Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:XX

pp. **X–XX**

# STABILISATION BY DELAY FEEDBACK CONTROL FOR HIGHLY NONLINEAR HYBRID STOCHASTIC DIFFERENTIAL EQUATIONS

Zhenyu Lu

School of Electronic and Information Engineering
Nanjing University of Information Science and Technology
Nanjing, Jiangsu 210044, China

Junhao Hu*

School of Mathematics and Statistics
South-Central University for Nationalities
Wuhan, Hubei 430074, China

Xuerong Mao

Department of Mathematics and Statistics
University of Strathclyde, Glasgow G1 1XH, U.K.

(Communicated by the associate editor Professor T. Caraballo)

Abstract. Given an unstable hybrid stochastic differential equation (SDE, also known as an SDE with Markovian switching), can we design a *delay* feedback control to make the controlled hybrid SDE become asymptotically stable? The paper [14] by Mao et al. was the first to study the stabilisation by *delay* feedback controls for hybrid SDEs, though the stabilization by *non-delay* feedback controls had been well studied. A critical condition imposed in [14] is that both drift and diffusion coefficients of the given hybrid SDE need to satisfy the linear growth condition. However, many hybrid SDE models in the real world do not fulfill this condition (namely, they are highly nonlinear) and hence there is a need to develop a new theory for these highly nonlinear SDE models. The aim of this paper is to design *delay* feedback controls in order to stabilise a class of highly nonlinear hybrid SDEs whose coefficients satisfy the polynomial growth condition.

1. **Introduction.** Hybrid stochastic differential equations (SDEs) driven by continuous-time Markov chains (also known as SDEs with Markovian switching) have frequently been used in many branches of science and industry. The hybrid SDEs can be described by

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t). \tag{1.1}$$

Here the state $x(t)$ takes values in $R^n$ and the mode $r(t)$ is a Markov chain taking values in a finite space $S = \{1, 2, \cdots, N\}$, $B(t)$ is a Brownian motion and $f$ and $g$ are referred to as the drift

2010 *Mathematics Subject Classification.* Primary: 60H10, 60J10; Secondary: 93D15.

*Key words and phrases.* Brownian motion, Markov chain, Asymptotic stability, Lypunov functional.

The authors would like to thank the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the Royal Society of Edinburgh (61294), the EPSRC (EP/K503174/1), the Natural Science Foundation of China (61773220, 61473334, 61876192, 61374085), the Ministry of Education (MOE) of China (MS2014DHDX020) for their financial support.

* Corresponding author: J. Hu.

and diffusion coefficient, respectively. One of the important issues in the study of hybrid SDEs is the analysis of stability (see, e.g., [4, 6, 15, 17, 20, 21, 23]). In particular, [12] is one of most cited papers (more than 560 Google citations) while [16] is the first book in this area (more than 800 Google citations).

Given an unstable hybrid SDE in the form of (1.1), it is classical to find a feedback control $u(x(t), r(t), t)$, based on the current state $x(t)$, for the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t), r(t), t)]dt + g(x(t), r(t), t)dB(t) \tag{1.2}$$

to become stable. However, taking into account a time lag $\tau$ ($> 0$) between the time when the observation of the state is made and the time when the feedback control reaches the system, it is more realistic that the control depends on a past state $x(t - \tau)$. Accordingly, the control should be of the form $u(x(t - \tau), r(t), t)$. Hence, the stabilisation problem becomes to design a delay feedback control $u(x(t - \tau), r(t), t)$ for the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t - \tau), r(t), t)]dt + g(x(t), r(t), t)dB(t) \tag{1.3}$$

to be stable. Mao et al. were the first to study this stabilisation problem in [14] by the delay feedback control for hybrid SDEs and there have been some further developments since then (see, e.g., [13, 22]), although the method of delay feedback controls has been well used in the area of ordinary differential equations (see, e.g., [1, 3, 19]). The common restrict condition imposed in these existing papers in the area of hybrid SDEs is that both drift coefficient $f$ and diffusion one $g$ need to satisfy the linear growth condition (namely bounded by linear functions). It is this restrict condition that excludes many SDE models in the real world, for example, the following scalar hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t), \tag{1.4}$$

where the coefficients $f$ and $g$ are defined by

$$f(x, 1, t) = x - 3x^3,\ f(x, 2, t) = x - x^3,\ g(x, 1, t) = x^2,\ g(x, 2, t) = 0.5x^2, \tag{1.5}$$

$B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with its generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix}. \tag{1.6}$$

This is a simple version of hybrid SDE models appeared frequently in finance and population systems (see, e.g., [2, 8]). It is therefore necessary and important to establish a new theory which shows how to design delay feedback controls in order to stabilise highly nonlinear hybrid SDEs. Let us begin to establish our new theory.

## 2. **Notation.**

Throughout this paper, unless otherwise specified, we use the following notation. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. For $x \in R^n$, $|x|$ denotes its Euclidean norm. If $A$ is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm. If $A$ is a symmetric real-valued matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. By $A \le 0$ and $A < 0$, we mean $A$ is non-positive and negative definite, respectively. Let $R_+ = [0, \infty)$. For $h > 0$, denote by $C([-h, 0]; R^n)$ the family of continuous functions $\varphi$ from $[-h, 0] \to R^n$ with the norm $\|\varphi\| = \sup_{-h \le u \le 0} |\varphi(u)|$. If both $a, b$ are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. If $A$ is a subset of $\Omega$, denote by $I_A$ its indicator function; that is, $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an $m$-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \ge 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \ne j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \ge 0$ is the transition rate from $i$ to $j$ if $i \ne j$ while

$$\gamma_{ii} = -\sum_{j \ne i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$. It is well known that almost all sample paths of $r(t)$ are piecewise constant except for a finite number of simple jumps in any finite subinterval of $R_+$. We stress that almost all sample paths of $r(t)$ are right continuous.

Suppose that the underlying system is described by a nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{2.1}$$

on $t \geq 0$ with the initial value $x(0) = x_0 \in R^n$, where

$$f : R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times S \times R_+ \to R^{n\times m}$$

are Borel measurable functions. The classical conditions for the existence and uniqueness of the global solution are the local Lipschitz condition and the linear growth condition (see, e.g., [9, 10, 11, 16]). In this paper, we need the local Lipschitz condition. However, we will consider highly nonlinear hybrid SDEs which, in general, do not satisfy the linear growth condition in this paper. We therefore impose the polynomial growth condition, instead of the linear growth condition. Let us state these conditions as an assumption for the use of this paper.

**Assumption 2.1.** *Assume that for any real number $b > 0$, there exists a positive constant $K_b$ such that*

$$|f(x, i, t) - f(\bar{x}, i, t)| \vee |g(x, i, t) - g(\bar{x}, i, t)| \leq K_b(|x - \bar{x}|) \tag{2.2}$$

*for all $x, \bar{x} \in R^n$ with $|x| \vee |\bar{x}| \leq b$ and all $(i, t) \in S \times R_+$. Assume moreover that there exist three constants $K > 0$, $q_1 \geq 1$ and $q_2 \geq 1$ such that*

$$|f(x, i, t)| \leq K(1 + |x|^{q_1}) \quad \text{and} \quad |g(x, i, t)| \leq K(1 + |x|^{q_2}) \tag{2.3}$$

*for all $(x, i, t) \in R^n \times S \times R_+$.*

Of course, if $q_1 = q_2 = 1$ then condition (2.3) is the familiar linear growth condition. However, let us stress once again that we are here interested in hybrid SDEs without the linear growth condition. In other words, we will either have $q_1 > 1$ or $q_2 > 1$. We will refer to condition (2.3) as the polynomial growth condition.

