Spatial asymptotics and strong comparison principle for some fractional stochastic heat equations.

Mohammud Foondun  
University of Strathclyde

Eulalia Nualart  
Pompeu Fabra University

Abstract

Consider the following stochastic heat equation,

\[ \frac{\partial u_t(x)}{\partial t} = -\nu(-\Delta)^{\alpha/2} u_t(x) + \sigma(u_t(x)) \dot{F}(t, x), \quad t > 0, \ x \in \mathbb{R}^d. \]

Here \(-\nu(-\Delta)^{\alpha/2}\) is the fractional Laplacian with \(\nu > 0\) and \(\alpha \in (0, 2]\), \(\sigma : \mathbb{R} \to \mathbb{R}\) is a globally Lipschitz function, and \(\dot{F}(t, x)\) is a Gaussian noise which is white in time and colored in space. Under some suitable additional conditions, we prove a strong comparison theorem and explore the effect of the initial data on the spatial asymptotic properties of the solution. This constitutes an important extension over a series of works most notably [8], [9], [5] and [4].

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1 Introduction and main results

Consider the following stochastic heat equation,

\[ \frac{\partial u_t(x)}{\partial t} = -\nu(-\Delta)^{\alpha/2} u_t(x) + \sigma(u_t(x)) \dot{F}(t, x), \quad t > 0, \ x \in \mathbb{R}^d, \]

where \(-\nu(-\Delta)^{\alpha/2}\) is the fractional Laplacian, that is, the infinitesimal generator of a symmetric \(\alpha\)-stable process with density \(p_t(x)\), where \(\alpha \in (0, 2]\), and \(\nu > 0\) is a viscosity constant. The noise \(\dot{F}(t, x)\) is white in time and colored in space satisfying

\[ \text{Cov}(\dot{F}(t, x), \dot{F}(s, y)) = \delta_0(t - s) f(x - y), \]

where \(f\) is the spatial correlation function which we take to be the Riesz kernel

\[ f(x) := \frac{1}{|x|^\beta}, \quad 0 < \beta < d. \]
The function $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz continuous function with $\sigma(0) = 0$, that is, there exists a constant $L_\sigma > 0$ such that

$$|\sigma(x)| \leq L_\sigma |x|, \quad \text{for all } x \in \mathbb{R}^d.$$ 

The initial condition $u_0$ is always going to be a nonnegative function in $\mathbb{R}^d$ such that

$$\bar{u}_0 := \sup_{x \in \mathbb{R}^d} u_0(x) < \infty.$$ 

Following Walsh [19], if one further assume that

$$\beta < \min(\alpha, d),$$ 

then (1.1) has a unique mild solution $\{u_t(x), t \geq 0, x \in \mathbb{R}^d\}$ which is adapted, jointly measurable and satisfies

$$u_t(x) = (p_t \ast u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u_s(y)) F(ds \, dy), \quad (1.2)$$

where

$$(p_t \ast u_0)(x) = \int_{\mathbb{R}^d} p_t(x-y)u_0(y)dy,$$

and

$$\sup_{x \in \mathbb{R}^d, t \in [0, T]} \mathbb{E}|u_t(x)|^k < \infty \quad \text{for all } k \geq 2 \quad \text{and} \quad T < \infty.$$ 

For more information about existence-uniqueness considerations, consult [19], [10] and [15]. This paper is motivated by two important results proved recently in [12]. The first one is the following weak comparison principle.

**Theorem 1.1.** [12] Suppose that $u$ and $v$ are two solutions to (1.2) with initial conditions $u_0$ and $v_0$ respectively such that $u_0 \leq v_0$. Then

$$\mathbb{P}(u_t(x) \leq v_t(x) \quad \text{for all } x \in \mathbb{R}^d, \quad t \geq 0) = 1.$$ 

Theorem 1.1 ensures nonnegativity of the solution, since the initial condition is assumed to be nonnegative. For the special case $\sigma(x) = x$ (known as the Parabolic Anderson model), this fact can be deduced from the Feynman-Kac representation of the solution. However, for the general non-linear case, this property for the solution to (1.2) was unknown until the work of [12].

The first aim of this paper is to use Theorem 1.1 in order to show the following strong comparison principle.
Theorem 1.2. Suppose that $u$ and $v$ are two solutions to (1.2) with initial conditions $u_0$ and $v_0$ respectively such that $u_0 < v_0$. Assume $\alpha \geq 1$. Then

$$P(u_t(x) < v_t(x) \text{ for all } x \in \mathbb{R}^d, \quad t \geq 0) = 1.$$  

The (strong) comparison principle for equation (1.1) with space-time white noise and $\alpha = 2$ is the well-known Mueller’s comparison principle (see [18]). Recently, several extensions have been developed. In [5] the authors extend Mueller’s result when the initial data is more general and there is a more general fractional differential operator than the fractional Laplacian. In [3] the authors consider the non-linear heat equation in $\mathbb{R}^d$ with a general spatial covariance and measured-valued initial data. The proof of our strong comparison principle uses the same strategy as in the papers mentioned above. But the presence of the fractional Laplacian and the colored noise makes it that we have to work a bit harder to prove our result. For the sake of conciseness, we only consider the Riesz kernel spatial covariance, but we believe that our method could be extended to general spatial covariances as in [3].

As another consequence of the weak comparison principle (Theorem 1.1), we show the next quantitative result on the strict positivity of the solution, which is an extension of [7, Theorem 5.1] (space-time white noise and $\alpha = 2$). See also [5, Theorem 1.4] and [3, Theorem 1.6]. Note that $\alpha$ is not required to be bigger than 1.

Theorem 1.3. Let $T > 0$ and $K \subset \mathbb{R}^d$ be a compact set contained in the support of the initial condition $u_0$. Then, there exist constants $c_1$ and $c_2$ depending on $T$ and $K$ such that for all $\epsilon > 0$, we have

$$P\left( \inf_{t \in [0, T]} \inf_{x \in K} u_t(x) < \epsilon \right) \leq c_2 \exp \left( -c_1 \{ \log \epsilon \log |\log \epsilon| \}^{\frac{2\alpha-\beta}{\alpha}} \right).$$

Let us now state the second motivation of this paper, which is the following moment comparison theorem.

Theorem 1.4. [12] Let $u$ and $v$ two solutions to (1.2), the first one with $\sigma$, the other with another globally Lipschitz continuous function $\tilde{\sigma}$ such that $\tilde{\sigma}(0) = \sigma(0) = 0$ and $\sigma(x) \geq \tilde{\sigma}(x) \geq 0$ for all $x \in \mathbb{R}_+$. Then for any $k \in \mathbb{N}$, $x \in \mathbb{R}^d$, and $t \geq 0$,

$$\mathbb{E}[u_t(x)^k] \geq \mathbb{E}[v_t(x)^k].$$

An important consequence of this result are the following sharp estimates on the moments of the solution to (1.2), when the initial condition is bounded below and under the additional assumption that there exists a constant $l_\sigma > 0$ such that

$$\sigma(x) \geq l_\sigma |x|, \quad \text{for all } x \in \mathbb{R}^d. \quad (1.3)$$

This was unknown until the work of [12].
**Theorem 1.5.** Let \( u \) be the solution to (1.2). Assume (1.3) and

\[
0 < u_0 := \inf_{x \in \mathbb{R}^d} u_0(x). \tag{1.4}
\]

Then there exists a positive constant \( A \) such that for all \( x \in \mathbb{R}^d \), \( t > 0 \), and \( k \geq 2 \),

\[
\frac{u^k}{Ak} \exp \left( \frac{1}{A} k^{\frac{2a-\beta}{a-\beta}} t^{\frac{\beta}{a-\beta}} \right) \leq E|u_t(x)|^k \leq A^k u_0^k \exp \left( Ak^{\frac{2a-\beta}{a-\beta}} t^{\frac{\beta}{a-\beta}} \right).
\]

For the Parabolic Anderson model, the above is given by [16, Lemma 4.1]. The scaling property of the heat kernel gives the dependence of the bounds on the parameter \( \nu \). An immediate consequence of Theorem 1.5 is that the solution to (1.1) is fully intermittent meaning that for all \( k \geq 2 \), the function

\[
k \to \frac{1}{k} \gamma(k) := \frac{1}{k} \lim_{t \to \infty} \log \sup_{x \in B(0,R)} u_t(x) \]

is strictly increasing.

Intuitively, this means that the solution develops many high peaks distributed over small \( x \)-intervals when \( t \) is large (see [11] and the references therein). The fact that the solution to (1.1) is intermittent was already known (see e.g. [13] and the references therein) but was shown by showing

\[
\gamma(2) > 0 \quad \text{and} \quad \gamma(k) < \infty, \text{ for all } k \geq 2.
\]

The previous results concern the moments of the solution to (1.1), but much less is known about the almost sure asymptotic behaviour of the solution, which is crucial to understand better its chaotic behaviour. This brings us to the second purpose of this paper is to explore how the almost surely spatial asymptotic behaviour of the solution to (1.1) depends on the initial function \( u_0 \). We start with the case that \( u_0 \) is bounded below as in Theorem 1.5. A first observation is that, since \( u_0 \) is also bounded above, then we can easily see that

\[
E u_t(x) \leq c,
\]

where \( c \) is the upper bound of \( u_0 \). Since \( u_0 \) is bounded below, it is not trivial to say more about \( \lim \inf_{|x| \to \infty} u_t(x) \) other than it is almost surely bounded above. This is in sharp contrast with the lim sup behaviour of the solution described by the next theorem.

**Theorem 1.6.** Let \( u \) be the unique solution to (1.2), and assume that (1.3) and (1.4) hold. Then there exist positive constants \( c_1, c_2 \) such that for every \( t > 0 \),

\[
c_1 \frac{t^{(\alpha-\beta)/(2a-\beta)}}{t^{\beta/(2a-\beta)}} \leq \lim_{R \to \infty} \frac{\log \sup_{x \in B(0,R)} u_t(x)}{(\log R)^{\alpha/(2a-\beta)}} \leq c_2 \frac{t^{(\alpha-\beta)/(2a-\beta)}}{t^{\beta/(2a-\beta)}} \quad \text{a.s.}
\]
This theorem is a major improvement of [8, Theorem 1.3] (space-time white noise case) and [9, Theorem 2.6] (Riesz kernel spatial covariance). See also [6] for exact spatial asymptotics when then noise is fractional in time and correlated in space. All these papers deal with the Parabolic Anderson model and the usual Laplacian ($\alpha = 2$). Moreover, in [8, 9] the dependence in time of the bounds is not explicit. The case $\sigma(x) = x$, fractional Laplacian and Riesz kernel spatial covariance is considered in the preprint [16, Theorem 1.2], without the dependence on $\nu$ and constant initial data. Obtaining the exact dependence on the viscosity constant $\nu$ is important to understand in which universality class the equation can be associated to. See [8, Remark 1.5].

