Integer-Digit Functions: An Example of Math-Art Integration

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"God made the integers, all else is the work of man."
L. Kronecker

Mathematics and the visual arts mutually reinforce one another [10]. On the one hand, many mathematical objects appear in artistic or decorative works [5, 6]. In particular, mathematical curves and art have a long-standing connection through the application of geometric principles [13, 20]. Simple curves such as the catenary are ubiquitous, for example in the work of the Catalan architect Antoni Gaudí, as well as in ancient [19] and in modern architecture [12]. On the other hand, many mathematical objects display artistic appeal *per se*. The vast list includes, among others: knots [1, 2], mosaics and tiles [3, 21], Fourier series [9], topological tori [15], and fractal curves [4], all of which produce visual patterns of undeniable beauty. Such connections between mathematics and the arts are explored and celebrated annually by the Bridges community formed by mathematicians and artists [8, 18].

This article introduces a new family of curves in the plane defined by functions transforming an integer according to sums of digit-functions. These transforms of an integer N are obtained by multiplying the function value $f(N)$ by the sum of $f(a_i)$, where a_i are the digits of N in a given base b and f is a standard function such as a trigonometric, logarithmic, or exponential one. These curves display a few attributes, such as beauty, symmetry, and resemblance to natural environments, which make them attractive for artistic purposes.

Digit Sum with "Memory"

A nonnegative integer N can be represented in a given base b according to the following expression

$$N = a_1 b^n + a_2 b^{n-1} + a_3 b^{n-2} + \cdots + a_{n-1} b + a_n, \qquad (1)$$

where a_i are nonnegative integers. We can represent a negative integer in a similar way if we multiply by -1 the right-hand-side part of the previous equation. For instance, the number $N = 257$, expressed in the base $b = 10$, is:

$$257 = 2 \cdot 10^2 + 5 \cdot 10 + 7.$$

A function explored in several contexts [14] is the "*digit sum*," or "*sum of digits*" of a given integer, defined by:

$$S_b(N) = \sum_{i=1}^{n} a_i = \sum_{k=0}^{\lfloor \log_b N \rfloor} \frac{1}{b^k} \left(N \bmod b^{k+1} - N \bmod b^k \right). \qquad (2)$$

For instance, for $b = 10$ the digit sum of 257 is $2 + 5 + 7 = 14$. The digit sum forms an integer sequence registered in the Online Encyclopedia of Integer Sequences (OEIS) [16, 17] by the sequence A007953. One characteristic of the digit sum function is that it does not "remember" the integer that gave rise to it. That is, we can find many different integers for which the digit sum is the same. For instance, $S_{10}(59) = S_{10}(77) = S_{10}(95) = S_{10}(149) = \cdots = S_{10}(257) = \cdots = 14$.

https://doi.org/10.1007/s00283-017-9726-x

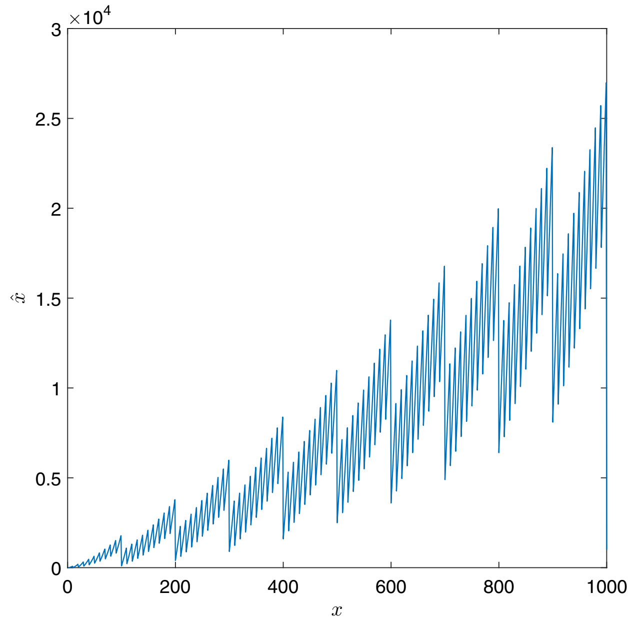

Figure 1. Plot of $\widehat{N}$ versus N for integers less than or equal to 1000.