Let us now suppose that the given SDE (2.1) is unstable and we are required to design a delay feedback control $u(x(t-\tau), r(t), t)$ in the drift part so that the controlled system

$$dx(t) = [f(x(t), r(t), t) + u(x(t-\tau), r(t), t)]dt + g(x(t), r(t), t)dB(t), \quad t \geq 0, \tag{2.4}$$

becomes stable. Of course, we assume that the controller function $u(x, i, t)$ is a Borel measurable function and is locally Lipschitz in $x$. This controlled system is a hybrid stochastic differential delay equation (SDDE). For an SDDE, it is required to know the initial data $x(t)$ on $t \in [-\tau, 0]$ in order for its solution to be well defined, although the given SDE (2.1) is non-delay and it only requires the initial value $x(0) \in R^n$. This can be interpreted as follows: the underlying equation (2.1) evolved from before, say from time $-\tau$, and we have observed the whole segment $\{x(t) : -\tau \leq t \leq 0\}$ by the current time $t = 0$. Starting from time zero on, we will then design the feedback control $u(x(t-\tau), r(t), t)$ to stabilise the hybrid system. We can hence impose the initial data

$$\{x(t) : -\tau \leq t \leq 0\} = \xi \in C([-\tau, 0]; R^n) \text{ and } r(0) = r_0 \in S. \tag{2.5}$$

It is known that Assumption 2.1 only guarantees that the hybrid SDDE (2.4) has a unique maximal local solution, which may explode to infinity at a finite time (see, e.g., [16]). To avoid such a possible explosion, we need to impose an additional condition in terms of Lyapunov functions. For this purpose, we need more notation.

Let $C^{2,1}(R^n \times S \times R_+; R_+)$ denote the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in R^n \times S \times R_+$ which are continuously twice differentiable in $x$ and once in $t$. For such a function $U(x, i, t)$, we will let

$$U_t(x, i, t) = \frac{\partial U(x, i, t)}{\partial t}, \quad U_x(x, i, t) = \Big(\frac{\partial U(x, i, t)}{\partial x_1}, \cdots, \frac{\partial U(x, i, t)}{\partial x_n}\Big),$$

and

$$U_{xx}(x, i, t) = \Big(\frac{\partial^2 U(x, i, t)}{\partial x_k \partial x_l}\Big)_{n\times n}.$$

Let $C(R^n \times [-\tau, \infty); R_+)$ denote the family of all continuous functions from $R^n \times [-\tau, \infty)$ to $R_+$. For a given $U \in C^{2,1}(R^n \times S \times R_+; R_+)$, we define a function $LU : R^n \times S \times R_+ \to R$ by

$$\begin{aligned} LU(x, i, t) &= U_t(x, i, t) + U_x(x, i, t)f(x, i, t) \\ &\quad + \frac{1}{2}\text{trace}[g^T(x, i, t)U_{xx}(x, i, t)g(x, i, t)] + \sum_{j=1}^N \gamma_{ij} U(x, j, t). \end{aligned} \tag{2.6}$$

Please note that $LU$ is a single function (not $L$ acting on $U$) associated with the given SDE (2.1) but not the controlled SDDE. We can now state another assumption.

**Assumption 2.2.** *Assume that there exists a pair of functions $\bar{U} \in C^{2,1}(R^n \times S \times R_+; R_+)$ and $\bar{U}_1 \in C(R^n \times [-\tau, \infty); R_+)$, as well as three constants $c_1 > 0$, $c_2 \in (0,1)$ and $q \ge 2(q_1 \vee q_2)$ (where $q_1$ and $q_2$ are the same as in Assumption 2.1), such that*

$$|x|^q \le \bar{U}(x,i,t) \le \bar{U}_1(x,t) \quad \forall (x,i,t) \in R^n \times S \times R_+ \tag{2.7}$$

*and*

$$L\bar{U}(x,i,t) + \bar{U}_x(x,i,t)u(y,i,t) \le c_1 - \bar{U}_1(x,t) + c_2\bar{U}_1(y,t-\tau) \tag{2.8}$$

*for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$.*

Let us now cite a theorem from [4], which shows the unique global solution of the SDDE (2.4) and its $q$-th moment property under the above assumptions.

**Theorem 2.3.** *Under Assumptions 2.1 and 2.2, the SDDE (2.4) with the initial data (2.5) has the unique global solution $x(t)$ on $t \ge -\tau$ and the solution has the property that*

$$\sup_{-\tau \le t < \infty} \mathbb{E}|x(t)|^q < \infty. \tag{2.9}$$

This theorem implies a number of nice properties of the solution. For example, for any $t \ge 0$, $x(t)$ is bounded in $L^p$ for any $p \in (0,q]$ while both $f(x(t), r(t), t)$ and $g(x(t), r(t), t)$ are in $L^2$. These properties will play their fundamental roles when we discuss the stabilisation of the SDDE (2.4) in the next section. Assumptions 2.1 and 2.2 will form our standing hypotheses in this paper. Let us emphasise that we will NOT explicitly mention Assumptions 2.1 and 2.2 in the next section in order for us to concentrate on our new assumptions to be imposed.

For the stability purpose of this paper, we naturally assume that

$$f(0,i,t) \equiv 0, \;\; u(0,i,t) \equiv 0, \;\; g(0,i,t) \equiv 0 \tag{2.10}$$

for all $(i,t) \in S \times R_+$. So the SDDE (2.4) admits a trivial solution $x(t) \equiv 0$.

3. **Stabilisation.** In this section, we will use the method of Lyapunov functionals to investigate the asymptotic stability of the controlled SDDE (2.4). To define a Lyapunov functional for the use of this paper, we define two segments $\hat{x}_t := \{x(t+s) : -2\tau \le s \le 0\}$ and $\hat{r}_t := \{r(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \le t < 2\tau$, we set $x(s) = \xi(-\tau)$ for $s \in [-2\tau, -\tau)$ and $r(s) = r_0$ for $s \in [-2\tau, 0)$. The Lyapunov functional used in this paper will be of the form

$$\begin{aligned} V(\hat{x}_t, \hat{r}_t, t) &= U(x(t), r(t), t) \\ &+ \theta \int_{-\tau}^{0} \int_{t+s}^{t} \Big[\tau |f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv ds \end{aligned} \tag{3.1}$$

for $t \ge 0$, where $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and $\theta$ is a positive number to be determined later while we set

$$f(x,i,v) = f(x,i,0), \quad u(x,i,v) = u(x,i,0), \quad g(x,i,v) = g(x,,i,0)$$

for $(x,i,v) \in R^n \times S \times [-2\tau, 0)$. The following lemma shows that $V(\hat{x}_t, \hat{r}_t, t)$ is an Itô process.

**Lemma 3.1.** *With the notation above, $V(\hat{x}_t, \hat{r}_t, t)$ is an Itô process on $t \ge 0$ with its Itô differential*

$$dV(\hat{x}_t, \hat{r}_t, t) = \mathbb{L}V(\hat{x}_t, \hat{r}_t, t)dt + dM(t), \tag{3.2}$$

*where $M(t)$ is a continuous local martingale with $M(0) = 0$ and*

$$\begin{aligned} &\mathbb{L}V(\hat{x}_t, \hat{r}_t, t) \\ &= LU(x(t), r(t), t) + U_x(x(t), r(t), t)u(x(t-\tau), r(t), t) \\ &\quad + \theta\tau\Big[\tau|f(x(t), r(t), t) + u(x(t-\tau), r(t), t)|^2 + |g(x(t), r(t), t)|^2\Big] \\ &\quad - \theta \int_{t-\tau}^{t} \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv. \end{aligned} \tag{3.3}$$

