Convergence of the split-step θ-method for stochastic age-dependent population equations with Markovian switching and variable delay

Shounian Deng[a,b], Weiyin Fei[b,*], Yong Liang[c], Xuerong Mao[d]

[a] School of Science, Nanjing University of Science and Technology, Nanjing, Jiangsu 210094, China
[b] School of Mathematics and Physics, Anhui Polytechnic University, Wuhu, Anhui 24100, China
[c] School of Mathematics, Shanghai University of Finance and Economics, Shanghai, 200433, China
[d] Department of Mathematics and Statistics, University of Strathclyde, Glasgow G1 1XH, U.K.

Abstract

We present a stochastic age-dependent population model that accounts for Markovian switching and variable delay. By using the approximate value at the nearest grid-point on the left of the delayed argument to estimate the delay function, we propose a class of split-step θ-method for solving stochastic delay age-dependent population equations (SDAPEs) with Markovian switching. We show that the numerical method is convergent under the given conditions. Numerical examples are provided to illustrate our results.

Keywords: Stochastic delay age-dependent population equations, Split-step θ-method, Strong convergence, Markovian swtiching, Itô foumula.

1. Introduction

Stochastic differential equations (SDEs) are becoming increasingly used to model real world phenomena in different fields, such as economics, biology, finance and population dynamics. As an important branch of SDEs, stochastic population equations have received a great deal of attention. In the present investigation, the random behavior described by different stochastic processes such as Markovian switching, Poisson jumps, and fractional Brownian motion is incorporated into the stochastic age-dependent population equations (SAPEs) (see e.g., [10, 11, 18, 19, 22, 28]). In these population dynamics, one assumes that the system is governed by a principle of causality, that is the future state of the system is determined solely by the present states. However, in real world population system, it takes certain time for individuals to mature as well as for infectious diseases to be cured. This motivates us to develop a more realistic model including some of the past states of the system called time delay to describe the relationship between the causes and their effects (see e.g., [9, 12, 13, 14, 16]). Then this stochastic delay age-dependent population

*Corresponding author
Email address: wyfei@ahpu.edu.cn (Weiyin Fei)

system with Markovian switching can be defined by the following form:

$$\begin{cases} d_t P_t = \left[-\dfrac{\partial P_t}{\partial a} - \mu(t,a)P_t + f(r(t), P_t, P_{\delta(t)}) \right] dt \\ \qquad\qquad + g(r(t), P_t, P_{\delta(t)}) dW_t, & (t,a) \in [0,T] \times [0,A] \\ P(t,a) = \varphi(t,a), & (t,a) \in [-\tau, 0] \times [0,A] \\ P(t,0) = \displaystyle\int_0^A \beta(t,a) P(t,a) da, & t \in [0,T] \end{cases} \tag{1.1}$$

where $\tau > 0$, $d_t P_t = \frac{\partial P_t}{\partial t} dt$, $\delta(t)$ stands for the time delay $(-\tau \leq \delta(t) \leq t)$, A denotes the maximum age, $r(t)$ is a Markov chain, W_t is a Wiener process, $P_t = P(t,a)$ denotes the population density of age a at time t, $P_{\delta(t)} = P(\delta(t), a)$ denotes the population density of age a at time $\delta(t)$, $\beta(t,a)$ denotes the fertility rate of females of age a at time t, $\mu(t,a)$ denotes the mortality rate of age a at time t, $f(r(t), P_t, P_{\delta(t)})$ denotes the effects of an external environment for the population system, $g(r(t), P_t, P_{\delta(t)})$ is a diffusion coefficient.

With its simple algebraic structure, cheap computational cost and acceptable convergence rate, Euler-Maruyama (EM) method has been widely used to solve SDEs (see e.g., [4, 5, 15, 25, 27]). Since most SAPEs cannot be solved explicitly, the constructions of efficient computational method have become essential. For example, Zhang et al. [11, 28, 29] investigated the convergence of numerical solutions to SAPEs, Li et al. [10] considered the convergence of numerical solutions to SAPEs with Markovian switching. In recent years, influenced by Higham, Mao and Stuart [6, 7], split-step θ-method (SSθ method) has attracted a lot of concern due to their advantages in dealing with the flexibility and the stability for the SDEs (see e.g., [2, 16, 23]). Researchers find that the SSθ method or its improved forms have desirable stability properties, convergence rates and structure-preserving properties (see e.g., [8, 17]). Tan et al. [22] constructed a class of SSθ method for SAPEs with Poisson jumps. In [19], Rathinasamy introduced a class of SSθ method for SAPEs with Markovian switching. In our stochastic population model (1.1), we allow the time delay to be a function of time, namely variable delay, which is more general than the constant delay. The main difficulty in dealing with variable delays by numerical method is that at the current time-step the delayed argument may not hit a precious time-step (see [16]). In order to overcome this difficulty we use the the approximate value at the nearest grid-point on the left of the delayed argument to estimate the delay function. We then present a class of SSθ method for SDAPEs with Markovian switching. The SSθ method includes the split-step forward method and the split-step backward Euler method by choosing $\theta = 0$ and $\theta = 1$ and it is more general than the two methods. The convergent result is proved under the given conditions.

The outline of the paper is organized as follows. In Section 2, we will introduce some basic preliminaries. A class of SSθ method for solving SDAPEs with Markovian switching will be proposed in Section 3. In Section 4, the SSθ method converge strongly to the exact solutions of SDAPEs with Markovian switching will be shown. Numerical experiments will be given in the final section.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions(i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\mathbb{R}^+$.

Let

$$V = H^1([0, A]) \equiv \left\{ \varphi | \, \varphi \in L^2([0, A]), \frac{\partial \varphi}{\partial x_i} \in L^2([0, A]) \right\}$$

where $\frac{\partial \varphi}{\partial x_i}$ is the generalized partial derivative and V is a Sobolev space. $H = L^2([0, A])$ such that $V \hookrightarrow H \equiv H' \hookrightarrow V'$. V' is the dual space of V. We denote by $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ the norms in V, H and V' respectively; by $\langle\cdot,\cdot\rangle$ the duality product between V and V', and by $(\cdot,\cdot)$ the scalar product in H. Let W_t be a Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and taking its values in the separable Hilbert space K, with increment covariance operator W. For an operator $B \in \mathcal{L}(K, H)$ being the space of all bounded linear operators from K to H, we denote by $\|B\|_2$ be the Hilbert-Schmidt norm, i.e. $\|B\|_2 = \mathrm{trace}(BWB^T)$. Let $C = C([-\tau, t]); H)$ be the space of all continuous function from $[-\tau, t]$ into H, with sup-norm $\|\phi_t\|_C = \sup_{-\tau\leq s\leq t} |\phi(s)|$, and $L^p_H = L^p([0, T]); H)$.

The integral version of (1.1) is given by the equations:

$$\begin{cases} P_t = P_0 - \int_0^t \frac{\partial P_s}{\partial a} ds - \int_0^t \mu(s, a) P_s ds + \int_0^t f(r(s), P_s, P_{\delta(s)}) ds \\ \qquad + \int_0^t g(r(s), P_s, P_{\delta(s)}) dW_s, & (t, a) \in [0, T] \times [0, A] \\ P(t, a) = \varphi(t, a), & (t, a) \in [-\tau, 0] \times [0, A] \\ P(t, 0) = \int_0^A \beta(t, a) P(t, a) da, & t \in [0, T] \end{cases} \tag{2.1}$$

where $f(i, \cdot, \cdot) : \mathbb{S} \times L^2_H \times L^2_H \to H$ be a family of non-linear operators, $\mathcal{F}_t$ measurable almost surely in t, $g(i, \cdot, \cdot) : \mathbb{S} \times L^2_H \times L^2_H \to \mathcal{L}(K, H)$ is the family of non-linear operators, $\mathcal{F}_t$ measurable almost surely in t, the other notions are defined in (1.1). As the standing hypotheses we always assume that:

- The Lipschitz continuous function $\delta : [0, +\infty) \to \mathbb{R}$ satisfies

$$-\tau \leq \delta(t) \leq t \quad \text{and} \quad |\delta(t) - \delta(s)| \leq \rho |t - s|, \quad \forall t, s \geq 0 \tag{2.2}$$

  for a positive constant ρ.

- There exist a constant $\bar{K} > 0$ such that for $-\tau \leq s < t \leq 0$, $0 \leq a \leq A$

$$\mathbb{E}\left[\sup_{-\tau\leq s<t\leq 0} |\, \varphi(t, a) - \varphi(s, a) \,|^2 \right] \leq \bar{K} \,|\, t - s \,| \,. \tag{2.3}$$

- For $k \geq 2$ and $0 \leq a \leq A$,

$$\mathbb{E}\left[\sup_{-\tau\leq t\leq 0}|\varphi(t,a)|^k\right] < \infty.$$

Moreover, we shall impose the following assumptions:
($\mathbb{H}1$) $f(i,0,0)=0,\ g(i,0,0)=0,\ \forall i \in S$.
($\mathbb{H}2$) There exists a positive constant L such that for $x,y \in C, i \in \mathbb{S}$,

$$|f(i,x,y)-f(i,\bar{x},\bar{y})| \vee \|g(i,x,y)-g(i,\bar{x},\bar{y})\|_2 \leq L(\|x-\bar{x}\|_C + \|y-\bar{y}\|_C).$$

($\mathbb{H}3$) $\mu(t,a), \beta(t,a)$ are continuous in $[0,T]\times[0,A]$ and there exist positive constants μ_0, $\bar{\alpha}$ and $\bar{\beta}$ such that

$$\begin{aligned}&0 \leq \mu_0 \leq \mu(t,a) \leq \overline{\alpha} < \infty,\\ &0 \leq \beta(t,a) \leq \overline{\beta} < \infty.\end{aligned}$$

From ($\mathbb{H}1$) and ($\mathbb{H}2$), we obtain that for $x,y \in C$

$$\begin{aligned}&|f(i,x,y)|^2 = |f(i,x,y)-f(i,0,0)|^2 \leq 2L^2(\|x\|_C^2 + \|y\|_C^2),\\ &\|g(i,x,y)\|_2^2 = \|g(i,x,y)-g(i,0,0)\|_2^2 \leq 2L^2(\|x\|_C^2 + \|y\|_C^2).\end{aligned} \tag{2.4}$$

These inequalitis will be very useful in what follows.

