\int_{T_i}^{t}(n-i)\,f_{i+1}(T_{i+1})dT_{i+1}-\int_{T_{i-1}}^{t} f_i(T_i)dT\right)
\end{aligned}
\tag{A6}
$$

which concludes the proof. □