A key ingredient of the proof of Theorem 1.6 are the moment bounds of Theorem 1.5, that will allow to obtain some tail estimates for the solution. Let us now consider the next observation where $u_0$ is not bounded below.

**Remark 1.7.** If $u_0(x) := 1_{B(0,1)}(x)$, then one can show that for $x \in B(0, R)^c$ and $R$ large enough, we have

$$E u_t(x) = (p_t * u_0)(x) \leq \frac{ct}{R^{d+\alpha}}.$$  

Then, a Borel-Cantelli argument shows that almost surely

$$\lim_{|R| \to \infty} \inf_{B(0, R^c)} u_t(x) = 0.$$

This motivated our next result.

The above remark can be seen as a motivation for us to drop the assumption that the initial function is bounded below. We have the following trichotomy result, that studies the amount of decay that the initial conditions needs to ensure that the solution is a bounded function a.s.

**Theorem 1.8.** Let $u$ be the unique solution to (1.2). Assume (1.3) and that $u_0(x)$ is a radial function satisfying

$$\lim_{x \to \infty} u_0(x) = 0 \quad \text{and} \quad u_0(x) \leq u_0(y) \quad \text{whenever} \quad x \geq y.$$

Set

$$\Lambda := \lim_{|x| \to \infty} \frac{|\log u_0(x)|}{(\log |x|)^{\alpha/(2\alpha-\beta)}}.$$

Then, if $0 < \Lambda < \infty$, there exists a random variable $T$ such that

$$P \left( \sup_{x \in \mathbb{R}^d} u_t(x) < \infty, \quad \forall t < T \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} u_t(x) = \infty, \quad \forall t > T \right) = 1.$$
Moreover, if $\Lambda = \infty$, then
\[
P \left( \sup_{x \in \mathbb{R}^d} u_t(x) < \infty, \quad \forall t > 0 \right) = 1.
\]

Finally, if $\Lambda = 0$, then
\[
P \left( \sup_{x \in \mathbb{R}^d} u_t(x) = \infty, \quad \forall t > 0 \right) = 1.
\]

This result is a major extension of [4, Theorem 1.1], where the case $\alpha = 2$ and space-time white noise is considered. The proof of their result is based on the technical Lemma [4, Lemma 2.3] which follows the ideas of [2]. Here, we improve the method of the proof, and our insensitivity theorem is based on a Gronwall’s type result (see Proposition 2.5 below). The latter result is one of the technical innovations of this paper. Theorem 1.8 shows precisely the effect of the fractional Laplacian and a smoother noise. In fact we can make the following observation.

**Remark 1.9.** We assume $\beta = 1$. Observe that for $\alpha < 2$, $\frac{2}{3} < \frac{\alpha}{2\alpha - 1}$. Choose $\epsilon$ such that $\frac{2}{3} < \epsilon < \frac{\alpha}{2\alpha - 1}$, and $u_0$ such that $|\log u_0(x)| \sim (\log |x|)^\epsilon$ as $|x| \to \infty$. Then $\Lambda = \infty$ if $\alpha = 2$ but $\Lambda = 0$ if $\alpha < 2$. Thus, for the same initial data and noise, the solution is bounded for the usual Laplacian but unbounded for the fractional Laplacian.

Observe that when $u_0$ has compact support corresponds to the case where $\Lambda = \infty$, and Theorem 1.8 shows that the solution is bounded for all times a.s.

We now give a plan of the article. In Section 2 we give some preliminary results needed throughout the paper. Section 3 is devoted to an approximation result needed for the proof of Theorems 1.6 and 1.8. These theorems are proved in Section 4. Finally, Section 5 gives the proof of Theorems 1.2 and 1.3.

## 2 Preliminary results

Let $X_t$ be the symmetric $\alpha$-stable process associated with the fractional Laplacian $-\nu(-\Delta)^{\alpha/2}$ and let $p_t(x)$ denote its heat kernel. We will frequently use the following properties.

- **Scaling property.** For any positive constant $a$, we have
  \[
p_t(x) = a^d p_{a^{-\alpha} t}(ax), \quad \text{for all } x \in \mathbb{R}^d, t > 0.
  \]

This property follows from
\[
p_t(x) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot z} e^{-t|z|^\alpha} \, dz.
\]
Heat kernel estimates (see [17] and references therein): For $0 < \alpha < 2$, there exist positive constants $c_1$ and $c_2$ such that for all $x \in \mathbb{R}^d$ and $t > 0$,

$$c_1 \left( \frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right) \leq p_t(x) \leq c_2 \left( \frac{1}{t^{d/\alpha}} \wedge \frac{t}{|x|^{d+\alpha}} \right).$$

**Remark 2.1.** The proofs of Theorems 1.6 and 1.8 will only use the upper bound

$$p_t(x) \leq c_2 \frac{t}{|x|^{d+\alpha}}, \quad \text{for sufficiently large } |x|,$$

while the proof of Theorems 1.2 and 1.3 will only use the lower bound

$$p_t(x) \geq c_1 \frac{1}{t^{d/\alpha}}, \quad \text{for sufficiently small } |x|.$$

Both are also valid for $\alpha = 2$.

The next result provides some estimates that involve the above heat kernel and the correlation function $f$. These estimates will be useful for the proof of our ‘insensitivity’ result; see Theorem 4.3.

**Lemma 2.2.** There exist positive constants $c_1, c_2$ and $c_3$ such that for all $t > 0$, $x \in \mathbb{R}^d$, and $R > 0$, we have

1. $\int_{B(x,R)^c \times B(x,R)^c} p_t(x-y)p_t(x-w)f(y-w)\,dy\,dw \leq c_1 \frac{t^2}{R^{2\alpha+\beta}}$, (2.3)
2. $\int_{B(x,R)^c \times B(x,R)} p_t(x-y)p_t(x-w)f(y-w)\,dy\,dw \leq c_2 \frac{t^{1-\beta/\alpha}}{R^\alpha}$, (2.4)
3. $\int_{\mathbb{R}^d \times \mathbb{R}^d} p_t(x-y)p_t(x-w)f(y-w)\,dy\,dw \leq c_3 t^{-\beta/\alpha}$. (2.5)

**Proof.** We start with (2.3).

$$\int_{B(x,R)^c \times B(x,R)^c} p_t(x-y)p_t(x-w)f(y-w)\,dy\,dw 
\leq \int_{B(0,R)^c \times B(0,R)^c} p_t(y)p_t(w)f(y-w)\,dy\,dw.$$ 

From (2.1), the above quantity is bounded by a constant times

$$\frac{t^2}{R^{2\alpha+\beta}} \int_{B(0,1)^c \times B(0,1)^c} \frac{1}{|y|^{d+\alpha}|w|^{d+\alpha}} \cdot \frac{1}{|y-w|^{\beta}}\,dw\,dy.$$
The above integral is finite so the proof of (2.3) is complete. For (2.4), we write
\[
\int_{B(x, R) \times B(x, R)} p_t(x - y)p_t(x - w)f(y - w) \, dy \, dw \\
\leq \int_{B(0, R) \times \mathbb{R}^d} p_t(y)p_t(w)f(y - w) \, dy \, dw.
\]
By the scaling property,
\[
\int_{\mathbb{R}^d} p_t(w)f(y - w) \, dw = \mathbb{E}^y |X_t|^{-\beta} \leq t^{-\beta/\alpha} \mathbb{E}^0 |X_1|^{-\beta}.
\]
Finally, proceeding as before, we get
\[
\int_{B(0, R)^c} p_t(y) \, dy \leq c \frac{t}{R^{\alpha}}.
\]
Combining the above estimates, we obtain (2.4). For (2.5), it suffices to use the semigroup property
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} p_t(x - y)p_t(x - w)f(y - w) \, dy \, dw = \int_{\mathbb{R}^d} p_{2t}(w)f(w) \, dw,
\]
and using the scaling property as before we obtain the desired bound.

We now return to \( u_t(x) \) the solution to (1.2). Our next property can be read from [1]. For any \( k \geq 2 \), there exists a positive constant \( c := c(k) \) such that for all \( s, t > 0 \), and \( x, y \in \mathbb{R}^d \)
\[
\mathbb{E} |u_s(x) - u_t(y)|^k \leq c \left( |x - y|^\eta k + |s - t|^{\tilde{\eta} k} \right),
\]
where \( \eta = \frac{\alpha - \beta}{2} \wedge \frac{1}{2} \) and \( \tilde{\eta} = \frac{\alpha - \beta}{2\alpha} \). The above together with the upper moment bound of Theorem 1.5 has the following consequence.

**Proposition 2.3.** \( u_t(x) \) has a continuous version, that is, for any \( k \geq 2 \), there exist positive constants \( c_1, c_2 := c_2(k) \) such that
\[
\mathbb{E} \left[ \sup_{x \neq y, s \neq t} \frac{|u_s(x) - u_t(y)|^k}{|x - y|^{\eta k} + |s - t|^{\tilde{\eta} k}} \right] \leq c_2 e^{c_1 k^{(2\alpha - \beta)/(\alpha - \beta)}/(\alpha - \beta)v^{-\beta/(\alpha - \beta)}/t}.
\]

**Proof.** The proof is very similar to Theorem 4.3 of [10] and is therefore omitted.

We also have the following property.
Lemma 2.4. Fix \( x \in \mathbb{R}^d \), then the solution \( u_t(x) \) satisfies the strong Markov Property.

Proof. We omit the proof since it is very similar to [18, Lemma 3.3].

As mentioned earlier, a key ingredient of the proof of Theorem 1.8 is an insensitivity theorem; see Theorem 4.3. Its proof hinges on the following proposition which is one of the main technical innovations of this paper. The proof follows that of [14, Lemma 7.1.1].