A simple modification of the digit sum function that "remembers" its origins better can be obtained by multiplying the digit sum of N by N [7]. Hereafter we represent this function by $\widehat{N}$. In the previous example we have $\widehat{257} = 257 \cdot 14 = 3598$. This function is not different for every integer, thus we cannot say that it remembers its origins perfectly, but at least it is not as degenerate as the digit sum. The corresponding integer sequence is given at the OEIS by A117570 [16]. In Figure 1 we illustrate the plot of $\widehat{N}$ versus N for the first 1000 integers.

Integer-Digit Functions

Now we extend this idea to other functions. Let $f(N)$ be a function of the integer N, for example, $\sin(N)$. Then we define the sum of digit-functions $\widehat{f(N)}$ of $f(N)$ by:

$$\widehat{f(N)} = (f(a_1) + f(a_2) + \cdots + f(a_n))f(N). \tag{3}$$

Here, once again, we are using the previous notation; that is, the nonnegative integers a_i are the digits of N in base b. For instance, $\widehat{\sin(257)} = (\sin(2) + \sin(5) + \sin(7)) \cdot \sin(257) \approx -0.2357$. For negative integers $-N$, if the function $f(-N)$ exists, we obtain

$$\widehat{f(-N)} = (f(a_1) + f(a_2) + \cdots + f(a_n))f(-N). \tag{4}$$

As an example, the plot of $\widehat{\sin(N)}$ versus N is given in Figure 2 (left panel), which mildly resembles the original sine function.

Our main interest here is, however, to make a complete transformation of both coordinate axes in terms of our digit-function. That is, we are interested in the plotting of $\widehat{f(N)}$ versus $\widehat{g(N)}$. For instance, when $\widehat{\sin(N)}$ is plotted versus $\widehat{N}$ in Figure 2 (right panel), a totally unexpected pattern emerges. This pattern does not resemble at all the original function and, if rotated by -90 degrees, it resembles a bush.

In the rest of this article we will be interested in the curves we obtain from certain combinations of $\widehat{f(N)}$ and $\widehat{g(N)}$. To represent the curves we use here the following method. For a given even number of points n, the curve we want is obtained for the integers $N \leq \frac{n}{2}$ using a parametric form of the equation,