55 ## 3. Split-step θ-method

For any given time $T>0$, there exist sufficiently large positive integers m and M, such that

$$0 < \Delta = \frac{\tau}{m} = \frac{T}{M} < 1.$$

Let $t_n = n\Delta$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$. We denote $\lfloor u \rfloor$ by the integer part of the real number u. For $t \in [0,T]$, there is an integer $n \in [0, M-1]$ such that $t \in [n\Delta, (n+1)\Delta)$. Recalling (2.2) that $-\tau \leq \delta(t) \leq t$, we obtain

$$-m\Delta = -\tau \leq \delta(n\Delta) \leq n\Delta.$$

Thus, we get

$$-m \leq \delta(n\Delta)/\Delta \leq n,$$

which means

$$-m \leq \lfloor \delta(n\Delta)/\Delta \rfloor \leq n.$$

Hence, we obtain

$$-\tau = -m\Delta \leq \lfloor \delta(n\Delta)/\Delta \rfloor \Delta \leq n\Delta.$$

Then the SSθ method is defined by the following:

$$\begin{cases} Q_n^* = Q_n - \dfrac{\partial Q_n}{\partial a}\Delta + (1-\theta)[-\mu(t_n,a)Q_n + f(r_n^\Delta, Q_n, Q_{\lfloor \delta(n\Delta)/\Delta \rfloor})]\Delta, \\ \qquad + \theta[-\mu(t_n,a)Q_n^* + f(r_n^\Delta, Q_n^*, Q_{\lfloor \delta(n\Delta)/\Delta \rfloor})]\Delta, & 0 \le n \le M-1 \\ Q_{n+1} = Q_n^* + g(r_n^\Delta, Q_n^*, Q_{\lfloor \delta(n\Delta)/\Delta \rfloor})\Delta W_n, & 0 \le n \le M-1 \\ Q_n = \varphi(t_n, a). & -m \le n \le 0 \end{cases} \tag{3.1}$$

with the initial value $Q_0 = \varphi(0,a)$, $r_0^\Delta = r_0$, $Q_n(t,0) = \int_0^A \beta(t,a)Q_n da$, $r_n^\Delta = r(n\Delta)$, for $0 \le n \le M-1$. Let Q_n be the approximation to $P(t_n, a)$ for $t_n = n\Delta$. We set $0 \le \theta \le 1$. The choice $\theta = 1$ gives the split-step backward Euler method. If $\theta = 0$, the SSθ method degenerates to the split-step forward method. The first equation in (3.1) is an implicit equation in Q_n^* that must be solved in order to obtain the intermediate approximation Q_n^*. Having obtained Q_n^*, substituting it into the second of (3.1) produces the next approximation Q_{n+1}. Using the classical Banach contraction mapping theorem, the first equation in (3.1) has a unique fixed Q_n^* (see [21]). We now present present the following lemma.

Lemma 3.1. *Assume that $f : \mathbb{R} \times \mathbb{R} \times [0,T] \longmapsto \mathbb{R}$ satisfies ($\mathbb{H}1$) and ($\mathbb{H}2$). Let $0 < \theta \le 1$ and $0 < \Delta < 1/(L\theta)$. Then for any a,b,c $\in \mathbb{R}$, the implicit equation $x = a + \Delta\theta f(b,x,c)$ has a unique solution x.*

We define the following step functions:

$$\begin{aligned} \bar{Z}_t &= \sum_{n=0}^{M-1} Q_n \mathbb{I}_{[n\Delta,(n+1)\Delta)}(t), \\ \widetilde{Z}_t &= \sum_{n=0}^{M-1} Q_{\lfloor \delta(n\Delta)/\Delta \rfloor} \mathbb{I}_{[n\Delta,(n+1)\Delta)}(t), \\ \hat{Z}_t &= \sum_{n=0}^{M-1} Q_n^* \mathbb{I}_{[n\Delta,(n+1)\Delta)}(t), \\ \bar{r}(t) &= \sum_{n=0}^{M-1} r_n^\Delta \mathbb{I}_{[n\Delta,(n+1)\Delta)}(t), \end{aligned} \tag{3.2}$$

where $\mathbb{I}$ is the indicator function. It is convenient to use the continuous-time approximation

$$\begin{cases} Q_t = P_0 - \displaystyle\int_0^t \frac{\partial Q_n}{\partial a} ds + \int_0^t (1-\theta)[-\mu(s,a)\bar{Z}_s + f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s)]ds \\ \qquad + \displaystyle\int_0^t \theta[-\mu(s,a)\hat{Z}_s + f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)]ds + \int_0^t g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s, & 0 \le t \le T \\ Q_t = \varphi(t,a), & -\tau \le t \le 0 \end{cases} \tag{3.3}$$

with $Q_0 = P_0 = \varphi(0,a)$, $Q(t,0) = \int_0^A \beta(t,a)Q_t da$, $Q_t = Q(t,a)$, $\bar{r}(0) = i_0$. Clearly, (3.3) can also be written as

$$Q_t = Q_n - \frac{\partial Q_n}{\partial a}(t-t_n) + (1-\theta)[-\mu(t_n,a)Q_n + f(r_n^\Delta, Q_n, Q_{\lfloor \delta(n\Delta)/\Delta \rfloor})](t-t_n)$$

$$+\theta[-\mu(t_n,a)Q_n^* + f(r_n^{\Delta}, Q_n^*, Q_{\lfloor\delta(n\Delta)/\Delta\rfloor})](t-t_n)$$
$$+ g(r_n^{\Delta}, Q_n^*, Q_{\lfloor\delta(n\Delta)/\Delta\rfloor})(W_t - W_{t_n}), \tag{3.4}$$

for $t_n \le t < t_{n+1}$. It is straightforward to check that $\bar{Z}(t_n, a) = Q_n = Q(t_n, a)$, for $-m \le n \le M-1$ and $P_t = Q_t = \varphi(t,a)$, for $-\tau \le t \le 0$.

4. Strong convergence

In this section, We shall discuss the strong convergence of SSθ approximate solution under the given conditions. We first give several lemmas which are useful for the main results.

Lemma 4.1. *Let P_t be the solution of* (2.1). *Then*

$$-\left\langle P_t, \frac{\partial P_t}{\partial a}\right\rangle \le \frac{1}{2}A\bar{\beta}^2|P_t|^2,$$

where $A, \bar{\beta}$ is the same as before.

Proof. It is easy to see that

$$-\left\langle P_t, \frac{\partial P_t}{\partial a}\right\rangle = -\int_0^A P_t \frac{\partial P_t}{\partial a} da \quad = -\int_0^A P(t,a) d_a P(t,a)$$
$$= -\frac{1}{2}\int_0^A d_a P^2(t,a) = -\frac{1}{2}\left(P^2(t,A) - P^2(t,0)\right).$$

Recalling that $P(t,a)$ denote the population density of the age a at time t, we have $P(t,A) = 0$ for the maximum age A and any $t \in [0,T]$. By (2.1) and the Cauchy-Schwarz inequality, we have

$$-\left\langle P_t, \frac{\partial P_t}{\partial a}\right\rangle = \frac{1}{2}P^2(t,0) = \frac{1}{2}\left(\int_0^A \beta(t,a)P(t,a)da\right)^2$$
$$\le \frac{1}{2}\int_0^A \beta^2(t,a)da \int_0^A P^2(t,a)da \le \frac{1}{2}A\bar{\beta}^2|P_t|^2.$$

Thus, the proof is complete. □

Remark 4.2. *If we denote $f_0(P_t) := -\frac{\partial P_t}{\partial a} - \mu(t,a)P_t$ for all $P_t \in V$. By Lemma 4.1, we have*

$$\left\langle f_0(P_t), P_t\right\rangle = \left\langle -\frac{\partial P_t}{\partial a} - \mu(t,a)P_t, P_t\right\rangle \le (\frac{1}{2}A\bar{\beta}^2 - \mu_0)|P_t|^2, \quad \forall P_t \in V \tag{4.1}$$

which means that f_0 is a linear operator in V and satisfies the one-sided linear growth condition. This viewpoint will be of great help in the following analysis.

Lemma 4.3. *Under the assumptions ($\mathbb{H}1$)-($\mathbb{H}3$), for any $k \ge 2$, there is a constants $C_1 > 0$ such that*

$$\mathbb{E}\left[\sup_{-\tau\le t\le T}\left|P_{t\wedge\tau_R}\right|^k\right] \le C_1,$$

where $\tau_R = \inf\{t \ge 0 : |P_t| \ge R\}$ and C_1 is independent of Δ.

Proof. Applying the Itô formula to $|P_{t\wedge\tau_R}|^k$ yields

$$\begin{aligned} |P_t|^k &= |P_0|^k + k\int_0^{t\wedge\tau_R} |P_s|^{k-2}\left\langle -\frac{\partial P_s}{\partial a}, P_s\right\rangle ds + k\int_0^{t\wedge\tau_R} |P_s|^{k-2}\left\langle -\mu(s,a)P_s, P_{\delta(s)}\right\rangle ds \\ &\quad + k\int_0^{t\wedge\tau_R} |P_s|^{k-2}(P_s, f(r(s), P_s, P_{\delta(s)}))ds + \frac{k}{2}\int_0^{t\wedge\tau_R} |P_s|^{k-2}\left\|g(r(s), P_s, P_{\delta(s)})\right\|_2^2 ds \\ &\quad + \frac{k(k-2)}{2}\int_0^{t\wedge\tau_R} |P_s|^{k-4}\left\|(P_s, g(r(s), P_s, P_{\delta(s)}))\right\|_2^2 ds + M(t\wedge\tau_R), \end{aligned} \tag{4.2}$$

where

$$M(t\wedge\tau_R) = k\int_0^{t\wedge\tau_R} |P_s|^{k-2}(P_s, g(r(s), P_s, P_{\delta(s)}))dW_s.$$

By Lemma 4.1, we have

$$\left\langle -\frac{\partial P_s}{\partial a}, P_s\right\rangle \le \frac{1}{2}A\bar{\beta}^2|P_s|^2. \tag{4.3}$$

By the assumptions ($\mathbb{H}$1)-($\mathbb{H}$3), we get that

$$k\int_0^{t\wedge\tau_R} |P_s|^{k-2}\left\langle -\mu(s,a)P_s, P_{\delta(s)}\right\rangle ds \le -k\mu_0\int_0^{t\wedge\tau_R} \sup_{-\tau\le u\le s}|P_u|^k ds, \tag{4.4}$$

$$k\int_0^{t\wedge\tau_R} |P_s|^{k-2}(P_s, f(r(s), P_s, P_{\delta(s)}))ds \le 2kL\int_0^{t\wedge\tau_R} \sup_{-\tau\le u\le s}|P_u|^k ds, \tag{4.5}$$

$$\begin{aligned} \frac{k}{2}\int_0^{t\wedge\tau_R} |P_s|^{k-2}\left\|g(r(s), P_s, P_{\delta(s)})\right\|_2^2 ds &\le kL^2\int_0^{t\wedge\tau_R} |P_s|^{k-2}(\|P_s\|_C^2 + \|P_{\delta(s)}\|_C^2)ds \\ &\le 2kL^2\int_0^{t\wedge\tau_R} \sup_{-\tau\le u\le s}|P_u|^k ds, \end{aligned} \tag{4.6}$$

$$\frac{k(k-2)}{2}\int_0^{t\wedge\tau_R} |P_s|^{k-4}\left\|(P_s, g(r(s), P_s, P_{\delta(s)}))\right\|_2^2 ds \le 2k(k-2)L^2\int_0^{t\wedge\tau_R} \sup_{-\tau\le u\le s}|P_u|^k ds. \tag{4.7}$$

Substituting (4.3)-(4.7) into (4.2) yields

$$|P_t|^k \le |P_0|^k + \frac{k}{2}(A\bar{\beta}^2 - 2\mu_0 + 4(k-2)L^2 + 4L)\int_0^{t\wedge\tau_R} \sup_{-\tau\le u\le s}|P_u|^k ds + M(t\wedge\tau_R). \tag{4.8}$$

Note that for $t\in[0,T]$,

$$\mathbb{E}\left[\sup_{-\tau\le u\le t}\left|P_{u\wedge\tau_R}\right|^k\right] = \mathbb{E}\left[\sup_{-\tau\le u\le 0}\left|P_{u\wedge\tau_R}\right|^k\right] \vee \mathbb{E}\left[\sup_{0\le u\le t}\left|P_{u\wedge\tau_R}\right|^k\right].$$