Proposition 2.5. Suppose that for \( R > 0 \), the function \( f_R(\cdot) \) is a non-negative non-decreasing locally integrable function on \([0, T]\) satisfying the following

\[
f_R(t) \leq A_R(t) + B \int_0^t \frac{f_{2R}(s)}{(t-s)^\gamma} \, ds,
\]

where \( A_R(\cdot) \) is also a locally integrable non-decreasing function \([0, T]\), \( B \) is a positive constant and \( \gamma < 1 \). If

\[
\sup_{n \geq 1} f_{2^n R}(t) < \infty \quad \text{and} \quad A_{(n+1)R}(t) \leq A_n R(t) \quad \text{for} \quad n \geq 1,
\]

then there exist positive constants \( c_1, c_2 \) such that for all \( t \in [0, T] \),

\[
f_R(t) \leq c_2 A_R(t) e^{c_1 t}.
\]

Proof. We begin by defining

\[
L \phi(t) := B \int_0^t \frac{\phi(s)}{(t-s)^\gamma} \, ds
\]

and \( L^{(n)} \phi(t) := L(L^{(n-1)} \phi(t)) \) for \( n \geq 1 \). The inequality stated in the statement of the proposition can therefore be written as

\[
f_R(t) \leq A_R(t) + L f_{2R}(t).
\]

Iterating the above and using the facts that \( A_R(s) \) is non-decreasing in \( s \) and satisfies \( A_{nR}(s) \leq A_{(n-1)R}(s) \), we have

\[
f_R(t) \leq A_R(t) \sum_{k=0}^{n-1} L^{(k)} 1(t) + R_n f_R(t),
\]

where \( 1(t) := 1 \). \( R_n f_R(t) \) is the remainder term given by

\[
R_n f_R(t) := \frac{1}{\Gamma(n(1-\gamma))} \int_0^t [B \Gamma((1-\gamma))]^n (t-s)^{n(1-\gamma)-1} f_{2^n R}(s) \, ds.
\]
Since $\sup_{n \geq 1} f_{2^n R}(t) < \infty$, we have that as $n \to \infty$, $R_n f_{R}(t) \to 0$. Some computations show that

$$\sum_{k=0}^{\infty} L^{(k)}(t) = \int_{0}^{t} \sum_{k=0}^{\infty} \frac{(B \Gamma(1 - \gamma))^{k} (t - s)^{k(1-\gamma) - 1}}{\Gamma(k(1 - \gamma))} \, ds \leq c_{2} e^{c_{1} t},$$

where $c_{1} := (B \Gamma(1 - \gamma))^{1/(1-\gamma)}$. Putting these estimates together, we obtain the desired result. \hfill \Box

3 An approximation result

Theorems 1.6 and 1.8 are almost sure limit theorems and rely on some Borel-Cantelli type arguments. To be able to carry out the proof, we will need to find an appropriate independent sequence of random variables and it is a priori not clear how to find such a sequence. We follow [8] and [9] where this issue was successfully resolved.

Let $n \geq 1$ and consider the following approximation $F_{n}(x)$ of the measure $F$ appearing in (1.2). Recall that the covariance of $\dot{F}$ is given by $f(x) = \frac{1}{|x|^{1+\beta}}$, $x \in \mathbf{R}^{d}$. This can be written as $f = h \ast \tilde{h}$, where $h(x) = \frac{1}{|x|^{1+\beta}}$ and $\tilde{h}(x) := h(-x)$. Define $h_{n}(x) := h(x)Q_{n}(x)$ and $f_{n}(x) = (h - h_{n}) \ast (\tilde{h} - \tilde{h}_{n})$, where

$$Q_{n}(x) = \prod_{j=1}^{d} \left( 1 - \frac{|x_{j}|}{n} \right)^{+}.$$

We take $\dot{F}_{n}(x)$ to be the noise satisfying

$$\text{Cov}(\dot{F}_{n}(t, x), \dot{F}_{n}(s, y)) = \delta_{0}(t - s)g_{n}(x - y),$$

where $g_{n} = h_{n} \ast \tilde{h}_{n}$.

By an argument similar to that of the proof of [9, Lemma 9.3], we have that for any $\gamma \in (0, \beta \wedge 1)$ there exists a positive constant $c$ such that for all $s > 0$ and $n \geq 1$,

$$(p_{s} \ast f_{n})(0) \leq c \frac{1}{n^{\gamma}} \frac{1}{s^{(\beta-\gamma)/\alpha}}.$$
As a consequence, using the semigroup property, we obtain that

\[
\int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(x-y)p_{t-s}(x-z)f_n(y-z)\,dy\,dz\,ds = \int_0^t \int_{\mathbb{R}^d} p_{2(t-s)}(z)f_n(z)\,dz\,ds = \int_0^t (p_{2r} * f_n)(0)\,dr \leq ct^{1-\frac{d-\alpha}{\alpha}} \frac{1}{n^\gamma}.
\]

(3.1)

Next, consider the following integral equation,

\[
U^{(n)}_t(x) = (p_t * u_0)(x) + \int_0^t \int_{B(x,(nt)^{1/\alpha})} p_{t-s}(x-y)\sigma(U^{(n)}_s(y))F^{(n)}(ds\,dy).
\]

(3.2)

The unique solution to this integral equation can be found via a standard fixed point argument. Fix \( n \geq 1 \). Set \( U^{(n,0)}_t := u_0 \) and for each \( j \geq 1 \), the \( j \)th Picard iteration is given by

\[
U^{(n,j)}_t(x) = \int_{\mathbb{R}^d} p_t(x-y)u_0(y)\,dy + \int_0^t \int_{B(x,(nt)^{1/\alpha})} p_{t-s}(x-y)\sigma(U^{(n,j-1)}_s(y))F^{(n)}(ds\,dy).
\]

Moreover, one can show that under our current standing conditions, the unique solution satisfies for all \( t > 0 \) and \( k \geq 2 \),

\[
\sup_{n \geq 1} \sup_{s \in [0, t]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|U^{(n)}_s(x)|^k \leq c_2 e^{c_1 k (2\alpha - \beta) / (\alpha - \beta) t},
\]

for some positive constants \( c_1, c_2(k) \). As a consequence, for all \( t > 0 \), \( k \geq 2 \), and sufficiently large \( n \),

\[
\sup_{s \in [0, t]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|U^{(n,n-1)}_s(x)|^k \leq c_2 e^{c_1 k (2\alpha - \beta) / (\alpha - \beta) t},
\]

(3.3)

for some positive constants \( c_1, c_2(k) \). We also have the following result which gives us the independent quantities we need.

**Lemma 3.1.** Let \( t > 0 \) and \( n \geq 1 \). Suppose that \( \{x_i\}_{i=1}^\infty \subset \mathbb{R}^d \) with \(|x_i - x_j| \geq 2n^{1+1/\alpha}t^{1/\alpha}\) for all \( i \neq j \). Then \( \{U^{(n,n)}_t(x_i)\}_{i=1}^\infty \) are independent random variables.

**Proof.** The proof is similar to that of [9, Lemma 5.4] and is omitted. \( \square \)

We will also need the fact that the random variables defined above approximate the solution to (1.1). We provide a proof of this next. Recall that \( \gamma < \beta \wedge 1 \).

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Lemma 3.2. For all $T > 0$ and $k \geq 2$, there exist positive constants $c_1$ and $c_2$ such that for large enough $n$,

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|u_t(x) - U_t^{(n,n)}(x)|^k \leq c_2 \frac{1}{n^{\gamma k/2}} e^{c_1 k (\frac{\alpha - \beta}{\alpha - \beta})}.$$ 

Proof. Consider the following integral equation,

$$V_t^{(n)}(x) = (p_t * u_0)(x) + \int_0^t \int B(x,(nt)^{1/\alpha})p_{t-s}(x-y)\sigma(V_s^{(n)}(y))F(dy)ds.$$ 

We first look at $V_t^{(n)}(x) - U_t^{(n,n)}(x)$ and its moments.

$$V_t^{(n)}(x) - U_t^{(n,n)}(x) = \int_0^t \int B(x,(nt)^{1/\alpha})p_{t-s}(x-y)\sigma(V_s^{(n)}(y))F(dy)ds - \int_0^t \int B(x,(nt)^{1/\alpha})p_{t-s}(x-y)\sigma(U_s^{(n,n-1)}(y))F(dy)ds.$$ 

We rewrite the above as

$$V_t^{(n)}(x) - U_t^{(n,n)}(x) = \int_0^t \int B(x,(nt)^{1/\alpha})p_{t-s}(x-y)[\sigma(V_s^{(n)}(y)) - \sigma(U_s^{(n,n-1)}(y))]F(dy)ds - \int_0^t \int B(x,(nt)^{1/\alpha})p_{t-s}(x-y)\sigma(U_s^{(n,n-1)}(y))[F(dy)ds - F(dy)]$$

$$:= I_1 + I_2.$$ 

We start by bounding $I_2$. Using Burkholder-Davis-Gundy inequality and Minkowski’s inequalities together with (3.3), we get that

$$\mathbb{E}|I_2|^k \leq c_2 e^{c_1 k (\frac{\alpha - \beta}{\alpha - \beta})} \left( \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(x-y)p_{t-s}(x-z)f_n(y-z)dydzds \right)^{k/2}.$$ 

Appealing to (3.1), we conclude that

$$\sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|I_2|^k \leq c_2 (T) \frac{e^{c_1 k (\frac{\alpha - \beta}{\alpha - \beta})}}{n^{\gamma k/2}}.$$ 

We next treat $I_1$. We look at $U_t^{(n,n)}(x) - U_t^{(n,n-1)}(x)$. Using Burkholder-Davis-Gundy and Minkowski’s inequalities together with Lemma 2.2(c), we obtain

$$\mathbb{E}|U_t^{(n,n)}(x) - U_t^{(n,n-1)}(x)|^k \leq c(k) \sup_{s \in [0,t]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|U_s^{(n,n-1)}(x) - U_s^{(n,n-2)}(x)|^k \left( \int_0^t s^{-\beta/\alpha}ds \right)^{k/2}.$$ 