*Proof.* Regarding the solution $x(t)$ of equation (2.4) as an Itô process and applying the generalised Itô formula (see, e.g., [16]) to $U(x(t), r(t), t)$, we get

$$dU(x(t), r(t), t) = \big[LU(x(t), r(t), t) + U_x(x(t), r(t), t)u(x(t-\tau), r(t), t)\big]dt + dM(t), \tag{3.4}$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this paper so we do not state it here but it can be found in [16, Theorem 1.45 on page 48]) and the function $LU$ has been defined in Section 2. On the other hand, the fundamental theory of calculus shows

$$\begin{aligned} &d\Big(\int_{-\tau}^{0}\int_{t+s}^{t}\Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dvds\Big) \\ &= \Big(\tau\Big[\tau|f(x(t), r(t), t) + u(x(v-\tau), r(v), v)|^2 + |g(x(t), r(t), t)|^2\Big] \\ &\quad - \int_{t-\tau}^{t}\Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv\Big)dt. \end{aligned} \tag{3.5}$$

Applying the generalised Itô formula (see, e.g., [16]) to the Lyapunov functional defined by (3.1) and using (3.4) and (3.5), we then get the required assertion (3.3). □

To study the asymptotic stability of the controlled SDDE (2.4), we need to impose a couple of new assumptions. First of all, recall that the delay $\tau$ is the time lag between the time when the observation of the state is made and the time when the feedback control reaches the system. If there is no time lag, namely the feedback control acts instantly when the state observation is made, then the controlled SDDE (2.4) becomes the controlled SDE (1.2) (the classical one). This indicates that the feedback control should at least be able to make the SDE (1.2) asymptotically stable and hence $LU(x, i, t) + U_x(x, i, t)u(x, i, t)$ should be negative-definite. However, our underlying controlled system is a highly nonlinear SDDE. To cope with the effect of the time lag and high nonlinearity, we need to impose a stronger assumption.

**Assumption 3.2.** *Assume that there is a function $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and positive constants $\beta_j$, $j = 0, 1, 2, 3$, such that*

$$\begin{aligned} LU(x, i, t) + U_x(x, i, t)u(x, i, t) + \beta_1|U_x(x, i, t)|^2 \\ + \beta_2|f(x, i, t)|^2 + \beta_3|g(x, i, t)|^2 \leq -\beta_0|x|^2 \end{aligned} \tag{3.6}$$

*for all $(x, i, t) \in R^n \times S \times R_+$.*

We next compare our controlled SDDE (2.4) with the SDE (1.2) and observe that it is the difference $u(x(t), r(t), t) - u(x(t-\tau), r(t), t)$ which makes the two controlled systems different. However, if the time lag $\tau$ is sufficiently small, we may hope the difference $u(x(t), r(t), t) - u(x(t-\tau), r(t), t)$ could be so small that the SDDE (2.4) remains stable and this would be the case if $u(x, i, t)$ is uniformly continuous in $x$. This motivates us to impose the other assumption in this section.

**Assumption 3.3.** *Assume that there exists a positive number $\beta$ such that*

$$|u(x, i, t) - u(y, i, t)| \leq \beta|x - y| \tag{3.7}$$

*for all $x, y \in R^n$, $i \in S$ and $t \geq 0$.*

This assumption, together with (2.10), implies

$$|u(x, i, t)| \leq \beta|x|, \quad \forall (x, i, t) \in R^n \times S \times R_+. \tag{3.8}$$

**Theorem 3.4.** *Let Assumptions 3.2 and 3.3 hold. Assume also that*

$$\tau < \frac{\sqrt{\beta_0\beta_1}}{\beta^2} \quad \text{and} \quad \tau \leq \frac{\sqrt{\beta_1\beta_2}}{\beta} \wedge \frac{2\beta_1\beta_3}{\beta^2}. \tag{3.9}$$

*Then for any given initial data (2.5), the solution of the SDDE (2.4) has the property that*

$$\int_0^\infty \mathbb{E}|x(t)|^2 dt < \infty. \tag{3.10}$$

*That is, the controlled SDDE (2.4) is $H_\infty$-stable in $L^2$.*

*Proof.* Fix the initial data $\xi \in C([-\tau, 0]; R^n)$ and $r_0 \in S$ arbitrarily. Let $k_0 > 0$ be a sufficiently large integer such that $\|\xi\| < k_0$. For each integer $k \geq k_0$, define the stopping time

$$\zeta_k = \inf\{t \geq 0 : |x(t)| \geq k\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). It is easy to see, by Theorem 2.3, that $\zeta_k$ is increasing to infinity with probability 1 as $k \to \infty$. By the generalised Itô formula (see, e.g., [16, Lemma 1.9 on page 49]), we obtain from Lemma 3.1 that

$$\mathbb{E}V(\hat{x}_{t\wedge\zeta_k}, \hat{r}_{t\wedge\zeta_k}, t\wedge\zeta_k) = V(\hat{x}_0, \hat{r}_0, 0) + \mathbb{E}\int_0^{t\wedge\zeta_k} \mathbb{L}V(\hat{x}_s, \hat{r}_s, s)ds \tag{3.11}$$

for any $t \ge 0$ and $k \ge k_0$.

We now let $\theta = \beta^2/(2\beta_1)$. (Please recall that $\theta$ is the free parameter in the definition of the Lyapunov functional.) By Assumption 3.3, it is easy to see that

$$\begin{aligned} &U_x(x(t), r(t), t)[u(x(t-\tau), r(t), t) - u(x(t), r(t), t)] \\ &\qquad \le \beta_1 |U_x(x(t), r(t), t)|^2 + \frac{\beta^2}{4\beta_1}|x(t) - x(t-\tau)|^2. \end{aligned} \tag{3.12}$$

By condition (3.9), we also have

$$2\theta\tau^2 \le \beta_2 \quad \text{and} \quad \theta\tau \le \beta_3. \tag{3.13}$$

It then follows from Lemma 3.1 that

$$\begin{aligned} &\mathbb{L}V(\hat{x}_s, \hat{r}_s, s) \\ &\le LU(x(s), r(s), s) + U_x(x(s), r(s), s)u(x(s), r(s), s) + \beta_1|U_x(x(s), r(s), s)|^2 \\ &\quad + \beta_2|f(x(s), r(s), s)|^2 + 2\theta\tau^2|u(x(s-\tau), r(s), s)|^2 + \beta_3|g(x(s), r(s), s)|^2 \\ &\quad + \frac{\beta^2}{4\beta_1}|x(s) - x(s-\tau)|^2 \\ &\quad - \frac{\beta^2}{2\beta_1}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv. \end{aligned}$$

By Assumption 3.2 and inequality (3.8), we then have

$$\begin{aligned} &\mathbb{L}V(\hat{x}_s, \hat{r}_s, s) \\ &\le -\beta_0|x(s)|^2 + 2\theta\tau^2\beta^2|x(s-\tau)|^2 + \frac{\beta^2}{4\beta_1}|x(s) - x(s-\tau)|^2 \\ &\quad - \frac{\beta^2}{2\beta_1}\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big]dv. \end{aligned}$$

Substituting this into (3.11) implies

$$\mathbb{E}V(\hat{x}_{t\wedge\zeta_k}, \hat{r}_{t\wedge\zeta_k}, t\wedge\zeta_k) \le V(\hat{x}_0, \hat{r}_0, 0) + \Psi_1 + \Psi_2 - \Psi_3, \tag{3.14}$$

where

$$\begin{aligned} \Psi_1 &= \mathbb{E}\int_0^{t\wedge\zeta_k} \big[-\beta_0|x(s)|^2 + 2\theta\tau^2\beta^2|x(s-\tau)|^2\big]ds, \\ \Psi_2 &= \frac{\beta^2}{4\beta_1}\mathbb{E}\int_0^{t\wedge\zeta_k} |x(s) - x(s-\tau)|^2 ds, \\ \Psi_3 &= \frac{\beta^2}{2\beta_1}\mathbb{E}\int_0^{t\wedge\zeta_k} \Big(\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 \\ &\qquad\qquad + |g(x(v), r(v), v)|^2\Big]dv\Big)ds. \end{aligned}$$