Now, it follows that

$$\begin{aligned}\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^k\right] &\le \mathbb{E}\left[\sup_{-\tau\le u\le 0}|\varphi(u,a)|^k\right]\\ &\quad+\frac{k}{2}(A\overline{\beta}^2-2\mu_0+4(k-2)L^2+4L)\int_0^{t\wedge\tau_R}E\left[\sup_{-\tau\le u\le s}|P_u|^k\right]ds\\ &\quad+k\mathbb{E}\left[\sup_{0\le u\le t}\int_0^{t\wedge\tau_R}|P_s|^{k-2}(P_s,g(r(s),P_s,P_{\delta(s)})dW_s)\right]. \qquad (4.9)\end{aligned}$$

By the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le s\le t}\int_0^{t\wedge\tau_R}|P_s|^{k-2}(P_s,g(r(s),P_s,P_{\delta(s)}))dW_s\right]\\ &\le 3\mathbb{E}\left[\int_0^{t\wedge\tau_R}\left||P_u|^{k-2}(P_s,g(r(s),P_s,P_{\delta(s)}))\right|^2ds\right]^{1/2}\\ &\le 3\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^{k/2}\left[\int_0^{t\wedge\tau_R}|P_s|^{k-2}\left\|g(r(s),P_s,P_{\delta(s)})\right\|_2^2ds\right]^{1/2}\right]\\ &\le \frac{1}{2k}\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^k\right]+K_0\mathbb{E}\left[\int_0^{t\wedge\tau_R}|P_s|^{k-2}\left\|g(r(s),P_s,P_{\delta(s)})\right\|_2^2ds\right]\\ &\le \frac{1}{2k}\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^k\right]+4K_0L^2\int_0^{t\wedge\tau_R}\mathbb{E}\left[\sup_{-\tau\le u\le s}|P_s|^k\right]ds\end{aligned}$$

for a positive constant K_0. Note that

$$\begin{aligned}\mathbb{E}\left[|P_s|^{k-2}\left\|g(r(s),P_s,P_{\delta(s)})\right\|_2^2\right] &\le 2L^2\mathbb{E}\left[|P_s|^{k-2}(\|P_s\|_C^2+\|P_{\delta(s)}\|_C^2)\right]\\ &\le 4L^2\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^k\right].\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^k\right] &\le \mathbb{E}\left[\sup_{-\tau\le u\le 0}|\varphi(u,a)|^k\right]+\frac{1}{2}\mathbb{E}\left[\sup_{-\tau\le u\le t}|P_u|^k\right]\\ &\quad+\frac{k}{2}(A\overline{\beta}^2-2\mu_0+4(k-2)L^2+4L+8K_0L^2)\int_0^{t\wedge\tau_R}\mathbb{E}\left[\sup_{-\tau\le u\le s}|P_u|^k\right]ds.\end{aligned}$$

Using the Gronwall inequality implies the required result with

$$C_1=2\mathbb{E}\left[\sup_{-\tau\le u\le 0}|\varphi(u,a)|^k\right]e^{k(A\overline{\beta}^2-2\mu_0+4(k-2)L^2+4L+8K_0L^2)T/2}.$$

Thus, the proof is complete. □

Lemma 4.4. *Under the assumptions ($\mathbb{H}1$)-($\mathbb{H}3$), let $\Delta<\min\left\{1,1/(\sqrt{12(\bar{\alpha}^2+2L^2)})\right\}$ and $\mathbb{E}\left|\frac{\partial Q_n}{\partial a}\right|^2<\infty$, $0\le n\le M-1$. Then there exist positive constants C_2, C_3, C_4, such that*

$$\mathbb{E}\left|Q_n^*\right|^2\le C_2\mathbb{E}|Q_n|^2+C_3\mathbb{E}\left|Q_{\lfloor\delta(n\Delta)/\Delta\rfloor}\right|^2+C_4,\quad 0\le n\le M-1$$

where Q_n, Q_n^* is defined in (3.1) and C_2, C_3, C_4 is independent of Δ.

Proof. Squaring both sides of the first equation in (3.1), we have

$$\begin{aligned}\left|Q_n^*\right|^2 &\le 3|Q_n|^2 + 3\left|\frac{\partial Q_n}{\partial a}\right|^2\Delta^2 + 3\Big|(1-\theta)[-\mu(t_n,a)Q_n + f(r_n^\Delta, Q_n, Q_{\lfloor\delta(n\Delta)/\Delta\rfloor})]\\
&\quad + \theta[-\mu(t_n,a)Q_n^* + f(r_n^\Delta, Q_n^*, Q_{\lfloor\delta(n\Delta)/\Delta\rfloor})]\Big|^2\Delta^2\\
&\le 3|Q_n|^2 + 3\left|\frac{\partial Q_n}{\partial a}\right|^2\Delta^2 + 12\Big(\bar{\alpha}^2|Q_n|^2 + \left|f(r_n^\Delta, Q_n, Q_{\lfloor\delta(n\Delta)/\Delta\rfloor})\right|^2\Big)\Delta^2\\
&\quad + 12\Big(\bar{\alpha}^2\left|Q_n^*\right|^2 + \left|f(r_n^\Delta, Q_n^*, Q_{\lfloor\delta(n\Delta)/\Delta\rfloor})\right|^2\Big)\Delta^2.\end{aligned}$$

By the assumptions ($\mathbb{H}1$)-($\mathbb{H}3$), we have

$$\begin{aligned}\left|Q_n^*\right|^2 &\le 3|Q_n|^2 + 3\left|\frac{\partial Q_n}{\partial a}\right|^2\Delta^2 + 12[\bar{\alpha}^2|Q_n|^2 + 2L^2|Q_n|^2 + 2L^2\left|Q_{\lfloor\delta(n\Delta)/\Delta\rfloor}\right|^2]\Delta^2\\
&\quad + 12[\bar{\alpha}^2\left|Q_n^*\right|^2 + 2L^2\left|Q_n^*\right|^2 + 2L^2\left|Q_{\lfloor\delta(n\Delta)/\Delta\rfloor}\right|^2)]\Delta^2\\
&= [3 + 12(\bar{\alpha}^2 + 2L^2)\Delta^2]|Q_n|^2 + 12(\bar{\alpha}^2 + 2L^2)\Delta^2\left|Q_n^*\right|^2\\
&\quad + 3\left|\frac{\partial Q_n}{\partial a}\right|^2\Delta^2 + 48L^2\Delta^2|Q_{\lfloor\delta(n\Delta)/\Delta\rfloor}|^2.\end{aligned}$$

Hence, we get

$$\begin{aligned}\mathbb{E}\left|Q_n^*\right|^2 &\le \frac{(3 + 12(\bar{\alpha}^2 + 2L^2)\Delta^2}{1 - 12(\bar{\alpha}^2 + 2L^2)\Delta^2}\mathbb{E}|Q_n|^2 + \frac{48L^2\Delta^2}{1 - 12(\bar{\alpha}^2 + 2L^2)\Delta^2}\mathbb{E}\left|Q_{\lfloor\delta(n\Delta)/\Delta\rfloor}\right|^2\\
&\quad + \frac{3\mathbb{E}\left|\frac{\partial Q_n}{\partial a}\right|^2\Delta^2}{1 - 12(\bar{\alpha}^2 + 2L^2)\Delta^2}\\
&\le C_2\mathbb{E}|Q_n|^2 + C_3\mathbb{E}\left|Q_{\lfloor\delta(n\Delta)/\Delta\rfloor}\right|^2 + C_4.\end{aligned}$$

Thus, the proof is complete. □

Lemma 4.5. *Under the assumptions ($\mathbb{H}1$)-($\mathbb{H}3$) and let $\Delta < \min\left\{1, 1/(\sqrt{12(\bar{\alpha}^2 + 2L^2)})\right\}$, there exist a positive constant C_5 independent of Δ such that*

$$\mathbb{E}\left[\sup_{-\tau\le t\le T}\left|Q_{t\wedge\sigma_R}\right|^2\right] \le C_5,$$

where $\sigma_R = \inf\{t \ge 0 : |Q_t| \ge R\}$.

Proof: Applying the Itô formula to $\left|Q_{t\wedge\sigma_R}\right|^2$ yields

$$\begin{aligned}\left|Q_{t\wedge\sigma_R}\right|^2 &= |Q_0|^2 + 2\int_0^{t\wedge\sigma_R}\left\langle -\frac{\partial Q_s}{\partial a}, Q_s\right\rangle ds\\
&\quad + 2\int_0^{t\wedge\sigma_R}((1-\theta)f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s) + \theta f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s), Q_s)ds\\
&\quad - 2\int_0^{t\wedge\sigma_R}(\mu(s,a)[(1-\theta)\bar{Z}_s + \theta\hat{Z}_s], Q_s)ds + \int_0^{t\wedge\sigma_R}\|g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)\|_2^2 ds\\
&\quad + 2\int_0^{t\wedge\sigma_R}(Q_s, g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s). \qquad (4.10)\end{aligned}$$

Recalling Lemma 4.1, we have

$$\left\langle -\frac{\partial Q_s}{\partial a}, Q_s \right\rangle \leq \frac{1}{2} A\bar{\beta}^2 |Q_s|^2. \tag{4.11}$$

By the elementary inequality $2ab \leq a^2 + b^2$ and ($\mathbb{H}1$)-($\mathbb{H}3$), we have

$$2\int_0^{t\wedge\sigma_R} ((1-\theta)f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s), Q_s)ds \leq \int_0^{t\wedge\sigma_R} |Q_s|^2 ds + 2L^2 \int_0^{t\wedge\sigma_R} (\|\bar{Z}_s\|_C^2 + \|\widetilde{Z}_s\|_C^2)ds, \tag{4.12}$$

$$2\int_0^{t\wedge\sigma_R} ((1-\theta)f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s), Q_s)ds \leq \int_0^{t\wedge\sigma_R} |Q_s|^2 ds + 2L^2 \int_0^{t\wedge\sigma_R} (\|\hat{Z}_s\|_C^2 + \|\widetilde{Z}_s\|_C^2)ds, \tag{4.13}$$

$$-2\int_0^{t\wedge\sigma_R} (\mu(s,a)(1-\theta)\bar{Z}_s, Q_s)ds \leq \bar{\alpha} \int_0^{t\wedge\sigma_R} (\left|\bar{Z}_s\right|^2 + |Q_s|^2)ds, \tag{4.14}$$

$$-2\int_0^{t\wedge\sigma_R} (\theta\mu(s,a)\hat{Z}_s, Q_s)ds \leq \bar{\alpha} \int_0^{t\wedge\sigma_R} (|Q_s|^2 + |\hat{Z}_s|^2)ds, \tag{4.15}$$

$$\int_0^{t\wedge\sigma_R} \|g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)\|_2^2 ds \leq 2L^2 \int_0^{t\wedge\sigma_R} (\|\hat{Z}_s\|_C^2 + \|\widetilde{Z}_s\|_C^2)ds. \tag{4.16}$$