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Iterating \( n \) times this procedure and choosing \( T \leq 1/2 \), we get

\[
\sup_{t \in [0,T]} \mathbb{E}|U^{(n,n)}_t(x) - U^{(n,n-1)}_t(x)|^k \leq c(k)T^{nk/2} \leq c(k) \left( \frac{1}{2} \right)^{nk/2} \leq c(k) \frac{1}{n^\gamma k/2}.
\]

Splitting the interval \([0, T]\) into subintervals of length \( 1/2 \), we deduce that for all \( T > 0 \),

\[
\sup_{t \in [0,T]} \mathbb{E}|U^{(n,n)}_t(x) - U^{(n,n-1)}_t(x)|^k \leq c(k, T) \frac{1}{n^\gamma k/2}.
\]

We next set

\[
\mathcal{D}_n^t := \sup_{x \in \mathbb{R}^d} \mathbb{E}|V^{(n)}_t(x) - U^{(n,n)}_t(x)|^k.
\]

Using Burkholder-Davis-Gundy inequality and Minkowski’s inequalities, together with Lemma 2.2(c), and adding and subtracting the term \( U^{(n,n)}_s(y) \), we obtain

\[
\mathbb{E}|I_1|^k \leq c(k, T) \int_0^t \frac{\mathcal{D}_n^{s} + n^{-\gamma k/2}}{(t-s)^{\beta/\alpha}} ds
\]

Using Proposition 2.5,

\[
\mathbb{E}|I_1|^k \leq c_2 \left( \int_0^t \frac{\mathcal{D}_n^{s}}{(t-s)^{\beta/\alpha}} ds + \frac{e^{c_1 k (2\alpha - \beta)/\alpha - \beta t}}{n^\gamma k/2} \right).
\]

Combining the bound for \( I_2 \) and \( I_1 \), we obtain

\[
\mathcal{D}_n^t \leq c_2 \left( \frac{e^{c_1 k (2\alpha - \beta)/\alpha - \beta t}}{n^\gamma k/2} + \int_0^t \frac{\mathcal{D}_n^{s}}{(t-s)^{\beta/\alpha}} ds \right).
\]

By an appropriate use of Proposition 2.5, we conclude that

\[
\mathcal{D}_n^t \leq c_2 \frac{e^{c_1 k (2\alpha - \beta)/\alpha - \beta t}}{n^\gamma k/2}.
\] (3.4)

We now look at \( u_t(x) - V^{(n)}_t(x) \) to obtain

\[
u_t(x) - V^{(n)}_t(x) = \int_0^t \int_{B(x,(nt)^{1/\alpha})} p_{t-s}(x-y)[\sigma(u_s(y) - V^{(n)}_s(y))]F(ds \, dy) + \int_0^t \int_{B(x,(nt)^{1/\alpha})} p_{t-s}(x-y)\sigma(u_s(y))F(ds \, dy),
\]
which gives us

\[\mathbb{E}|u_t(x) - V_s^{(n)}(x)|^k \leq c \left( \int_0^t \int_{B(x, (nt)^{1/\alpha})} p_{t-s}(x-y)[\sigma(u_s(y) - V_s^{(n)}(y))]|F(ds \, dy) \right)^k \]

\[+ \mathbb{E} \left( \int_0^t \int_{B(x, (nt)^{1/\alpha})} p_{t-s}(x-y)\sigma(u_s(y))|F(ds \, dy) \right)^k := I_1 + I_2.\]

We bound the second term first. Using the bound on the moments of the solution together with Lemma 2.2(a), we obtain

\[I_2 \leq c_2 e^{c_1 k(2\alpha - \beta)/(\alpha - \beta) t} \left( \int_0^t \int_{B(x, (nt)^{1/\alpha})} p_{t-s}(x-y)p_{t-s}(x-w)f(y-w) \, dy \, dw \, ds \right)^{k/2} \leq c_2 \frac{1}{n(2+\beta/\alpha)k/2} e^{c_1 k(2\alpha - \beta)/(\alpha - \beta) t}.\]

We now consider the first term.

\[I_1 \leq c \int_0^t \sup_{y \in \mathbb{R}^d} \mathbb{E}|u_s(y) - V_s^{(n)}(y)|^k \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(x-y)p_{t-s}(x-w) \, f(y-w) \, dy \, dw \, ds \leq c \int_0^t \sup_{y \in \mathbb{R}^d} \mathbb{E}|u_s(y) - V_s^{(n)}(y)|^k \frac{1}{(t-s)^{\beta/\alpha}} \, ds.\]

Putting these two bounds together and using Proposition 2.2, we obtain

\[\sup_{s \in [0,T]} \sup_{x \in \mathbb{R}^d} \mathbb{E}|u_s(x) - V_s^{(n)}(x)|^k \leq c_2 \frac{1}{n(2+\beta/\alpha)k/2} e^{c_1 k(2\alpha - \beta)/(\alpha - \beta) t}. \tag{3.5}\]

Combining the estimates (3.4) and (3.5) and using the fact that \(\gamma < 2 + \beta/\alpha\) we obtain the required result.

### 4 Proof of the spatial asymptotic results

In this section we give the proofs of Theorems 1.6 and 1.8. We start with several preliminary results.

#### 4.1 Tail estimates I

This subsection is devoted to the proof of two tail estimates which are a consequence of the sharp moment estimates in Theorem 1.5.
**Lemma 4.1.** There exists a constant $c_{A,\alpha,\beta} > 0$ such that for all $\lambda > 0$ and $t > 0$,

$$\sup_{x \in \mathbb{R}^d} P(u_t(x) > \lambda) \leq \exp \left( - \frac{c_{A,\alpha,\beta} \nu^{\beta/\alpha}}{\nu^{(\alpha-\beta)/\alpha}} \log \left| \frac{\lambda}{A\bar{u}_0} \right|^{(2\alpha-\beta)/\alpha} \right),$$

where $A$ and $\bar{u}_0$ are defined in Theorem 1.5.

**Proof.** We start by using Chebyshev’s inequality to obtain,

$$P(u_t(x) > \lambda) \leq \frac{1}{k} \mathbb{E}|u_t(x)|^k \leq A^k \bar{u}_0 \nu^{-k} e^{A^{2\alpha-\beta} / (\alpha-\beta) \nu^{-\beta/2}} - k \log \left( \frac{\lambda}{A\bar{u}_0} \right).$$

The function $F(k) := A^{2\alpha-\beta} / (\alpha-\beta) \nu^{-\beta/2} - k \log \left( \frac{\lambda}{A\bar{u}_0} \right)$ is optimised at the point

$$k^* = \left[ \frac{\nu^{(\alpha-\beta) / 2}}{A} \left( \frac{\alpha - \beta}{2\alpha - \beta} \right) \left( \log \left( \frac{\lambda}{A\bar{u}_0} \right) \right)^{(\alpha-\beta) / \alpha} \right].$$

Some computations then give

$$P(u_t(x) > \lambda) \leq \exp \left( - \frac{c_{A,\alpha,\beta} \nu^{\beta/\alpha}}{\nu^{(\alpha-\beta)/\alpha}} \log \left| \frac{\lambda}{A\bar{u}_0} \right|^{(2\alpha-\beta)/\alpha} \right),$$

where $c_{A,\alpha,\beta} = \frac{\nu}{2\alpha-\beta} \left[ \frac{1}{4} \left( \frac{\alpha - \beta}{2\alpha - \beta} \right)^{(\alpha-\beta) / \alpha} \right]$. \qed

**Lemma 4.2.** Fix $t > 0$. Set $\lambda := \frac{\nu}{2\alpha-\beta} u_0^{(\alpha-\beta) / \nu^{(\alpha-\beta)/2}}$ for $k \geq 2$. Then there exists a constant $\tilde{c}_{A,\alpha,\beta} > 0$,

$$\inf_{x \in \mathbb{R}^d} P(u_t(x) > \lambda) \geq \frac{1}{4} \exp \left( - \frac{\tilde{c}_{A,\alpha,\beta} \nu^{\beta/\alpha}}{\nu^{(\alpha-\beta)/\alpha}} \log \frac{2A}{\bar{u}_0} \right)^{(2\alpha-\beta)/\alpha} \left( 1 + \frac{1}{\log \frac{2A}{\bar{u}_0}} \right),$$

where the quantities $A$ and $\bar{u}_0$ are defined in Theorem 1.5.

**Proof.** By Paley-Zygmund inequality, we have for all $k \geq 2$,

$$P(u_t(x) \geq \frac{1}{2} \|u_t(x)\|_{L^2(\Omega)}) \geq \frac{(\mathbb{E}|u_t(x)|^{2k})^2}{4\mathbb{E}|u_t(x)|^{4k}}.$$
Set $\lambda := \frac{\tilde{u}_0}{2A} e^{tk/(2-\beta)} A/(2-\beta)$. Taking into account the bounds on the moments, we obtain

$$
\mathbb{P}(u_t(x) \geq \lambda) \geq \frac{1}{4} \exp \left( -c_{A,\alpha,\beta} k(2\alpha-\beta)/(2-\beta) t \right) - k \log \left( \frac{\tilde{u}_0}{A^8} \right),
$$

where $c_{A,\alpha,\beta} := 2(2\alpha-\beta)/[(2\alpha-\beta) - 2] + \frac{2}{A}$ and $\tilde{u}_0 = \frac{u_0}{\tilde{u}_0}$. Finally, some computations we get the desired bound. \[\square\]

### 4.2 Insensitivity analysis

The next theorem is crucial in the proof the of spatial asymptotic result when the initial condition is not bounded below. Intuitively, we study how the solution is sensible to changes to the initial data, and we conclude that when $R$ is large, the values of the solution in a given ball of radius $R$ are insensitive to the changes of the initial value outside the ball.