$$x = \widehat{f(N)}, \tag{5}$$

$$y = \widehat{g(N)}. \tag{6}$$

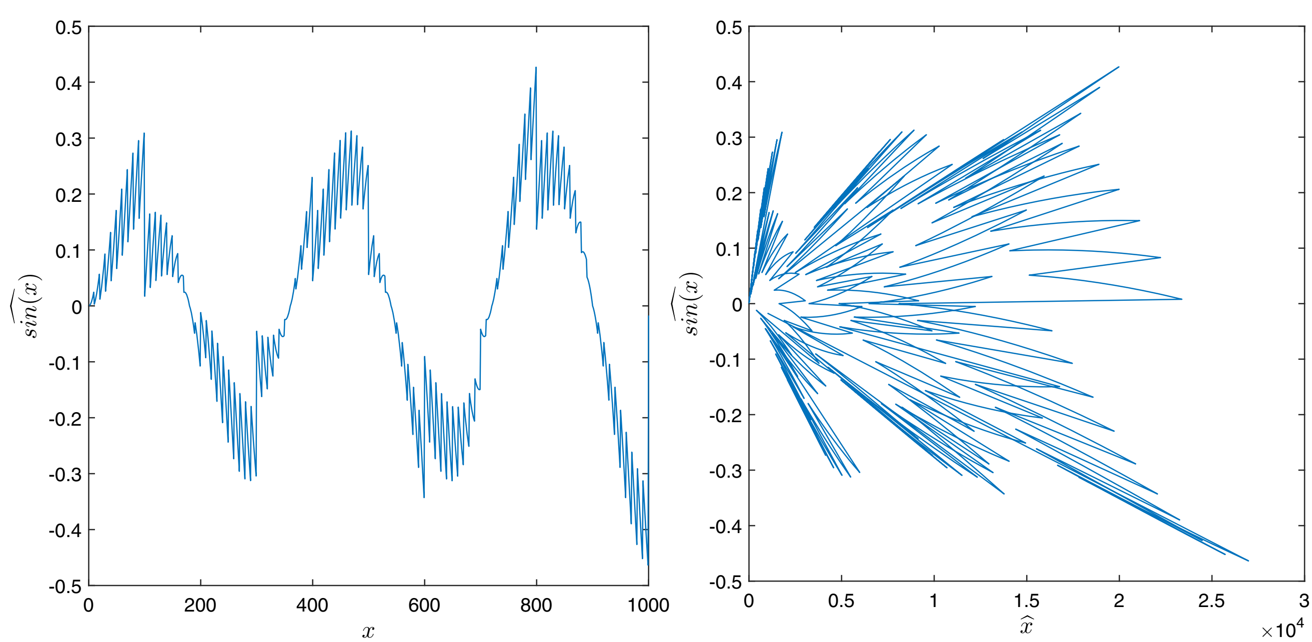

Figure 2. Semideconstruction of the sine function (*left*) and total deconstruction of the same (*right*).

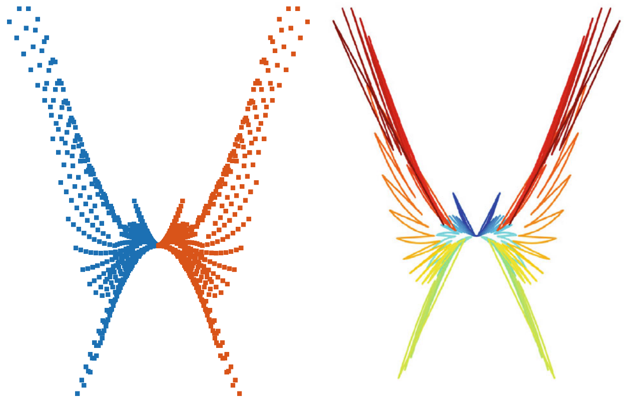

Figure 3. Curve defined by Equations (7)–(8) with $n = 1000$ points. The left panel illustrates the discrete character of the curve. Note also that negative inputs are distinguishable from positive inputs. In the image on the left, red is used for negative values of N and blue for positive ones. The image on the right is obtained by joining the dots with lines to give a more fluid view. In the right panel we used a "Jet" colormap to color the curve.

If the functions $\widehat{f(N)}$ and $\widehat{g(N)}$ are also defined for negative arguments, the corresponding transforms are obtained for $-\frac{n}{2} \leq N$.

All trigonometric functions are obtained in degrees. For instance, let

$$x = \widehat{N}, \tag{7}$$

$$y = \widehat{\sin N} - \widehat{N} \cdot \widehat{\cos N}. \tag{8}$$

Then for $n = 1000$, we obtain the curve illustrated in the left panel of Figure 3. Notice that the curves are discrete because we only evaluate our functions at integer values. However, we can join the points to create a more fluid form; see the right panel of Figure 3, in which we used the `colormap` function available in Matlab®.[1]

Artistic Creation Using Integer-Digit Functions

Now we play with this construction and transform some well-known curves, such as sine, involute of a circle, astroid, and others. Let us start by the simple sine function in which we make the transformation

$$x = \widehat{N}, \tag{9}$$

$$y = \widehat{\sin N}. \tag{10}$$

This function is illustrated in Figure 4 for $n = 2000$ points. The artistic representation of the curve given on the right-hand-side of the figure clearly resembles a butterfly and can serve as the basis for more artistic inspirations.

We now explore the transformation of both coordinate axes for the involute of a circle. This function is given by the following parametric equation

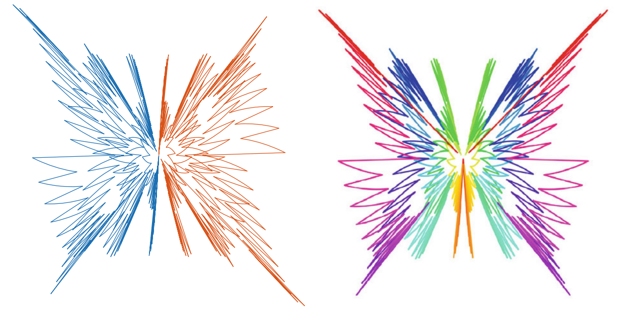

Figure 4. Sine after the integer-digit transform using $n = 2000$ points (*left*) and another view of it (*right*) using colormap "Jet" for artistic impression. To prepare the second image, we simply rotate the negative part of the curve to obtain a mirror symmetry. On the left-hand side we use red for negative values of N and blue for positive ones.