Inserting (4.11)-(4.16) into (4.10) gives

$$\begin{aligned}\left|Q_{t\wedge\sigma_R}\right|^2 \leq |Q_0|^2 + K_1 \int_0^{t\wedge\sigma_R} |Q_s|^2 ds + K_2 \int_0^{t\wedge\sigma_R} \|\bar{Z}_s\|_C^2 ds \\ + K_3 \int_0^{t\wedge\sigma_R} \|\hat{Z}_s\|_C^2 ds + K_4 \int_0^{t\wedge\sigma_R} \|\widetilde{Z}_s\|_C^2 ds + M_1(t\wedge\sigma_R),\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}&K_1 = A^2\bar{\beta}^2 + 2\bar{\alpha} + 2, K_2 = 2L^2 + \bar{\alpha}, K_3 = 4L^2 + \bar{\alpha}, \\ &K_4 = 6L^2, M_1(t\wedge\sigma_R) = 2\int_0^{t\wedge\sigma_R} (Q_s, g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s)\end{aligned}$$

By Lemma 4.4, it follows that for any $t_1 \in [0, T]$

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq t\leq t_1} \left|Q_{t\wedge\sigma_R}\right|^2\right] \\ &\leq \mathbb{E}|Q_0|^2 + K_1\mathbb{E}\left[\sup_{0\leq t\leq t_1} \int_0^{t\wedge\sigma_R} |Q_s|^2 ds\right] + (K_2 + K_3C_2)\mathbb{E}\left[\sup_{0\leq t\leq t_1} \int_0^{t\wedge\sigma_R} \left|\bar{Z}_s\right|^2 ds\right] \\ &\quad + (K_4 + K_3C_3)\mathbb{E}\left[\sup_{0\leq t\leq t_1} \int_0^{t\wedge\sigma_R} |\widetilde{Z}_s|^2 ds\right] + K_2C_4T + 2\mathbb{E}\left[\sup_{0\leq t\leq t_1} M_1(t\wedge\sigma_R)\right].\end{aligned} \tag{4.18}$$

For $s \in [0, T]$, we have

$$\begin{aligned}\mathbb{E}\|\bar{Z}_s\|_C^2 \leq \mathbb{E}\left[\sup_{-\tau\leq u\leq s} \left|Q_u\right|^2\right], \\ \mathbb{E}\|\widetilde{Z}_s\|_C^2 \leq \mathbb{E}\left[\sup_{-\tau\leq u\leq s} \left|Q_u\right|^2\right].\end{aligned}$$

Inserting this into (4.18) gives

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le t_1}\left|Q_{t\wedge\sigma_R}\right|^2\right]\\
&\le \mathbb{E}|Q_0|^2 + K_5\int_0^{t_1\wedge\sigma_R}\mathbb{E}\left[\sup_{-\tau\le u\le s}|Q_u|^2 ds\right] + K_2C_4T + 2\mathbb{E}\left[\sup_{0\le t\le t_1} M_1(t\wedge\sigma_R)\right], \qquad (4.19)
\end{aligned}$$

where

$$K_5 = K_1 + K_2 + K_3C_2 + K_4 + K_3C_3.$$

By the Burkholder-Davis-Gundy inequality and the Young inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le t_1}\int_0^{t\wedge\sigma_R}(Q_s, g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)dW_s)\right]\\
&\le 3\mathbb{E}\left|\int_0^{t_1\wedge\sigma_R}|Q_s|^2\|g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)\|^2 ds\right|^{1/2}\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le t_1}|Q_t|^2\right] + 9\mathbb{E}\int_0^{t\wedge\sigma_R}\left\|g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)\right\|_2^2 dt\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le t_1}|Q_t|^2\right] + 18L^2\int_0^{t\wedge\sigma_R}\mathbb{E}(\|\hat{Z}_s\|_C^2 + \|\widetilde{Z}_s\|_C^2)dt. \qquad (4.20)
\end{aligned}$$

Substituting this into (4.19) yields

$$\begin{aligned}
\mathbb{E}\left[\sup_{-\tau\le t\le t_1}\left|Q_{t\wedge\sigma_R}\right|^2\right] &\le \mathbb{E}\left[\sup_{-\tau\le s\le 0}|\varphi(s,a)|^2\right] + (K_5 + 36L^2(C_2+C_3))\int_0^{t_1\wedge\sigma_R}\mathbb{E}\left[\sup_{-\tau\le u\le s}|Q_u|^2 ds\right]\\
&\quad + \frac{1}{2}\mathbb{E}\left[\sup_{-\tau\le t\le t_1}\left|Q_{t\wedge\sigma_R}\right|^2\right] + (36L^2 + K_2)C_4T. \qquad (4.21)
\end{aligned}$$

Applying the Gronwall inequality, we get

$$\mathbb{E}\left[\sup_{-\tau\le t\le t_1}\left|Q_{t\wedge\sigma_R}\right|^2\right] \le C_5, \quad t_1\in[0,T].$$

Thus, the proof is complete. □

Next, we shall employ the technique in [27] to bound the errors of replacing the right-continuous Markov chain by the interpolation of the discrete time Markov chain.

Lemma 4.6. *Let* $\Delta < \min\left\{1, 1/(\sqrt{12(\bar{\alpha}^2+2L^2)})\right\}$, *for any* $t\in[0,T]$

$$\mathbb{E}\int_0^{t\wedge\nu_R}|f(\bar{r}(s),\bar{Z}_s,\widetilde{Z}_s) - f(r(s),\bar{Z}_s,\widetilde{Z}_s)|^2 ds \le C_6\Delta,$$

$$\mathbb{E}\int_0^{t\wedge\nu_R}|f(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s) - f(r(s),\hat{Z}_s,\widetilde{Z}_s)|^2 ds \le C_7\Delta,$$

$$\mathbb{E}\int_0^{t\wedge\nu_R}\|g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s) - g(r(s),\hat{Z}_s,\widetilde{Z}_s)\|_2^2 ds \le C_8\Delta,$$

where $\tau_R = \inf\{t\ge 0 : |P_t|\ge R\}$, $\sigma_R = \inf\{t\ge 0 : |Q_t|\ge R\}$, $\nu_R = \tau_R\wedge\sigma_R$ *and* C_6, C_7, C_8 *is independent of* Δ.

Proof. Using ($\mathbb{H}1$)-($\mathbb{H}3$), we have

$$\mathbb{E}\int_0^{t\wedge\nu_R} |f(\bar r(s),\bar Z_s,\widetilde Z_s)-f(r(s),\bar Z_s,\widetilde Z_s)|^2 ds \le \sum_{k=0}^{\lfloor t/\Delta\rfloor} M_k, \tag{4.22}$$

where

$$\begin{aligned}
M_k &:= \mathbb{E}\int_{t_k\wedge\nu_R}^{t_{k+1}\wedge\nu_R} \left|f(\bar r(s),Q_k,Q_{\lfloor\delta(k\Delta)/\Delta\rfloor})-f(r(s),Q_k,Q_{\lfloor\delta(k\Delta)/\Delta\rfloor})\right|^2 ds\\
&= \mathbb{E}\int_{t_k\wedge\nu_R}^{t_{k+1}\wedge\nu_R} \left|f(\bar r(s),Q_k,Q_{\lfloor\delta(k\Delta)/\Delta\rfloor})-f(r(s),Q_k,Q_{\lfloor\delta(k\Delta)/\Delta\rfloor})\right|^2 \mathbb{I}_{\{r(s)\neq r(t_k)\}}(s) ds\\
&\le 4L^2\mathbb{E}\int_{t_k\wedge\nu_R}^{t_{k+1}\wedge\nu_R} [|Q_k|^2+\left|Q_{\lfloor\delta(k\Delta)/\Delta\rfloor}\right|^2]\mathbb{I}_{\{r(s)\neq r(t_k)\}}(s) ds\\
&= 4L^2\int_{t_k\wedge\nu_R}^{t_{k+1}\wedge\nu_R} \mathbb{E}[\mathbb{E}[|Q_k|^2+\left|Q_{\lfloor\delta(k\Delta)/\Delta\rfloor}\right|^2]\big|r(t_k)]\mathbb{E}[\mathbb{I}_{\{r(s)\neq r(t_k)\}}(s)\big|r(t_k)] ds.
\end{aligned} \tag{4.23}$$

In the last step we use the fact that $Q_k, Q_{\lfloor\delta(k\Delta)/\Delta\rfloor}$ and $r(s)\neq r(t_k)$ are conditionally independent with respect to the σ-algebra generated by $r(t_k)$. But, by the Markov property,

$$\begin{aligned}
\mathbb{E}[\mathbb{I}_{\{r(s)\neq r(t_k)\}}(s)\big|r(t_k)] &= \sum_{i\in S} P(r(t_k)=i)P(r(s)\neq r(t_k)\big|r(t_k)=i)\\
&= \sum_{i\in S} P(r(t_k)=i)\sum_{j\neq i} P(r(s)=j\big|r(t_k)=i)\\
&\le \sum_{i\in S} P(r(t_k)=i)\sum_{j\neq i}(\gamma_{ij}\Delta+o(\Delta))\\
&= \sum_{i\in S} P(r(t_k)=i)((-\gamma_{ii})\Delta+o(\Delta))\\
&\le \hat\gamma\Delta,
\end{aligned}$$

where $\hat\gamma=[N\max_{1\le i\le N}(-\gamma_{ii})]$. Substituting this into (4.23) along with Lemma 4.5 gives

$$M_k \le 4L^2\Delta\int_{t_k\wedge\nu_R}^{t_{k+1}\wedge\nu_R} \mathbb{E}[|Q_k|^2+\left|Q_{\lfloor\delta(k\Delta)/\Delta\rfloor}\right|^2] ds \le 8L^2C_5\hat\gamma\Delta^2. \tag{4.24}$$

Combining (4.22) and (4.24), we obtain that

$$\mathbb{E}\int_0^{t\wedge\nu_R} |f(\bar r(s),\bar Z_s,\widetilde Z_s)-f(r(s),\bar Z_s,\widetilde Z_s)|^2 ds \le 8L^2C_5\hat\gamma T\Delta = C_6\Delta.$$

Similarly, we can prove that

$$\mathbb{E}\int_0^{t\wedge\nu_R} |f(\bar r(s),\bar Z_s,\widetilde Z_s)-f(r(s),\hat Z_s,\widetilde Z_s)|^2 ds \le C_7\Delta,$$

$$\mathbb{E}\int_0^{t\wedge\nu_R} \|g(\bar r(s),\bar Z_s,\widetilde Z_s)-g(r(s),\hat Z_s,\widetilde Z_s)\|_2^2 ds \le C_8\Delta.$$

Thus, the proof is complete. □

Lemma 4.7. *Under the assumptions($\mathbb{H}1$)-($\mathbb{H}3$) and let $\Delta < \min\left\{1, 1/(\sqrt{12(\bar{\alpha}^2 + 2L^2)})\right\}$, for any $k \geq 2$, there is a positive constant C_9 such that*

$$\mathbb{E}\left[\sup_{-\tau \leq t \leq T} \left|Q_{t\wedge\sigma_R}\right|^k\right] \leq C_9,$$

where $\sigma_R = \inf\{t \geq 0 : |Q_t| \geq R\}$ and C_9 is independent of Δ.

The proof is similar to that of Lemma 4.3.