Noting that

$$\int_0^{t\wedge\zeta_k} |x(s-\tau)|^2 ds \le \int_{-\tau}^{t\wedge\zeta_k} |x(s)|^2 ds,$$

we have

$$\Psi_1 \le 2\theta\tau^3\beta^2\|\xi\|^2 - (\beta_0 - 2\theta\tau^2\beta^2)\mathbb{E}\int_0^{t\wedge\zeta_k} |x(s)|^2 ds, \tag{3.15}$$

Substituting this into (3.14) and recalling (3.1), we obtain

$$\mathbb{E}U(x(t\wedge\zeta_k), r(t\wedge\zeta_k), t\wedge\zeta_k) + (\beta_0 - 2\theta\tau^2\beta^2)\mathbb{E}\int_0^{t\wedge\zeta_k} |x(s)|^2 ds \le C_1 + \Psi_2 - \Psi_3, \tag{3.16}$$

where $C_1 = V(\hat{x}_0, \hat{r}_0, 0) + 2\theta\tau^3\beta^2\|\xi\|^2$. Applying the well-known Fatou lemma and recalling the paragraph below Theorem 2.3, we can let $k \to \infty$ in (3.16) to get

$$(\beta_0 - 2\theta\tau^2\beta^2)\mathbb{E}\int_0^t |x(s)|^2 ds \le C_1 + \bar{\Psi}_2 - \bar{\Psi}_3, \tag{3.17}$$

where

$$\bar{\Psi}_2 = \frac{\beta^2}{4\beta_1}\mathbb{E}\int_0^t |x(s) - x(s-\tau)|^2 ds,,$$
$$\bar{\Psi}_3 = \frac{\beta^2}{2\beta_1}\mathbb{E}\int_0^t \Big(\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv\Big) ds.$$

But, by the well-known Fubini theorem,

$$\bar{\Psi}_2 = \frac{\beta^2}{4\beta_1}\int_0^t \mathbb{E}|x(s) - x(s-\tau)|^2 ds.$$

For $t \in [0, \tau]$, we clearly have

$$\bar{\Psi}_2 \le \frac{\beta^2}{2\beta_1}\int_0^\tau (\mathbb{E}|x(s)|^2 + \mathbb{E}|x(s-\tau)|^2) ds \le \frac{\tau\beta^2}{\beta_1}\Big(\sup_{-\tau \le v \le \tau} \mathbb{E}|x(v)|^2\Big) =: C_2,$$

where, as usual, =: means 'denoted by'. For $t > \tau$, we have

$$\bar{\Psi}_2 \le C_2 + \frac{\beta^2}{4\beta_1}\int_\tau^t \mathbb{E}|x(s) - x(s-\tau)|^2 ds.$$

On the other hand, it follows from the SDDE (2.4) that, for $s \ge \tau$,

$$\begin{aligned}
&\mathbb{E}|x(s) - x(s-\tau)|^2 \\
&= \mathbb{E}\Big|\int_{s-\tau}^s [f(x(v), r(v), v) + u(x(v-\tau), r(v), v)] dv + \int_{s-\tau}^s g(x(v), r(v), v) dB(v)\Big|^2 \\
&\le 2\mathbb{E}\int_{s-\tau}^s \Big(\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big) dv.
\end{aligned}$$

Hence

$$\begin{aligned}
\bar{\Psi}_2 &\le C_2 + \frac{\beta^2}{2\beta_1}\int_\tau^t \mathbb{E}\Big(\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 \\
&\qquad + |g(x(v), r(v), v)|^2\Big] dv\Big) ds \\
&\le C_2 + \frac{\beta^2}{2\beta_1}\mathbb{E}\int_0^t \Big(\int_{s-\tau}^s \Big[\tau|f(x(v), r(v), v) + u(x(v-\tau), r(v), v)|^2 \\
&\qquad + |g(x(v), r(v), v)|^2\Big] dv\Big) ds \\
&= C_2 + \bar{\Psi}_3,
\end{aligned}$$

where the Fubini theorem has been used once again. In other words, we always have

$$\bar{\Psi}_2 \le C_2 + \bar{\Psi}_3, \quad \forall t \ge 0. \tag{3.18}$$

Substituting this into (3.17) yields

$$(\beta_0 - 2\theta\tau^2\beta^2)\mathbb{E}\int_0^t |x(s)|^2 ds \le C_1 + C_2. \tag{3.19}$$

Noting that $\beta_0 - 2\theta\tau^2\beta^2 = \beta_0 - \beta^4\tau^2/\beta_1 > 0$ by condition (3.9), we see from the above inequality that

$$\mathbb{E}\int_0^t |x(s)|^2 ds \le \frac{C_1 + C_2}{\beta_0 - \beta^4\tau^2/\beta_1}.$$

Letting $t \to \infty$ and then using the Fubini theorem we obtain the assertion (3.10). The proof is therefore complete. □

In general, it does not follow from (3.10) that $\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0$. However, in our case, this is possible. We state this as our second theorem in this section.

**Theorem 3.5.** *Under the same assumptions of Theorem 3.4, the solution of the controlled hybrid SDDE (2.4) satisfies*

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^2 = 0$$

*for any given initial data (2.5). That is, the controlled system (2.4) is asymptotically stable in mean square.*

*Proof.* Fix the initial data (2.5) arbitrarily. By Theorem 2.3 and conditions (2.3) and (3.8), we can apply the Itô formula to show

$$\begin{aligned} &\big|\mathbb{E}|x(t_2)|^2 - \mathbb{E}|x(t_1)|^2\big| \\ =&\Big|\mathbb{E}\int_{t_1}^{t_2}\Big(2x(t)[f(x(t),r(t),t)+u(x(t-\tau),r(t),t)] + |g(x(t),r(t),t)|^2\Big)dt\Big| \\ \le& C_3(t_2-t_1), \end{aligned}$$

where $C_3$ is a constant independent of $t_1$ and $t_2$. That is, $\mathbb{E}|x(t)|^2$ is uniformly continuous in $t$ on $R_+$. It then follows from (3.10) that $\lim_{t\to\infty}\mathbb{E}|x(t)|^2 = 0$ as required. □

In general, we cannot imply $\lim_{t\to\infty}|x(t)| = 0$ a.s. from (3.10). But, in our case, this is once again possible with an additional condition. We should also point out that You et al. [22] showed this *under the linear growth condition* on the coefficients of the underlying SDDE. Our new proof given below does not only overcome the difficulty without the linear growth condition but is also much simplified.

**Theorem 3.6.** *In addition to the assumptions of Theorem 3.4, assume that*

$$\lim_{k\to\infty}\Big(\inf\{U(x,i,t) : |x|\ge k,\ (i,t)\in S\times R_+\}\Big) = \infty. \tag{3.20}$$

*Then the solution of the controlled hybrid SDDE (2.4) satisfies*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{3.21}$$

*for any given initial data (2.5). That is, the controlled system (2.4) is almost surely asymptotically stable.*

*Proof.* Again fix any initial data (2.5). We first observe that (3.10) is equivalent to that

$$C_4 := \mathbb{E}\int_0^\infty |x(t)|^2 dt < \infty \tag{3.22}$$

by the well-known Fubini theorem. This implies that $\int_0^\infty |x(t)|^2dt < \infty$ a.s. and hence

$$\liminf_{t\to\infty}|x(t)| = 0 \quad a.s. \tag{3.23}$$

But this is not the required assertion (3.21) yet. Let us now assume that the assertion were not true. There is then a positive number $\varepsilon\in(0,1/4)$ such that

$$\mathbb{P}\Big(\limsup_{t\to\infty}|x(t)| > 2\varepsilon\Big) \ge 4\varepsilon. \tag{3.24}$$