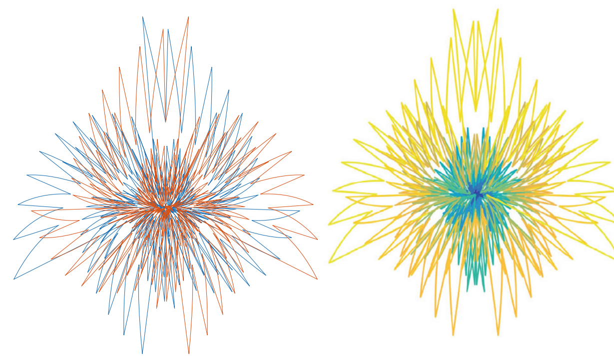

Figure 5. Involute of the circle after the integer-digit transform using $n = 4000$ points (*left*) and a coloring of it using "Parula" color map (*right*).

$$x = \widehat{\sin N} - \widehat{N} \cdot \widehat{\cos N}, \tag{11}$$

$$y = \widehat{\cos N} + \widehat{N} \cdot \widehat{\sin N}. \tag{12}$$

and the corresponding curve is illustrated in Figure 5.

The analogue of the astroid using the integer-digit transform is given by

$$x = \left(\widehat{\cos N}\right)^3, \tag{13}$$

$$y = \left(\widehat{\sin N}\right)^3, \tag{14}$$

and the corresponding curve for $n = 10,000$ points is illustrated in Figure 6.

The cardioid curve can be transformed using the following parametric equations

$$x = 2\widehat{\cos N} + \widehat{\cos(2N)}, \tag{15}$$

$$y = 2\widehat{\sin N} + 3\widehat{\sin(2N)}. \tag{16}$$

[1]All images in this article were created in Matlab®. Any interested reader can obtain the code from the author via e-mail.

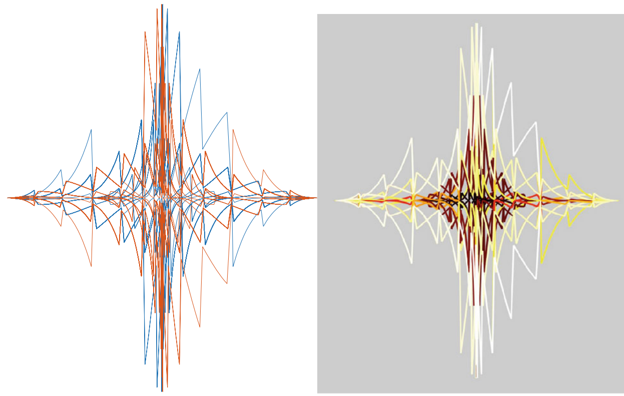

Figure 6. Astroid after the integer-digit transform using $n = 10,000$ points (*left*) and a coloring of it using the colormap "Hot" and a grey background (*right*).

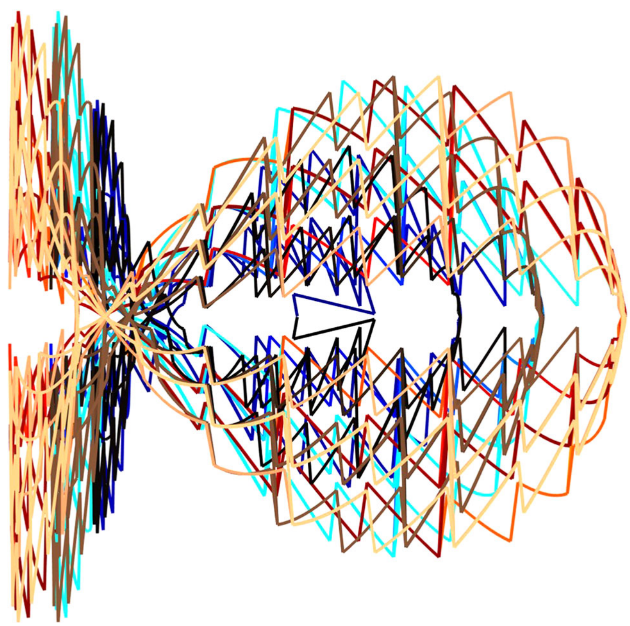

Figure 7. Cardioid after the integer-digit transform using $n = 50,000$ points and using a combination of "Jet" and "Cooper" colormaps.