Lemma 4.8. *Under the Assumptions ($\mathbb{H}1$)-($\mathbb{H}3$) and let $\Delta < \min\left\{1, 1/(\sqrt{12(\bar{\alpha}^2 + 2L^2)})\right\}$, $\mathbb{E}\left|\frac{\partial Q_n}{\partial a}\right|^2 < \infty$, there exist positive constants C_{10} and C_{11} independent of Δ such that*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |Q_t - \bar{Z}_t|^2\Big] \leq C_{10}\Delta,$$

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |Q_t - \hat{Z}_t|^2\Big] \leq C_{11}\Delta.$$

Proof. For $t \in [0, T]$, there exists an integer k such that $t \in [k\Delta, (k+1)\Delta)$, we have

$$\begin{aligned} Q_t - \bar{Z}_t = &- \int_{k\Delta}^t \frac{\partial Q_k}{\partial a} ds + \int_{k\Delta}^t ((1-\theta) f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s) + \theta f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s))ds \\ &- \int_{k\Delta}^t \mu(s, a)((1-\theta)\bar{Z}_s + \theta\hat{Z}_s)ds + \int_{k\Delta}^t g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s. \end{aligned} \tag{4.25}$$

Applying $(a + b + c + d)^2 \leq 4a^2 + 4b^2 + 4c^2 + 4d^2$ and conditions ($\mathbb{H}1$)-($\mathbb{H}2$), we find that

$$\begin{aligned} |Q_t - \bar{Z}_t|^2 \leq\, & 4\Big|\int_{k\Delta}^t \frac{\partial Q_k}{\partial a} ds\Big|^2 + 4\Big|\int_{k\Delta}^t ((1-\theta) f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s) + \theta f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s))ds\Big|^2 \\ &+ 4\Big|\int_{k\Delta}^t (\mu(s, a)((1-\theta)\bar{Z}_s + \theta\hat{Z}_s))ds\Big|^2 + 4\Big|\int_{k\Delta}^t g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s\Big|^2 \\ \leq\, & 4\Delta \int_{k\Delta}^t \Big|\frac{\partial Q_k}{\partial a}\Big|^2 ds + 4\Delta \int_{k\Delta}^t \Big|((1-\theta) f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s) + \theta f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s))\Big|^2 ds \\ &+ 4\bar{\alpha}^2\Delta \int_{k\Delta}^t \Big|(1-\theta)\bar{Z}_s + \theta\hat{Z}_s\Big|^2 ds + 4\Big|\int_{k\Delta}^t g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s\Big|^2 \\ \leq\, & 4\Delta \int_{k\Delta}^t \Big|\frac{\partial Q_k}{\partial a}\Big|^2 ds + 8L^2\Delta \int_{k\Delta}^t (\|\bar{Z}_s\|_C^2 + \|\widetilde{Z}_s\|_C^2 + \|\hat{Z}\|_C^2 + \|\widetilde{Z}_s\|_C^2)ds \\ &+ 4\bar{\alpha}^2\Delta \int_{k\Delta}^t (|\bar{Z}_s|^2 + |\hat{Z}_s|^2)ds + 4\Big|\int_{k\Delta}^t g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s\Big|^2. \end{aligned}$$

Taking expectations, we have

$$\begin{aligned} \mathbb{E}\Big[\sup_{0\leq t\leq T} |Q_t - \bar{Z}_t|^2\Big] \leq\, & \mathbb{E}\Big[\max_{0\leq k\leq M-1} \sup_{k\Delta\leq u\leq(k+1)\Delta} |Q_u - \bar{Z}_u|^2\Big] \\ \leq\, & \max_{0\leq k\leq M-1} \Big(4\Delta^2\mathbb{E}\Big|\frac{\partial Q_k}{\partial a}\Big|^2 + 4\bar{\alpha}^2\Delta \int_{k\Delta}^{(k+1)\Delta} (\mathbb{E}|\bar{Z}_s|^2 + \mathbb{E}|\hat{Z}_s|^2)ds \\ &+ 8L^2\Delta \int_{k\Delta}^{(k+1)\Delta} (\mathbb{E}\|\bar{Z}_s\|_C^2 + \mathbb{E}\|\widetilde{Z}_s\|_C^2 + \mathbb{E}\|\hat{Z}\|_C^2 + \mathbb{E}\|\widetilde{Z}_s\|_C^2)ds\Big) \\ &+ 4\mathbb{E}\Big[\max_{0\leq k\leq M-1} \sup_{k\Delta\leq u\leq(k+1)\Delta} \Big|\int_{k\Delta}^u g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)dW_s\Big|^2\Big]. \end{aligned}$$

Using the Doob martingale inequality (see e.g. [6, 11]), Lemma 4.4 and 4.5, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|Q_t-\bar{Z}_t|^2\Big] &\le 16\max_{0\le k\le M-1}\Big[\int_{k\Delta}^{(k+1)\Delta}\mathbb{E}\|g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)\|_2^2ds\Big]+o(\Delta)\\ &\le 32L^2\max_{0\le k\le M-1}\Big[\int_{k\Delta}^{(k+1)\Delta}(\mathbb{E}\|\hat{Z}_s\|_C^2+\mathbb{E}\|\widetilde{Z}_s\|_C^2)ds\Big]+o(\Delta)\\ &\le 32L^2\Delta\max_{0\le k\le M-1}(C_2\mathbb{E}|Q_k|^2+(C_3+1)\mathbb{E}|Q_{\lfloor\delta(k\Delta)/\Delta\rfloor}|^2+C_4)+o(\Delta)\\ &\le 32L^2((C_2+C_3+1)C_9+C_4)\Delta+o(\Delta).\end{aligned}$$

A similar analysis gives

$$\mathbb{E}\Big[\sup_{0\le t\le T}|Q_t-\hat{Z}_t|^2\Big]\le C_{11}\Delta.$$

Thus, the proof is complete. □

Next, we will employ the method due to Mao [16] to prove the following lemma.

Lemma 4.9. *Under the assumptions ($\mathbb{H}1$)-($\mathbb{H}3$) and let $\Delta<\min\Big\{1,1/(\sqrt{12(\bar{\alpha}^2+2L^2)})\Big\}$, $\mathbb{E}\Big|\frac{\partial Q_n}{\partial a}\Big|^2<\infty$, there exists a positive constant C_{12} independent of Δ such that*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|Q_{\delta(t)}-\widetilde{Z}_t|^2\Big]\le C_{12}\Delta.$$

Proof. For $t\in[0,T]$, there exists an integer k such that $t\in[k\Delta,(k+1)\Delta)$. By (2.2) and the triangle inequality, we yield

$$\begin{aligned}|\delta(t)-\lfloor\delta(k\Delta)/\Delta\rfloor\Delta| &\le|\delta(t)-\delta(k\Delta)|+|\delta(k\Delta)-\lfloor\delta(k\Delta)/\Delta\rfloor\Delta|\\ &\le\rho\Delta+|\delta(k\Delta)/\Delta-\lfloor\delta(k\Delta)/\Delta\rfloor|\Delta\\ &\le(\rho+1)\Delta.\end{aligned}\tag{4.26}$$

To show the desired result, let us consider the following four possible cases:

- If $\delta(t)\ge\lfloor\delta(k\Delta)/\Delta\rfloor\Delta\ge0$ or $\lfloor\delta(k\Delta)/\Delta\rfloor\Delta\ge\delta(t)\ge0$, we have

$$\begin{aligned}&\big|Q_{\delta(t)}-\widetilde{Z}_t\big|\\ &=\Big|-\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}\frac{\partial Q_k}{\partial a}ds+\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}((1-\theta)f(\bar{r}(s),\bar{Z}_s,\widetilde{Z}_s)+\theta f(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s))ds\\ &\quad-\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}\mu(s,a)((1-\theta)\bar{Z}_s+\theta\hat{Z}_s)ds+\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)dW_s\Big|.\end{aligned}$$

  Applying $(a+b+c+d)^2\le4a^2+4b^2+4c^2+4d^2$ and (4.26), we have

$$\begin{aligned}&|Q_{\delta(t)}-\widetilde{Z}_t|^2\\ &\le4\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}\frac{\partial Q_k}{\partial a}ds\Big|^2+4\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}((1-\theta)f(\bar{r}(s),\bar{Z}_s,\widetilde{Z}_s)+\theta f(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s))ds\Big|^2\\ &\quad+4\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}(\mu(s,a)((1-\theta)\bar{Z}_s+\theta\hat{Z}_s))ds\Big|^2+4\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)dW_s\Big|^2\end{aligned}$$

$$\leq 4(\rho+1)\Delta\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}\Big|\frac{\partial Q_k}{\partial a}\Big|^2 ds + 8L^2(\rho+1)\Delta\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}(\|\bar{Z}_s\|_C^2 + 2\|\widetilde{Z}_s\|_C^2 + \|\hat{Z}\|_C^2)ds$$
$$+ 4\bar{\alpha}^2(\rho+1)\Delta\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)}(|\bar{Z}_s|^2 + |\hat{Z}_s|^2)ds + 4\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(t)} g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)dW_s\Big|^2.$$

Taking expectations and using Lemma 4.4 and 4.5 as well as (4.26), we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T}|Q_{\delta(t)} - \widetilde{Z}_t\Big|^2]\\
&\leq \mathbb{E}\Big[\max_{0\leq k\leq M-1}\sup_{k\Delta\leq u\leq (k+1)\Delta}|Q_{\delta(u)} - \widetilde{Z}_u|^2\Big]\\
&\leq \max_{0\leq k\leq M-1}\Big(4(\rho+1)^2\Delta^2\mathbb{E}\Big|\frac{\partial Q_k}{\partial a}\Big|^2 + 4\bar{\alpha}^2(\rho+1)\Delta\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta+(\rho+1)\Delta}(\mathbb{E}|\bar{Z}_s|^2 + \mathbb{E}|\hat{Z}_s|^2)ds\\
&\quad + 8L^2(\rho+1)\Delta\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta+(\rho+1)\Delta}(\mathbb{E}\|\bar{Z}_s\|_C^2 + 2\mathbb{E}\|\widetilde{Z}_s\|_C^2 + \mathbb{E}\|\hat{Z}\|_C^2)ds\Big)\\
&\quad + 4\mathbb{E}\Big[\max_{0\leq k\leq M-1}\sup_{k\Delta\leq u\leq (k+1)\Delta}\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(u)} g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)dW_s\Big|^2\Big]\\
&\leq 4\mathbb{E}\Big[\max_{0\leq k\leq M-1}\sup_{k\Delta\leq u\leq (k+1)\Delta}\Big|\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\delta(u)} g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)dW_s\Big|^2\Big] + o(\Delta).
\end{aligned}$$

Using the Doob martingale inequality along with Lemma 4.4 and 4.5, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T}|Q_{\delta(t)} - \widetilde{Z}_t|^2\Big]\\
&\leq 16\max_{0\leq k\leq M-1}\Big[\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta+(\rho+1)\Delta}\mathbb{E}\|g(\bar{r}(s),\hat{Z}_s,\widetilde{Z}_s)\|_2^2 ds\Big] + o(\Delta)\\
&\leq 32L^2\max_{0\leq k\leq M-1}\Big[\int_{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta}^{\lfloor\delta(k\Delta)/\Delta\rfloor\Delta+(\rho+1)\Delta}(\mathbb{E}\|\hat{Z}_s\|_C^2 + \mathbb{E}\|\widetilde{Z}_s\|_C^2)ds\Big] + o(\Delta)\\
&\leq 32L^2(\rho+1)\Delta\max_{0\leq k\leq M-1}(C_2\mathbb{E}|Q_k|^2 + (C_3+1)\mathbb{E}|Q_{\lfloor\delta(k\Delta)/\Delta\rfloor}|^2 + C_4) + o(\Delta)\\
&\leq 32L^2(\rho+1)((C_2+C_3+1)C_9 + C_4)\Delta + o(\Delta).
\end{aligned}$$