**Theorem 4.3.** Let $a \in \mathbb{R}^d$ and $R > 1$. Let $u$ and $v$ be the solution to (1.2) with respective initial conditions $u_0$ and $v_0$. Suppose that on $B(a, 2R)$, $u_0(x) = v_0(x)$. Then for all $k \geq 2$ there exist positive constants $c_1, c_2$ such that for all $t > 0$,

$$
\sup_{x \in B(a,R)} \mathbb{E} |u_t(x) - v_t(x)|^k \leq c_2 \|u_0 - v_0\|_{L^\infty(\Omega)} \frac{1}{R^{\alpha k/2}} e^{c_1 k(2\alpha-\beta)/(2-\beta) t}.
$$

**Proof.** From the mild solution, we have

$$
u_t(x) - v_t(x) = (\mathcal{G} u_0)_t(x) - (\mathcal{G} v_0)_t(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)[\sigma(u_s(y)) - \sigma(v_s(y))] F(dy \, ds),
$$

where $(\mathcal{G} u_0)_t(x) := (p_t * u_0)(x)$. We obtain

$$
\|u_t(x) - v_t(x)\|^2_{L^k(\Omega)} \leq c \left( \| (\mathcal{G} u_0)_t(x) - (\mathcal{G} v_0)_t(x) \|^2_{L^k(\Omega)} + \| \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)[\sigma(u_s(y)) - \sigma(v_s(y))] F(dy \, ds) \|^2_{L^k(\Omega)} \right).
$$

We bound $I_1$ first. Noting that $x \in B(a, R)$ and $y \in B(a, 2R)^c$, we have

$$
I_1 \leq \|u_0 - v_0\|^2_{L^\infty(\mathbb{R}^d)} \left[ \int_{B(a,2R)^c} p_t(x-y) \, dy \right]^2 \leq c_1 \|u_0 - v_0\|^2_{L^\infty(\mathbb{R}^d)} \frac{l^2}{R^{2\alpha}} \leq c_1 \|u_0 - v_0\|^2_{L^\infty(\mathbb{R}^d)} \frac{e^{c_2 k(2\alpha-\beta)/(2-\beta) t}}{R^{2\alpha}}.
$$
We use Burkholder-Davis-Gundy inequality and Minkowski’s inequalities to bound the second term as follows

\[
I_2 \leq ck \left\{ \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(x-y_1)p_{t-s}(x-y_1)[u_s(y_1)-v_s(y_1)][u_s(y_2)-v_s(y_2)]f(y_1, y_2)dy_1 dy_2 ds \right] \right\}^{k/2} \\
\leq k \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t-s}(x-y_1)p_{t-s}(x-y_1)[u_s(y_1)-v_s(y_1)][u_s(y_2)-v_s(y_2)]\|f(y_1, y_2)\|_{L^{k/2}(\Omega)} dy_1 dy_2 ds.
\]

We split the integral on the right hand side as follows

\[
\mathbb{E} \left[ \int_0^t \int_{B(x, R) \times B(x, R)} [\cdots] dy_1 dy_2 ds \right] + \mathbb{E} \left[ \int_0^t \int_{B(x, R)^c \times B(x, R)} [\cdots] dy_1 dy_2 ds \right] \\
+ \mathbb{E} \left[ \int_0^t \int_{B(x, R) \times B(x, R)^c} [\cdots] dy_1 dy_2 ds \right] + \mathbb{E} \left[ \int_0^t \int_{B(x, R)^c \times B(x, R)^c} [\cdots] dy_1 dy_2 ds \right] \\
= I_3 + I_4 + I_5 + I_6.
\]

Since \( x \in B(a, R) \), we can bound \( I_3 \) as follows,

\[
I_3 \leq \int_0^t \sup_{y \in B(x, R)} \|u_s(y) - v_s(y)\|^2_{L^k(\Omega)} \int_{B(x, R) \times B(x, R)} p_{t-s}(x-y_1)p_{t-s}(x-y_1)f(y_1, y_2)dy_1 dy_2 ds \\
\leq c \int_0^t \sup_{y \in B(a, 2R)} \|u_s(y) - v_s(y)\|^2_{L^k(\Omega)} \frac{1}{(t-s)^{\beta/\alpha}} ds.
\]

We now use Lemma 2.2(b) to obtain

\[
I_4 \leq \int_0^t \sup_{y \in \mathbb{R}^d} \|u_s(y) - v_s(y)\|^2_{L^k(\Omega)} \int_{B(x, R)^c \times B(x, R)} p_{t-s}(x-y_1)p_{t-s}(x-y_1)f(y_1, y_2)dy_1 dy_2 ds \\
\leq c_1 \frac{e^{c_2k(2\alpha-\beta)/(\alpha-\beta)t}}{R^\alpha}.
\]

Similarly, we can use Lemma 2.2(b) again to

\[
I_5 \leq c_1 \frac{e^{c_2k(2\alpha-\beta)/(\alpha-\beta)t}}{R^\alpha} \quad \text{and} \quad I_6 \leq c_1 \frac{e^{c_2k(2\alpha-\beta)/(\alpha-\beta)t}}{R^\alpha}.
\]

Combining those bounds, we obtain

\[
\sup_{y \in B(a, R)} \|u_t(y) - v_t(y)\|^2_{L^k(\Omega)} \\
\leq c_1 \|u_0 - v_0\|^2_{L^\infty(\mathbb{R}^d)} \frac{e^{c_2k(2\alpha-\beta)/(\alpha-\beta)t}}{R^\alpha} + kc_3 \int_0^t \sup_{y \in B(a, 2R)} \|u_s(y) - v_s(y)\|^2_{L^k(\Omega)} \frac{1}{(t-s)^{\beta/\alpha}} ds.
\]
We set
\[ f_R(t) := \sup_{y \in B(a, R)} \| u_t(y) - v_t(y) \|_{L^k(\Omega)}^2 \]
and
\[ A_R(t) := \| u_0 - v_0 \|_{L^\infty(\mathbb{R}^d)}^2 e^{c^2 k (2\alpha - \beta)/((\alpha - \beta) t)} \frac{e^{c_1 k (2\alpha - \beta)/((\alpha - \beta) t)}}{R^3}. \]

We now use Proposition 2.5 to arrive at the result. \[\square\]

4.3 Tail Estimates II

In this subsection we are going to prove tail estimates when the initial condition is not bounded below. The next result is an extension of [4, Theorem 2.4]

**Theorem 4.4.** Suppose that \( u \) and \( u_0 \) are as in Theorem 1.8. Then, there exist positive constants \( K_1, K_2 \) such that for all \( \lambda > 0 \),

\[
-K_1 \frac{\Lambda^{(2\alpha - \beta)/\alpha}}{t^{(\alpha - \beta)/\alpha}} \leq \liminf_{|x| \to \infty} \frac{\log P(u_t(x) > \lambda)}{\log |x|} \leq \limsup_{|x| \to \infty} \frac{\log P(u_t(x) > \lambda)}{\log |x|} \leq -K_2 \frac{\Lambda^{(2\alpha - \beta)/\alpha}}{t^{(\alpha - \beta)/\alpha}},
\]

uniformly for all \( t \) in every fixed compact subset of \((0, \infty)\).

**Proof.** We prove the lower bound first. Fix \( a \in \mathbb{R}^d \). Let \( w_t \) be the solution to (1.1) when the initial condition is given by the following

\[
w_0(x) := u_0(|x| \vee (3|a|)) \quad \text{for all} \quad x \in \mathbb{R}^d.
\]

Since \( w_0 \leq u_0 \), the weak comparison principle Theorem 1.1 tells us that for all \( t > 0 \) and \( x \in \mathbb{R}^d \),

\[
w_t(x) \leq u_t(x).\]

This means that finding a lower bound on the tail distribution of \( u_t(x) \) amounts to finding a lower bound for the corresponding distribution of \( w_t(x) \).

Now, let \( u_t^a(x) \) be the solution to (1.1) when initial condition \( u_0(3|a|) \). Fix \( \lambda > 0 \). Then, by Theorem 4.3, whenever \( R = |a| > 1 \),

\[
\sup_{x \in B(a, |a|)} P(|u_t^a(x) - w_t(x)| > \lambda) \leq \sup_{x \in B(a, |a|)} \frac{\mathbb{E}|u_t^a(x) - w_t(x)|^k}{\lambda^k} \leq c_2 \frac{1}{\lambda^k |a|^{\alpha k/2}} e^{c_1 k (2\alpha - \beta)/((\alpha - \beta) t)}.
\]

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Recall that \( u_0(3|a|) \) is decreasing in \( a \) and
\[
\lim_{a \to \infty} u_0(3|a|) = 0.
\]
Therefore, we can then take \(|a|\) large enough so that
\[
k := \left( \frac{A\nu}{t} \log \left( \frac{4A\lambda}{u_0(3|a|)} \right) \right)^{\frac{\alpha - \beta}{\alpha}} \geq 2,
\]
and
\[
\frac{1}{2} \leq \left| \log \frac{4\lambda A}{u_0(3|a|)} \right|.
\]
We now use Lemma 4.2 to obtain
\[
\inf_{x \in B(a,|a|)} P(u_t^a(x) \geq 2\lambda) \geq \frac{1}{4} \exp \left( -\frac{\bar{c}_{A,\alpha,\beta} \nu^{\beta/\alpha}}{t^{(\alpha-\beta)/\alpha}} \log \left( \frac{2\lambda A}{u_0} \right)^{(2\alpha-\beta)/\alpha} \right).
\]
Upon taking \(|a|\) larger if required so that
\[
\left| \log \frac{4\lambda A}{u_0(3|a|)} \right| \leq 2|\log u_0(3|a|)|,
\]
we can use the above together with the definition of \( \Lambda \) to write
\[
\inf_{x \in B(a,|a|)} P(u_t(x) \geq \lambda) \geq \inf_{x \in B(a,|a|)} P(u_t^a(x) \geq 2\lambda) - \sup_{x \in B(a,|a|)} P(|u_t^a(x) - w_t(x)| > \lambda)
\geq \frac{1}{4} \exp \left( -\frac{c_{A,\alpha,\beta} \nu^{\beta/\alpha} \Lambda^{(2\alpha-\beta)/\alpha}}{t^{(\alpha-\beta)/\alpha}} \log |a| \right) - c_2 \frac{1}{\lambda^{k|a|^{\alpha/2}}} c_{1,\nu}^{(2\alpha-\beta)/\alpha} t^{(\alpha-\beta)/\alpha}.
\]
For any fixed \( t > 0 \), we choose \( k \) large enough so that for \( a \) large, we obtain
\[
\inf_{x \in B(a,|a|)} P(u_t(x) \geq \lambda) \geq \frac{1}{8|a|^{K_1^{\alpha/2}}} \Lambda^{(2\alpha-\beta)/\alpha}.
\]
The above immediately gives the lower bound needed. We now turn our attention to the upper bound. The proof uses a similar strategy as the above. We look at \( w_t \), the solution to (1.1) but this time, the initial condition is defined by
\[
w_0(x) := u_0(|x| \land 2|a|),
\]
so that now we have \( w_0(x) \geq u_0(x) \) which gives us \( w_t(x) \geq u_t(x) \) by the weak comparison principle. Now consider \( u_t^a \) a solution with constant initial condition given by
\[
z_0(x) := u_0(2|a|).
\]
We choose \( a \) large enough such that
\[
\left| \log \frac{\lambda}{2A u_0(2|a|)} \right| \geq \left| \log u_0(2|a|) \right|.
\]
Then, by Theorem 4.3 and Lemma 4.1, for \( |a| \) large enough
\[
\sup_{x \in B(a, |a|)} P(u_t(x) > \lambda) \leq \sup_{x \in B(a, |a|)} P(w_t(x) > \lambda)
\leq \sup_{x \in B(a, |a|)} P(u_t^a(x) > \lambda/2) + \sup_{x \in B(a, |a|)} P(|u_t^a(x) - w_t(x)| > \lambda/2)
\leq \exp\left(- \frac{c_{A,\alpha,\beta} \Lambda}{t^{(\alpha-\beta)/\alpha}} \log |a| \right) + c_2 \frac{1}{A^k |a|^{\alpha k/2}} e^{c_1 k^2(2\alpha-\beta)/(\alpha-\beta) t}.
\]
By choosing \( k \) large, we obtain upon taking \( a \) large enough,
\[
\sup_{x \in B(a, |a|)} P(u_t(x) > \lambda) \leq \frac{1}{|a|^2 \Lambda^{(2\alpha-\beta)/\alpha}}.
\]
This finishes the proof.