For each $k\ge\|\xi\|$, let $\zeta_k$ be the same stopping time as defined in the proof of Theorem 3.4 and set

$$\phi_k = \inf\{U(x,i,t) : |x|\ge k,\ (i,t)\in S\times R_+\}.$$

It follows from (3.16) that

$$\phi_k\mathbb{P}(\zeta_k\le t) \le C_1 + \Psi_2 - \Psi_3, \quad \forall t\ge 0.$$

This, together with (3.18), implies

$$\limsup_{k\to\infty}\phi_k\mathbb{P}(\zeta_k\le t) \le C_1 + C_2, \quad \forall t\ge 0. \tag{3.25}$$

As this holds for any $t\ge 0$, we must have

$$\limsup_{k\to\infty}\phi_k\mathbb{P}(\zeta_k<\infty) \le C_1 + C_2.$$

Thus, there is a positive integer $k_1$ large enough for

$$\phi_k\mathbb{P}(\zeta_k<\infty) \le C_1 + C_2 + 1, \quad \forall k\ge k_1.$$

We can then choose a particular $k\ge k_1$, which will be fixed from now on, sufficiently large for $(C_1+C_2+1)/\phi_k\le\varepsilon$ to get $\mathbb{P}(\zeta_k<\infty)\le\varepsilon$. This means that

$$\mathbb{P}(|x(t)|<k \text{ for } \forall t\ge-\tau) \ge 1-\varepsilon. \tag{3.26}$$

Combining (3.24) and (3.26) together gives

$$\mathbb{P}(\bar{\Omega}) \geq 3\varepsilon, \tag{3.27}$$

where

$$\bar{\Omega} = \Big\{ \limsup_{t\to\infty} |x(t)| > 2\varepsilon \text{ and } |x(t)| < k \text{ for } \forall t \geq -\tau \Big\}.$$

Define the stopped process $\bar{x}(t) = x(t \wedge \zeta_k)$ for $t \geq 0$. Clearly, $\bar{x}(t)$ is an Itô process of the form

$$d\bar{x}(t) = \bar{f}(t)dt + \bar{g}(t)dB(t), \tag{3.28}$$

where

$$\begin{aligned} \bar{f}(t) &= [f(x(t), r(t), t) + u(x(t-\tau), r(t), t)] I_{[0,\zeta_k)}(t), \\ \bar{g}(t) &= g(x(t), r(t), t) I_{[0,\zeta_k)}(t). \end{aligned}$$

By Assumptions 2.2, 3.3 as well as condition (2.10), we see that $\bar{f}(t)$ and $\bar{g}(t)$ are bounded processes, say

$$|\bar{f}(t)| \vee |\bar{g}(t)| \leq C_5 \quad a.s. \tag{3.29}$$

for all $t \geq 0$. Let us now define a sequence of stopping times

$$\begin{aligned} \mu_1 &= \inf\{t \geq 0 : |\bar{x}(t)| \geq 2\varepsilon\}, \\ \mu_{2l} &= \inf\{t \geq \mu_{2l-1} : |\bar{x}(t)| \leq \varepsilon\}, \ l = 1, 2, \cdots, \\ \mu_{2l+1} &= \inf\{t \geq \mu_{2l} : |\bar{x}(t)| \geq 2\varepsilon\}, \ l = 1, 2, \cdots. \end{aligned}$$

By (3.23) and the definition of $\bar{\Omega}$, we have

$$\bar{\Omega} \subset \{\mu_l < \infty\}, \ l = 1, 2, \cdots. \tag{3.30}$$

Choose a small positive number $\nu$ and a large positive integer $\bar{l}$ such that

$$C_5(\nu + 4\sqrt{2\nu}) \leq \varepsilon^2 \quad \text{and} \quad C_4 < \varepsilon^3 \nu \bar{l}. \tag{3.31}$$

By (3.27) and (3.30), we can further choose a sufficiently large number $T$ for

$$\mathbb{P}(\mu_{2\bar{l}} \leq T) \geq 2\varepsilon. \tag{3.32}$$

In particular, if $\mu_{2\bar{l}} \leq T$, $|\bar{x}(\mu_{2\bar{l}})| = \varepsilon$ and hence $\mu_{2\bar{l}} < \zeta_k$ by the definition of $\bar{x}(t)$ (otherwise $|\bar{x}(\mu_{2\bar{l}})| = |\bar{x}(\zeta_k)| = k$, a contradiction). In other words, we have

$$\bar{x}(t,\omega) = x(t,\omega) \text{ for all } 0 \leq t \leq \mu_{2\bar{l}} \text{ and } \omega \in \{\mu_{2\bar{l}} \leq T\}. \tag{3.33}$$

By the Burkholder-Davis-Gundy inequality (see, e.g., [16, Theorem 2.13 on page 70]), we can then derive from (3.28) that, for $1 \leq l \leq \bar{l}$,

$$\begin{aligned} &\mathbb{E}\Big( \sup_{0\leq t\leq\nu} \big||\bar{x}(\mu_{2l-1}\wedge T + t)| - |\bar{x}(\mu_{2l-1}\wedge T)|\big| \Big) \\ \leq &\mathbb{E}\Big( \sup_{0\leq t\leq\nu} |\bar{x}(\mu_{2l-1}\wedge T + t) - \bar{x}(\mu_{2l-1}\wedge T)| \Big) \\ \leq &\mathbb{E}\int_{\mu_{2l-1}\wedge T}^{\mu_{2l-1}\wedge T+\nu} |\bar{f}(s)|ds + 4\sqrt{2}\mathbb{E}\Big( \int_{\mu_{2l-1}\wedge T}^{\mu_{2l-1}\wedge T+\nu} |\bar{g}(s)|^2 ds \Big)^{1/2} \\ \leq &C_5(\nu + 4\sqrt{2\nu}). \end{aligned}$$

This, together with (3.31), implies

$$\mathbb{P}\Big( \sup_{0\leq t\leq\nu} \big||\bar{x}(\mu_{2l-1}\wedge T + t)| - |\bar{x}(\mu_{2l-1}\wedge T)|\big| \geq \varepsilon \Big) \leq \varepsilon.$$

Noting that $\mu_{2l-1} \leq T$ if $\mu_{2\bar{l}} \leq T$, we can derive from (3.32) and the above inequality that

$$\begin{aligned} &\mathbb{P}\Big( \{\mu_{2\bar{l}} \leq T\} \cap \Big\{ \sup_{0\leq t\leq\nu} \big||\bar{x}(\mu_{2l-1} + t)| - |\bar{x}(\mu_{2l-1})|\big| < \varepsilon \Big\} \Big) \\ &= \mathbb{P}(\mu_{2\bar{l}} \leq T) - P\Big( \{\mu_{2\bar{l}} \leq T\} \cap \Big\{ \sup_{0\leq t\leq\nu} \big||\bar{x}(\mu_{2l-1}\wedge T + t)| - |\bar{x}(\mu_{2l-1}\wedge T)|\big| \geq \varepsilon \Big\} \Big) \\ &\geq \mathbb{P}(\mu_{2\bar{l}} \leq T) - \mathbb{P}\Big( \sup_{0\leq t\leq\nu} \big||\bar{x}(\mu_{2l-1}\wedge T + t)| - |\bar{x}(\mu_{2l-1}\wedge T)|\big| \geq \varepsilon \Big) \\ &\geq \varepsilon. \end{aligned}$$

This implies easily that

$$\mathbb{P}\Big( \{\mu_{2\bar{l}} \leq T\} \cap \{\mu_{2l} - \mu_{2l-1} \geq \nu\} \Big) \geq \varepsilon. \tag{3.34}$$