- If $\delta(t)\leq\lfloor\delta(k\Delta)/\Delta\rfloor\Delta\leq 0$ or $\lfloor\delta(k\Delta)/\Delta\rfloor\Delta\leq\delta(t)\leq 0$. Then, by (3.1) and (3.2), we have

$$Q_{\delta(t)} - \widetilde{Z}_t = \varphi(\delta(t),a) - \varphi(\lfloor\delta(k\Delta)/\Delta\rfloor\Delta,a).$$

By (2.2), (2.3) and (4.26), we get

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\leq t\leq T}|Q_{\delta(t)} - \widetilde{Z}_t|^2\Big]\\
&\leq \mathbb{E}\Big[\max_{0\leq k\leq M-1}\sup_{k\Delta\leq u\leq (k+1)\Delta}|Q_{\delta(u)} - \widetilde{Z}_u|^2\Big]\\
&\leq \mathbb{E}\Big[\max_{0\leq k\leq M-1}\sup_{-\tau\leq\delta(u),\lfloor\delta(k\Delta)/\Delta\rfloor\Delta\leq 0}|\varphi(\delta(t),a) - \varphi(\lfloor\delta(k\Delta)/\Delta\rfloor\Delta,a)|^2\Big]\\
&\leq \bar{K}\Big|\delta(t) - \lfloor\delta(k\Delta)/\Delta\rfloor\Delta\Big|\\
&\leq \bar{K}(\rho+1)\Delta.
\end{aligned}$$

- If $\lfloor \delta(k\Delta)/\Delta \rfloor \Delta \le 0 \le \delta(t)$. Then, $-\lfloor \delta(k\Delta)/\Delta \rfloor \Delta \le (\rho+1)\Delta,\ \delta(t) \le (\rho+1)\Delta$. Hence

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} |Q_{\delta(t)} - \widetilde{Z}_t|^2\Big] \\
&\le 2\mathbb{E}\Big[\sup_{0\le t\le T} |Q_{\delta(t)} - Q_0|^2\Big] + 2\mathbb{E}\Big[\sup_{0\le t\le T} |\widetilde{Z}_t - \varphi(0,a)|^2\Big] \\
&\le 2\mathbb{E}\Big[\sup_{0\le t\le T} |Q_{\delta(t)} - Q_0|^2\Big] + 2\mathbb{E}\Big[\max_{0\le k\le M-1} \sup_{-\tau\le\lfloor \delta(k\Delta)/\Delta\rfloor\Delta\le 0} |\varphi(0,a) - \varphi(\lfloor \delta(k\Delta)/\Delta \rfloor\Delta, a)|^2\Big] \\
&\le 64L^2(\rho+1)((C_2+C_3+1)C_9 + C_4)\Delta + 2\bar{K}(\rho+1)\Delta + o(\Delta).
\end{aligned}$$

- If $\delta(t) \le 0 \le \lfloor \delta(k\Delta)/\Delta \rfloor \Delta$. Then, we have

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le t\le T} |Q_{\delta(t)} - \widetilde{Z}_t|^2\Big] \\
&\le 2\mathbb{E}\Big[\sup_{0\le t\le T} |Q_{\delta(t)} - Q_0|^2\Big] + 2\mathbb{E}\Big[\sup_{0\le t\le T} |\widetilde{Z}_t - \varphi(0,a)|^2\Big] \\
&\le 2\mathbb{E}\Big[\sup_{-\tau\le\delta(t)\le 0} |\varphi(\delta(0),a) - \varphi(\delta(t),a)|^2\Big] + 2\mathbb{E}\Big[\sup_{0\le t\le T} |\widetilde{Z}_t - Q_0|^2\Big] \\
&\le 2\bar{K}(\rho+1)\Delta + 64L^2(\rho+1)((C_2+C_3+1)C_9 + C_4)\Delta + o(\Delta).
\end{aligned}$$

Combining these different cases together, we get

$$\mathbb{E}\Big[\sup_{0\le t\le T} |Q_{\delta(t)} - \widetilde{Z}_t|^2\Big] \le C_{12}\Delta,$$

as required. □

Theorem 4.10. *Under the assumptions (*$\mathbb{H}1$*)-(*$\mathbb{H}3$*) and let* $\Delta < \min\Big\{1, 1/(\sqrt{12(\bar{\alpha}^2 + 2L^2)})\Big\}$*, then*

$$\mathbb{E}\left[\sup_{0\le t\le T} \left|P_{t\wedge\nu_R} - Q_{t\wedge\nu_R}\right|^2\right] \le C_{13}\Delta,$$

where $\tau_R = \inf\{t\ge 0 : |P_t| \ge R\}$, $\sigma_R = \inf\{t \ge 0 : |Q_t| \ge R\}$, $\nu_R = \tau_R \wedge \sigma_R$ and C_{13} is independent of Δ.

Proof. By (2.1) and (3.3), we have

$$\begin{aligned}
P_{t\wedge\nu_R} - Q_{t\wedge\nu_R} = &-\int_0^{t\wedge\nu_R} \frac{\partial(P_s - Q_s)}{\partial a} ds - \int_0^{t\wedge\nu_R} \mu(s,a)[(1-\theta)(P_s - \bar{Z}_s) + \theta(P_s - \hat{Z}_s]ds \\
&+ \int_0^{t\wedge\nu_R} [(1-\theta)(f(r(s), P_s, P_{\delta(s)}) - f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s))]ds \\
&+ \int_0^{t\wedge\nu_R} \theta(f(r(s), P_s, P_{\delta(s)}) - f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)ds \\
&+ \int_0^{t\wedge\nu_R} (g(r(s), P_s, P_{\delta(s)}) - g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s))dW_s.
\end{aligned}$$

Using the generalized Itô formula yields

$$
\begin{aligned}
\left|P_{t\wedge\nu_R}-Q_{t\wedge\nu_R}\right|^2 = &-2\int_0^{t\wedge\nu_R}\left\langle P_s-Q_s,\frac{\partial(P_s-Q_s)}{\partial a}\right\rangle ds\\
&-2\int_0^{t\wedge\nu_R}(P_s-Q_s,\mu(s,a)[(1-\theta)(P_s-\bar Z_s)+\theta(P_s-\hat Z_s)]ds)\\
&+2\int_0^{t\wedge\nu_R}(P_s-Q_s,(1-\theta)(f(r(s),P_s,P_{\delta(s)})-f(\bar r(s),\bar Z_s,\widetilde{Z}_s)))ds\\
&+2\int_0^{t\wedge\nu_R}(P_s-Q_s,\theta(f(r(s),P_s,P_{\delta(s)})-f(\bar r(s),\hat Z_s,\widetilde{Z}_s)))ds\\
&+\int_0^{t\wedge\nu_R}\|g(r(s),P_s,P_{\delta(s)})-g(\bar r(s),\hat Z_s,\widetilde{Z}_s)\|_2^2ds\\
&+2\int_0^{t\wedge\nu_R}(P_s-Q_s,(g(r(s),P_s,P_{\delta(s)})-g(\bar r(s),\hat Z_s,\widetilde{Z}_s))dW_s). \qquad (4.27)
\end{aligned}
$$

Let

$$
\begin{aligned}
J_4(s) &:= f(r(s),\bar Z_s,\bar Z_{\delta(s)})-f(\bar r(s),\bar Z_s,\widetilde{Z}_s),\\
J_5(s) &:= f(r(s),\hat Z_s,\bar Z_{\delta(s)})-f(\bar r(s),\hat Z_s,\widetilde{Z}_s),\\
J_6(s) &:= g(r(s),\hat Z_s,\bar Z_{\delta(s)})-g(\bar r(s),\hat Z_s,\widetilde{Z}_s).
\end{aligned}
$$

Lemma 4.1 gives

$$
2\int_0^{t\wedge\nu_R}\left\langle P_s-Q_s,\frac{\partial(P_s-Q_s)}{\partial a}\right\rangle ds\le A\overline{\beta}^2\int_0^{t\wedge\nu_R}|P_s-Q_s|^2ds. \qquad (4.28)
$$

By ($\mathbb{H}$1)-($\mathbb{H}$3) and the elementary inequalities

$$
2\langle u,v\rangle\le|u|^2+|v|^2,\quad |(1-\theta)u+\theta v|^2\le|u|^2+|v|^2,\quad u,v\in\mathbb{R}^n,
$$

we get

$$
\begin{aligned}
&-2\int_0^{t\wedge\nu_R}(P_s-Q_s,\mu(s,a)[(1-\theta)(P_s-\bar Z_s)+\theta(P_s-\hat Z_s)])ds\\
&\le\bar\alpha\int_0^{t\wedge\nu_R}(|P_s-Q_s|^2+|P_s-\bar Z_s|^2+|P_s-\hat Z_s|^2)ds\\
&\le\bar\alpha\int_0^{t\wedge\nu_R}(|P_s-Q_s|^2+2|P_s-Q_s|^2+2|Q_s-\bar Z_s|^2+2|P_s-Q_s|^2+2|Q_s-\hat Z_s|^2)ds\\
&\le5\bar\alpha\int_0^{t\wedge\nu_R}|P_s-Q_s|^2ds+2\bar\alpha\int_0^{t\wedge\nu_R}\left|Q_s-\bar Z_s\right|^2ds+2\bar\alpha\int_0^{t\wedge\nu_R}|Q_s-\hat Z_s|^2ds. \qquad (4.29)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&2\int_0^{t\wedge\nu_R}(P_s-Q_s,(1-\theta)(f(r(s),P_s,P_{\delta(s)})-f(\bar r(s),\bar Z_s,\widetilde{Z}_s)))ds\\
&\le\int_0^{t\wedge\nu_R}(|P_s-Q_s|^2+2|f(r(s),P_s,P_{\delta(s)})-f(r(s),\bar Z_s,\widetilde{Z}_s)|^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 2|f(r(s), \bar{Z}_s, \widetilde{Z}_s) - f(\bar{r}(s), \bar{Z}_s, \widetilde{Z}_s)|^2 \,)ds \\
\le& \int_0^{t\wedge\nu_R} (|P_s - Q_s|^2 + 4L^2\|P_s - \bar{Z}_s\|_C^2 + 4L^2\|P_{\delta(s)} - \widetilde{Z}_s\|_C^2 + 2|J_4(s)|^2)ds \\
\le& \int_0^{t\wedge\nu_R} (|P_s - Q_s|^2 + 8L^2\|P_s - Q_s\|_C^2 + 8L^2\|Q_s - \bar{Z}_s\|_C^2 \\
&+ 8L^2\|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 + 8L^2\|Q_{\delta(s)} - \widetilde{Z}_s\|_C^2 + 2|J_4(s)|^2)ds, \qquad (4.30)
\end{aligned}
$$

$$
\begin{aligned}
&2\int_0^{t\wedge\nu_R} (P_s - Q_s, \theta(f(r(s), P_s, P_{\delta(s)}) - f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)))ds \\
\le& \int_0^{t\wedge\nu_R} (|P_s - Q_s|^2 + 2|f(r(s), P_s, P_{\delta(s)}) - f(r(s), \hat{Z}_s, \widetilde{Z}_s)|^2 \\
&+ 2|f(r(s), \hat{Z}_s, \widetilde{Z}_s) - f(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)|^2 \,)ds \\
\le& \int_0^{t\wedge\nu_R} (|P_s - Q_s|^2 + 4L^2\|P_s - \hat{Z}_s\|_C^2 + 4L^2\|P_{\delta(s)} - \widetilde{Z}_s\|_C^2 + 2|J_5(s)|^2)ds \\
\le& \int_0^{t\wedge\nu_R} (|P_s - Q_s|^2 + 8L^2\|P_s - Q_s\|_C^2 + 8L^2\|Q_s - \hat{Z}_s\|_C^2 \\
&+ 8L^2\|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 + 8L^2\|Q_{\delta(s)} - \widetilde{Z}_s\|_C^2 + 2|J_5(s)|^2)ds \qquad (4.31)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^{t\wedge\nu_R} \|g(r(s), P_s, P_{\delta(s)}) - g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)\|_2^2 ds \\
\le& \int_0^{t\wedge\nu_R} (2\|g(r(s), P_s, P_{\delta(s)}) - g(r(s), \hat{Z}_s, \widetilde{Z}_s)\|_2^2 \\
&+ 2\|g(r(s), \hat{Z}_s, \widetilde{Z}_s) - g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)\|_c^2 \,)ds \\
\le& \int_0^{t\wedge\nu_R} (4L^2\|P_s - \hat{Z}_s\|_C^2 + 4L^2\|P_{\delta(s)} - \widetilde{Z}_s\|_C^2 + 2|J_6(s)|^2)ds \\
\le& \int_0^{t\wedge\nu_R} (8L^2\|P_s - Q_s\|_C^2 + 8L^2\|Q_s - \hat{Z}_s\|_C^2 \\
&+ 8L^2\|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 + 8L^2\|Q_{\delta(s)} - \widetilde{Z}_s\|_C^2 + 2|J_6(s)|^2)ds. \qquad (4.32)
\end{aligned}
$$