4.4 Proof of Theorem 1.6

Proof of Theorem 1.6. Let \( t > 0 \) and set
\[
L := \frac{u_0}{6A} \exp \left( \delta_1 t^{(\alpha-\beta)/(2\alpha-\beta)} |\log R|^{\alpha/(2\alpha-\beta)} \nu^{-\beta/(2\alpha-\beta)} \log \left| A \right| \right),
\]
where \( \delta_1 \) be a positive constant. Then, we choose
\[
k = (A\delta_1)^{(\alpha-\beta)/\alpha} \left( \frac{|\log (R)|}{t} \right)^{(\alpha-\beta)/(2\alpha-\beta)} \nu^{\beta/(2\alpha-\beta)},
\]
so that \( L \) becomes
\[
L := \frac{u_0}{6A} e^{k^{\alpha/(\alpha-\beta)} \nu^{-\beta/(\alpha-\beta)} / A}.
\]
We now apply inequality (4.1) to obtain for sufficiently large \( R \),
\[
P(u_t(x) > 3L)
\geq \frac{1}{4} \exp \left(-c_{A,\alpha,\beta} (A\delta_1)^{(2\alpha-\beta)/\alpha} |\log R| - \log \left( A^8 \right) (A\delta_1)^{(\alpha-\beta)/\alpha} \left( |\log R| / t \right)^{(\alpha-\beta)/(2\alpha-\beta)} \nu^{\beta/(2\alpha-\beta)} \right)
\geq \frac{1}{4 R^{\delta_2}},
\]
where $\delta_1$ is chosen such that $\delta_2 < 2$.

Let $N > 0$ and choose $x_1, x_2, \ldots, x_N \in \mathbb{R}^d$ such that $|x_i - x_j| \geq 2n^{1+1/\alpha} t^{1/\alpha}$ for $i \neq j$. Lemma 3.1 then implies that the $U_t^{(n,n)}(x_i)$’s are independent for large enough $n$. We have

$$P(\max_{1 \leq i \leq N} u_t(x_i) < L) \leq P(\max_{1 \leq i \leq N} |U_t^{(n,n)}(x_i)| < 2L)$$

$$+ P(|u_t(x_i) - U_t^{(n,n)}(x_i)| > L \text{ for some } 1 \leq i \leq N)$$

$$:= I_1 + I_2.$$ 

We will look at the second term first. By Lemma 3.2, for all $k \geq 2$ and large $n$,

$$I_2 \leq \frac{N\mathbb{E}|u_t(x_i) - U_t^{(n,n)}(x_i)|^k}{L^k} \leq c_2 \frac{N}{n^{\gamma k/2}} e^{c_1 k(2\alpha - \beta)/(\alpha - \beta) t},$$

where we have chosen $R$ large enough such that $L \geq 1$.

We now choose $n \geq N^{10/(3\gamma)}$ so that we have

$$I_2 \leq c_2 \frac{1}{N^{2/3}} e^{c_1 k(2\alpha - \beta)/(\alpha - \beta) t}.$$ 

Upon choosing $N$ to be an integer greater than $R^3$, we obtain

$$I_2 \leq c(T, k) \frac{1}{R^2}.$$ 

To bound $I_1$, we have for large enough $R$,

$$P(U_t^{(n,n)}(x_i) \geq 2L) \geq P(|u_t(x_i)| \geq 3L) - P(|u_t(x_i) - U_t^{(n,n)}(x_i)| \geq L)$$

$$\geq c \left( \frac{1}{R^{8/2}} - \frac{1}{R^2} \right) \geq \frac{c}{R^2}.$$ 

By independence, we have

$$I_1 \leq \left( 1 - P(U_t^{(n,n)}(x_i) \geq 2L) \right)^N.$$ 

Combining the above and bearing in mind that $N$ is larger than $R^3$, we obtain

$$P(\max_{1 \leq i \leq N} u_t(x_i) < L) \leq \left( 1 - P(U_t^{(n,n)}(x_i) \geq 2L) \right)^N + \frac{c}{R^2}$$

$$\leq \frac{c}{R^2},$$

for $R$ large enough. And hence by a standard monotonicity argument, we have

$$P\left( \sup_{x \in B(0,R)} u_t(x) \leq \frac{u_0}{6A} \exp \left[ \delta_1 t^{(\alpha - \beta)/(2\alpha - \beta)} (\log R)^{\alpha/(2\alpha - \beta) - \beta/(\alpha - \beta)} \right] \right) \leq \frac{c}{R^2}.$$
We now use Borel Cantelli lemma to obtain that almost surely, for \( R \to \infty \), we have
\[
\sup_{x \in B(0, R)} u_t(x) \geq \frac{\mu_0}{6A} \exp \left[ \delta_1 t^{(\alpha-\beta)/(2\alpha-\beta)} (\log R)^{\alpha/(2\alpha-\beta)} \right],
\]
which concludes the proof of the lower bound. We now prove the upper bound. Set
\[
U := A\bar{u}_0 \exp \left[ \delta_3 t^{(\alpha-\beta)/(2\alpha-\beta)} (\log R)^{\alpha/(2\alpha-\beta)} \right],
\]
for some positive constant \( \delta_3 \). For \( x \in \mathbb{Z}^d \), denote the cube of side length 1 by \( Q_x \). Let \( R \) be a positive integer and decompose \([-R, R]^d\) into cubes of the form \( Q_x \) so that \([-R, R]^d = \bigcup_{x \in S} Q_x \) where \( S \) is some finite set. By Proposition 2.3, for any \( x \in \mathbb{R}^d \) and \( k \geq 2 \), we have
\[
E \left[ \sup_{w, y \in Q_x} |u_t(w) - u_t(y)|^k \right] \leq c_2 \exp \left[ c_1 k^{(2\alpha-\beta)/(\alpha-\beta)} t \right]. \tag{4.2}
\]
We can now write
\[
P \left( \sup_{x \in [-R, R]^d} u_t(x) \geq 2U \right) \leq P \left( \max_{x \in S} u_t(x) \geq U \right) + P \left( \max_{x \in S} \sup_{y \in Q_x} |u_t(y) - u_t(x)| \geq U \right)
= I_1 + I_2.
\]
To bound \( I_1 \), we use Lemma 4.1 to obtain
\[
I_1 \leq |S| P(u_t(x) \geq U) \leq \frac{c}{R^{\delta_4}},
\]
where the constant \( \delta_4 \) is chosen so that \( \delta_4 > 1 \). We now bound \( I_2 \) by making use of (4.2),
\[
I_2 \leq |S| P \left( \sup_{y \in Q_x} |u_t(y) - u_t(x)| \geq U \right) \leq \frac{c_2 |S| \exp(c_1 k^{(2\alpha-\beta)/(\alpha-\beta)} t)}{\exp(k \delta_3 t^{(\alpha-\beta)/(2\alpha-\beta)} (\log R)^{\alpha/(2\alpha-\beta)} \nu^{\beta/(2\alpha-\beta)})}.
\]
We now set \( k = \delta_5 \left( \frac{\log R}{t} \right)^{(\alpha-\beta)/(2\alpha-\beta)} \nu^{\beta/(2\alpha-\beta)} \) to obtain
\[
I_2 \leq \frac{c}{R^{\delta_6}},
\]
where we choose \( \delta_6 \) so that \( \delta_6 > 1 \). We can conclude that
\[
\sum_{R=1}^{\infty} P \left( \sup_{x \in [-R, R]^d} u_t(x) > 2U \right) < \infty.
\]
We can now use Borel-Cantelli and the fact that \( B(0, R) \subset [-R, R]^d \) to finish the proof. \( \square \)
4.5 Proof of Theorem 1.8.