Finally, by (3.22), (3.33) and (3.34), we derive

$$\begin{aligned} C_4 &= \mathbb{E}\int_0^\infty |x(t)|^2 dt \\ &\geq \sum_{l=1}^{\bar{l}} \mathbb{E}\Big(I_{\{\mu_{2\bar{l}}\leq T\}}\int_{\mu_{2l-1}}^{\mu_{2l}} |\bar{x}(t)|^2 dt\Big) \\ &\geq \varepsilon^2 \sum_{l=1}^{\bar{l}} \mathbb{E}\Big(I_{\{\mu_{2\bar{l}}\leq T\}}(\mu_{2l}-\mu_{2l-1})\Big) \\ &\geq \varepsilon^2\nu \sum_{l=1}^{\bar{l}} \mathbb{P}\Big(\{\mu_{2\bar{l}}\leq T\}\cap\{\mu_{2l}-\mu_{2l-1}\geq\nu\}\Big)\Big) \\ &\geq \varepsilon^3\nu\bar{l}. \end{aligned}$$

But this contradicts the second inequality in (3.31). Therefore the required assertion (3.21) must hold. The proof is complete. □

4. **Examples.** As pointed out in Section 1, our key aim in this paper is to design a *delay* feedback control in order to stabilise a given unstable hybrid SDE whose drift and diffusion coefficients are highly nonlinear. As far as we know, the paper [14] by Mao et al. was the first to study the stabilisation by *delay* feedback controls for hybrid SDEs and is the only paper on this topic so far. However, a critical condition imposed in [14] is that both drift and diffusion coefficients of the given unstable hybrid SDE need to satisfy the linear growth condition. Our new result in this paper has removed this restrictive condition, whence our new result enables us to design a *delay* feedback control in order to stabilise a given unstable hybrid SDE. On the other hand, the use of our result depends on the construction of the Lyapunov function $U(x,i,t)$ and the control function $u(x,i,t)$ for conditions (3.6) and (3.7) to hold. We realise that, in general, there is no theory on the construction of the Lyapunov functions. Of course, this is not the problem of the Lyapunov method and, in fact, it is because there is no such a general theory that the Lyapunov method has been one of most powerful methods in the stability study for more than 100 years. Nevertheless, to apply our new result, we may first construct the Lyapunov function $U(x,i,t)$ and the control function $u(x,i,t)$ for the following hybrid SDE (not SDDE)

$$dx(t) = [f(x(t),r(t),t) + u(x(t),r(t),t)]dt + g(x(t),r(t),t)dB(t)$$

to be stable. There is a great amount of literature on hybrid SDEs (see, e.g., [16] and the reference therein). For our purpose, we need require that

$$LU(x,i,t) + U_x(x,i,t)u(x,i,t) \leq -\beta_0|x|^2$$

as well as (3.7) to hold. In order to overcome the high nonlinearity of the coefficients $f(x,i,t)$ and $g(x,i,t)$, we may next modify them in order for them to satisfy the stronger condition (3.6). Finally, we just restrict the time delay $\tau$ sufficiently small for (3.9) to hold.

We should point out that the control function $u(x,i,t)$ used in this paper is allowed to depend on the mode $i$ (i.e., the state of the Markov chain). There are two reasons for this. One is because it is easier to design the control function $u(x,i,t)$, which is the key for our theory to be applied, if we take the different system structure in different mode into account. The other is because the underlying system may not be observable in some modes. In this case, the feedback control could not be used when the system are operating in those modes (namely we have to set $u(x,i,t)=0$ for those $i$) and the feedback control could only be designed for the other modes where the system is observable. This situation is illustrated in our Example 4.2 fully.

To illustrate our theoretical results, we return to the hybrid SDE (1.4), where the coefficients $f$ and $g$ are defined by (1.5), $B(t)$ is a scalar Brownian motion and $r(t)$ is a Markov chain on $S=\{1,2\}$ with the generator $\Gamma$ defined by (1.6). As we mentioned in Section 1, this is a simple version of hybrid SDE models appeared frequently in finance and population systems (see, e.g., [2, 8]). The reason why we use this simplified SDE model is not only to avoid the verification of the assumptions imposed becoming too long but also be able to illustrate our theory fully. We consider two cases.

**Example 4.1.** We first consider the case where the system is fully observable and controllable in both mode 1 and 2. That is, we could use a delay feedback control in both modes to stabilise the given unstable hybrid SDE (1.4). In our notation, we assume that the controlled hybrid SDDE has the form

$$dx(t) = [f(x(t), r(t), t) + u(x(t-\tau), r(t), t)]dt + g(x(t), r(t), t)dB(t) \tag{4.1}$$

while the delay feedback control is defined by

$$u(x(t-\tau), r(t), t) = \begin{cases} -2x(t-\tau) & \text{if } r(t) = 1, \\ -x(t-\tau) & \text{if } r(t) = 2. \end{cases} \tag{4.2}$$

To apply our results in order to get the bound on $\tau$, we need to verify the assumptions imposed in Sections 2 and 3. It is easy to see that Assumption 2.1 is satisfied with $q_1 = 3$ and $q_2 = 2$. To verify Assumption 2.2, we define $\bar{U}(x, i, t) = |x|^6$ for $(x, i, t) \in R \times S \times R_+$. It is straightforward to show that, for $(x, y, i, t) \in R \times R \times S \times R_+$,

$$L\bar{U}(x, i, t) + \bar{U}_x(x, i, t)u(y, i, t) = \begin{cases} 6x^6 - 3x^8 - 12x^5 y & \text{if } i = 1, \\ 6x^6 - 2.25x^8 - 6x^5 y & \text{if } i = 2. \end{cases}$$

Noting that $6x^5 y \le 5x^6 + y^6$, we then have

$$L\bar{U}(x, i, t) + \bar{U}_x(x, i, t)u(y, i, t) \le c_1 - 4x^6 + 2y^6,$$

where $c_1 = \sup_{x \in R} \big[(20x^6 - 3x^8) \vee (10x^6 - 2.25x^8)\big] < \infty$. That is, Assumption 2.2 is fulfilled with $\bar{U}_1(x, t) = 4x^6$, $c_2 = 0.5$ and $q = 6$.

To verify Assumption 3.2, we define

$$U(x, i, t) = \begin{cases} 0.75(x^2 + x^4) & \text{if } i = 1, \\ x^2 + x^4 & \text{if } i = 2 \end{cases}$$

for $(x, i, t) \in R \times S \times R_+$. It is easy to show that

$$LU(x, i, t) + U_x(x, i, t)u(x, i, t) = \begin{cases} -1.25x^2 - 6.5x^4 - 4.5x^6 & \text{if } i = 1, \\ -1.25x^2 - 3x^4 - 2.5x^6 & \text{if } i = 2. \end{cases}$$

Denote the left hand side of inequality (3.6) by $LHS$. Then

$$\begin{aligned} LHS &= -1.25x^2 - 6.5x^4 - 4.5x^6 + \beta_1(1.5x + 3x^3)^2 + \beta_2(x - 3x^3)^2 + \beta_3 x^4 \\ &= -(1.25 - 2.25\beta_1 - \beta_2)x^2 - (6.5 - 9\beta_1 + 6\beta_2 - \beta_3)x^4 - (4.5 - 9\beta_1 - 9\beta_2)x^6 \end{aligned}$$

when $i = 1$, and

$$\begin{aligned} LHS &= -1.25x^2 - 3x^4 - 2.5x^6 + \beta_1(2x + 4x^3)^2 + \beta_2(x - x^3)^2 + 0.25\beta_3 x^4 \\ &= -(1.25 - 4\beta_1 - \beta_2)x^2 - (3 - 16\beta_1 + 2\beta_2 - 0.25\beta_3)x^4 - (2.5 - 16\beta_1 - \beta_2)x^6 \end{aligned}$$

when $i = 2$. Choosing $\beta_1 = \beta_2 = 0.1$ and $\beta_3 = 5.8$, we obtain

$$LHS = \begin{cases} -0.825x^2 - 0.4x^4 - 2.7x^6 & \text{if } i = 1, \\ -0.75x^2 - 0.15x^4 - 0.8x^6 & \text{if } i = 2. \end{cases}$$

Thus we always have

$$LHS \le -0.75x^2.$$

In other words, we have just verified Assumption 3.2 with $\beta_0 = 0.75$. Moreover, by definition (4.2) of the delay feedback control, we see easily that Assumption 3.3 holds with $\beta = 2$. Finally, condition (3.9) becomes $\tau \le 0.06847$. By Theorems 3.4, 3.5 and 3.6, we can therefore conclude that if we use the delay feedback control (4.2) and make sure the time delay $\tau \le 0.06847$, then the controlled hybrid SDDE (4.1) is not only $H_\infty$-stable in $L^2$ but also asymptotically stable in $L^2$ and almost surely as well.