Substituting (4.28)-(4.32) into (4.27), we have

$$
\begin{aligned}
\left|P_{t\wedge\nu_R} - Q_{t\wedge\nu_R}\right|^2 \le& \mu_3 \int_0^{t\wedge\nu_R} \|P_s - Q_s\|_C^2 ds + \mu_4 \int_0^{t\wedge\nu_R} \|Q_s - \bar{Z}_s\|_C^2 ds \\
&+ \mu_5 \int_0^{t\wedge\nu_R} \|Q_s - \hat{Z}_s\|_C^2 ds + \mu_6 \int_0^{t\wedge\nu_R} \|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 ds \\
&+ \mu_7 \int_0^{t\wedge\nu_R} \|Q_{\delta(s)} - \widetilde{Z}_s\|_C^2 ds + 2\int_0^{t\wedge\nu_R} (J_4(s)|^2 + J_5(s)|^2 + J_6(s)|^2)ds \\
&+ 2\int_0^{t\wedge\nu_R} (P_s - Q_s, (g(r(s), P_s, P_{\delta(s)}) - g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s))dW_s), \qquad (4.33)
\end{aligned}
$$

where

$$\mu_3 = A\bar{\beta}^2 + 5\bar{\alpha} + 24L^2 + 2, \qquad \mu_4 = 2\bar{\alpha} + 8L^2,$$
$$\mu_5 = 2\bar{\alpha} + 16L^2, \qquad \mu_6 = 24L^2, \qquad \mu_7 = 24L^2.$$

By the Bukholder-Davis-Gundy inequality, the Young inequality and (4.32), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\int_0^{s\wedge\nu_R}(P_s - Q_s, (g(r(s), P_s, P_{\delta(s)}) - g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s))dW_s)\right]\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le s\le t}\left|P_{s\wedge\nu_R} - Q_{s\wedge\nu_R}\right|^2\right] + \frac{\mu_2}{4}\mathbb{E}\left[\int_0^{t\wedge\nu_R}\|g(r(s), P_s, P_{\delta(s)}) - g(\bar{r}(s), \hat{Z}_s, \widetilde{Z}_s)\|_2^2 ds\right]\\
&\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le s\le t}\left|P_{s\wedge\nu_R} - Q_{s\wedge\nu_R}\right|^2\right] + \int_0^{t\wedge\nu_R}(2\mu_2 L^2\mathbb{E}\|P_s - Q_s\|_C^2 + 2\mu_2 L^2\mathbb{E}\|Q_s - \hat{Z}_s\|_C^2\\
&\quad + 2\mu_2 L^2\mathbb{E}\|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 + 2\mu_2 L^2\mathbb{E}\|Q_{\delta(s)} - \widetilde{Z}_s\|_C^2 + \frac{\mu_2}{2}\mathbb{E}|J_6(s)|^2)ds, \qquad (4.34)
\end{aligned}$$

where μ_2 is a positive constant. Combining (4.33) with (4.34), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le t_1}\left|P_{t\wedge\nu_R} - Q_{t\wedge\nu_R}\right|^2\right]\\
&\le \mu_3\int_0^{t_1\wedge\nu_R}\mathbb{E}\left[\sup_{0\le u\le s}|P_u - Q_u|^2\right]ds + \mu_4\int_0^{t_1\wedge\nu_R}\mathbb{E}\|Q_s - \bar{Z}_s\|_C^2 ds\\
&+ \mu_5\int_0^{t_1\wedge\nu_R}\mathbb{E}\|Q_s - \hat{Z}_s\|_C^2 ds + \mu_6\int_0^{t_1\wedge\nu_R}\mathbb{E}\|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 ds\\
&+ \mu_7\int_0^{t_1\wedge\nu_R}\mathbb{E}\|Q_{\delta(s)} - \bar{Z}_{\delta(s)}\|_C^2 ds + 2\int_0^{t\wedge\nu_R}(\mathbb{E}|J_4(s)|^2 + \mathbb{E}|J_5(s)|^2 + \mathbb{E}|J_6(s)|^2)ds\\
&+ \frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le t_1}\left|P_{t\wedge\nu_R} - Q_{t\wedge\nu_R}\right|^2\right] + \int_0^{t_1\wedge\nu_R}(4\mu_2 L^2\mathbb{E}\|P_s - Q_s\|_C^2 + 4\mu_2 L^2\mathbb{E}\|Q_s - \hat{Z}_s\|_C^2\\
&\quad + 4\mu_2 L^2\mathbb{E}\|P_{\delta(s)} - Q_{\delta(s)}\|_C^2 + 4\mu_2 L^2\mathbb{E}\|Q_{\delta(s)} - \widetilde{Z}_s\|_C^2 + \mu_2\mathbb{E}|J_6(s)|^2)ds.
\end{aligned}$$

By Lemma 4.6, 4.8 and 4.9, we have

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{0\le t\le t_1}\left|P_{t\wedge\nu_R} - Q_{t\wedge\nu_R}\right|^2\right]\\
&\le (\mu_3 + \mu_6 + 8\mu_2 L^2)\int_0^{t_1}\mathbb{E}\left[\sup_{0\le u\le s}|P_{u\wedge\nu_R} - Q_{u\wedge\nu_R}|^2\right]ds + \mu_4\int_0^{t_1\wedge\nu_R}\mathbb{E}\|Q_s - \bar{Z}_s\|_C^2 ds\\
&\quad + (\mu_5 + 4\mu_2 L^2)\int_0^{t_1\wedge\nu_R}\mathbb{E}\|Q_s - \hat{Z}_s\|_C^2 ds + (\mu_7 + 4\mu_2 L^2)\int_0^{t_1\wedge\nu_R}\mathbb{E}\|Q_{\delta(s)} - \bar{Z}_{\delta(s)}\|_C^2 ds\\
&\quad + 2(C_6 + C_7 + C_8)\Delta + \mu_2 C_8\Delta\\
&\le (\mu_3 + \mu_6 + 8\mu_2 L^2)\int_0^{t_1}\mathbb{E}\left[\sup_{0\le u\le s}|P_{u\wedge\nu_R} - Q_{u\wedge\nu_R}|^2\right]ds\\
&+ (\mu_4 C_{10}T + (\mu_5 + 4\mu_2 L^2)C_{11}T + (\mu_7 + 4\mu_2 L^2)C_{12}T + 2(C_6 + C_7 + C_8) + \mu_2 C_8)\Delta.
\end{aligned}$$

Applying the Gronwall inequality, we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|P_{t\wedge\nu_R} - Q_{t\wedge\nu_R}\right|^2\right] \le C_{13}\Delta.$$

Thus, the proof is complete. □

Letting $R \to \infty$, we have the following theorem.

Theorem 4.11. *Suppose that the preceding assumptions hold, then*

$$\lim_{\Delta \to 0} \mathbb{E}\Big[\sup_{0 \le t \le T} |P_t - Q_t|^2\Big] = 0.$$

Remark 4.12. *The result is known without time delay in [19], but it is new in the case of variable delay. As we stated in the Introduction section, so far most of the existing strong convergence result for numerical methods requires the coefficients of the SAPEs to be globally Lipschitz continuous (see, e.g, [19, 22, 23]) and little is yet known about the convergence for numerical solution to SAPEs under the local Lipschitz condition. As sequels to this work, we shall discuss the strong convergence result for numerical methods under the local Lipschitz condition in future work.*

5. Numerical examples

In this section, we consider the numerical solution and the strong convergence of some S-DAPEs with Markovian switching by the SSθ method given in (3.4). We use sample average to approximate the expectation. More precisely, we measure the maximum norms errors by

$$\epsilon = \frac{1}{\hat{M}} \sum_{i=1}^{\hat{M}} \max_{0 \le n \le M-1} \left| x^i(t_n) - x_n^i \right|^2,$$

where $\hat{M}$, $x^i(t_n)$, x_n^i denote the number of sample paths, the ith true solution at time t_n and the ith numerical solution at time t_n, respectively. In the following simulations, we set $\hat{M} = 1000$.

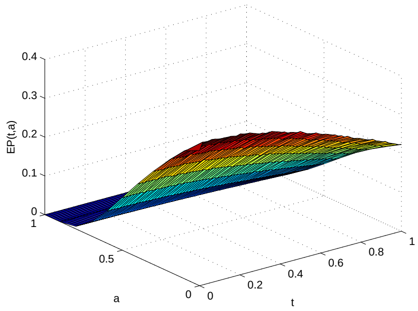
Table 1: The maximum norms errors of SSθ and EM method for solving (5.1)

Δt	$\theta = 0$	$\theta = 0.2$	$\theta = 0.5$	$\theta = 0.8$	$\theta = 1$	EM
2^{-4}	*	5.8779e-002	5.3161e-002	5.2947e-002	5.0462e-002	*
2^{-5}	4.7722e-002	4.9001e-002	4.7601e-002	4.8312e-002	4.7538e-002	5.0948e-002
2^{-6}	4.7309e-002	4.7939e-002	4.7372e-002	4.7252e-002	4.7173e-002	4.9077e-002
2^{-7}	4.7076e-002	4.7545e-002	4.6437e-002	4.7967e-002	4.5905e-002	4.6863e-002

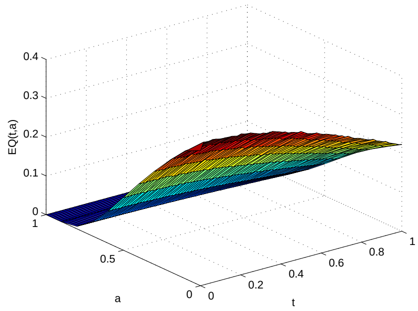
Example 5.1. Consider the following SAPEs without Markovian switching as in [11]

$$\begin{cases} d_t P_t = \left[-\dfrac{\partial P_t}{\partial a} - \dfrac{P_t}{(1-a)^2} - tP_t \right] dt + P_t dW_t, & (t,a) \in (0,1) \times (0,1) \\ P(0,a) = \varphi(a), & a \in (0,1) \\ P(t,0) = \displaystyle\int_0^1 \frac{P(t,a)}{(1-a)^2} da, & t \in (t,a) \in (0,1) \times (0,1) \end{cases} \quad (5.1)$$