Proof of Theorem 1.8. We split the proof into two parts. In the first part we assume that \( \Lambda > 0 \). Consider the sequence \( \{x_n\}_{n \geq 1} \subset \mathbb{R}^d \) such that \( |x_n| = n^{1/2} \) and all \( x_n \) lie on a straight line through the origin. We next choose

\[
\lambda \in (0, \Lambda),
\]

and consider

\[
t(j, n) := \frac{jT}{n}, \quad \text{for} \quad j \in \left[\frac{n\tau}{T}, n\right] \cap \mathbb{Z}.
\]

We look at the following parameters \( \tau \) and \( T \) such that

\[
0 < \tau < T := \frac{K_2^{\alpha/(\alpha-\beta)} \lambda(2\alpha-\beta)/(\alpha-\beta)}{2\alpha/(\alpha-\beta)},
\]

where \( K_2 \) is the constant in the statement of Theorem 4.4. Then, by Theorem 4.4, for all \( \theta > 0 \), \( t \in (\tau, T) \) and large enough \( n \),

\[
P\left( \max_{j \in \left[\frac{n\tau}{T}, n\right] \cap \mathbb{Z}} u_t(j, n) (x_n) > \theta \right) \leq c \sum_{j \in \left[\frac{n\tau}{T}, n\right] \cap \mathbb{Z}} P\left( u_t(j, n) (x_n) > \theta \right)
\]

\[
\leq cn \exp\left( -\frac{K_2 \lambda(2\alpha-\beta)/\alpha}{t(j, n)(\alpha-\beta)/\alpha} \log |x_n| \right)
\]

\[
\leq c n^{3/2}.
\]

An application of Borel-Cantelli lemma gives us

\[
\lim_{n \to \infty} \max_{j \in \left[\frac{n\tau}{T}, n\right] \cap \mathbb{Z}} u_t(j, n) (x_n) = 0 \quad \text{a.s.}
\]

We now use Proposition 2.3 to obtain for all \( \theta > 0 \),

\[
P\left\{ \sup_{t \in (\tau, T)} \min_{j \in \left[\frac{n\tau}{T}, n\right] \cap \mathbb{Z}} |u_t(j, n)(x_n) - u_t(x_n)| > \theta \right\}
\]

\[
\leq P\left\{ \sup_{t \in (\tau, T)} |u_s(x_n) - u_t(x_n)| > \theta \right\}
\]

\[
\leq \frac{c_{T,k}}{n^{\delta_k}}.
\]
By choosing \( k \) large enough, we can apply Borel-Cantelli and use the above to see that
\[
\lim_{n \to \infty} \sup_{t \in (\tau, T)} u_t(x_n) = 0 \quad \text{a.s.} \quad (4.3)
\]

We next use Proposition 2.3, to get for all \( \theta > 0, \)
\[
P \left\{ \sup_{t \in (\tau, T)} \sup_{x \in [x_n, x_{n+1}]} |u_t(x) - u_t(y)| \geq \theta \right\} \leq cn^{4/\alpha}P \left\{ \sup_{t \in (\tau, T)} \sup_{|x-y| \leq \frac{1}{n}} |u_t(x) - u_t(y)| \geq \theta \right\} \leq c n^{1/2} n^k.
\]

We then take \( k \) large enough, use Borel-Cantelli again and (4.3) to conclude that
\[
\lim_{|x| \to \infty} \sup_{t \in (\tau, T)} u_t(x) = 0 \quad \text{a.s.}
\]

where in the above, \( x \) tends to infinity along a fixed straight line. Since the line is arbitrary and \( u \) is almost surely continuous (Proposition 2.3), it follows that
\[
P \left( \sup_{x \in \mathbb{R}^d} u_t(x) < \infty, \text{ for all } t \in (\tau, T) \right) = 1.
\]

The above is valid when \( \Lambda = \infty \) in which case we can take \( T \) as large as we want. Taking \( \tau \) close to zero finishes the second part of the proof. We next assume that \( \Lambda < \infty \). Fix \( \theta > 0 \) and set
\[
E_t(x) := \{ \omega \in \Omega : u_t(x) \leq \theta \} \quad \text{for every } t > 0, x \in \mathbb{R}^d.
\]

We will show that solution is almost surely unbounded for large enough times. Let
\[
\tau > (\Lambda^{(2\alpha-\beta)/\alpha}2K_1/\alpha)^{\alpha/(\alpha-\beta)} \quad \text{and} \quad T > \tau.
\]

According to Theorem 4.4, for every \( \lambda \in (\Lambda, (\alpha\tau^{(\alpha-\beta)/\alpha}/(2K_1))^{\alpha/(2\alpha-\beta)}) \), we can find a real number \( n(\lambda, \theta) > 1 \) such that
\[
P(E_t(x)) \leq \left( 1 - |x|^{-K_1\lambda^{(2\alpha-\beta)/\alpha}1/(\alpha-\beta)/\alpha} \right) \leq \left( 1 - \frac{1}{|x|^\alpha/2} \right), \quad (4.4)
\]
uniformly for all \( |x| \geq n(\lambda, \theta) \) and \( t \in (\tau, T) \). Consider the events
\[
E_t^{(n)}(x) := \{ \omega \in \Omega : U_{x}^{(n,n)}(x) \leq 2\theta \} \quad \text{for every } x \in \mathbb{R}^d, n \geq 1.
\]

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By Lemma 3.2, we get
\[
\sup_{t \in (\tau, T)} \mathbb{P}(E_t(x) \setminus E_t^n(x)) \leq \sup_{t \in (\tau, T)} \mathbb{P}\left(\left| u_t(x) - U_t^{(n)}(x) \right| \geq \theta \right)
\leq C_{T,k} \frac{\gamma_k}{n^{\gamma_k/2}}.
\] (4.5)

Therefore,
\[
\mathbb{P}\left( \bigcap_{x \in [n^{a+\frac{1}{2}},2n^{a+\frac{1}{2}}]^d} E_t(x) \right) \leq \mathbb{P}\left( \bigcap_{\ell \in [n^{a+\frac{1}{2}},2n^{a+\frac{1}{2}}] \cap \mathbb{Z}^d} E_t(\ell) \right)
\leq \mathbb{P}\left( \bigcap_{\ell \in [n^{a+\frac{1}{2}},2n^{a+\frac{1}{2}}] \cap \mathbb{Z}^d} E_t^{(n)}(\ell) \right) + \frac{c}{n^{\gamma_k/2}},
\]
uniformly for all \( n \geq 1 \) and \( t \in (\tau, T) \), where \( a > 1 \) is constant. We will now look at the first term of the above display. Set \( x_1 := (n^{a+\frac{1}{2}}, \ldots, n^{a+\frac{1}{2}}) \in \mathbb{R}^d \) and define iteratively for \( j \geq 1 \),
\[
x_{j+1} := x_j + (2n^{1+\frac{1}{a}l^{1/\alpha}}, \ldots, 2n^{1+\frac{1}{a}l^{1/\alpha}}).
\]
Let
\[
\gamma_n := \max \left\{ j \geq 1 : x_{j,i} \leq 2n^{a+\frac{1}{2}}, \text{ for all } i = 1, \ldots, d \right\},
\]
where \( x_j = (x_{j,1}, \ldots, x_{j,d}) \). Observe that
\[
\gamma_n \geq \frac{n^{a-1}}{2T^{1/a}}.
\]
By independence (Lemma 3.1), (4.5) and (4.4), we get
\[
\mathbb{P}\left( \bigcap_{\ell \in [n^{a+\frac{1}{2}},2n^{a+\frac{1}{2}}] \cap \mathbb{Z}^d} E_t^{(n)}(\ell) \right) \leq \mathbb{P}\left( \bigcap_{j=1}^{\gamma_n} E_t^{(n)}(x_j) \right) = \prod_{j=1}^{\gamma_n} \mathbb{P}(E_t^{(n)}(x_j))
\leq \prod_{j=1}^{\gamma_n} \left[ \mathbb{P}(E_t(x_j)) + \frac{c}{n^{\gamma_k/2}} \right]
\leq \left[ 1 - \frac{1}{n^{a_2+\frac{1}{2}}} + \frac{c}{n^{\gamma_k/2}} \right]^{\gamma_n}.
\]
We now take $k$ larger if necessary to obtain

$$P\left( \bigcap_{\ell \in [n^{a+\frac{1}{\alpha}}, 2n^{a+\frac{1}{\alpha}}]} E_t^{(n)}(\ell) \right) \leq \exp\left( -c_1 n^{-\frac{a}{2} - \frac{1}{2} + a - 1} \right).$$

We choose $a > \frac{5}{2(\alpha - 1)}$, so that we have

$$P\left( \bigcap_{\ell \in [n^{a+\frac{1}{\alpha}}, 2n^{a+\frac{1}{\alpha}}]} E_t^{(n)}(\ell) \right) \leq \exp(-c_1 n).$$

Combining the above estimates, we have for large enough $k$,

$$\sup_{t \in (\tau, T)} P\left( \sup_{x \in [n^{a+\frac{1}{\alpha}}, 2n^{a+\frac{1}{\alpha}}]} u_t(x) \leq \theta \right) \leq \frac{c}{n^{\gamma k/2}}. \quad (4.6)$$

We next write

$$P\left( \inf_{t \in (\tau, T)} \sup_{x \in [n^{a+\frac{1}{\alpha}}, 2n^{a+\frac{1}{\alpha}}]} u_t(x) \leq \theta \right) \leq P\left( \inf_{1 \leq i \leq n} \sup_{x \in [n^{a+\frac{1}{\alpha}}, 2n^{a+\frac{1}{\alpha}}]} u_t(x) \leq 2\theta \right) + P\left( \inf_{|t-s| < 1/n} \sup_{x \in [n^{a+\frac{1}{\alpha}}, 2n^{a+\frac{1}{\alpha}}]} |u_s(x) - u_t(x)| \geq \theta \right) =: I_1 + I_2.$$  

From (4.6), we can bound the first term as follows

$$I_1 \leq \frac{cn}{n^{\gamma k/2}}.$$  

We now look at the second term. Using Proposition 2.3 we obtain that

$$I_2 \leq \sum_{k=1}^{1+n^{a+1/\alpha}} P\left( \inf_{|t-s| < 1/n} \sup_{x \in [k, k+1]^d} |u_s(x) - u_t(x)| \geq \theta \right) \leq \frac{c}{n^\kappa},$$

where $\kappa$ can be made as large as possible. Combining the above estimates, we conclude that

$$P\left( \inf_{t \in (\tau, T)} \sup_{x \in \mathbb{R}^d} u_t(x) < \theta \right) = 0.$$
If $\Lambda = 0$, then we can choose $\tau$ as close to zero as we want and hence the final part of the theorem is proved. For each $N \geq 1$, set

$$T_N := \inf \{ t > 0 : \sup_{x \in \mathbb{R}^d} u_t(x) \geq N \}$$

and let $T := \lim_{N \to \infty} T_N$. The first part of the theorem is then proved by using exactly the same argument as in the main theorem of [4]. We leave it to the reader to fill in the details.