We perform a computer simulation with the time-delay $\tau = 0.06$ for all $t \ge 0$ and the initial data $x(u) = 5 + \sin(u)$ for $u \in [-0.06, 0]$ and $r(0) = 2$. The sample paths of the Markov chain and the solution of the SDDE (4.1) are plotted in Figure 4.1. The simulation supports our theoretical results clearly.

**Example 4.2.** We now consider the case where the system is observable only in mode 1 but not in mode 2 so we could only use a delay feedback control in mode 1. In our notation, we assume that the controlled hybrid SDDE has the form of (4.1) and the delay feedback control is defined by

$$u(x(t-\tau), r(t), t) = \begin{cases} -5x(t-\tau) & \text{if } r(t) = 1, \\ 0 & \text{if } r(t) = 2. \end{cases} \tag{4.3}$$

To apply our results in order to get the bound on $\tau$, we need to verify the assumptions imposed in Sections 2 and 3. As before, Assumption 2.1 is satisfied with $q_1 = 3$ and $q_2 = 2$. To verify

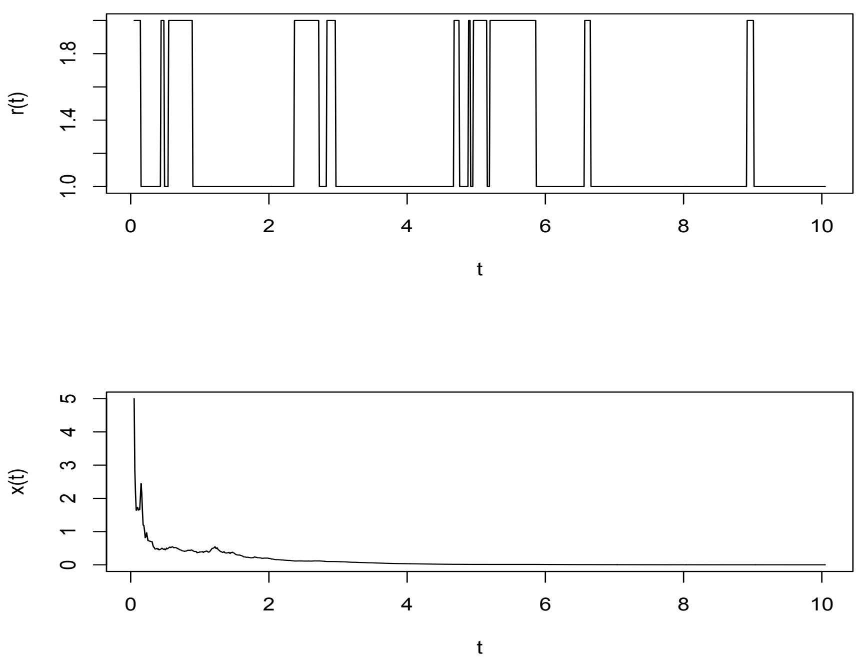


Figure 4.1: The computer simulation of the sample paths of the Markov chain and the SDDE (2.4) with control (4.2) and $\tau = 0.06$ using the Euler–Maruyama method with step size $10^{-4}$.

Assumption 2.2, we still define $\bar{U}(x,i,t) = |x|^6$ for $(x,i,t) \in R \times S \times R_+$. It is straightforward to show that, for $(x,y,i,t) \in R \times R \times S \times R_+$,

$$L\bar{U}(x,i,t) + \bar{U}_x(x,i,t)u(y,i,t) = \begin{cases} 6x^6 - 3x^8 - 30x^5y & \text{if } i = 1, \\ (16x^6 - 2.25x^8) - 10x^6 & \text{if } i = 2. \end{cases}$$

Noting that $30x^5y \le 25x^6 + 5y^6$, we then have

$$L\bar{U}(x,i,t) + \bar{U}_x(x,i,t)u(y,i,t) \le c_1 - 10x^6 + 5y^6,$$

where $c_1 = \sup_{x\in R}\big[(41x^6 - 3x^8) \vee (16x^6 - 2.25x^8)\big] < \infty$. That is, Assumption 2.2 is fulfilled with $\bar{U}_1(x,t) = 10x^6$, $c_2 = 0.5$ and $q = 6$.

To verify Assumption 3.2, we define

$$U(x,i,t) = \begin{cases} 0.25(x^2 + x^4) & \text{if } i = 1, \\ x^2 + x^4 & \text{if } i = 2 \end{cases}$$

for $(x,i,t) \in R \times S \times R_+$. It is easy to show that

$$LU(x,i,t) + U_x(x,i,t)u(x,i,t) = \begin{cases} -1.25x^2 - 4.5x^4 - 1.5x^6 & \text{if } i = 1, \\ -1.75x^2 - 1.5x^4 - 2.5x^6 & \text{if } i = 2. \end{cases}$$

Still denote the left hand side of inequality (3.6) by $LHS$. Then

$$\begin{aligned} LHS &= -1.25x^2 - 4.5x^4 - 1.5x^6 + \beta_1(0.5x + x^3)^2 + \beta_2(x - 3x)^2 + \beta_3 x^4 \\ &= -(1.25 - 0.25\beta_1 - \beta_2)x^2 - (4.5 - \beta_1 + 6\beta_2 - \beta_3)x^4 - (1.5 - \beta_1 - 9\beta_2)x^6 \end{aligned}$$

when $i = 1$, and

$$\begin{aligned} LHS &= -1.75x^2 - 1.5x^4 - 2.5x^6 + \beta_1(2x + 4x^3)^2 + \beta_2(x - x^3)^2 + 0.25\beta_3 x^4 \\ &= -(1.75 - 4\beta_1 - \beta_2)x^2 - (1.5 - 16\beta_1 + 2\beta_2 - 0.25\beta_3)x^4 - (2.5 - 16\beta_1 - \beta_2)x^6 \end{aligned}$$

when $i = 2$. Choosing $\beta_1 = \beta_2 = 0.1$ and $\beta_3 = 0.4 + 8\sqrt{0.1})$, we obtain

$$LHS = \begin{cases} -1.125x^2 - (4.6 - 8\sqrt{0.1})x^4 - 0.5x^6 & \text{if } i = 1, \\ -1.25x^2 + 2\sqrt{0.1}x^4 - 0.8x^6 & \text{if } i = 2. \end{cases}$$

As

$$-0.125x^2 + 2\sqrt{0.1}x^4 - 0.8x^6 = -0.125x^2(1 - 16\sqrt{0.1}x^2 + 6.4x^4) = -0.125x^2(1 - 8\sqrt{0.1}x^2)^2 \le 0,$$

we then always have

$$LHS \le -1.125x^2.$$

In other words, we have just verified Assumption 3.2 with $\beta_0 = 1.125$. Moreover, by definition (4.3) of the delay feedback control, we see easily that Assumption 3.3 holds with $\beta = 5$. Finally, condition (3.9) becomes $\tau \le 0.013416$. By Theorems 3.4, 3.5 and 3.6, we can therefore conclude that if we use the delay feedback control (4.3) and make sure the time delay $\tau \le 0.013416$, then the controlled hybrid SDDE (2.4) is not only $H_\infty$-stable in $L^2$ but also asymptotically stable in $L^2$ and almost surely as well.