Figure 8. Composition of integer-digit transform (*left*) and another view of the same composition (*right*).

In Figure 7 we plot the curve corresponding to the transformed cardioid, which resembles a fish.

Although "*beauty is in the eye of the beholder*" there is no doubt that these plots have certain artistic attributes that overall make them in some way unexpectedly beautiful. In the next section we explore some further possibilities for artistic creation.

Artistic Compositions

Let us start by seeing what we get when we compose functions. We combine the function:

$$x = \widehat{\sin N} - \widehat{N} \cdot \widehat{\cos N}, \tag{17}$$

$$y = \widehat{N}, \tag{18}$$

with

$$x = 3000\widehat{\sin N} - 1000, \tag{19}$$

$$y = \widehat{N}, \tag{20}$$

and

$$x = -3000\widehat{\sin N} + 1000, \tag{21}$$

$$y = \widehat{N}, \tag{22}$$

to make the composition illustrated in the Figure 8.

Next we introduce logarithmic functions into the mix. The function

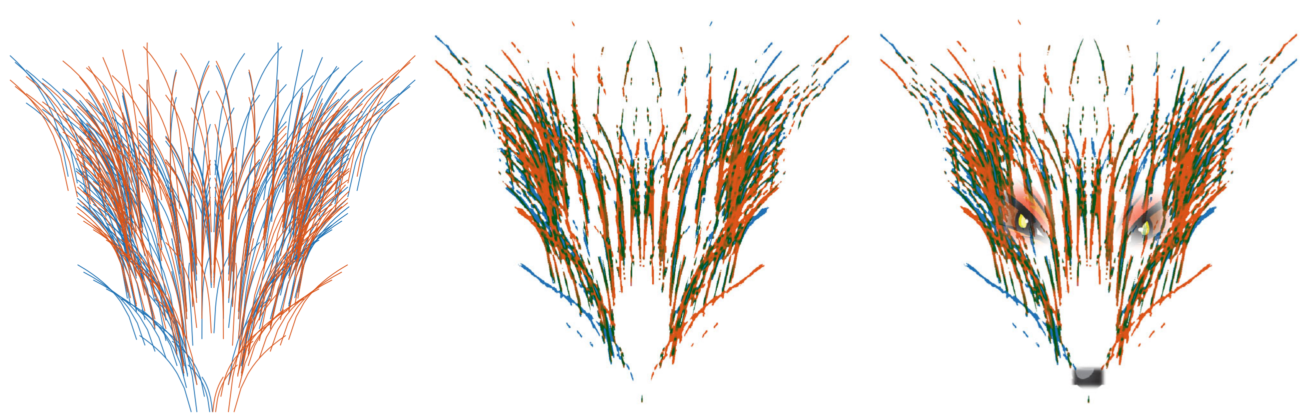

Figure 9. Curve obtained by using logarithmic integer-digit transform for $n = 5000$ points (*left*) and two modified views of it. Red is used for negative values of t and blue for positive ones.

Figure 10. Combination of integer-digit curves to produce a landscape and a seascape based on the examples here.

$$x = \widehat{\sin N}, \quad (23)$$

$$y = \widehat{\ln N}, \quad (24)$$

is illustrated in the left-hand-side of Figure 9. The second view was created with some of the filters available in PowerPoint. Here, as the log function is not defined for negative numbers we plot $-\widehat{\sin N}$ versus $\widehat{\ln N}$, and $\widehat{\sin N}$ versus $\widehat{\ln N}$ to produce a symmetric image. A very simple modification in the last image produces what in the eyes of the beholder resembles the face of a fox.

Finally in Figure 10, we present two naive combinations of some of the previously obtained integer-digit transforms to produce a landscape and a seascape. In the top panel we show the combination of functions used for producing Figures 4 and 8. In the bottom panel we use the function

$$x = \widehat{N}, \quad (25)$$

$$y = \widehat{\ln N}, \quad (26)$$

to produce the plants and then combine it with the function used to produce Figure 7 to make the final composition.

The artistic compositions presented here may seem simple and naive. But is it not surprising that they were entirely produced by a mathematical procedure based only on integers– with just a little extra imagination?

ACKNOWLEDGMENTS

The author thanks artist Puri Pereira for useful discussions. He also thanks the Royal Society of London for a Wolfson Research Merit Award. Finally he thanks Gizem Karaali for her editorial assistance.

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