5 Proof of the comparison principle and strict positivity

In order to prove the strong comparison principle (Theorem 1.2), we need the next two preliminary results which are extensions of [3, Lemmas 7.1 and 7.2] (see also [5, Lemmas 4.1 and 4.3]). In particular, the proof of the next proposition is new compared with that of [3, Lemma 7.1] or [5, Lemma 4.1].

**Proposition 5.1.** Let $M > 0$. For all $R > 0$ and $t > 0$, there exist constants $0 < c_R < 1$ and $1 < m_0(t, R) < \infty$ such that for all $m \geq m_0$,

$$\int_{B(0, R)} p_s(x - y) \, dy \geq c_R \quad \text{for all} \quad (s, x) \in A_{m, t, R},$$

where

$$A_{m, t, R} := \{(s, x) : x \in B(0, R + M (t/m)^{1/\alpha}) \quad \text{and} \quad t \frac{m}{2} \leq s \leq t \frac{m}{m} \}.$$

**Proof.** We take $m$ large enough so that $(\frac{t}{m})^{1/\alpha} \leq R$. Then, using the the lower bound (2.1), we obtain

$$\int_{B(0, R)} p_s(x - y) \, dy \geq \int_{B(0, R) \cap B(x, 2M(t/m)^{1/\alpha})} p_s(x - y) \, dy$$

$$\geq c \left( \frac{t}{m} \right)^{d/\alpha} s^{-d/\alpha}$$

$$\geq c$$

where the constant $c$ might depend on $R$ and $M$ but can be chosen to be strictly less than 1.

**Proposition 5.2.** Fix $R > 0$, $t > 0$ and $M > 0$ and assume that

$$u_0(x) \geq 1_{B(0, R)}(x).$$
Then there exist positive constants $c_1(R)$, $c_2(R)$, and $m_0(t, R)$ such that for all $m \geq m_0$,

$$
P(u_s(x) \geq c_11_{B(0, R+M(t/m)^{1/\alpha})}(x) \quad \text{for all} \quad \frac{t}{2m} \leq s \leq \frac{t}{m} \quad \text{and} \quad x \in \mathbb{R}^d)
$$

$$
\geq 1 - c_m,
$$

where

$$
c_m := \exp \left(-c_2m^{(\alpha-\beta)/\alpha}\log m^{(2\beta-\alpha)/\alpha}\right).
$$

**Proof.** From the mild formulation of the solution of the equation, we have

$$
u_s(x) = \int_{\mathbb{R}^d} p_s(x - y)u_0(y) \, dy + \int_0^s \int_{\mathbb{R}^d} p_{s-t}(x - y)\sigma(u_t(y))F(\, dy \, dl).
$$

By Proposition 5.1, there exists a $0 < c_1 < 1$ such that for large enough $m$,

$$
\int_{\mathbb{R}^d} p_s(x - y)u_0(y) \, dy \geq 2c_11_{B(0, R+M(t/m)^{1/\alpha})}(x) \quad \text{for all} \quad x \in \mathbb{R}^d \quad \text{and} \quad \frac{t}{2m} \leq s \leq \frac{t}{m}.
$$

By using the mild formulation and the above, we obtain

$$
P(u_s(x) \leq c_11_{B(0, R+M(t/m)^{1/\alpha})}(x) \quad \text{for some} \quad \frac{t}{2m} \leq s \leq \frac{t}{m})
$$

$$
\leq P\left(\int_0^s \int_{\mathbb{R}^d} p_{s-t}(x - y)\sigma(u_t(y))F(\, dy \, dl) < -c_1 \quad \text{for some} \quad (s, x) \in A_{m,t,R}\right)
$$

$$
\leq P\left(\left|\int_0^s \int_{\mathbb{R}^d} p_{s-t}(x - y)\sigma(u_t(y))F(\, dy \, dl)\right| > c_1 \quad \text{for some} \quad (s, x) \in A_{m,t,R}\right).
$$

The term in the above display can now be bounded by

$$
c_1^{-k}\mathbb{E}\sup_{(s,x)\in A_{m,t,R}} \left|\int_0^s \int_{\mathbb{R}^d} p_{s-t}(x - y)\sigma(u_t(y))F(\, dy \, dl)\right|^k.
$$

The above in turn can be bounded using Proposition 2.3 to obtain

$$
\mathbb{E}\sup_{(s,x)\in A_{m,t,R}} \left|\int_0^s \int_{\mathbb{R}^d} p_{s-t}(x - y)\sigma(u_t(y))F(\, dy \, dl)\right|^k
$$

$$
\leq c\rho^{\tilde{\eta}k}\exp(Ak^{(2\alpha-\beta)/(\alpha-\beta)}\rho),
$$

where $\rho := t/m$ and $\tilde{\eta} := \frac{\alpha-\beta}{2\alpha}$. We now optimise the above quantity with respect to $k$ and combine all our estimates to end up with the result. See [3] and [5] for details.
5.1 Proof of Theorem 1.2

We next prove the strong comparison principle. We leave it to the reader to consult [18] for the original idea and to [3] and [5] for further details.

**Proof.** It suffices to show that if \( u_0 \) has finite support then \( u_t(x) > 0 \) for all \( t > 0 \) and \( x \in \mathbb{R}^d \) a.s. The general case will follow as in [3] and [5]. Assume that \( u_0(x) = 1_{B(0,R)}(x) \), for some \( R > 0 \). Choose \( M > 0, t > 0 \) and \( m > 0 \). Define for \( k \geq 0 \),

\[
A_k := \left\{ u_s(x) \geq c_1^{k+1}1_{B^m_k}(x) \quad \text{for all} \quad s \in \left[ \frac{(2k+1)t}{2m}, \frac{(k+1)t}{m} \right] \quad \text{and} \quad x \in \mathbb{R}^d \right\}.
\]

For \( k \geq 1 \), set

\[
\tilde{A}_k := \left\{ u_s(x) \geq c_1^{k+1}1_{B^m_k}(x) \quad \text{for all} \quad s \in \left[ \frac{kt}{m}, \frac{(2k+1)t}{2m} \right] \quad \text{and} \quad x \in \mathbb{R}^d \right\}.
\]

Finally, we define

\[
\tilde{A}_0 := \left\{ u_{\frac{t}{2m}}(x) \geq c_11_{B^m_0}(x) \quad \text{for all} \quad x \in \mathbb{R}^d \right\},
\]

where \( B^m_k = B(0, R + kM(t/m)^{1/\alpha}) \) and \( c_1 \) is as in Proposition 5.2. It is clear that if \( \alpha > 1 \), then as \( m \) gets large, the sets \( B^m_k \) cover the whole space. For \( \alpha = 1 \), the sets \( B^m_k \) cover \( B(0, R + M t^{1/\alpha}) \). Since \( M \) is arbitrary, we have for any \( M \) large enough,

\[
P(u_s(x) > 0 \quad \text{for all} \quad t/2 \leq s \leq t \quad \text{and} \quad x \in B(0, M/2)) \geq \lim_{m \to \infty} P(\bigcap_{0 \leq k \leq m-1} A_k \cap \tilde{A}_k)
\]

(5.1)

Proposition 5.2 can be used to obtain

\[
P(A_0) \geq 1 - c_m,
\]

(5.2)

whenever \( m \) is large enough. On the other hand, on the event \( A_{k-1}, k \geq 1 \),

\[
u_{\frac{t}{m}}(x) \geq c_1^k1_{B^m_{k-1}}(x), \quad \text{for all} \quad x \in \mathbb{R}^d.
\]

By the Markov property, \( \{u_{s + \frac{t}{m}}(x), s \geq 0, x \in \mathbb{R}^d\} \) solves (1.1) with the time-shifted noise \( \hat{F}_k(s, x) := \hat{F}(s + \frac{kt}{m}, x) \) starting from \( u_{\frac{t}{m}}(x) \). Let \( \{v_s^{(k)}(x), s \geq 0, x \in \mathbb{R}^d\} \) be the solution to (1.1) with the time-shifted noise \( \hat{F}_k(s, x) \), \( \sigma \) replaced by \( \sigma_k(x) = c_1^{-k}\sigma(c_1^kx) \), and initial condition \( 1_{B^m_{k-1}}(x) \). On one hand, by Proposition 5.2 we get that

\[
P(v_{\frac{t}{m}}^{(k)}(x) \geq c_11_{B^m_k}(x) \quad \text{for all} \quad s \in \left[ \frac{t}{2m}, \frac{t}{m} \right] \quad \text{and} \quad x \in \mathbb{R}^d) \geq 1 - c_m,
\]

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whenever $m$ is large enough. On the other hand, by Markov property and the weak comparison principle (Theorem 1.1) we see that on $A_{k-1}$, $u_{s+kt/m}(x) \geq c^k v_s^{(k)}(x)$ for all $x \in \mathbb{R}^d$ and $s \geq 0$. We therefore have for all $k = 1, \ldots, m-1$,

$$P(A_k|F_{kt/m}) \geq 1 - c_m \text{ on } A_{k-1}.$$ 

And hence

$$P(A_k|A_{k-1} \cap \cdots \cap A_0) \geq 1 - c_m.$$ 

Similarly, we have

$$P(\tilde{A}_k|\tilde{A}_{k-1} \cap \cdots \cap \tilde{A}_0) \geq 1 - c_m.$$ 

Combining the above estimates as in [3] or [5], we have

$$P(u_s(x) > 0 \text{ for all } t/2 \leq s \leq t \text{ and } x \in B(0, M/2)) \geq 2(1 - c_m)^m - 1 \to 1,$$

as $m \to \infty$. This finishes the proof since $t > 0$ is arbitrary and $M$ can be taken as large as possible.

\section{Proof of Theorem 1.3}

\textit{Proof.} The proof is very similar to those in [3], [5] and [7], using the strong Markov property (Lemma 2.4) and the weak comparison principle (Theorem 1.1). So we omit it. \hfill \square

\textbf{Remark 5.3.} As mentioned in the introduction, the above comparison theorem and strict positivity results are shown under the assumption that the initial conditions are bounded functions. But this can be relaxed to a wider class of initial conditions as studied in [3] and [5]. We leave it to the reader to check the details. \hfill \square

\section*{References}