We should point out that with the delay feedback control (4.3), the controlled system (4.1) has its form

$$dx(t) = [x(t) - 3x^3(t) - 5x(t - \tau)]dt + x^2(t)dB(t)$$

in mode $i = 1$, and

$$dx(t) = [x(t) - x^3(t)]dt + 0.5x^2(t)dB(t)$$

in mode $i = 2$. We observe that this controlled system is unstable in mode 2 while it is stable in mode 1 when the delay $\tau$ is sufficiently small. But, overall, the feedback controlled system (4.1) is stable as long as $\tau < 0.013416$.

We perform a computer simulation with the time-delay $\tau = 0.01$ for all $t \ge 0$ and the initial data $x(u) = 5 + \sin(u)$ for $u \in [-0.01, 0]$ and $r(0) = 2$. The sample paths of the Markov chain and the solution of the SDDE (4.1) are plotted in Figure 4.2. The simulation supports our theoretical results.

## 5. Conclusion.

In this paper we have discussed the stabilisation of highly nonlinear hybrid SDEs by delay feedback controls. We pointed out that the existing results on the stabilisation of nonlinear hybrid SDEs require the coefficients of the underlying SDEs satisfy the linear growth condition. On the other hand, many hybrid SDE models in the real world do not fulfill this linear growth condition (namely, they are highly nonlinear). There is hence a need to develop a new theory on the stabilisation by delay feedback controls for the highly nonlinear SDE models. In this paper we have successfully used the method of Lyapunov functionals to study this stabilisation problem by delay feedback controls. We have showed that a class of highly nonlinear unstable hybrid SDEs whose coefficients satisfy the polynomial growth condition can be stabilised by delay feedback controls. A couple of examples and computer simulations have been used to motivate our work and to illustrate our theory as well.

**Acknowledgments.** The authors wish to thank the referees for their detailed comments and helpful suggestions.

## REFERENCES


[1] A. Ahlborn and U. Parlitz, Stabilizing unstable steady states using multiple delay deedback control, *Physical Review Letters*, **93**, 264101 (2004).

[2] A. Bahar and X. Mao, Stochastic delay population dynamics, *Journal of International Applied Mathematics*, **11(4)** (2004), 377–400.

[3] J. Cao, H.X. Li and D.W.C. Ho, Synchronization criteria of Lur's systems with time-delay feedback control, *Chaos, Solitons and Fractals*, **23** (2005), 1285–1298.

[4] L. Hu, X. Mao and Y. Shen, Stability and boundedness of nonlinear hybrid stochastic differential delay equations, *Systems & Control Letters*, **62** (2013), 178–187.

[5] G.S. Ladde and V. Lakshmikantham, *Random Differential Inequalities*, Academic Press, 1980.

[6] Y. Ji and H.J. Chizeck, Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Transaction on Automatic Control*, **35** (1990), 777–788.

[7] V.B. Kolmanovskii and V.R. Nosov, *Stability of Functional Differential Equations*, Academic Press, 1986.

[8] A.L. Lewis, *Option Valuation under Stochastic Volatility: with Mathematica Code*, Finance Press, 2000.

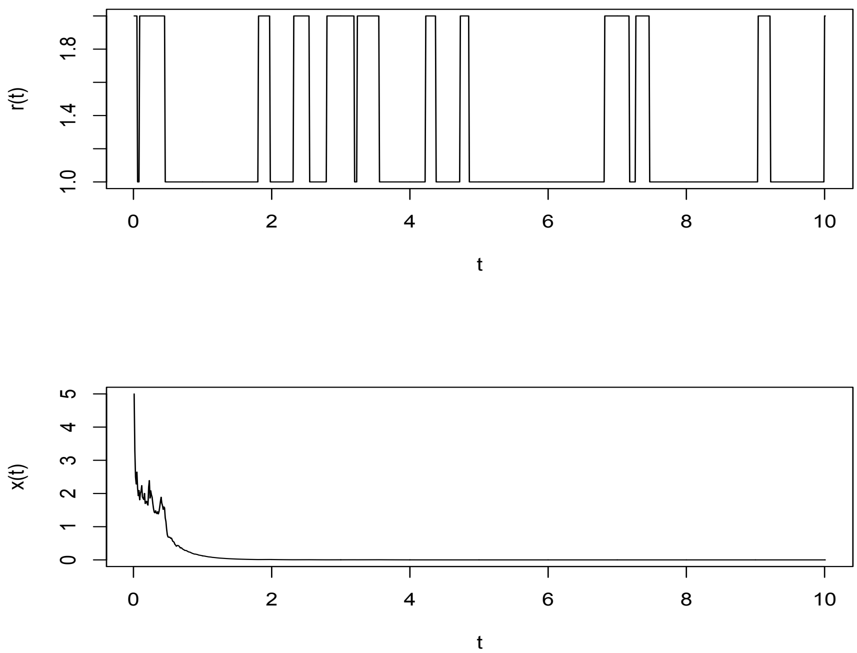


Figure 4.2: The computer simulation of the sample paths of the Markov chain and the SDDE (2.4) with control (4.3) and $\tau = 0.01$ using the Euler–Maruyama method with step size $10^{-4}$.


[9] X. Mao, *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, 1991.

[10] X. Mao, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, 1994.

[11] X. Mao, *Stochastic Differential Equations and Their Applications*, $2^{nd}$ edition, Horwood Publishing Limited, Chichester, 2007.

[12] X. Mao, Stability of stochastic differential equations with Markovian switching, *Stochastic Processes and Their Applications*,**79** (1999), 45–67.

[13] X. Mao, Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control, *Automatica*, **49** (2013), 3677–3681.

[14] X. Mao, J. Lam and L. Huang, Stabilisation of hybrid stochastic differential equations by delay feedback control, *Systems & Control Letters*, **57** (2008), 927–935.

[15] X. Mao, A. Matasov and A.B. Piunovskiy, Stochastic differential delay equations with Markovian switching, *Bernoulli*, **6** (2000), 73–90.

[16] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, 2006.

[17] M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, 1990.

[18] S.-E.A. Mohammed, *Stochastic Functional Differential Equations*, Longman Scientific and Technical, 1984.

[19] K. Pyragas, Control of chaos via extended delay feedback, *Physics Letters A*, **206** (1995), 323–330.

[20] L. Shaikhet, Stability of stochastic hereditary systems with Markov switching, *Theory of Stochastic Processes*, **2** (1996), 180–184.

[21] L. Wu, X. Su and P. Shi, Sliding mode control with bounded $L_2$ gain performance of Markovian jump singular time-delay systems, *Automatica*, **48** (2012), 1929–1933.

[22] S. You, W. Liu, J. Lu, X. Mao and Q. Qiu,, Stabilization of hybrid systems by feedback control based on discrete-time state observations, *SIAM Journal on Control and Optimization*, **53** (2015), 905–925.

[23] D. Yue and Q. Han, Delay-dependent exponential stability of stochastic systems with time-varying delay, nonlinearity, and Markovian switching, *IEEE Transaction on Automatic Control*, **50** (2005),217–222.

Received xxxx 20xx; revised xxxx 20xx.

*E-mail address*: Zhenyu Lu: luzhenyu76@163.com
*E-mail address*: Junhao Hu: junhaohu74@163.com
*E-mail address*: Xuerong Mao: x.mao@strath.ac.uk