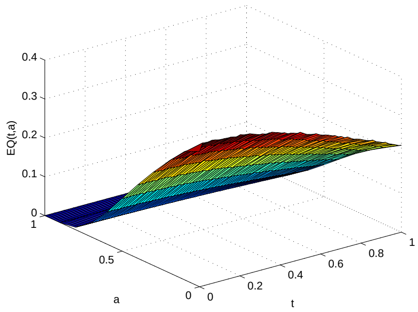
where W_t is a scalar Brownian motion, with $A = 1$, $T = 1$, $\mu(t,a) = \beta(t,a) = \frac{1}{(1-a)^2}$, $f(t,P) = -tP$, $g(t,P) = P$, $\varphi = \exp(\frac{-1}{1-a})$. Clearly, the operators f and g satisfy conditions ($\mathbb{H}$1)-($\mathbb{H}$2), $\mu(t,a)$,



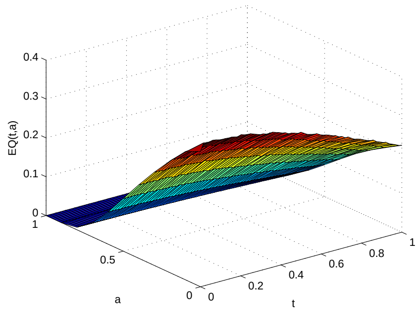
(a) Exact solution without perturbation



(b) EM method



(c) SSθ method with $\theta = 0.5$



(d) SSθ method with $\theta = 1$

Fig. 1. Expectation simulations for (5.1)

$\beta(t, a)$ satisfy condition ($\mathbb{H}3$). By Theorem 4.11, the numerical solution will converge to the exact solution in the sense of mean square. Note that the explicit solution to (5.1) without perturbation is

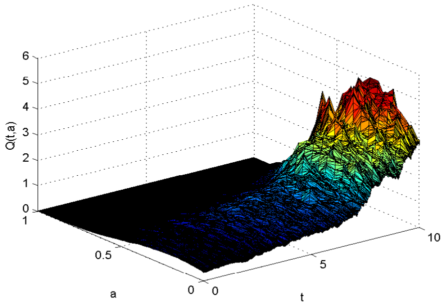
$$\mathbb{E}P(t, a) = \exp(\frac{-1}{1-a} - \frac{t^2}{2}).$$

It is difficult to obtain the true explicit solution to (5.1), so the explicit solution to (5.1) can be replaced by

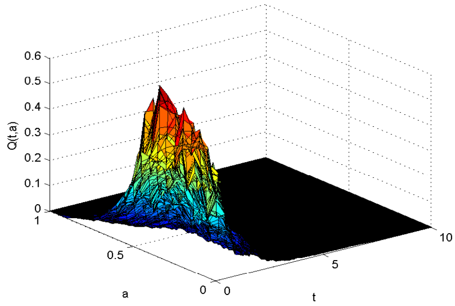
$$\exp(\frac{-1}{1-a} - \frac{t^2}{2})(1 + \Delta W_t)$$

(see e.g.,[11, 22, 24]). Setting step size $\Delta t = 0.005$ and $\Delta a = 0.05$, we simulate the expected value of the exact solution to (5.1) and the numerical solution by EM and SSθ method with $\theta = 0.5$, $\theta = 1$, respectively (see Fig.1), where

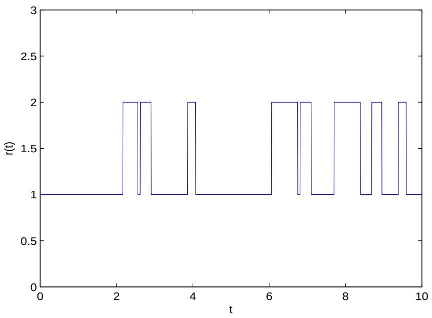
$$\mathbb{E}Q(t, a) \approx \frac{1}{\hat{M}} \sum_{k=1}^{\hat{M}} Q_k(t, a).$$



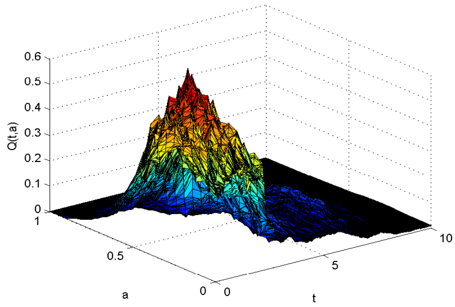
(a) State trajectories of subsystem 1



(b) State trajectories of subsystem 2



(c) State trajectories of Markov chian $r(t)$



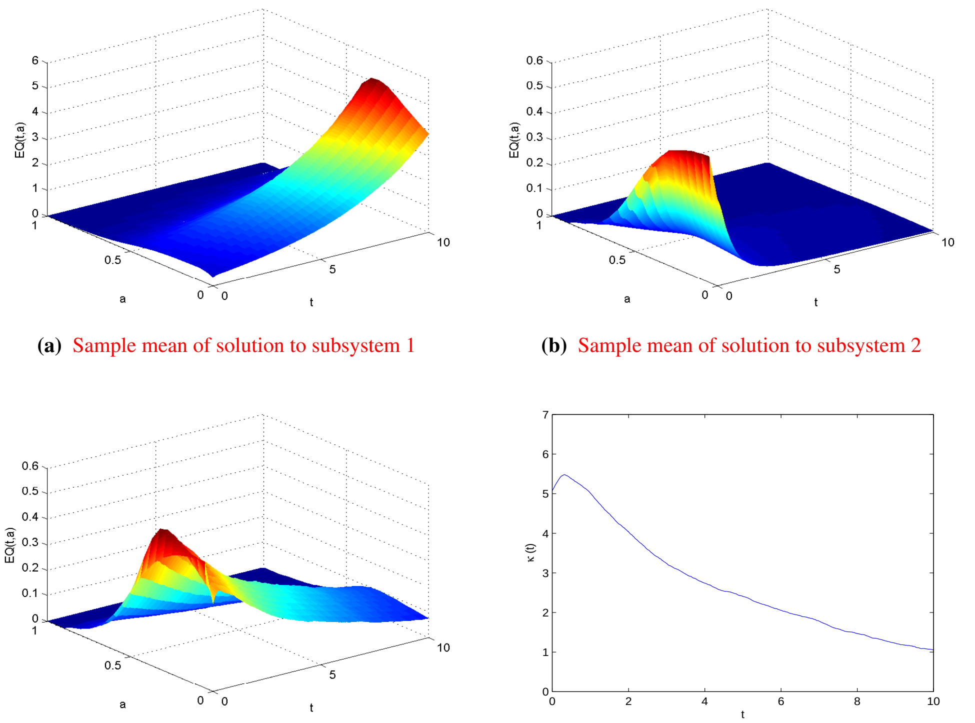
(d) State trajectories of switched system

Fig. 2. Single simulation for (5.2) by SSθ method with $\theta = 1$

The maximum norms errors of SSθ and EM method for solving (5.1) are shown in Table 1 with the same age step size $\Delta a = 0.1$. Numerical experiments show the smaller error of SSθ method in comparison with the explicit EM method.

Example 5.2. Let W_t be a scalar Brownian motion and $r(t)$ be a right-continuous Markov chain taking values in $\mathbb{S} = \{1, 2\}$ with generator $\Gamma = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}$. SDAPEs with Markovian switching (5.2) are considered to be a class of hybrid systems, which consist of two distinct subsystems, denoted by subsystem 1 and subsystem 2. Note that subsystem 1 differs from subsystem 2 only in the drift and diffusion coefficients, namely, f and g.

$$\begin{cases} d_t P_t = \left[-\dfrac{\partial P_t}{\partial a} - \mu(t,a)P_t + f(r(t), P_t, P_{\delta(t)}) \right] dt \\ \qquad\qquad + g(r(t), P_t, P_{\delta(t)}) dW_t, & (t,a) \in (0,3) \times (0,1) \\ P(t,a) = \varphi(t,a), r(0) = 1, & (t,a) \in [-\tau, 0] \times (0,1) \\ P(t,0) = \displaystyle\int_0^1 \beta(t,a) P(t,a) da, & (t,a) \in (0,3) \times (0,1) \end{cases} \tag{5.2}$$



(a) Sample mean of solution to subsystem 1

(b) Sample mean of solution to subsystem 2

(c) Sample mean of solution to switched system

(d) Sample mean of total population for switched system

Fig. 3. Expectation simulations for (5.2) by SSθ method with $\theta = 1$

where

$$\delta(t) = t - \tau, \qquad \tau = 0.1, \tag{5.3}$$

$$f(1, x, y) = 1.5x - 0.2y, \qquad g(1, x, y) = 0.5x, \tag{5.4}$$

$$f(2, x, y) = -0.5x + 0.1y, \qquad g(2, x, y) = -\sin x, \tag{5.5}$$

$$\beta(t, a) = \begin{cases} 2, & \text{if } 0.2 \le a \le 0.6 \\ 0, & \text{otherwise} \end{cases} \tag{5.6}$$

$$\mu(t, a) = \frac{10\exp(10(a - 0.5))}{\exp(10(a - 0.5)) + \exp(-10(a - 0.5))}, \tag{5.7}$$

$$\varphi(t, a) = \frac{0.5}{\exp(10(a - 0.5)) + 1}. \tag{5.8}$$

For switched system (5.2), we take the birth term $\beta(t, a)$ to have the form (5.6) which means that individuals are fecund if they are not too old or too young. The death modulus (5.7) we used corresponds to a situation where mortality is low until around a certain age, at which point mortality increases dramatically (see, e.g., [1, 3]). The initial population distribution (5.8) is an expansive shape, which is typical for fast-growing countries where each birth cohort (a group of people born in the same year or years period) is larger than the previous one (Latin American, Africa) (see, e.g.,[20, 26]). Obviously, the conditions ($\mathbb{H}1$)-($\mathbb{H}3$) are satisfied. Applying Theorem 4.11, we can deduce that the numerical solution will tend to the exact solution in the mean square sense.

We set $\Delta a = 0.05$, $\Delta t = 0.005$ and $\theta = 1$. Fig 2.(c) shows the sample trajectories of Markov chian $r(t)$, which determine the switching rule in the single simulation. Fig 2.(a), (b) and (d) give the state trajectories of the solutions to subsystem 1, subsystem 2 and switched system by SSθ method with $\theta = 1$, respectively. It seems that subsystem 1 is unstable and subsystem 2 is stable, but the switched system (5.2) is stable. We also plot the sample mean of the solutions to the three systems in Fig 3.(a), (b) and (c), respectively, where $\mathbb{E}Q(t, a)$ is defined in Example 5.1. The sample mean of total population density for switched system (5.2) is illustrated in Fig 3.(d), where total population density $\kappa(t)$ at time t is defined by

$$\kappa(t) = \sum_{a=0}^{A} \mathbb{E}Q(t, a).$$

This clearly reveals the population dynamics tendency.

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